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THE KERNEL OF THE DETERMINANT MAP ON CERTAIN SIMPLE C^*-ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a unital separable simple C^*-algebra such that either

(i) $\mathcal{A}$ has real rank zero, strict comparison and cancellation of projections; or

(ii) $\mathcal{A}$ is TAI (tracially approximate interval).

Let $\Delta_T : GL^0(\mathcal{A}) \to E_u/T(K_0(\mathcal{A}))$ be the universal determinant of de la Harpe and Skandalis.

Then for all $x \in GL^0(\mathcal{A})$, $\Delta_T(x) = 0$ if and only if x is the product of 8 multiplicative commutators in $GL^0(\mathcal{A})$. We also have results for the unitary case and other cases.

KEYWORDS: *Real rank zero, tracially approximate interval algebra.*

MSC (2010): 46L35, 46L05.

1. INTRODUCTION

Let $\mathcal{A}$ be a unital C^*-algebra and let $x \in \mathcal{A}$.

QUESTION. (1) When is x a finite sum of additive commutators? i.e., when is x a sum of finitely many elements of the form $ab - ba$ where $a, b \in \mathcal{A}$?

(2) If x is invertible (unitary) when is x a finite product of multiplicative commutators? I.e., when is x a product of finitely many elements of the form $yzy^{-1}z^{-1}$ where $y, z \in \mathcal{A}$ are invertible (respectively unitary) elements?

The first question has a long history, is connected to basic questions about the structure of C^*-algebras, and is still a subject matter of recent papers. (E.g., see [2], [4], [5], [9] [10], [11], [12], [26], [27], and the references therein.)

In this paper, we focus on the second question. The first result in this direction is due to Brown and Pearcy who proved that every unitary operator on a separable infinite dimensional Hilbert space is a multiplicative commutator of unitaries, i.e., has the form vwv^*w^* where v, w are unitary operators on the Hilbert space ([3]). This was generalized by M. Broise who proved that a von Neumann factor $\mathcal{M}$ is not of finite type I if and only if every unitary operator in $\mathcal{M}$ is a finite product of multiplicative commutators of unitaries ([1]).

In [10], Fack and de la Harpe proved that if $\mathcal{M}$ is a type II_1 factor and $x \in \mathcal{M}$ is invertible, then x has Fuglede–Kadison determinant one if and only if x is a finite product of multiplicative commutators, i.e., a finite product of elements of the form $yzy^{-1}z^{-1}$ where $y, z \in \mathcal{M}$ are invertibles. (See Proposition 2.5. of [10].)

De la Harpe, Skandalis and Thomsen generalized the above results to classes of C^*-algebras that are not necessarily von Neumann algebras. For a unital C^*-algebra $\mathcal{A}$, let $GL^0(\mathcal{A}), U^0(\mathcal{A})$ denote the connected component of the identity of the invertible group of $\mathcal{A}$ and the connected component of the identity of the unitary group of $\mathcal{A}$ respectively; let $DGL^0(\mathcal{A}), DU^0(\mathcal{A})$ denote the commutator subgroups of $GL^0(\mathcal{A})$ and $U^0(\mathcal{A})$ respectively; and let Δ_T denote the universal determinant of $\mathcal{A}$, introduced by de la Harpe and Skandalis in [14]. (More information about Δ_T and basic references can be found at the end of this introduction.) In Theorem 6.6 of [15], de la Harpe and Skandalis proved that if $\mathcal{A}$ is a unital simple infinite dimensional AF-algebra and $x \in GL^0(\mathcal{A})$, then $\Delta_T(x) = 0$ if and only if x is the product of four multiplicative commutators in $GL^0(\mathcal{A})$. They have a similar result for $U^0(\mathcal{A})$, when $\mathcal{A}$ is simple AF ([15], Proposition 6.7). Moreover, when $\mathcal{A}$ is simple AF, $DGL^0(\mathcal{A})$ and $DU^0(A)$ are both simple modulo their centres ([16]). Finally, when $\mathcal{A}$ is a unital simple properly infinite C^*-algebra, both $GL^0(\mathcal{A})$ and $U^0(\mathcal{A})$ are perfect groups ([15], Theorem 7.5 and Propositon 7.7).

In [30], Thomsen generalized de la Harpe and Skandalis' results to the class of unital C^*-algebras $\mathcal{A}$ which have the following properties:

(i) $\mathcal{A}$ is an inductive limit where the building blocks have the form $\mathbb{M}_{n_1}(C(X_1)) \oplus \mathbb{M}_{n_2}(C(X_2)) \oplus \cdots \oplus \mathbb{M}_{n_k}(C(X_k))$ such that each X_j is a compact metric space with covering dimension $\dim(X_j) \leqslant 2$ and $H^2(X_j, \mathbb{Z}) = 0$.

(ii) $K_0(\mathcal{A})$ has large denominators.

Henceforth, we will call the above class of C^*-algebras *Thomsen's class*.

In Theorem 3.4 of [30], using fundamental results in classification theory that Thomsen developed, it was proven that for a C^*-algebra $\mathcal{A}$ in Thomsen's class, for $x \in GL^0(\mathcal{A})$ (or $x \in U^0(\mathcal{A})$), $\Delta_T(x) = 0$ if and only if x is a finite product of commutators in $GL^0(\mathcal{A})$ (respectively in $U^0(\mathcal{A})$). We note that these results (unlike the result of, say, Theorem 6.6 of [15] which gives four) does not give a bound on the number of commutators. Moreover, the argument itself does not give such a bound. Finally, in Theorem 4.1 and Theorem 4.3 of [30], it was proven that for a C^*-algebra $\mathcal{A}$ in Thomsen's class, $DGL^0(\mathcal{A})$ and $DU^0(\mathcal{A})$ are both simple modulo their centres.

In this paper, we generalize the results of [15] and [30] to the class of simple TAI-algebras and the class of simple unital C^*-algebras with real rank zero, strict comparison and cancellation of projections. (The definition of *TAI-algebra* is in Definition 1.1.) These are large classes of C^*-algebras which have been important in the classification program. (E.g., the C^*-algebras in [6] and [8] belong to these classes.) These classes also include the classes in [15] and [30] (in the simple finite case). Our main result is the following: Let $\mathcal{A}$ be a unital simple separable C^*-algebra such that either (a) $\mathcal{A}$ is TAI or (b) $\mathcal{A}$ has real rank zero, strict comparison and cancellation of projections. Let $x \in GL^0(\mathcal{A})$. Then $\Delta_T(x) = 0$ if and only if x is the product of eight multiplicative commutators in $GL^0(\mathcal{A})$. (See Theorem 3.5 and Theorem 4.2.) We note that unlike the (nonetheless interesting) results in [30], there is a bound (eight) on the number of commutators. It is an open question whether we can reduce the bound. We also have results in the unitary case. (See also Theorem 2.18 and Theorem 4.1.)

The arguments in our paper extensively use techniques from classification theory, including a difficult uniqueness theorem from the literature (Theorem 2.10).

We end this section by giving some basic references and fixing some notation and definitions which we will use throughout this paper.

A basic reference for the de la Harpe–Skandalis determinant is [14]. A good summary can also be found in [13]. A basic reference for TAI-algebras is [20].

We now fix some notation and definitions. We refer the reader to the references given above for more details. For a unital C^*-algebra, $\mathcal{A}$ and for $n \in \{1, 2, \ldots\} \cup \{\infty\}$, let $U_n(\mathcal{A}), U_n^0(\mathcal{A}), GL_n(\mathcal{A}), GL_n^0(\mathcal{A})$ be the unitary group, the connected component of the identity of the unitary group, the group of invertibles, and connected component of the identity of the group of invertibles respectively of $\mathbb{M}_n(\mathcal{A})$. Oftentimes, we use $U(\mathcal{A}), U^0(\mathcal{A}), GL(\mathcal{A}), GL^0(\mathcal{A})$ to abbreviate $U_1(\mathcal{A}), U_1^0(\mathcal{A}), GL_1(\mathcal{A}), GL_1^0(\mathcal{A})$ respectively. Also, for a group G and for $x, y \in G$, we let (x, y) denote the multiplicative commutator $(x, y) =_{\mathrm{df}} xyx^{-1}y^{-1}$. We let DG denote the commutator subgroup of G, i.e., the subgroup of G generated by the multiplicative commutators (x, y) where $x, y \in G$. (E.g., $DU^0(\mathcal{A})$ is the commutator subgroup of $U^0(\mathcal{A})$.)

For a Banach space E, a tracial continuous linear function $\tau : \mathcal{A} \to E$, and for a piecewise continuously differentiable curve $\xi : [t_0, t_1] \to GL_\infty^0(\mathcal{A})$, we let $\widetilde{\Delta}_\tau(\xi) =_{\mathrm{df}} \frac{1}{2\pi \mathrm{i}} \int_{t_0}^{t_1} \tau(\xi'(t)\xi(t)^{-1})\mathrm{d}t \in E$ (Section 1 of [14]; see also Section 6 of [13]). By Lemma 1(c) of [14] (also Lemma 10(iii) of [13]), $\widetilde{\Delta}_\tau(\xi)$ depends only on the homotopy class of ξ (with endpoints fixed). This (and a form of Bott periodicity) then induces a group homomorphism $\Delta_\tau : GL_\infty^0(\mathcal{A}) \to E/\tau(K_0(\mathcal{A}))$ ([14], Proposition 2; also [13], Theorem 13).

Let E_u denote the Banach space quotient of $\mathcal{A}$ by the closed linear span of the additive commutators $[a, b] =_{\mathrm{df}} ab - ba$, $a, b \in \mathcal{A}$, i.e., $E_u =_{\mathrm{df}} \mathcal{A}/\overline{[\mathcal{A}, \mathcal{A}]}$. Let $T : \mathcal{A} \to E_u$ denote the natural quotient map. (T is called the *universal tracial continuous linear map*.) From the above, we have a group homomorphism $\Delta_T : GL_\infty^0(\mathcal{A}) \to E_u/T(K_0(\mathcal{A}))$ which is called the *universal de la Harpe–Skandalis determinant*. Throughout this paper, we will simply call Δ_T the *de la Harpe–Skandalis* determinant. (We note that this determinant has been useful in classification theory. See, for example, [21], [25].)

Next, for a unital C^*-algebra, we let $T(\mathcal{A})$ denote the simplex of tracial states on $\mathcal{A}$.

We let $\mathbb{T}$ denote the unit circle of the complex plane; i.e., $\mathbb{T} =_{\mathrm{df}} \{z \in \mathbb{C} : |z| = 1\}$.

Throughout this paper, we let $\mathbb{INT}$ denote the class of C^*-algebras of the form $\bigoplus\limits_{j=1}^{m} \mathcal{B}_j$, where for each j, $\mathcal{B}_j \cong \mathbb{M}_{n_j}$ or $\mathcal{B}_j \cong \mathbb{M}_{n_j}(C[0,1])$ for some positive integer n_j.

The following notion is due to Lin:

DEFINITION 1.1. A unital simple C^*-algebra $\mathcal{A}$ is said to be *tracially approximate interval* (TAI) if for any $\varepsilon > 0$, for any finite subset $\mathcal{F} \subset \mathcal{A}$, and for any nonzero positive element $a \in \mathcal{A}_+$, there exists a projection $p \in \mathcal{A}$ and a C^*-subalgebra $\mathcal{I} \in \mathbb{INT}$ with $1_{\mathcal{I}} = p$ such that:

(i) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{a\mathcal{A}a}$.

(ii) $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$, and

(iii) pxp is within ε of an element of $\mathcal{I}$, for all $x \in \mathcal{F}$.

(In the above definition, "AI" abbreviates "approximately interval".)

Every simple unital TAI-algebra is quasidiagonal, has real rank at most one, stable rank one, property (SP), and strict comparison (of projections by tracial states). The K_0 group of a simple unital TAI-algebra has weak unperforation and the Riesz interpolation property. Many simple C^*-algebras are TAI; in particular every simple unital AH-algebra with bounded dimension growth is TAI. (E.g., the algebras in [6] and [8] are TAI.) For these and other basic results about TAI-algebras, we refer the reader to [20].

REMARK 1.1. By Corollary 3.3 of [20], for the C^*-algebra $\mathcal{I}$ in Definition 1.1, the matrix sizes of the summands of $\mathcal{I}$ can be taken to be arbitrarily large; i.e., for every $L \geqslant 1$, we can find an $\mathcal{I}$ satisfying the conditions in Definition 1.1 such that every irreducible representation of $\mathcal{I}$ has dimension greater than L (i.e., the image of any irreducible representation of $\mathcal{I}$ has the form $\mathbb{M}_k$ with $k \geqslant L$).

In the results that follow, we will often state the result in general, but only prove it in the infinite dimensional case.

2. THE TAI CASE

LEMMA 2.1. *There exist two continuous functions $v, w : (-\pi/2, \pi/2) \to SU(2)$ such that, for all $t \in (-\pi/2, \pi/2)$,*

$$(v(t), w(t)) = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}, \quad \|v(t) - 1\|, \|w(t) - 1\| \leqslant |e^{it} - 1|^{1/2}, \quad \textit{and} \quad v(0) = w(0) = 1.$$

The proof follows from Lemma 5.13 of [15]. (Note that that $v_j(0) = 1$ for $j = 1, 2$ follows from the inequalities.)

COROLLARY 2.2. *Let $\alpha \in \mathbb{T}$.*
Then there exist unitaries $v, w \in \mathbb{M}_2(\mathbb{C})$ such that

$$(v, w) = \begin{bmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{bmatrix}.$$

If, in addition, $|\alpha - 1| < \sqrt{2}$, then we may choose v, w so that

$$\|v - 1\|,\ \|w - 1\| \leqslant |\alpha - 1|^{1/2}.$$

Proof. If $|\alpha - 1| < \sqrt{2}$ (i.e., the principal argument of α is in $(-\pi/2, \pi/2)$), then the result follows from Lemma 2.1.

For general $\alpha \in \mathbb{T}$, we note that

$$\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \overline{\alpha} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{bmatrix}. \quad \blacksquare$$

We now fix a notation. Let X be a metric space and let $S \subseteq X$ be a subset. For every $\delta > 0$, let $N(S, \delta)$ denote the *δ-neighbourhood* of S; i.e., $N(S, \delta) =_{\mathrm{df}} \{t \in X : \mathrm{dist}(t, S) < \delta\}$.

LEMMA 2.3. *Let $\theta : [0, 1] \to \mathbb{R}$ be a continuous map.*
Then there exists $v_1, w_1, v_2, w_2, v_3, w_3, v_4, w_4 \in U(\mathbb{M}_2(C[0, 1]))$ such that, for all $s \in [0, 1]$,

$$(v_1(s), w_1(s))(v_2(s), w_2(s))(v_3(s), w_3(s))(v_4(s), w_4(s)) = \begin{bmatrix} e^{i\theta(s)} & 0 \\ 0 & e^{-i\theta(s)} \end{bmatrix}.$$

Moreover, if there exists an open set $G \subseteq [0, 1]$ such that $\theta(s) = 0$ for all $s \in [0, 1] - G$, then for every $\delta > 0$, we can choose the unitaries so that $w_1 = w_3 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, and $v_k(s) = w_j(s) = 1$ for all $s \in [0, 1] - N(G, \delta)$ and for $1 \leqslant k, j \leqslant n$ with $j \neq 1, 3$.

Proof. Let $O_1, O_2, \ldots, O_n$ be an open covering of $[0, 1]$ such that for each j with $1 \leqslant j \leqslant n$, there exists an angle θ_i so that $\theta(s) \in \theta_j + [-\pi/4, \pi/4]$ for all $s \in O_j$.

Since $[0, 1]$ has covering dimension one, taking a refinement of the open cover if necessary, we may assume that each point in $[0, 1]$ is contained in at most two of the O_j. Moreover, rearranging the O_j if necessary, we may assume that for j, j' such that $|j - j'| \geqslant 2$, $O_j \cap O_{j'} = \emptyset$.

Let $\{f_j\}_{j=1}^n$ be a partition of unity for $[0, 1]$ subordinate to $\{O_j\}_{j=1}^n$.

We have that for $1 \leqslant j \leqslant n$, $\theta(s) - \theta_j \in [-\pi/4, \pi/4]$ for all $s \in O_j$. Hence, $f_j(s)(\theta(s) - \theta_j) \in [-\pi/4, \pi/4]$ for all $s \in [0, 1]$. Let $v, w : (-\pi/2, \pi/2) \to SU(2)$ be the continuous functions from Lemma 2.1. Let $\widetilde{v}_j(s) =_{\mathrm{df}} v(f_j(s)(\theta(s) - \theta_j))$ and $\widetilde{w}_j(s) =_{\mathrm{df}} w(f_j(s)(\theta(s) - \theta_j))$ for $s \in [0, 1]$. Hence, by Lemma 2.1, we have for all $s \in [0,1]$,

$$(\widetilde{v}_j(s), \widetilde{w}_j(s)) = \begin{bmatrix} e^{if_j(s)(\theta(s)-\theta_j)} & 0 \\ 0 & e^{-if_j(s)(\theta(s)-\theta_j)} \end{bmatrix}.$$

Note that for $1 \leqslant j \leqslant n$ and for all $s \in [0,1]$,

$$\begin{bmatrix} e^{if_j(s)\theta_j} & 0 \\ 0 & e^{-if_j(s)\theta_j} \end{bmatrix} = \left(\begin{bmatrix} e^{if_j(s)\theta_j} & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$

Also, note that $\prod_{j=1}^{n} e^{if_j(s)\theta_j} e^{if_j(s)(\theta(s)-\theta_j)} = e^{i\theta(s)}$ for all $s \in [0,1]$.

Thus, we can take $v_1 =_{\mathrm{df}} \prod_{j\,\mathrm{odd}} \mathrm{diag}(e^{if_j\theta_j}, 1)$, $w_1 = w_3 =_{\mathrm{df}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $v_2 =_{\mathrm{df}} \prod_{j\,\mathrm{odd}} \widetilde{v}_j$, $w_2 =_{\mathrm{df}} \prod_{j\,\mathrm{odd}} \widetilde{w}_j$, $v_3 =_{\mathrm{df}} \prod_{j\,\mathrm{even}} \mathrm{diag}(e^{if_j\theta_j}, 1)$, $v_4 =_{\mathrm{df}} \prod_{j\,\mathrm{even}} \widetilde{v}_j$, and $w_4 =_{\mathrm{df}} \prod_{j\,\mathrm{even}} \widetilde{w}_j$.

Suppose, in addition, that $G \subseteq [0,1]$ is an open subset such that $\theta(s) = 0$ for all $s \in [0,1] - G$. Let $\delta > 0$ be given. Let $g : [0,1] \to [0,\infty)$ be a continuous function with $0 \leqslant g \leqslant 1$ such that (i) $g(s) = 1$ for all $s \in G$ and (ii) $g(s) = 0$ for all $s \in [0,1] - N(G, \delta/2)$. In the definitions of $v_j, \widetilde{v}_j, w_j, \widetilde{w}_j$ $(1 \leqslant j \leqslant n)$ above, replace every occurrence of f_j with the (pointwise product) gf_j $(1 \leqslant j \leqslant n)$. Then $v_k(s) = w_j(s) = 1$, for all $s \in [0,1] - N(G,\delta)$ and for $1 \leqslant k, j \leqslant n$ with $j \neq 1, 3$. ∎

LEMMA 2.4. *Let $\phi_k : [0,1] \to \mathbb{R}$ $(1 \leqslant k \leqslant m)$ be continuous maps such that*

$$\phi_1(s) \leqslant \phi_2(s) \leqslant \phi_3(s) \leqslant \cdots \leqslant \phi_m(s) \quad \text{and} \quad \sum_{k=1}^{m} \phi_k(s) = 0$$

for all $s \in [0,1]$.

Then we have the following:

(i) *There exist $v_j, w_j \in U^0(\mathbb{M}_m(C[0,1]))$ $(1 \leqslant j \leqslant 16)$ such that*

$$\mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_m}) = \prod_{j=1}^{16} (v_j, w_j).$$

(Here, $\prod_{j=1}^{16} (v_j, w_j) = (v_1, w_1)(v_2, w_2) \cdots (v_{16}, w_{16})$.)

(ii) *Suppose, in addition, that* $\mathrm{ran}(\phi_k) \subset (\pi/2, \pi/2)$ *for $1 \leqslant k \leqslant m$. Then there exist $v_1, w_1, v_2, w_2, v_3, w_3, v_4, w_4 \in U^0(\mathbb{M}_m(C[0,1]))$ such that*

$$\mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_m}) = (v_1, w_1)(v_2, w_2)(v_3, w_3)(v_4, w_4) \quad \text{and}$$
$$\|v_j - 1_{\mathcal{A}}\|, \ \|w_j - 1_{\mathcal{A}}\| \leqslant \sqrt{2} \|u - 1_{\mathcal{A}}\|^{1/2}$$

for $1 \leqslant j \leqslant 4$, where $\mathcal{A} =_{\mathrm{df}} \mathbb{M}_m(C[0,1])$ and $u =_{\mathrm{df}}$ $\mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_m})$.

Proof. The proof is a modification of the arguments of Lemma 2.7 and Lemma 2.8 in [30], where we additionally use Lemma 2.1 and Lemma 2.3. (Indeed, the proof of part (ii) is contained in the proof of Lemma 2.8. in [30].) We provide the argument for the convenience of the reader.

If $\mathrm{diag}(\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{\mathrm{i}\phi_m}) = 1_{\mathcal{A}}$ then we are done. Hence, let us assume that $\mathrm{diag}(\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{\mathrm{i}\phi_m}) \neq 1_{\mathcal{A}}$.

Choose $\delta > 0$ small enough so that $\delta < \pi/4$, $|\mathrm{e}^{\mathrm{i}\delta} - 1| < \|u - 1\|$ and $|\mathrm{e}^{-\mathrm{i}\delta} - 1| < \|u - 1\|$. Moreover, if $\mathrm{ran}(\phi_j) \subset (-\pi/2, \pi/2)$ we require $\mathrm{ran}(\phi_j) \pm \delta \subset (-\pi/2, \pi/2)$.

Firstly, note that for each $s \in [0,1]$, since $\sum\limits_{k=1}^{m} \phi_k(s) = 0$, there is a permutation σ of $\{1, 2, \ldots, m\}$ (σ is dependent on s) such that, for $1 \leqslant l \leqslant m$,

$$\phi_1(s) \leqslant \sum_{k=1}^{l} \phi_{\sigma(k)}(s) \leqslant \phi_m(s).$$

Since $[0,1]$ is compact, let $\{O_j\}_{j=1}^{n}$ be an open covering of $[0,1]$ and for $1 \leqslant j \leqslant n$, let $x_j \in U(\mathbb{M}_m)$ be a permutation unitary and let σ_j be a permutation of $\{1, 2, \ldots, m\}$ such that, for $1 \leqslant l \leqslant m$ and for all $s \in O_j$,

$$x_j \mathrm{diag}(\phi_1, \phi_2, \ldots, \phi_m) x_j^* = \mathrm{diag}(\phi_{\sigma_j(1)}, \phi_{\sigma_j(2)}, \ldots, \phi_{\sigma_j(m)}) \quad \text{and}$$

$$\phi_1(s) - \delta \leqslant \sum_{k=1}^{l} \phi_{\sigma_j(k)}(s) \leqslant \phi_m(s) + \delta.$$

Let $\gamma > 0$ be given. Since $[0,1]$ has covering dimension one, taking refinements, permuting and contracting the O_js and contracting $\gamma > 0$ if necessary, we may assume that if $|j - j'| \geqslant 2$ then $N(O_j, \gamma) \cap N(O_{j'}, \gamma) = \varnothing$.

Let $\{f_j\}_{j=1}^{n}$ be a partition of unity of $[0,1]$ subordinate to $\{O_j\}_{j=1}^{n}$. For $1 \leqslant j \leqslant n$, let $a_j \in \mathbb{M}_m(C[0,1])$ be the self-adjoint element given by

$$a_j =_{\mathrm{df}} f_j \mathrm{diag}(\phi_1, \phi_2, \ldots, \phi_m)$$

and for $1 \leqslant k \leqslant m$, let

$$\psi_{j,k} =_{\mathrm{df}} f_j \sum_{l=1}^{k} \phi_{\sigma_j(l)}.$$

Hence,

$$\mathrm{diag}(\phi_1, \phi_2, \ldots, \phi_m) = \sum_{j=1}^{n} a_j$$

and for $1 \leqslant j \leqslant n$,

$$x_j a_j x_j^* = \mathrm{diag}(\psi_{j,1}, -\psi_{j,1}, \psi_{j,3}, -\psi_{j,3}, \ldots) + \mathrm{diag}(0, \psi_{j,2}, -\psi_{j,2}, \psi_{j,4}, -\psi_{j,4}, \ldots) \tag{2.1}$$

where the first diagonal ends with zero if m is odd, and the second diagonal ends with zero if m is even.

We consider the two cases in the statement of the lemma.

Case 1 *or part* (ii). Suppose that $\mathrm{ran}(\phi_k) \subset (-\pi/2, \pi/2)$ for $1 \leqslant k \leqslant m$.

By (2.1) and by Lemma 2.1, we have that for $1 \leqslant j \leqslant n$ and $1 \leqslant l \leqslant 2$, there exist unitaries $v_{j,l}, w_{j,l} \in \mathbb{M}_m(C[0,1])$ such that

$$x_j^* \mathrm{diag}(\mathrm{e}^{\mathrm{i}\psi_{j,1}}, \mathrm{e}^{-\mathrm{i}\psi_{j,1}}, \mathrm{e}^{\mathrm{i}\psi_{j,3}}, \mathrm{e}^{-\mathrm{i}\psi_{j,3}}, \ldots)x_j = (v_{j,1}, w_{j,1}),$$
$$x_j^* \mathrm{diag}(1, \mathrm{e}^{\mathrm{i}\psi_{j,2}}, \mathrm{e}^{-\mathrm{i}\psi_{j,2}}, \mathrm{e}^{\mathrm{i}\psi_{j,4}}, \mathrm{e}^{-\mathrm{i}\psi_{j,4}}, \ldots)x_j = (v_{j,2}, w_{j,2}),$$

and for $l = 1, 2$,

$$v_{j,l}(s) = w_{j,l}(s) = 1,$$

for all $s \in [0,1] - O_j$. Moreover, by Lemma 2.1 and by our choice of δ, for $1 \leqslant j \leqslant n$ and $l = 1, 2$,

$$\begin{aligned}
&\|v_{j,l} - 1\| \\
&\leqslant \max\{|\mathrm{e}^{\mathrm{i}(\phi_1(s)-\delta)} - 1|^{1/2}, |\mathrm{e}^{\mathrm{i}(\phi_m(s)+\delta)} - 1|^{1/2} : s \in O_j\} \\
&\leqslant \max\{|\mathrm{e}^{\mathrm{i}\phi_1(s)}(\mathrm{e}^{-\mathrm{i}\delta}-1)|+|\mathrm{e}^{\mathrm{i}\phi_1(s)}-1|, |\mathrm{e}^{\mathrm{i}\phi_m(s)}(\mathrm{e}^{\mathrm{i}\delta}-1)|+|\mathrm{e}^{\mathrm{i}\phi_m(s)}-1| : s \in O_j\}^{1/2} \\
&\leqslant \sqrt{2}\|u - 1\|^{1/2}.
\end{aligned}$$

Similarly,

$$\|w_{j,l} - 1\| \leqslant \sqrt{2}\|u - 1\|^{1/2}.$$

Now let $v_1 =_{\mathrm{df}} \prod_{j\,\mathrm{odd}} v_{j,1}$, $w_1 =_{\mathrm{df}} \prod_{j\,\mathrm{odd}} w_{j,1}$, $v_2 =_{\mathrm{df}} \prod_{j\,\mathrm{even}} v_{j,1}$, $w_2 =_{\mathrm{df}} \prod_{j\,\mathrm{even}} w_{j,1}$, $v_3 =_{\mathrm{df}} \prod_{j\,\mathrm{odd}} v_{j,2}$, $w_3 =_{\mathrm{df}} \prod_{j\,\mathrm{odd}} w_{j,2}$, $v_4 =_{\mathrm{df}} \prod_{j\,\mathrm{even}} v_{j,2}$, $w_4 =_{\mathrm{df}} \prod_{j\,\mathrm{even}} w_{j,2}$.

Then for $1 \leqslant l \leqslant 4$, $\|v_l - 1\| \leqslant \sqrt{2}\|u-1\|^{1/2}$ and $\|w_l - 1\| \leqslant \sqrt{2}\|u-1\|^{1/2}$. Also, as required,

$$u = \prod_{l=1}^{4}(v_l, w_l).$$

Case 2 *or part* (i). General case.

The proof for this case is the same as that of Case 1, except that we replace Lemma 2.1 with Lemma 2.3 and we get sixteen commutators (instead of four). (We also do not get a norm estimate for the unitaries that make up the commutators.) ∎

LEMMA 2.5. *Let $\phi_k : [0,1] \to \mathbb{R}$ ($1 \leqslant k \leqslant m$) be continuous maps such that, for all $s \in [0,1]$,*

$$\phi_1(s) \leqslant \phi_2(s) \leqslant \phi_3(s) \leqslant \cdots \leqslant \phi_m(s) \quad \textit{and} \quad \sum_{k=1}^{m} \phi_k(s) = 0.$$

Then there exist $x_j, y_j \in GL^0(\mathbb{M}_m(C[0,1]))$ ($1 \leqslant j \leqslant 4$) such that

$$\mathrm{diag}(\mathrm{e}^{\phi_1}, \mathrm{e}^{\phi_2}, \ldots, \mathrm{e}^{\phi_m}) = \prod_{j=1}^{4}(x_j, y_j) \quad \textit{and} \quad \|x_j - 1_{\mathcal{A}}\|,\ \|y_j - 1_{\mathcal{A}}\| \leqslant 2\|z - 1_{\mathcal{A}}\|^{1/2}$$

for $1 \leqslant j \leqslant 4$, *where* $\mathcal{A} =_{\mathrm{df}} \mathbb{M}_m(C[0,1])$ *and* $z =_{\mathrm{df}} \mathrm{diag}(\mathrm{e}^{\phi_1}, \mathrm{e}^{\phi_2}, \dots, \mathrm{e}^{\phi_m})$.

The proof is essentially the same as Lemma 2.7 of [30]. Alternatively, the proof is the same as Lemma 2.4 part (ii), but with Lemma 2.1 replaced with Lemma 2.6 of [30].

Recall the definitions of "TAI" and "$\mathbb{INT}$" from the end of the Introduction.

LEMMA 2.6. *Let* $\mathcal{A}$ *be a unital separable simple* TAI-*algebra and let* $\{c_n\}_{n=1}^{\infty}$ *be a countable dense subset of the closed unit ball of* $\mathcal{A}$.

Let $\{\mathcal{I}_n\}_{n=1}^{\infty}$ *be a sequence of* C^*-*subalgebras of* $\mathcal{A}$, *with* $\mathcal{I}_n \in \mathbb{INT}$ *for all n and let* $\{p_n\}_{n=1}^{\infty}$ *be a sequence of projections in* $\mathcal{A}$ *with* $1_{\mathcal{I}_n} = p_n$ *for all* $n \geqslant 1$ *such that for all* $n \geqslant 1$, *the following hold:*

(i) $\tau(1 - p_n) < 1/n$ *for all* $\tau \in T(\mathcal{A})$,

(ii) $\|p_n c_k - c_k p_n\| < 1/n$ *for all* $k \leqslant n$, *and*

(iii) $p_n c_k p_n$ *is within* $1/n$ *of an element of* $\mathcal{I}_n$ *for all* $k \leqslant n$.

Suppose that $a \in \mathcal{A}$ *is an element such that* $|\tau(a)| < \varepsilon$ *for all* $\tau \in \mathcal{A}$. *For all* $n \geqslant 1$, *let* $a_n \in \mathcal{I}_n$ *such that* $\|p_m a p_m - a_m\| \to 0$ *as* $m \to \infty$.

Then there exists $N \geqslant 1$ *such that for all* $n \geqslant N$, *for all* $\tau \in T(\mathcal{I}_n)$, $|\tau(a_n)| < \varepsilon$.

Proof. Firstly, since the map $T(\mathcal{A}) \to \mathbb{C} : \tau \mapsto \tau(a)$ is a continuous function on the compact set $T(\mathcal{A})$, let $0 \leqslant \delta < \varepsilon$ be such that $\delta = \max\{|\tau(a)| : \tau \in T(\mathcal{A})\}$; i.e., $|\tau(a)| \leqslant \delta < \varepsilon$ for all $\tau \in T(\mathcal{A})$.

Suppose, to the contrary, that $\{n_l\}_{l=1}^{\infty}$ is a subsequence of the positive integers and for all $l \geqslant 1$, $\tau_l \in T(\mathcal{I}_{n_l})$ is such that $|\tau_l(a_{n_l})| \geqslant \varepsilon$.

Let $\prod\limits_{l=1}^{\infty} \mathcal{I}_{n_l}$ and $\sum\limits_{l=1}^{\oplus\,\infty} \mathcal{I}_{n_l}$ be the (l_∞) direct product and (c_0) direct sum respectively. For each $k \geqslant 1$, τ_k induces an element $\widetilde{\tau}_k \in T(\prod_{l=1}^{\infty} \mathcal{I}_{n_l})$ in the following manner: for $\{b_l\}_{l=1}^{\infty} \in \prod\limits_{l=1}^{\infty} \mathcal{I}_{n_l}$, $\widetilde{\tau}_k(\{b_l\}_{l=1}^{\infty}) =_{\mathrm{df}} \tau_k(b_k)$.

Since $T(\prod_{l=1}^{\infty} \mathcal{I}_{n_l})$ is compact, $\{\widetilde{\tau}_l\}_{l=1}^{\infty}$ must have a converging subnet $\{\widetilde{\tau}_{l_\alpha}\}$. Suppose that $\lim\limits_{\alpha} \widetilde{\tau}_{l_\alpha} = \mu \in T(\prod_{l=1}^{\infty} \mathcal{I}_{n_l})$. Note that $\sum\limits_{l=1}^{\infty}{}^{\oplus} \mathcal{I}_{n_l}$ is contained in the kernel of μ. Hence, μ naturally induces a trace in $T(\prod\limits_{l=1}^{\infty} \mathcal{I}_{n_l} / \sum\limits_{l=1}^{\infty}{}^{\oplus} \mathcal{I}_{n_l})$, which we also denote by "μ".

Let $\Phi : \mathcal{A} \to \prod\limits_{l=1}^{\infty} \mathcal{I}_{n_l} / \sum\limits_{l=1}^{\infty}{}^{\oplus} \mathcal{I}_{n_l}$ be the unital $*$-embedding that is defined as follows:

Let $d \in \mathcal{A}$ be given. Then

$$\Phi(d) =_{\mathrm{df}} [\{d_l\}_{l=1}^{\infty}]$$

where $d_l \in \mathcal{I}_{n_l}$ for all $l \geqslant 1$, $\|p_{n_l} d p_{n_l} - d_l\| \to 0$ as $l \to \infty$, and $[\{d_l\}_{l=1}^{\infty}]$ is the equivalence class of $\{d_l\}_{l=1}^{\infty}$ in $\prod\limits_{l=1}^{\infty} \mathcal{I}_{n_l} / \sum\limits_{l=1}^{\infty}{}^{\oplus} \mathcal{I}_{n_l}$. (It is clear that Φ is a well-defined unital $*$-homomorphism; in particular, $\Phi(d)$ is independent of the choice of the sequence $\{d_l\}$ with the above properties.)

Then $\mu \circ \Phi \in T(\mathcal{A})$. Then $|\mu \circ \Phi(a)| = |\mu([\{a_l\}_{l=1}^{\infty}])| = \lim_{\alpha} |\widetilde{\tau}_{l_\alpha}(\{a_l\}_{l=1}^{\infty})| = \lim_{\alpha} |\tau_{l_\alpha}(a_{l_\alpha})| \geqslant \varepsilon$. This contradicts our assumption that $|\tau(a)| \leqslant \delta < \varepsilon$ for all $\tau \in T(\mathcal{A})$. ∎

The next lemma is a straightforward computation.

LEMMA 2.7. *Let $n \in \mathbb{Z}_+ \cup \{\infty\}$. Let $\mathcal{A}$ be a unital C^*-algebra, let V be a Banach space and let $\tau : \mathcal{A} \to V$ be a tracial continuous linear map. Let $\xi : [t_0, t_1] \to GL_n^0(\mathcal{A})$ (or $U_n^0(\mathcal{A})$) be a piecewise continuously differentiable curve with $\xi(t_0) = 1$.*

For every $\varepsilon > 0$, there exists $\delta > 0$ such that the following hold:

If $x \in GL_n^0(\mathcal{A})$ (respectively $U_n^0(\mathcal{A})$) is such that $\|x - \xi(t_1)\| < \delta$, then there exists a piecewise continuously differentiable curve $\eta : [t_0, t_1] \to GL_n^0(\mathcal{A})$ (respectively $U_n^0(\mathcal{A})$) with $\eta(t_0) = 1$ and $\eta(t_1) = x$ such that

$$\|\widetilde{\Delta}_\tau(\xi) - \widetilde{\Delta}_\tau(\eta)\| < \varepsilon.$$

For a unital C^*-algebra $\mathcal{A}$, recall that E_u is the Banach space quotient $E_u =_{\mathrm{df}} \mathcal{A}/\overline{[\mathcal{A},\mathcal{A}]}$. Viewing E_u as a metric group (with metric induced by the norm), $T(K_0(\mathcal{A})) \subseteq E_u$ is a (not necessarily closed) topological subgroup of E_u. The metric on E_u induces a pseudometric d on the quotient group $E_u/T(K_0(\mathcal{A}))$; i.e., for all $a, b \in E_u$,

$$d([a],[b]) =_{\mathrm{df}} \inf\{\|a - b + c\| : c \in T(K_0(\mathcal{A}))\},$$

where $\|\cdot\|$ is the norm on E_u and $[a], [b]$ are the equivalence classes of a, b (respectively) in $E_u/T(K_0(\mathcal{A}))$.

LEMMA 2.8. *Let $\mathcal{A}$ be a unital C^*-algebra and let d be the pseudometric on $E_u/T(K_0(\mathcal{A}))$ induced by the metric (or norm) on E_u. Let $y \in GL_\infty^0(\mathcal{A})$ (respectively $U_\infty^0(\mathcal{A})$) be such that $\Delta_T(y) = 0$.*

Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in GL_\infty^0(\mathcal{A})$ (respectively $U_\infty^0(\mathcal{A})$) is such that $\|x - y\| < \delta$ then

$$d(\Delta_T(y), 0) < \varepsilon.$$

Proof. Since $\Delta_T(y) = 0$, there exists a piecewise continuously differentiable curve $\xi : [0,1] \to GL_\infty^0(\mathcal{A})$ ($U_\infty^0(\mathcal{A})$ respectively) such that $\xi(0) = 1$, $\xi(1) = y$ and $\widetilde{\Delta}_T(\xi) = 0$. Now apply Lemma 2.7. ∎

Next, we consider some results about the closure of the commutator subgroup. For a topological group G, recall that DG is the commutator subgroup of G and $\overline{DG}$ is its closure. For a unital C^*-algebra $\mathcal{A}$, $\overline{DU(\mathcal{A})}$ and $\overline{DU^0(\mathcal{A})}$ will be the closures in the *norm* topology.

LEMMA 2.9. *Let $\mathcal{A}$ be a unital separable simple* TAI*-algebra. For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every self-adjoint element $a \in \mathcal{A}$ with $|\tau(a)| < \delta$ for all*

$\tau \in T(\mathcal{A})$,

$$\mathrm{dist}(e^{i2\pi a}, \overline{DU^0(\mathcal{A})}) =_{\mathrm{df}} \inf\{\|e^{i2\pi a} - u\| : u \in \overline{DU^0(\mathcal{A})}\} < \varepsilon.$$

Proof. This follows from [31] which gives a topological group isomorphism:

$$\Phi : U^0(\mathcal{A})/\overline{DU^0(\mathcal{A})} \to \mathrm{Aff}(T(\mathcal{A}))/\overline{T(K_0(\mathcal{A}))}.$$

The map Φ is the map induced by the de la Harpe–Skandalis determinant (with universal trace). Note that for a self-adjoint $a \in \mathcal{A}$, $\Phi([e^{i2\pi a}]) = a + \overline{T(K_0(\mathcal{A}))}$. ∎

We will need a uniqueness result of Lin's. Towards this, we fix some notation. For a unital C^*-algebra $\mathcal{A}$ and for a unitary $u \in U(\mathcal{A})$, let $\overline{u}$ denote the image of u in $U(\mathcal{A})/\overline{DU(\mathcal{A})}$. For $\overline{u}, \overline{v} \in U(\mathcal{A})/\overline{DU(\mathcal{A})}$, let

$$\mathrm{dist}(\overline{u}, \overline{v}) =_{\mathrm{df}} \inf\{\|x - y\| : x, y \in U(\mathcal{A}) \text{ and } \overline{x} = \overline{u}, \overline{y} = \overline{v}\}.$$

It follows that

$$\mathrm{dist}(\overline{u}, \overline{v}) = \inf\{\|uv^* - x\| : x \in \overline{DU(\mathcal{A})}\} = \mathrm{dist}(uv^*, \overline{DU(\mathcal{A})}).$$

If $\mathcal{A}, \mathcal{B}$ are unital C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is a unital $*$-homomorphism, then ϕ brings $U(\mathcal{A})$ to $U(\mathcal{B})$, and brings $\overline{DU(\mathcal{A})}$ to $\overline{DU(\mathcal{B})}$. Hence, ϕ induces a topological group homomorphism $\phi^{\ddagger} : U(\mathcal{A})/\overline{DU(\mathcal{A})} \to U(\mathcal{B})/\overline{DU(\mathcal{B})}$. Also, ϕ induces a map $[\phi] : \underline{K}(\mathcal{A}) \to \underline{K}(\mathcal{B})$. (Here, $\underline{K}$ is total K-theory. See, for example, Definition 5.8.13. of [18].) Finally, if X is a compact metric space and $\tau \in T(C(X))$ (tracial state) then, by the Riesz representation theorem, τ induces a Borel probability measure μ_τ on X.

The following is a result of Lin in [23]. (Also, a generalized version, with the space X being an arbitrary compact metric space, can be found in [24].)

THEOREM 2.10. *Let X be a compact metric space such that either X is a finite* CW*-complex with dimension no more than one or $X = [0,1]^n$ (n-cube) or $X = \mathbb{T}^n$ (n-torus). Let $\varepsilon > 0$, let $\mathcal{F} \subset C(X)$ be a finite subset and let $F : (0,1) \to (0,1)$ be a nondecreasing map. Then there exist $\eta > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ and a finite subset $\mathcal{U} \subset U(\mathbb{M}_\infty(C(X)))$ satisfying the following:*

Suppose that $\mathcal{A}$ is a unital separable simple TAI*-algebra and $\phi, \psi : C(X) \to \mathcal{A}$ are two unital $*$-homomorphisms such that:*

$$\mu_{\tau\circ\phi}(O_s) \geqslant F(s)$$

for all $s \geqslant \eta$, for all open balls O_s in X with radius s and all $\tau \in T(\mathcal{A})$;

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta$$

for all $g \in \mathcal{G}$ and all $\tau \in T(\mathcal{A})$; and

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \quad \textit{and} \quad \mathrm{dist}(\phi^{\ddagger}(\overline{z}), \psi^{\ddagger}(\overline{z})) < \delta$$

for all $z \in \mathcal{U}$.

Then there exists a unitary $u \in \mathcal{A}$ such that, for all $f \in \mathcal{F}$,

$$\|\phi(f) - u\psi(f)u^*\| < \varepsilon.$$

The proof follows from Theorem 10.8 of [23].

LEMMA 2.11. *Let $\mathcal{A}$ be a unital C^*-algebra. Let $x \in GL^0(\mathcal{A})$ have polar decomposition $x = u|x|$. (So u is a unitary and $|x|$ is a positive invertible.)*
Suppose that $\Delta_T(x) = 0$.
Then $\Delta_T(u) = \Delta_T(|x|) = 0$. Moreover, $\tau(\mathrm{Log}(|x|)) = 0$ for all $\tau \in T(\mathcal{A})$.

The proof follows from the (short) argument of Proposition 2 d) in[14].

Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism. Then for every $n \geqslant 1$, the map $\mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_n(\mathcal{B}) : [a_{i,j}] \mapsto [\phi(a_{i,j})]$ is a $*$-homomorphism, which will also denote by "ϕ".

LEMMA 2.12. *Let $\mathcal{A}$ be a unital separable simple* TAI*-algebra.*
(i) *If $u \in U^0(\mathcal{A})$ is a unitary such that $\Delta_T(u) = 0$, then for every $\varepsilon > 0$, there exist unitaries $x_j, y_j \in U^0(\mathcal{A})$, $1 \leqslant j \leqslant 18$, such that*

$$\Big\| u - \prod_{j=1}^{18} (x_j, y_j) \Big\| < \varepsilon.$$

(Here, $\prod_{j=1}^{18} (x_j, y_j) = (x_1, y_1)(x_2, y_2) \cdots (x_{18}, y_{18})$.)
(ii) *If $x \in GL^0(\mathcal{A})$ is an invertible such that $\Delta_T(x) = 0$, then for every $\varepsilon > 0$, there exist invertibles $x_j, y_j \in GL^0(\mathcal{A})$, $1 \leqslant j \leqslant 24$, such that*

$$\Big\| x - \prod_{j=1}^{24} (x_j, y_j) \Big\| < \varepsilon.$$

Proof. We prove part (i). The proof of part (ii) is similar.

Let $X \subseteq \mathbb{T}$ be the compact subset given by

$$X =_{\mathrm{df}} \{t \in \mathbb{T} : |t - 1| \leqslant 2\|u - 1\|\}.$$

(Note that $1 \in X$, and X is either $\mathbb{T}$ or homeomorphic to $[0, 1]$.)

Let $\varepsilon > 0$ be given. Contracting ε if necessary, we may assume that $0 < \varepsilon < 1/10$ and that $\varepsilon > 0$ is small enough so that for every unitary $v \in \mathcal{A}$, if $\|u - v\| < \varepsilon$ then $\mathrm{sp}(v) \subseteq X$.

Let $\delta_1 > 0$ be such that for all self-adjoint elements $c, c' \in \mathcal{A}$ if $\|c - c'\| < \delta_1$ then $\|\mathrm{e}^{\mathrm{i}2\pi c} - \mathrm{e}^{\mathrm{i}2\pi c'}\| < \varepsilon/10$. We may assume that $\delta_1 < \varepsilon/10$. Plug $\delta_1/10$ (for ε) into Lemma 2.8 to get $\delta_2 > 0$. We may assume that $\delta_2 < \varepsilon/10$.

By Theorem 3.3 of [22], there exists a self-adjoint element $a \in \mathcal{A}$ such that

$$\|u - \mathrm{e}^{\mathrm{i}2\pi a}\| < \delta_2. \tag{2.2}$$

By our choice of δ_2, we must have that

$$d(\Delta_T(\mathrm{e}^{\mathrm{i}2\pi a}), 0) < \delta_1/10$$

where d is the pseudometric on $\mathrm{Aff}(T(\mathcal{A}))/K_0(\mathcal{A})$ induced by the (uniform) metric on $\mathrm{Aff}(T(\mathcal{A}))$.

Since $\Delta_T(\mathrm{e}^{\mathrm{i}2\pi a}) = [a]$ (where $[a]$ is the equivalence class of a in $\mathrm{Aff}(T(\mathcal{A}))/K_0(\mathcal{A})$), there exist projections $q, r \in \mathbb{M}_\infty(\mathcal{A})$ such that, for all $\tau \in T(\mathcal{A})$,

$$|\tau(a) - \tau(q) + \tau(r)| < \frac{\delta_1}{10}. \tag{2.3}$$

Let $F : (0,1) \to (0,1)$ be the nondecreasing map given by $F(t) =_{\mathrm{df}} t/10$ for all $t \in (0,1)$. Let $\mathcal{F} \subset C(X)$ be a finite subset that contains the identity function $h(t) =_{\mathrm{df}} t$ ($t \in X$).

Plug X, $\varepsilon/10$ (for ε), $\mathcal{F}$ and F into Theorem 2.10 to get $\eta_1 > 0$, $\delta_3 > 0$, a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ and a finite subset $\mathcal{U} \subset U(\mathbb{M}_\infty(C(X))$ satisfying the conclusions of Theorem 2.10.

Note that X is closed under complex conjugates. Hence, let $S =_{\mathrm{df}} \{1, t_1, \bar{t}_1, t_2, \bar{t}_2, \ldots, t_N, \bar{t}_N\} \subset X$ be a finite collection of $2N+1$ distinct points and $\eta_2 > 0$ such that for all $s \geqslant \eta_1$ for all open balls O_s in X with radius s,

$$\mathrm{card}(O_s \cap S)\frac{1-\eta_2}{2N+1} > F(s), \tag{2.4}$$

where $\mathrm{card}(O_s \cap S)$ is the cardinality of $O_s \cap S$.

Let $\Phi : C(X) \to \mathcal{A}$ be the unital $*$-homomorphism given by $\Phi(h) =_{\mathrm{df}} \mathrm{e}^{\mathrm{i}2\pi a}$, where $h \in C(X)$ is the identity map (i.e., $h(t) = t$ for all $t \in X$). Note that by our assumption on ε and by (2.2), the spectrum of $\mathrm{e}^{\mathrm{i}2\pi a}$ is contained in X; so Φ is well-defined. Let $N_1 \geqslant 1$ be an integer so that $\mathcal{U} \subset \mathbb{M}_{N_1}(C(X))$. Let $M_1 \geqslant 1$ be an integer and let $\mathcal{F}_1 \subset \mathbb{M}_{N_1}(\mathcal{A})$ be a finite set of self-adjoint elements so that for all $v \in \mathcal{U}$, there exist self–adjoint elements $a_{v,1}, a_{v,2}, \ldots, a_{v,M_1} \in \mathcal{F}_1$ (repetitions allowed) so that $\Phi(v) = \mathrm{e}^{\mathrm{i}2\pi a_{v,1}}\mathrm{e}^{\mathrm{i}2\pi a_{v,2}} \cdots \mathrm{e}^{\mathrm{i}2\pi a_{v,M_1}}$. (Note that $K_1(\Phi) = 0$.)

Choose $\delta_4 > 0$ such that if $u_1, u_2, \ldots, u_{M_1} \in U(\mathbb{M}_{N_1}(\mathcal{A}))$ are unitaries such that $\mathrm{dist}(u_j, \overline{DU(\mathbb{M}_{N_1}(\mathcal{A}))}) < \delta_4$ for $1 \leqslant j \leqslant M_1$ then $\mathrm{dist}(u_1 u_2 \cdots u_{M_1}, \overline{DU(\mathbb{M}_{N_1}(\mathcal{A}))}) < \delta_3/10$. We may assume that $\delta_4 < \delta_3/10$.

Plug $\delta_4/10$ (for ε) and $\mathbb{M}_{N_1}(\mathcal{A})$ (for $\mathcal{A}$) into Lemma 2.9 to get $\delta_5 > 0$.

Choose integer $N_2 \geqslant 1$ such that $1/N_2 < \eta_2/10$. Also choose $N_3 \geqslant 1$ such that $N_3 \geqslant \max\{\|b\| : b \in \mathcal{G} \cup \mathcal{F}_1\}$.

Since $\mathcal{A}$ is TAI and by Lemma 2.6 and Remark 1.1, let $p \in \mathcal{A}$ be a projection and let $\mathcal{I} \in \mathbb{INT}$ be a C^*-subalgebra of $\mathcal{A}$ with $1_{\mathcal{I}} = p$ such that the following hold:

(a) $\tau(1_{\mathcal{A}} - p) < \min\{\delta_3/(10(1+N_3)), \delta_5/(10(1+N_3)), \eta_2/(10(1+N_3))\}$ for all $\tau \in T(\mathcal{A})$.

(b) Each summand in $\mathcal{I}$ has matrix size at least $N_2(2N+1)$. (Equivalently, every irreducible represention of $\mathcal{I}$ has image with the form M_n with $n \geqslant N_2(2N+1)$.)

(c) There exists $a_1 \in \mathcal{I}$ such that $\|a - ((1-p)a(1-p) + a_1)\| < \delta_1/10$ and $\|\mathrm{e}^{\mathrm{i}2\pi a} - \mathrm{e}^{\mathrm{i}2\pi((1-p)a(1-p)+a_1)}\| < \varepsilon/10$.

(d) There exist projections $q', r' \in \mathbb{M}_\infty(\mathcal{I})$ such that $|\tau(a_1) - \tau(q') + \tau(r')| < \delta_1/10$ for all $\tau \in T(\mathcal{I})$.

(e) Let $\psi_0 : C(X) \to (1-p)\mathcal{A}(1-p)$ be the unital $*$-homomorphism given by $\psi_0(h) =_{\mathrm{df}} (1-p)\mathrm{e}^{\mathrm{i}2\pi(1-p)a(1-p)}(1-p)$, where $h \in C(X)$ is the identity map (i.e., $h(t) = t$ for all $t \in X$). (Note that by (2.2), by (c) and our assumptions on ε, p can be chosen so that the spectrum of $\mathrm{e}^{\mathrm{i}2\pi(1-p)a(1-p)}$ is contained in X; so the map ψ_0 is well-defined.)

Then for all $v \in \mathcal{U}$,

$$\|\psi_0(v) - (1-p)\mathrm{e}^{\mathrm{i}2\pi(1-p)a_{v,1}(1-p)}\mathrm{e}^{\mathrm{i}2\pi(1-p)a_{v,2}(1-p)} \cdots \mathrm{e}^{\mathrm{i}2\pi(1-p)a_{v,M_1}(1-p)}(1-p)\| < \frac{\delta_3}{10}.$$

(Here, we identify $1_{\mathcal{A}} - p$ with $(1_{\mathcal{A}} - p) \otimes 1_{\mathbb{M}_{N_1}} \in \mathbb{M}_{N_1}(\mathcal{A})$.)

We denote the above statements by "$(*)$".

Since $\mathcal{I} \in \mathbb{INT}$, let us suppose, to simplify notation, that $\mathcal{I}$ has the form

$$\mathcal{I} = \bigoplus_{j=1}^{N_4} \mathbb{M}_{m_j}(C[0,1])$$

where $N_4 \geqslant 1$. The proof for the other cases are similar.

We now construct two unital $*$-homomorphisms $\phi_1, \phi_2 : C(X) \to \mathcal{A}$.

By $(*)$, we have that for $1 \leqslant j \leqslant N_4$, $m_j \geqslant N_2(2N+1)$. For each j, let $\psi_j : C(X) \to \mathbb{M}_{m_j}(C[0,1])$ be the (finite rank) unital $*$-homomorphism given by

$$\begin{aligned}\psi_j(f) =_{\mathrm{df}} \mathrm{diag}(&f(1), f(t_1), f(\bar{t}_1), f(t_2), f(\bar{t}_2), \ldots, f(t_N), f(\bar{t}_N), f(1), f(t_1),\\ &f(\bar{t}_1), f(t_2), f(\bar{t}_2), \ldots, f(t_N), f(\bar{t}_N), f(1), f(t_1), f(\bar{t}_1), \ldots)\end{aligned}$$

for all $f \in C(X)$, where the tail of the diagonal either has the form "$\ldots f(t_l), f(\bar{t}_l))$" or has the form "$\ldots f(t_l), f(\bar{t}_l), f(1))$".

Let $h \in C(X)$ be the identity function, i.e., $h(t) = t$ for all $t \in X$.

We define the unital $*$-homomorphisms $\phi_1, \phi_2 : C(X) \to \mathcal{A}$ in the following manner:

$$\phi_1(h) =_{\mathrm{df}} \psi_0(h) \oplus \bigoplus_{j=1}^{N_4} \psi_j(h) \quad \text{and} \quad \phi_2(h) =_{\mathrm{df}} (1-p) \oplus \bigoplus_{j=1}^{N_4} \psi_j(h).$$

From $(*)$, (2.4) and our choices of N_2, N_3 and η_2, we have the following statements:

$$\begin{aligned}&\text{(i) } \mu_{\tau\circ\phi_2}(O_s) \geqslant F(s) \text{ for all } s \geqslant \eta_1, \text{ for all open balls } O_s \in X\\ &\qquad\qquad \text{with radius } s \text{ and for all } \tau \in T(\mathcal{A}).\\ &\text{(ii) } |\tau\circ\phi_1(f) - \tau\circ\phi_2(f)| < \delta_3/2 \text{ for all } f \in \mathcal{G} \text{ and for all } \tau \in T(\mathcal{A}).\end{aligned} \tag{2.5}$$

Next, since X is either $\mathbb{T}$ or homeomorphic to $[0,1]$ and since the image of h (under both ϕ_1 and ϕ_2) is contained in $U^0(\mathcal{A})$,

$$\underline{K}(\phi_1) = \underline{K}(\phi_2). \tag{2.6}$$

Finally, from $(*)$(a), we have that $|\tau((1-p)b(1-p))| < \delta_5/10$ for all $b \in \mathcal{F}_1$ and for all $\tau \in T(\mathbb{M}_{N_1}(\mathcal{A}))$. It follows, from the definition of δ_5 and Lemma 2.9, that $\mathrm{dist}(e^{i2\pi(1-p)b(1-p)}, \overline{DU(\mathbb{M}_{N_1}(\mathcal{A}))}) < \delta_4/10$ for all $b \in \mathcal{F}_1$. From the definition of δ_4 and the definition of $\mathcal{F}_1$, it follows that for all $v \in \mathcal{U}$,

$$\mathrm{dist}(e^{i2\pi(1-p)a_{v,1}(1-p)}e^{i2\pi(1-p)a_{v,2}(1-p)}\dots e^{i2\pi(1-p)a_{v,M_1}(1-p)}, \overline{DU(\mathbb{M}_{N_1}(\mathcal{A}))}) < \frac{\delta_3}{10}.$$

From this and $(*)$(e), we have that for all $v \in \mathcal{U}$,

$$\mathrm{dist}(\psi_0(v) \oplus p, \overline{DU(\mathbb{M}_{N_1}(\mathcal{A}))}) < \frac{\delta_3}{5}. \tag{2.7}$$

Also, $(1-p)\phi_2(\mathbb{M}_{N_1}(C(X)))(1-p) \subseteq \mathbb{M}_{N_1}(\mathbb{C}(1-p))$. Hence, for all $v \in \mathcal{U}$, there exists self–adjoint $c \in \mathbb{M}_{N_1}(\mathbb{C}(1-p))$ with $\|c\| \leqslant 1$ such that $(1-p)\phi_2(v)(1-p) \oplus p = e^{i2\pi c}$. Note that this and $(*)$(a) implies that $|\tau(c)| < \delta_5/10$ for all $\tau \in T(\mathbb{M}_{N_1}(\mathcal{A}))$. From this, the definition of δ_5 and since $\delta_4 < \delta_3/10$, we have that for all $v \in \mathcal{U}$, $\mathrm{dist}((1-p)\phi_2(v)(1-p) \oplus p, \overline{DU(\mathbb{M}_{N_1}(\mathcal{A}))}) < \delta_3/10$. From this, the definitions of ϕ_1, ϕ_2 and (2.7), we have that for all $v \in \mathcal{U}$,

$$\mathrm{dist}(\phi_1^{\ddagger}(\overline{v}), \phi_2^{\ddagger}(\overline{v})) < \delta_3. \tag{2.8}$$

From (2.5), (2.6), (2.8) and from Theorem 2.10, there exists a unitary $w \in \mathcal{A}$ such that for all $f \in \mathcal{F}$,

$$\|\phi_1(f) - w\phi_2(f)w^*\| < \frac{\varepsilon}{10}.$$

Since the identity function h (i.e., $h(t) =_{\mathrm{df}} t$ for all $t \in X$) is an element of $\mathcal{F}$, it follows that

$$\left\|\Big((1-p)e^{i2\pi(1-p)a(1-p)}(1-p) \oplus \bigoplus_{j=1}^{N_4}\psi_j(h)\Big) - w\Big((1-p) \oplus \bigoplus_{j=1}^{N_4}\psi_j(h)\Big)w^*\right\| < \frac{\varepsilon}{10}.$$

From this and Corollary 2.2, there exist unitaries $x_1, y_1, x_2, y_2 \in \mathcal{A}$ such that

$$\left\|(x_1, y_1) - \Big((1-p)e^{i2\pi(1-p)a(1-p)}(1-p) \oplus \bigoplus_{j=1}^{N_4}\psi_j(h)\Big)\right\| < \frac{\varepsilon}{10} \quad \text{and} \tag{2.9}$$

$$(x_2, y_2) = (1-p) \oplus \bigoplus_{j=1}^{N_4}\overline{\psi_j(h)}. \tag{2.10}$$

By Lemma 1.9 of [29], there exist real-valued continuous functions $\theta_{j,k} : [0,1] \to \mathbb{R}$ $(1 \leqslant j \leqslant N_4,\ 1 \leqslant k \leqslant m_j)$, and there exist pairwise orthogonal minimal projections $p_{j,k} \in \mathbb{M}_{m_j}(C[0,1])$ (again $1 \leqslant j \leqslant N_4$, $1 \leqslant k \leqslant m_j$) with

$\sum_{k=1}^{m_j} p_{j,k} = 1_{\mathbb{M}_{m_j}(C[0,1])}$ for $1 \leqslant j \leqslant N_4$ such that (a) $\theta_{j,1} \leqslant \theta_{j,2} \leqslant \cdots \leqslant \theta_{j,m_j}$ for $1 \leqslant j \leqslant N_4$ and (b) $\sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k}$ is approximately unitarily equivalent to a_1 in $\mathcal{I}$. Note that the spectrum of $\mathcal{I}$ is $\widehat{\mathcal{I}} = \bigsqcup_{j=1}^{N_4} \widehat{\mathbb{M}_{m_j}(C[0,1])} = \bigsqcup_{j=1}^{N_4} [0,1]$; and so, for all $s \in \widehat{\mathcal{I}}$, the spectrum of $a_1(s)$ is $\{\theta_{j,k}(s) : 1 \leqslant j \leqslant N_4 \text{ and } 1 \leqslant k \leqslant m_j\}$.

Hence, replacing $\sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k}$ by a unitarily equivalent (in $\mathcal{I}$) self-adjoint element if necessary, we may assume that

$$\Big\| a_1 - \sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k} \Big\| < \frac{\delta_1}{10}. \tag{2.11}$$

(Note that a unitary equivalence is the same as simultaneously replacing the projections $p_{j,k}$ by unitarily equivalent projections, with the same unitary for all the projections. In particular, the eigenvalue functions $\theta_{j,k}$ stay the same.)

Moreover, by $(*)$(d) and our assumptions on $\theta_{j,k}$, $p_{j,k}$, for all $\tau \in T(\mathcal{I})$,

$$\Big| \tau\Big(\sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k} \Big) - \tau(q') + \tau(r') \Big| < \frac{\delta_1}{10}. \tag{2.12}$$

Let $g : \widehat{\mathcal{I}} = \bigsqcup_{j=1}^{N_4} \widehat{\mathbb{M}_{m_j}(C[0,1])} \to \mathbb{R}$ be the continuous function defined as follows:

For $s \in \widehat{\mathbb{M}_{m_j}(C[0,1])} \cong [0,1]$,

$$g(s) =_{\mathrm{df}} (1/m_j) \sum_{k=1}^{m_j} \theta_{j,k}(s) - (1/m_j)\mathrm{Tr}(q'(s)) + (1/m_j)\mathrm{Tr}(r'(s)),$$

where Tr is the (nonnormalized) trace on $\mathbb{M}_\infty$. (Note that q', r' must, by definition of $\mathbb{M}_\infty(\mathcal{I})$, sit in some big matrix algebra over $\mathcal{I}$.)

Hence, $g1_{\mathcal{I}} \in \mathcal{I}$ is a self-adjoint element, and by (2.12),

$$\|g1_{\mathcal{I}}\| < \frac{\delta_1}{10} \quad \text{and} \tag{2.13}$$

$$\tau\Big(\sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k} \Big) - \tau(q') + \tau(r') - \tau(g1_{\mathcal{I}}) = 0 \tag{2.14}$$

for all $\tau \in T(\mathcal{I})$.

For $1 \leqslant j \leqslant N_4$, fix $\tau_j \in T(\mathbb{M}_{m_j}(C[0,1]))$. Let $a_2 =_{\mathrm{df}} \sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k} - g1_{\mathcal{I}}$ and let $a_2 = \sum_{j=1}^{N_4} a_{2,j}$, where $a_{2,j} \in \mathbb{M}_{m_j}(C[0,1])$ for $1 \leqslant j \leqslant N_4$.

Let $q' = \sum_{j=1}^{N_4} q_j$ and $r' = \sum_{j=1}^{N_4} r_j$ where for $1 \leqslant j \leqslant N_4$, $q_j, r_j \in \mathbb{M}_\infty \otimes \mathbb{M}_{m_j}(C[0,1])$ are projections. Suppose that, for $1 \leqslant j \leqslant N_4$, q_j,r_j are the sums of L_j and L'_j minimal projections in $\mathbb{M}_\infty \otimes \mathbb{M}_{m_j}(C[0,1])$ respectively. To simplify notation, let us assume that $L'_j \geqslant L_j$ for $1 \leqslant j \leqslant N_4$. Then for $1 \leqslant j \leqslant N_4$, $\mathrm{e}^{\mathrm{i}2\pi(a_{2,j}+(L'_j-L_j)p_{j,m_j})} = \mathrm{e}^{\mathrm{i}2\pi a_{2,j}}$ and hence,

$$\mathrm{e}^{\mathrm{i}2\pi(a_2+\sum_{j=1}^{N_4}(L'_j-L_j)p_{j,m_j})} = \mathrm{e}^{\mathrm{i}2\pi a_2}. \tag{2.15}$$

Let $a_3 =_{\mathrm{df}} a_2 + \sum_{j=1}^{N_4}(L'_j - L_j)p_{j,m_j} \in \mathcal{I}$. By (2.14), we have that $\tau(a_3) = 0$ for all $\tau \in T(\mathcal{I})$. Hence, by Lemma 2.4(i) and by Lemma 1.9 of [29], there exist unitaries $x_3, y_3, x_4, y_4, \ldots, x_{18}, y_{18}$ in $\mathcal{A}$ such that

$$\|\mathrm{e}^{\mathrm{i}2\pi a_3} - (x_3, y_3)(x_4, y_4)\cdots(x_{18}, y_{18})\| < \frac{\varepsilon}{10}. \tag{2.16}$$

(Actually, for $3 \leqslant j \leqslant 18$, $px_jp, py_jp \in \mathcal{I}$, $x_j = px_jp \oplus (1-p)$ and $y_j = py_jp \oplus (1-p)$.)

From the definition of δ_1 and by (2.11), (2.13), (2.15) and (2.16),

$$\Big\|\mathrm{e}^{\mathrm{i}2\pi a_1} - \prod_{j=3}^{18}(x_j, y_j)\Big\| < \frac{\varepsilon}{5}.$$

From this, (2.2), (∗) statement (c), (2.9) and (2.10), we have that

$$\Big\|u - \prod_{j=1}^{18}(x_j, y_j)\Big\| < \varepsilon.$$

We now prove part (ii). Say that $x = u|x|$ is the polar decomposition of x. Then by Lemma 2.11, $\Delta_T(u) = \Delta_T(|x|) = 0$.

By part (i), let $x_j, y_j \in U^0(\mathcal{A})$ be unitaries such that

$$\Big\|u - \prod_{j=1}^{18}(x_j, y_j)\Big\| < \frac{\varepsilon}{2}. \tag{2.17}$$

Hence, to complete the proof, it suffices to prove the following claim:

Claim. There exist invertibles $x_j, y_j \in GL^0(\mathcal{A})$, $19 \leqslant j \leqslant 24$, such that

$$\Big\||x| - \prod_{j=19}^{24}(x_j, y_j)\Big\| < \frac{\varepsilon}{2}.$$

Sketch of proof of the Claim. The proof of the Claim is very similar to the proof of part (i) of this lemma. The main differences are the following:

(i) Since $|x| \geqslant 0$, $|x|$ automatically has the form $|x| = \mathrm{e}^a$ where $a \in \mathcal{A}_{sa}$.

(ii) Set $X =_{\mathrm{df}} [-2\|a\|, 2\|a\|]$. The set S will have the form $S = \{0, t_1, -t_1, t_2, -t_2, \ldots, t_N, -t_N\}$. (Of course, at some point, one would need to exponentiate the images of corresponding maps $C(X) \to \mathcal{A}$.)

(iii) In the proof of part (i), replace Corollary 2.2 and Lemma 2.4 with Lemma 2.6 in [30] and (this paper) Lemma 2.5 respectively.

(iv) In the proof of part (i), replace every occurrence of $\Delta_T(|x|) = 0$ and every occurrence of equation (2.3) with the condition $\tau(\mathrm{Log}(|x|) = 0$ for all $\tau \in T(\mathcal{A})$. (See Lemma 2.11.)

End of sketch of proof of the Claim and of the lemma. ∎

LEMMA 2.13. *Let $\mathcal{A}$ be a unital separable simple* TAI-*algebra.*

(i) *Suppose that $u \in U^0(\mathcal{A})$ is a unitary such that $\Delta_T(u) = 0$. Then for every $\varepsilon > 0$, there exist unitaries $x_j, y_j \in U^0(\mathcal{A})$, with $1 \leqslant j \leqslant 20$, and there exists a self-adjoint element $a \in \mathcal{A}$ such that, for all $\tau \in T(\mathcal{A})$,*

$$u = \Big(\prod_{j=1}^{20}(x_j, y_j)\Big)\mathrm{e}^{\mathrm{i}2\pi a} \quad \|a\| < \varepsilon \text{ and } \tau(a) = 0.$$

(ii) *Suppose that $x \in GL^0(\mathcal{A})$ is an invertible such that $\Delta_T(x) = 0$. Then for every $\varepsilon > 0$, there exist invertibles $x_j, y_j \in GL^0(\mathcal{A})$, with $1 \leqslant j \leqslant 26$, and there exists an element $d \in \mathcal{A}$ such that, for all $\tau \in T(\mathcal{A})$,*

$$u = \Big(\prod_{j=1}^{26}(x_j, y_j)\Big)\mathrm{e}^{d} \quad \|d\| < \varepsilon \text{ and } \tau(d) = 0.$$

Proof. We firstly prove part (i). The proof of part (ii) is similar.

Choose an integer $N \geqslant 10$ such that if $c_1, c_2, c_3 \in \mathcal{A}$ are self-adjoint elements such that $\|c_j\| < 1/N$ for $1 \leqslant j \leqslant 3$ then $\|\mathrm{e}^{\mathrm{i}2\pi c_1}\mathrm{e}^{\mathrm{i}2\pi c_2}\mathrm{e}^{\mathrm{i}2\pi c_3} - 1\| < 1$ and $(1/2\pi)\|\mathrm{Log}(\mathrm{e}^{\mathrm{i}2\pi c_1}\mathrm{e}^{\mathrm{i}2\pi c_2}\mathrm{e}^{\mathrm{i}2\pi c_3})\| < \varepsilon$.

Choose a $\delta > 0$, with $\delta < 1$, such that for any unitary $v \in \mathcal{A}$, if $\|v - 1\| < \delta$ then $(1/2\pi)\|\mathrm{Log}(v)\| < 1/(2N)$.

By Lemma 2.12 part (i), there exist unitaries $x_j, y_j \in U^0(\mathcal{A})$ with $1 \leqslant j \leqslant 18$ and there exists a unitary $w \in U^0(\mathcal{A})$ such that

$$u = (x_1, y_1)(x_2, y_2)\cdots(x_{18}, y_{18})w \tag{2.18}$$

and $\|w - 1\| < \delta$.

By our choice of δ, there exists a self-adjoint element $b \in \mathcal{A}$ with $\|b\| < 1/(2N)$ such that $w = \mathrm{e}^{\mathrm{i}2\pi b}$. Since $\Delta_T(w) = 0$, there exist projections $p_0, q_0 \in \mathbb{M}_\infty(\mathcal{A})$ such that

$$\tau(b) - \tau(p_0) + \tau(q_0) = 0$$

for all $\tau \in T(\mathcal{A})$.

Since $\mathcal{A}$ is simple TAI and since $\|b\| < 1/(2N)$, we can replace p_0, q_0 by projections $p, q \in \mathcal{A}$ with $\tau(p), \tau(q) < 1/(2N)$ and

$$\tau(b) - \tau(p) + \tau(q) = 0$$

for all $\tau \in T(\mathcal{A})$.

Since $\tau(p), \tau(q) < 1/(2N)$ for all $\tau \in T(\mathcal{A})$ and since $\mathcal{A}$ has strict comparison, there exist pairwise orthogonal projections $p_1, p_2, \ldots, p_N, q_1, q_2, \ldots, q_N \in \mathcal{A}$ such that $p_j \sim p$ and $q_j \sim q$ for $1 \leqslant j \leqslant N$. Hence, for all $\tau \in T(\mathcal{A})$,

$$\tau(b) - \tau\Big(\frac{1}{N}\sum_{j=1}^{N} p_j\Big) + \tau\Big(\frac{1}{N}\sum_{j=1}^{N} q_j\Big) = 0. \tag{2.19}$$

We have that

$$\begin{aligned} \mathrm{e}^{\mathrm{i}2\pi b} = {} & \mathrm{e}^{\mathrm{i}2\pi(1/N)\sum_{j=1}^{N} p_j}\mathrm{e}^{-\mathrm{i}2\pi(1/N)\sum_{j=1}^{N} q_j} \\ & \cdot (\mathrm{e}^{-\mathrm{i}2\pi(1/N)\sum_{j=1}^{N} p_j}\mathrm{e}^{\mathrm{i}2\pi(1/N)\sum_{j=1}^{N} q_j}\mathrm{e}^{\mathrm{i}2\pi b}). \end{aligned} \tag{2.20}$$

By Lemma 2.1 of [30], there exist unitaries $x_{19}, y_{19}, x_{20}, y_{20} \in U^0(\mathcal{A})$ such that

$$\mathrm{e}^{\mathrm{i}2\pi(1/N)\sum_{j=1}^{N} p_j} = (x_{19}, y_{19}) \quad \text{and} \tag{2.21}$$

$$\mathrm{e}^{-\mathrm{i}2\pi(1/N)\sum_{j=1}^{N} q_j} = (x_{20}, y_{20}). \tag{2.22}$$

Also, by our choice of N, there exists a self-adjoint element $a \in \mathcal{A}$ such that

$$\mathrm{e}^{\mathrm{i}2\pi a} = \mathrm{e}^{-\mathrm{i}2\pi(1/N)\sum_{j=1}^{N} p_j}\mathrm{e}^{\mathrm{i}2\pi(1/N)\sum_{j=1}^{N} q_j}\mathrm{e}^{\mathrm{i}2\pi b}. \tag{2.23}$$

Moreover, $\|a\| < \varepsilon$; by Lemma 3(b) of [14], (2.23) and (2.19),

$$\tau(a) = \tau(b) - \tau\Big(\frac{1}{N}\sum_{j=1}^{N} p_j\Big) + \tau\Big(\frac{1}{N}\sum_{j=1}^{N} q_j\Big) = 0$$

for all $\tau \in T(\mathcal{A})$. Finally, by (2.18), (2.20), (2.21), (2.22) and (2.23),

$$u = (x_1, y_1)(x_2, y_2)(x_3, y_3)\cdots(x_{20}, y_{20})\mathrm{e}^{\mathrm{i}2\pi a}.$$

The proof of part (ii) is very similar to the proof of part (i). The main difference is that we replace Lemma 2.12 part (i) with Lemma 2.12 part (ii). ∎

LEMMA 2.14. *Let $\mathcal{A}$ be a unital separable simple* TAI-*algebra.*

(i) *Suppose that $ux \in U^0(\mathcal{A})$ is a unitary with $\|u - 1\| < \sqrt{2}/100$ and $\tau(\mathrm{Log}(u)) = 0$ for all $\tau \in T(\mathcal{A})$. Then for every $\varepsilon > 0$, there exist unitaries $x_j, y_j, z \in U^0(\mathcal{A})$, $1 \leqslant j \leqslant 6$, such that*

$$u = \Big(\prod_{j=1}^{6}(x_j, y_j)\Big)z \quad \|z - 1\| < \varepsilon, \ \tau(\mathrm{Log}(z)) = 0$$

for all $\tau \in T(\mathcal{A})$, and

$$\|x_j - 1\|, \ \|y_j - 1\| < 2\sqrt{2}\|u - 1\|^{1/2}$$

for $1 \leqslant j \leqslant 6$.

(ii) *Suppose that* $x \in GL^0(\mathcal{A})$ *is an invertible with* $\|x-1\| < 1/1000$ *and* $\tau(\mathrm{Log}(x)) = 0$ *for all* $\tau \in T(\mathcal{A})$. *Then for every* $\varepsilon > 0$, *there exist invertibles* $x_j, y_j, z \in GL^0(\mathcal{A})$, $1 \leqslant j \leqslant 12$, *such that*

$$x = \Big(\prod_{j=1}^{12}(x_j, y_j)\Big)z \quad \|z-1\| < \varepsilon,\ \tau(\mathrm{Log}(z)) = 0$$

for all $\tau \in T(\mathcal{A})$, *and*

$$\|x_j - 1\|,\ \|y_j - 1\| < 24\|x-1\|^{1/2}$$

for $1 \leqslant j \leqslant 12$.

Proof. The argument of (i) is a variation on the argument of Lemma 2.12(i), where we need to control the norm distance to the unit of the operators that make up the commutators. We go through the proof for the convenience of the reader.

Let $X \subseteq \mathbb{T}$ be the compact subset given by

$$X =_{\mathrm{df}} \{t \in \mathbb{T} : |t-1| \leqslant 2\|u-1\|\}.$$

(Note that $1 \in X$ and X is homeomorphic to $[0,1]$.)

Let $\varepsilon > 0$ be given. Contracting ε if necessary, we may assume that $0 < \varepsilon < \min\{1/100, \|u-1\|\}$ and that $\varepsilon > 0$ is small enough so that for every unitary $v \in \mathcal{A}$, if $\|u - v\| < \varepsilon$ then $\mathrm{sp}(v) \subseteq X$.

Since $\|u-1\| < \sqrt{2}/10$, $a =_{\mathrm{df}} (1/(\mathrm{i}2\pi))\mathrm{Log}(u) \in \mathcal{A}_{sa}$ and

$$u = \mathrm{e}^{\mathrm{i}2\pi a}. \tag{2.24}$$

Hence, $\tau(a) = 0$ for all $\tau \in T(\mathcal{A})$. Also, $\mathrm{sp}(a) \subset (-\pi/2, \pi/2)$.

Let $\delta_1 > 0$ be such that for all self-adjoint elements $c, c' \in \mathcal{A}$ if $\|c - c'\| < \delta_1$ then $\|\mathrm{e}^{\mathrm{i}2\pi c} - \mathrm{e}^{\mathrm{i}2\pi c'}\| < \varepsilon/10$. We may assume that $\delta_1 < \varepsilon/10$ and that for all $0 < \delta_1' \leqslant \delta_1$, $(\delta_1' + \mathrm{sp}(a)) \cup (-\delta_1' + \mathrm{sp}(a)) \subset (-\pi/2, \pi/2)$.

Let $F : (0,1) \to (0,1)$ be the nondecreasing map given by $F(t) =_{\mathrm{df}} t/10$ for all $t \in (0,1)$. Let $\mathcal{F} \subset C(X)$ be a finite subset that contains the identity function $h(t) =_{\mathrm{df}} t$ $(t \in X)$.

Plug X, $\varepsilon/10$ (for ε), $\mathcal{F}$ and F into Theorem 2.10 to get $\eta_1 > 0$, $\delta_3 > 0$, a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ and a finite subset $\mathcal{U} \subset U(\mathbb{M}_\infty(C(X)))$ satisfying the conclusions of Theorem 2.10.

Note that X is closed under complex conjugates. Hence, let $S =_{\mathrm{df}} \{1, t_1, \bar{t}_1, t_2, \bar{t}_2, \ldots, t_N, \bar{t}_N\} \subset X$ be a finite collection of $2N+1$ distinct points and $\eta_2 > 0$ such that for all $s \geqslant \eta_1$ for all open balls O_s in X with radius s,

$$\mathrm{card}(O_s \cap S)\frac{1-\eta_2}{2N+1} > F(s), \tag{2.25}$$

where $\mathrm{card}(O_s \cap S)$ is the cardinality of $O_s \cap S$.

Let $\Phi : C(X) \to \mathcal{A}$ be the unital $*$-homomorphism given by $\Phi(h) =_{\mathrm{df}} \mathrm{e}^{\mathrm{i}2\pi a} = u$, where $h \in C(X)$ is the identity map (i.e., $h(t) = t$ for all $t \in X$). Note that the spectrum of $u = \mathrm{e}^{\mathrm{i}2\pi a}$ is contained in X; so Φ is well-defined. Let

$N_1 \geqslant 1$ be an integer so that $\mathcal{U} \subset \mathbb{M}_{N_1}(C(X))$. Note that since X is homeomorphic to $[0,1]$, every unitary in $\mathbb{M}_{N_1}(C(X))$ can be approximated arbitrarily close by exponentials. Hence, let $\mathcal{F}_1 \subset \mathbb{M}_{N_1}(\mathcal{A})$ be a finite set of self-adjoint elements so that for all $v \in \mathcal{U}$, there exists a self–adjoint element $a_v \in \mathcal{F}_1$ so that $\|\Phi(v) - \mathrm{e}^{\mathrm{i}2\pi a_v}\| < \delta_3/10$.

Plug $\delta_3/10$ (for ε) and $\mathbb{M}_{N_1}(\mathcal{A})$ (for $\mathcal{A}$) into Lemma 2.9 to get $\delta_4 > 0$.

Choose integer $N_2 \geqslant 1$ such that $1/N_2 < \eta_2/10$. Also choose $N_3 \geqslant 1$ such that $N_3 \geqslant \max\{\|b\| : b \in \mathcal{G} \cup \mathcal{F}_1\}$.

Since $\mathcal{A}$ is TAI and by Lemma 2.6 and Remark 1.1, let $p \in \mathcal{A}$ be a projection and let $\mathcal{I} \in \mathbb{INT}$ be a C^*-subalgebra of $\mathcal{A}$ with $1_{\mathcal{I}} = p$ such that the following hold:

(a) $\tau(1_{\mathcal{A}} - p) < \min\{\delta_3/(10(1+N_3)), \delta_4/(10(1+N_3)), \eta_2/(10(1+N_3))\}$ for all $\tau \in T(\mathcal{A})$.

(b) Each summand in $\mathcal{I}$ has matrix size at least $N_2(2N+1)$. (Equivalently, every irreducible represention of $\mathcal{I}$ has image with the form $\mathbb{M}_n$ with $n \geqslant N_2(2N+1)$.)

(c) There exists $a_1 \in \mathcal{I}$ such that $\|a - ((1-p)a(1-p) + a_1)\| < \delta_1/10$ and $\|\mathrm{e}^{\mathrm{i}2\pi a} - \mathrm{e}^{\mathrm{i}2\pi((1-p)a(1-p)+a_1)}\| < \varepsilon/10$. Note that for $0 < \delta_1' \leqslant \delta_1$, $\mathrm{sp}(a_1) \cup (\delta_1'/10 + \mathrm{sp}(a_1)) \cup (-\delta_1'/10 + \mathrm{sp}(a_1)) \subset (-\pi/2, \pi/2)$.

(d) $|\tau(a_1)| < \delta_1/10$ for all $\tau \in T(\mathcal{I})$.

(e) Let $\psi_0 : C(X) \to (1-p)\mathcal{A}(1-p)$ be the unital $*$-homomorphism given by $\psi_0(h) =_{\mathrm{df}} (1-p)\mathrm{e}^{\mathrm{i}2\pi(1-p)a(1-p)}(1-p)$, where $h \in C(X)$ is the identity map (i.e., $h(t) = t$ for all $t \in X$). (Note that by (2.24), by (c) and our assumptions on ε, p can be chosen so that the spectrum of $\mathrm{e}^{\mathrm{i}2\pi(1-p)a(1-p)}$ is contained in X; so the map ψ_0 is well-defined.)

Then for all $v \in \mathcal{U}$,

$$\|\psi_0(v) - (1-p)\mathrm{e}^{\mathrm{i}2\pi(1-p)a_v(1-p)}(1-p)\| < \frac{\delta_3}{10}.$$

(Here, we identify $1_{\mathcal{A}} - p$ with $(1_{\mathcal{A}} - p) \otimes 1_{\mathbb{M}_{N_1}} \in \mathbb{M}_{N_1}(\mathcal{A})$.)

We denote the above statements by "$(*)$".

Since $\mathcal{I} \in \mathbb{INT}$, let us suppose, to simplify notation, that $\mathcal{I}$ has the form

$$\mathcal{I} = \bigoplus_{j=1}^{N_4} \mathbb{M}_{m_j}(C[0,1])$$

where $N_4 \geqslant 1$. The proofs for the other cases are similar.

We now construct two unital $*$-homomorphisms $\phi_1, \phi_2 : C(X) \to \mathcal{A}$.

By $(*)$, we have that for $1 \leqslant j \leqslant N_4$, $m_j \geqslant N_2(2N+1)$. For each j, let $\psi_j : C(X) \to \mathbb{M}_{m_j}(C[0,1])$ be the (finite rank) unital $*$-homomorphism given by

$$\begin{aligned}\psi_j(f) =_{\mathrm{df}} \mathrm{diag}(&f(1), f(t_1), f(\bar{t}_1), f(t_2), f(\bar{t}_2), \ldots, f(t_N), f(\bar{t}_N), f(1), f(t_1),\\ &f(\bar{t}_1), f(t_2), f(\bar{t}_2), \ldots, f(t_N), f(\bar{t}_N), f(1), f(t_1), f(\bar{t}_1), \ldots)\end{aligned}$$

for all $f \in C(X)$, where the tail of the diagonal either has the form "$\ldots f(t_l), f(\bar{t}_l))$" or has the form "$\ldots f(t_l), f(\bar{t}_l), f(1))$".

Let $h \in C(X)$ be the identity function, i.e., $h(t) = t$ for all $t \in X$.

We define the unital $*$-homomorphisms $\phi_1, \phi_2 : C(X) \to \mathcal{A}$ in the following manner:

$$\phi_1(h) =_{\text{df}} \psi_0(h) \oplus \bigoplus_{j=1}^{N_4} \psi_j(h) \quad \text{and} \quad \phi_2(h) =_{\text{df}} (1-p) \oplus \bigoplus_{j=1}^{N_4} \psi_j(h).$$

From $(*)$, (2.25) and our choices of N_2, N_3 and η_2, we have the following statements:

$$\begin{aligned} &\text{(i) } \mu_{\tau\circ\phi_2}(O_s) \geqslant F(s) \text{ for all } s \geqslant \eta_1, \text{ for all open balls } O_s \in X \\ &\qquad \text{with radius } s \text{ and for all } \tau \in T(\mathcal{A}). \\ &\text{(ii) } |\tau \circ \phi_1(f) - \tau \circ \phi_2(f)| < \delta_3/2 \text{ for all } f \in \mathcal{G} \text{ and for all } \tau \in T(\mathcal{A}). \end{aligned} \tag{2.26}$$

Next, since X is homeomorphic to $[0,1]$ and since the image of h (under both ϕ_1 and ϕ_2) is contained in $U^0(\mathcal{A})$,

$$\underline{K}(\phi_1) = \underline{K}(\phi_2). \tag{2.27}$$

Finally, from $(*)$(a), we have that $|\tau((1-p)b(1-p))| < \delta_4/10$ for all $b \in \mathcal{F}_1$ and for all $\tau \in T(\mathbb{M}_{N_1}(\mathcal{A}))$. It follows, from the definition of δ_4 and Lemma 2.9, that $\text{dist}(e^{i2\pi(1-p)b(1-p)}, \overline{DU(\mathbb{M}_{N_1}(\mathcal{A}))}) < \delta_3/10$ for all $b \in \mathcal{F}_1$. From this and $(*)$(e), we have that for all $v \in \mathcal{U}$,

$$\text{dist}(\psi_0(v) \oplus p, \overline{DU(\mathbb{M}_{N_1}(\mathcal{A}))}) < \frac{\delta_3}{5}. \tag{2.28}$$

Also, $(1-p)\phi_2(\mathbb{M}_{N_1}(C(X)))(1-p) \subseteq \mathbb{M}_{N_1}(\mathbb{C}(1-p))$. Hence, for all $v \in \mathcal{U}$, there exists self–adjoint $c \in \mathbb{M}_{N_1}(\mathbb{C}(1-p))$ with $\|c\| \leqslant 1$ such that $(1-p)\phi_2(v)(1-p) \oplus p = e^{i2\pi c}$. Note that this and $(*)$(a) implies that $|\tau(c)| < \delta_4/10$ for all $\tau \in T(\mathbb{M}_{N_1}(\mathcal{A}))$. From this and the definition of δ_4, we have that for all $v \in \mathcal{U}$, $\text{dist}((1-p)\phi_2(v)(1-p) \oplus p, \overline{DU(\mathbb{M}_{N_1}(\mathcal{A}))}) < \delta_3/10$. From this, the definitions of ϕ_1, ϕ_2 and (2.28), we have that for all $v \in \mathcal{U}$,

$$\text{dist}(\phi_1^{\ddagger}(\overline{v}), \phi_2^{\ddagger}(\overline{v})) < \delta_3. \tag{2.29}$$

From (2.26), (2.27), (2.29) and from Theorem 2.10, there exists a unitary $w \in \mathcal{A}$ such that for all $f \in \mathcal{F}$,

$$\|\phi_1(f) - w\phi_2(f)w^*\| < \frac{\varepsilon}{10}.$$

Since the identity function h (i.e., $h(t) =_{\text{df}} t$ for all $t \in X$) is an element of $\mathcal{F}$, it follows that

$$\left\| \Big((1-p)e^{i2\pi(1-p)a(1-p)}(1-p) \oplus \bigoplus_{j=1}^{N_4} \psi_j(h) \Big) - w\Big((1-p) \oplus \bigoplus_{j=1}^{N_4} \psi_j(h) \Big) w^* \right\| < \frac{\varepsilon}{10}.$$

From this and Corollary 2.2, there exist unitaries $x_1, y_1, x_2, y_2 \in \mathcal{A}$ such that

$$\Big\|(x_1, y_1) - \Big((1-p)\mathrm{e}^{\mathrm{i}2\pi(1-p)a(1-p)}(1-p) \oplus \bigoplus_{j=1}^{N_4} \psi_j(h)\Big)\Big\| < \frac{\varepsilon}{10} \quad \text{and} \tag{2.30}$$

$$(x_2, y_2) = (1-p) \oplus \bigoplus_{j=1}^{N_4} \overline{\psi_j(h)}. \tag{2.31}$$

Moreover, since $|t-1| \leqslant 2\|u-1\| < \sqrt{2}$ for all $t \in X$, it follows, by Corollary 2.2, that $\|x_j - 1\|, \|y_j - 1\| \leqslant \sqrt{2}\|u-1\|^{1/2}$ for $j = 1, 2$.

Finally, by inspection (and the definition of X), we see that there exist $b_1, b_2 \in \mathcal{A}_{sa}$ with $(x_j, y_j) = \mathrm{e}^{\mathrm{i}2\pi b_j}$, $\|b_j\| < 1$, $\tau(b_j) = 0$ for all $\tau \in T(\mathcal{A})$, and $\|\mathrm{e}^{\mathrm{i}2\pi b_j} - 1\| \leqslant 2\|u-1\| \leqslant \sqrt{2}/50$, for $j = 1, 2$.

By Lemma 1.9 of [29], there exist real-valued continuous functions $\theta_{j,k} : [0,1] \to \mathbb{R}$ ($1 \leqslant j \leqslant N_4$, $1 \leqslant k \leqslant m_j$), and there exist pairwise orthogonal minimal projections $p_{j,k} \in \mathbb{M}_{m_j}(C[0,1])$ (again $1 \leqslant j \leqslant N_4$, $1 \leqslant k \leqslant m_j$) with $\sum_{k=1}^{m_j} p_{j,k} = 1_{\mathbb{M}_{m_j}(C[0,1])}$ for $1 \leqslant j \leqslant N_4$ such that (a) $\theta_{j,1} \leqslant \theta_{j,2} \leqslant \cdots \leqslant \theta_{j,m_j}$ for $1 \leqslant j \leqslant N_4$ and (b) $\sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k}$ is approximately unitarily equivalent to a_1 in $\mathcal{I}$. Note that the spectrum of $\mathcal{I}$ is $\widehat{\mathcal{I}} = \bigsqcup_{j=1}^{N_4} \widehat{\mathbb{M}_{m_j}(C[0,1])} = \bigsqcup_{j=1}^{N_4} [0,1]$; and so, for all $s \in \widehat{\mathcal{I}}$, the spectrum of $a_1(s)$ is $\{\theta_{j,k}(s) : 1 \leqslant j \leqslant N_4 \text{ and } 1 \leqslant k \leqslant m_j\}$.

Hence, replacing $\sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k}$ by a unitary equivalent (in $\mathcal{I}$) self-adjoint element if necessary, we may assume that

$$\Big\|a_1 - \sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k}\Big\| < \frac{\delta_1}{10}. \tag{2.32}$$

(Note that a unitary equivalence is the same as simultaneously replacing the projections $p_{j,k}$ by unitarily equivalent projections, with the same unitary for all the projections. In particular, the eigenvalue functions $\theta_{j,k}$ stay the same.)

Moreover, by $(*)$(d) and our assumptions on $\theta_{j,k}$, $p_{j,k}$, for all $\tau \in T(\mathcal{I})$,

$$\Big|\tau\Big(\sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k}\Big)\Big| < \frac{\delta_1}{10}. \tag{2.33}$$

Let $g : \widehat{\mathcal{I}} = \bigsqcup_{j=1}^{N_4} \widehat{\mathbb{M}_{m_j}(C[0,1])} \to \mathbb{R}$ be the continuous function defined as follows:

For $s \in \widehat{\mathbb{M}_{m_j}(C[0,1])} \cong [0,1]$,

$$g(s) =_{\mathrm{df}} \frac{1}{m_j} \sum_{k=1}^{m_j} \theta_{j,k}(s).$$

Hence, $g1_{\mathcal{I}} \in \mathcal{I}$ is a self-adjoint element, and by (2.33), for all $\tau \in T(\mathcal{I})$,

$$\|g1_{\mathcal{I}}\| < \frac{\delta_1}{10} \quad \text{and} \tag{2.34}$$

$$\tau\Big(\sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k}\Big) - \tau(g1_{\mathcal{I}}) = 0. \tag{2.35}$$

Let $a_2 =_{\mathrm{df}} \sum_{j=1}^{N_4} \sum_{k=1}^{m_j} \theta_{j,k} p_{j,k} - g1_{\mathcal{I}}$. By $(*)$(c) and (2.34), $\mathrm{sp}(a_2) \subset (-\pi/2, \pi/2)$. Also, by (2.35), we have that $\tau(a_2) = 0$ for all $\tau \in T(\mathcal{I})$.

Hence, by Lemma 2.4(ii) (and by conjugating with an appropriate permutation unitary if necessary) there exist unitaries $x_3, y_3, x_4, y_4, x_5, y_5, x_6, y_6$ in $\mathcal{A}$ such that

$$e^{i2\pi a_2} = (x_3, y_3)(x_4, y_4)(x_5, y_5)(x_6, y_6) \tag{2.36}$$

and for $3 \leqslant j \leqslant 6$, $\|x_j - 1\| \leqslant \sqrt{2}\|e^{i2\pi a_2} - 1\|^{1/2}$. Note that by the definition of δ_1 and our assumptions on ε, $\|e^{i2\pi a_2} - 1\| \leqslant \|e^{i2\pi a_2} - e^{i2\pi a_1}\| + \|e^{i2\pi a_1} - 1\| \leqslant \varepsilon/10 + \|e^{i2\pi((1-p)a(1-p))+a_1)} - 1\| \leqslant (1/10)\|u-1\| + \|e^{i2\pi((1-p)a(1-p))+a_1)} - e^{i2\pi a}\| + \|e^{i2\pi a} - 1\| \leqslant (1/10)\|u - 1\| + \varepsilon/10 + \|u-1\| \leqslant 2\|u-1\|$.

Hence, $\|e^{i2\pi a_2} - 1\| \leqslant \sqrt{2}/50$ and $\|x_j - 1\| \leqslant 2\|u-1\|^{1/2}$. Similarly, $\|y_j - 1\| \leqslant 2\|u-1\|^{1/2}$ for $3 \leqslant j \leqslant 6$.

From the definitions of a_2 and δ_1,

$$\Big\|e^{i2\pi a_1} - \prod_{j=3}^{6}(x_j, y_j)\Big\| < \frac{\varepsilon}{5}.$$

From this, (2.24), $(*)$ statement (c), (2.30) and (2.31), we have that

$$\Big\|u - \prod_{j=1}^{6}(x_j, y_j)\Big\| < \varepsilon$$

and $\|x_j - 1\|, \|y_j - 1\| \leqslant 2\sqrt{2}\|u-1\|^{1/2}$ for $1 \leqslant j \leqslant 6$.

Hence, $u = \Big(\prod_{j=1}^{6}(x_j, y_j)\Big)z$ where $z \in U^0(\mathcal{A})$ and $\|z - 1\| < \varepsilon$ (which is $< 1/100$ by our hypotheses on ε). Hence, $e^{i2\pi a} = e^{i2\pi b_1}e^{i2\pi b_2}e^{i2\pi a_2}z$. But $(1 - \|e^{i2\pi b_1} - 1\|)(1 - \|e^{i2\pi b_2} - 1\|)(1 - \|e^{i2\pi a_2} - 1\|)(1 - \|z - 1\|) \geqslant (1 - \sqrt{2}/50)^3$ $(99/100) > 1/2$. Hence, by Lemma 3(b) of [14], $\tau(a) = \tau(b_1) + \tau(b_2) + \tau(a_2) + (1/(2\pi i))\tau(\mathrm{Log}(z))$ for all $\tau \in T(\mathcal{A})$. Hence, $\tau(\mathrm{Log}(z)) = 0$ for all $\tau \in T(\mathcal{A})$.

Next, we sketch the proof of (ii).

Let $x = u|x|$ be the polar decomposition of x. Since $\|x - 1\| < 1/1000$, $\||x| - 1\| < 2001/1000000$ and $\|u - 1\| < 3001/1000000 < \sqrt{2}/100$.

Also, by Lemma 2.11, $\Delta_T(|x|) = \Delta_T(u) = 0$, and $\tau(\mathrm{Log}(|x|)) = 0$ for all $\tau \in T(\mathcal{A})$. Hence, by Lemma 3(b) of [14], $\tau(\mathrm{Log}(u)) = 0$ for all $\tau \in T(\mathcal{A})$.

Hence, by (i), there exist unitaries $x'_j, y'_j, v \in U^0(\mathcal{A})$, $1 \leqslant j \leqslant 6$, such that

$$u = \Big(\prod_{j=1}^{6}(x'_j, y'_j)\Big)v, \quad \|v - 1\| < \min\{\varepsilon/10, 1/100\}, \quad \tau(\mathrm{Log}(v)) = 0$$

for all $\tau \in T(\mathcal{A})$, and

$$\|x'_j - 1\|,\ \|y'_j - 1\| < 2\sqrt{2}\|u - 1\|^{1/2} \leqslant 4\sqrt{2}\|x - 1\|^{1/2}$$

for $1 \leqslant j \leqslant 6$.

Claim. There exist invertibles $x_j, y_j, z' \in GL^0(\mathcal{A})$, $7 \leqslant j \leqslant 12$, such that

$$|x| = \Big(\prod_{j=7}^{12}(x_j, y_j)\Big)z', \quad \|z' - 1\| < \min\{\varepsilon/10, 1/100\},\ \tau(\mathrm{Log}(z')) = 0$$

for all $\tau \in T(\mathcal{A})$, and

$$\|x_j - 1\|,\ \|y_j - 1\| < 8\||x| - 1\|^{1/2}$$

for $7 \leqslant j \leqslant 12$.

Sketch of proof of Claim.

The proof is similar to the proof of (i) (also similar to the proof of Lemma 2.12). Here are the main differences:

(1) Since $|x| \geqslant 0$, $|x|$ has the form $|x| = \mathrm{e}^a$ where $a \in \mathcal{A}_{sa}$. Choose $\delta_0 > 0$ so that $|\mathrm{e}^{\|a\|+\delta_0} - 1|,\ |\mathrm{e}^{-\|a\|-\delta_0} - 1| \leqslant 2\|\mathrm{e}^a - 1\|$. Then take $X =_{\mathrm{df}} [-\|a\| - \delta_0, \|a\| + \delta_0]$. (At some point, one would need to exponentiate the images of corresponding maps $C(X) \to \mathcal{A}$.)

(2) In the proof of (i), Corollary 2.2 and Lemma 2.4 (ii) should be replaced with Lemma 2.6 of [30] and (this paper) Lemma 2.5 respectively.

End of sketch of proof of the Claim.

Note that from the Claim, it follows that for $7 \leqslant j \leqslant 12$,

$$\|x_j - 1\| \leqslant 24\|x - 1\|^{1/2} \quad \text{and} \quad \|y_j - 1\| \leqslant 24\|x - 1\|^{1/2}.$$

From the above, we have that

$$x = u|x| = \Big(\prod_{j=1}^{6}(x'_j, y'_j)\Big)v\Big(\prod_{j=7}^{12}(x_j, y_j)\Big)z' = \Big(\prod_{j=1}^{6}(x'_j, y'_j)\Big)\Big(\prod_{j=7}^{12}(vx_jv^*, vy_jv^*)\Big)vz'.$$

Let $z =_{\mathrm{df}} vz'$. Then $\|z - 1\| < \varepsilon$. Also, $(1 - \|v - 1\|)(1 - \|z' - 1\|) \geqslant (99/100)^2 > 1/2$. Hence, by Lemma 3(b) of [14], $\tau(\mathrm{Log}(z)) = \tau(\mathrm{Log}(v)) + \tau(\mathrm{Log}(z')) = 0$ for all $\tau \in T(\mathcal{A})$. ∎

Next, towards the proof of Theorem 2.18, we slightly reword Lemma 5.17 of [15] for the case of interest:

LEMMA 2.15. *Let $\mathcal{A}$ be a unital C^*-algebra with cancellation of projections and two projections $p, q \in \mathcal{A}$ with $p + q = 1$ and $u \in \mathcal{A}$ a partial isometry such that $u^*u = p$ and $uu^* \leqslant q$.*

Say that $x \in U^0(\mathcal{A})$ with $x - 1 \in p\mathcal{A}p$ and $\|x - 1\| < 1$. Then there exist $v, w \in U^0(\mathcal{A})$ and $y \in U^0(\mathcal{A})$ with $y - 1 \in q\mathcal{A}q$ such that $x = (v, w)y$, $\|y - 1\| = \|x - 1\|$, $\max\{\|v - 1\|, \|w - 1\|\} \leqslant \|x - 1\|^{1/2}$, and $T(\mathrm{Log}(y)) = T(\mathrm{Log}(x))$.

Proof. This follows immediately from the statement of Lemma 5.17 in [15], taking $qyq =_{\mathrm{df}} uxu^* + q - uu^*$ and noting that y is unitarily equivalent to x. (Here cancellation is used.) ∎

REMARK 2.16. In the proof of Theorem 2.18, we will repeatedly use Lemma 2.15, and Proposition 5.18 of [15]. We note that in the latter, if the C^*-algebra $\mathcal{A}$ is infinite dimensional simple TAI, and if the starting unitary x satisfies $\|x - 1\| < 1/4$ then all the unitaries and partial unitaries in the statement are in the connected component of the identity. This follows immediately from the statement of Proposition 5.18 in [15], and from the assumption that $\mathcal{A}$ is simple TAI.

Next, Theorem 2.18 will also require a lemma concerning the existence of projections in simple nonelementary TAI-algebras. Since no extra effort is required, we also prove it for the case of simple real rank zero C^*-algebras with strict comparison and cancellation. The next lemma generalizes Lemma 3.6 of [9]. (See also Proposition 6.1 of [15] (the proof) and Lemma 1.7 of [30].)

LEMMA 2.17. *Let $\mathcal{A}$ be a unital separable simple nonelementary C^*-algebra such that either*

(i) *$\mathcal{A}$ is* TAI, *or*

(ii) *$\mathcal{A}$ has real rank zero, strict comparison and cancellation.*

Then there exist projections p_n, q_n, r_n in $\mathcal{A}$ ($n \geqslant 1$) such that the following hold:

(a) *$p_1 + q_1 + r_1 = 1_{\mathcal{A}}$.*

(b) *$p_n \preceq q_n \preceq r_n$ ($n \geqslant 1$).*

(c) *$r_n \perp r_m$ for $n \neq m$.*

(d) *$r_n = p_{n+1} + q_{n+1}$ ($n \geqslant 1$).*

Proof. Firstly, we show that $\mathcal{A}$ is weakly divisible; i.e., for every nonzero projection $p \in \mathcal{A}$, for all $n \geqslant 2$, there is a unital embedding of $\mathbb{M}_n \oplus \mathbb{M}_{n+1}$ into $p\mathcal{A}p$.

If $\mathcal{A}$ has real rank zero then $\mathcal{A}$ is weakly divisible by Proposition 5.3 of [28].

Suppose that $\mathcal{A}$ is TAI. Then $\mathcal{A}$ has the ordered K_0 group of a simple unital real rank zero C^*-algebra (see Theorem 4.8 of [20] and Theorem 4.18 of [7]). Hence, since simple infinite dimensional nonelementary real rank zero C^*-algebras are weakly divisible ([28], Proposition 5.3), $\mathcal{A}$ is weakly divisible.

Whichever the case, we have that $\mathcal{A}$ is weakly divisible.

We now inductively construct the projections p_n, q_n, r_n $(n \geqslant 1)$.

Basis step $n = 1$.

Since $\mathcal{A}$ is weakly divisible, there is a unital embedding of $\phi : \mathbb{M}_2 \oplus \mathbb{M}_3 \to \mathcal{A}$. Let $\{e_{i,j}\}_{1 \leqslant i,j \leqslant 2}$ and $\{f_{k,l}\}_{1 \leqslant k,l \leqslant 3}$ be systems of matrix units for $\mathbb{M}_2$ and $\mathbb{M}_3$ respectively. Let $r_1 \in \mathcal{A}$ be the projection that is given by $r_1 =_{\text{df}} \phi(e_{1,1}) + \phi(f_{1,1})$. Then clearly

$$2[r_1] < [1_{\mathcal{A}}] < 3[r_1]$$

in $(K_0(\mathcal{A}), K_0(\mathcal{A})_+)$.

Hence, $2\tau(r_1) < 1 < 3\tau(r_1)$ for all $\tau \in T(\mathcal{A})$. Hence,

$$0 < \frac{\tau(r_1)}{2} < \frac{1 - \tau(r_1)}{2} < \tau(r_1) \tag{2.37}$$

for all $\tau \in T(\mathcal{A})$. Find an integer $N \geqslant 2$ such that

$$\frac{1}{N} < \inf\Big\{\tau(r_1) - \frac{1 - \tau(r_1)}{2} : \tau \in T(\mathcal{A})\Big\}. \tag{2.38}$$

(This integer exists since the map $\tau \mapsto \tau(r_1)$ is a continuous function on the compact set $T(\mathcal{A})$.)

Since $\mathcal{A}$ is weakly divisible, let $\phi_1 : \mathbb{M}_{2N} \oplus \mathbb{M}_{2N+1} \to (1 - r_1)\mathcal{A}(1 - r_1)$ be a unital embedding. Let $\{e_{1,i,j}\}_{1 \leqslant i,j \leqslant 2N}$ and $\{f_{1,k,l}\}_{1 \leqslant k,l \leqslant 2N+1}$ be systems of matrix units for $\mathbb{M}_{2N}$ and $\mathbb{M}_{2N+1}$ respectively.

Let $p_1 =_{\text{df}} \sum_{j=1}^{N} (\phi_1(e_{1,j,j}) + \phi_1(f_{1,j,j}))$. Let $q_1 =_{\text{df}} 1 - r_1 - p_1$. Then $p_1 + q_1 + r_1 = 1_{\mathcal{A}}$; and by (2.37), (2.38) and the definitions of p_1 and q_1, $\tau(p_1) < \tau(q_1) < \tau(r_1)$ for all $\tau \in T(\mathcal{A})$. Hence, by strict comparison (see Theorem 4.7 of [20] for the TAI case), $p_1 \preceq q_1 \preceq r_1$.

Induction step. Say that p_k, q_k, r_k have been constructed for $k \leqslant n$. We now construct $p_{n+1}, q_{n+1}, r_{n+1}$.

Let $r' \in \mathcal{A}$ be the projection given by $r' =_{\text{df}} \sum_{j=1}^{n-1} r_j$. (By induction hypothesis, the addends are pairwise orthogonal.)

By induction hypothesis, we have that

$$2[r_n] < [1 - r'] < 3[r_n]$$

in $(K_0(\mathcal{A}), K_0(\mathcal{A})_+)$. Hence,

$$[r_n] < [1 - r' - r_n] \leqslant 2[r_n],$$

and thus, for all $\tau \in T(\mathcal{A})$,

$$\frac{\tau(r_n)}{2} < \frac{\tau(1 - r' - r_n)}{2}. \tag{2.39}$$

Choose an integer $M \geqslant 2$ so that

$$\frac{1}{M} < \inf\Big\{\frac{\tau(1 - r' - r_n) - \tau(r_n)}{2} : \tau \in T(\mathcal{A})\Big\}. \tag{2.40}$$

(This integer exists since that maps $\tau \mapsto \tau(r')$ and $\tau \mapsto \tau(r_n)$ are continuous functions on the compact set $T(\mathcal{A})$.)

Since $\mathcal{A}$ is weakly divisible, there is a unital embedding $\phi_2 : \mathbb{M}_{2M} \oplus \mathbb{M}_{2M+1} \to (1 - r' - r_n)\mathcal{A}(1 - r' - r_n)$. Let $\{e_{2,i,j}\}_{1 \leqslant i,j \leqslant 2M}$, $\{f_{2,k,l}\}_{1 \leqslant k,l \leqslant 2M+1}$ be systems of matrix units for $\mathbb{M}_{2M}$, $\mathbb{M}_{2M+1}$ respectively.

Let $r_{n+1} \in \mathcal{A}$ be the projection that is given by $r_{n+1} =_{\mathrm{df}} \sum_{j=1}^{M} (\phi_2(e_{2,j,j}) + \phi_2(f_{2,j,j}))$.

Clearly, for $k \leqslant n$,

$$r_{n+1} \perp r_k.$$

Also, in $(K_0(\mathcal{A}), K_0(\mathcal{A})_+)$,

$$2[r_{n+1}] < [1 - r' - r_n] < 3[r_{n+1}].$$

Moreover, by (2.39), (2.40) and the definition of r_{n+1}, we have that

$$\frac{\tau(r_n)}{2} < \tau(r_{n+1}) \tag{2.41}$$

for all $\tau \in T(\mathcal{A})$. Hence, let $L \geqslant 2$ be an integer such that

$$\frac{1}{L} < \inf\Big\{\tau(r_{n+1}) - \frac{\tau(r_n)}{2} : \tau \in T(\mathcal{A})\Big\}. \tag{2.42}$$

Since $\mathcal{A}$ is weakly divisible, there exists a unital embedding $\phi_3 : \mathbb{M}_{2L} \oplus \mathbb{M}_{2L+1} \to r_n \mathcal{A} r_n$. Let $\{e_{3,i,j}\}_{1 \leqslant i,j \leqslant 2L}$ and $\{f_{3,k,l}\}_{1 \leqslant k,l \leqslant 2L+1}$ be systems of matrix units for $\mathbb{M}_{2L}$ and $\mathbb{M}_{2L+1}$ respectively.

Let $p_{n+1}, q_{n+1} \in \mathcal{A}$ be the projections given by $p_{n+1} =_{\mathrm{df}} \sum_{j=1}^{L} (\phi_3(e_{3,j,j}) + \phi_3(f_{3,j,j}))$ and $q_{n+1} =_{\mathrm{df}} r_n - p_{n+1}$. Then clearly,

$$r_n = p_{n+1} + q_{n+1} \quad \text{and} \quad p_{n+1} \preceq q_{n+1}.$$

Finally, by (2.41), (2.42) and the definition of q_{n+1}, we have that

$$q_{n+1} \preceq r_{n+1}.$$

This completes the inductive construction of p_n, q_n, r_n for $n \geqslant 1$. ∎

THEOREM 2.18. *Let $\mathcal{A}$ be a unital separable simple* TAI-*algebra.*

(i) *Suppose that $u \in U^0(\mathcal{A})$ is a unitary such that $\Delta_T(u) = 0$. Then there exist unitaries $x_j, y_j \in U^0(\mathcal{A})$, $1 \leqslant j \leqslant 34$, such that*

$$u = \prod_{j=1}^{34} (x_j, y_j).$$

(ii) *Suppose that* $x \in GL^0(\mathcal{A})$ *is an invertible such that* $\Delta_T(x) = 0$. *Then there exist invertibles* $x_j, y_j \in GL^0(\mathcal{A})$, $1 \leqslant j \leqslant 46$, *such that*

$$x = \prod_{j=1}^{46} (x_j, y_j).$$

Proof. The proof is a modification of the arguments of [15] (see also [30]), subtituting our lemmas in the appropriate places. (It is also the multiplicative version of Thierry Fack's result in [9] for additive commutators.) For the convenience of the reader, we provide the proof.

By Lemma 2.17, there exist projections p_n, q_n, r_n $(n \geqslant 1)$ in $\mathcal{A}$ which have the following properties:

(i) $p_1 + q_1 + r_1 = 1$.
(ii) $p_n \preceq q_n \preceq r_n$, $n \geqslant 1$.
(iii) $r_m \perp r_n$ when $m \neq n$.
(iv) $r_n = p_{n+1} + q_{n+1}$, $n \geqslant 1$.

By Lemma 2.13(i), there exist 20 commutators C_j $(1 \leqslant j \leqslant 20)$ in $U^0(\mathcal{A})$ and $a \in \mathcal{A}_{sa}$ with $\|e^{i2\pi a} - 1\| < 1/51200$ and $\tau(a) = 0$ for all $\tau \in T(\mathcal{A})$ such that $u = \Big(\prod_{j=1}^{20} C_j\Big) e^{i2\pi a}$. By Lemma 5.18 of [15] (and also Remark 2.16) to $e^{i2\pi a}$, there exist commutators C_{21}, C_{22} in $U^0(\mathcal{A})$ and a unitary $x_0' \in U^0(\mathcal{A})$ such that $u = \Big(\prod_{j=1}^{22} C_j\Big) x_0'$, $x_0' - 1 \in (q_1 + r_1)\mathcal{A}(q_1 + r_1)$, $\|x_0' - 1\| < 1/6400$ and $T(\mathrm{Log}(x_0')) = 0$. By applying Lemma 5.18 of [15] to x_0', there exist commutators C_{23}, C_{24} in $U^0(\mathcal{A})$ and a unitary $x_0 \in U^0(\mathcal{A})$ such that $u = \Big(\prod_{j=1}^{24} C_j\Big) x_0$, $x_0 - 1 \in r_1 \mathcal{A} r_1$, $\|x_0 - 1\| < 1/800$ and $T(\mathrm{Log}(x_0)) = 0$.

Following the argument of Proposition 6.1 of [15], we now construct (by induction) unitaries x_n, y_n^j, z_n^j $(n \geqslant 1, 1 \leqslant j \leqslant 9)$ in $U^0(\mathcal{A})$ with $x_1 = x_0$ such that the following hold:

(2.43)
(i) $\|x_n - 1\| < 1/(100n^2)$, $T(\mathrm{Log}(x_n)) = 0$ and $x_n - 1 \in r_n \mathcal{A} r_n$.
(ii) $\|y_n^j - 1\|, \|z_n^j - 1\| < 2/n$ $(1 \leqslant j \leqslant 9$.
(iii) $y_n^j - 1, z_n^j - 1 \in r_n \mathcal{A} r_n$ $(1 \leqslant j \leqslant 8)$.
(iv) $y_n^9 - 1, z_n^9 - 1 \in (r_n + r_{n+1})\mathcal{A}(r_n + r_{n+1})$.
(v) $x_n = \Big(\prod_{j=1}^{9} (y_n^j, z_n^j)\Big) x_{n+1}$.

Suppose that the unitaries $\{x_m\}_{m=1}^{n}, \{y_m^j\}_{m=1}^{n-1}, \{z_m^j\}_{m=1}^{n-1}$ $(1 \leqslant j \leqslant 9)$ have already been constructed with $x_1 = x_0$.

Apply Lemma 2.14(i) to $x_n + r_n - 1$ to get $x_n', y_n^j, z_n^j \in U^0(\mathcal{A})$ $(1 \leqslant j \leqslant 6)$ such that, for $1 \leqslant j \leqslant 6$, the following hold:

$$x_n = \Big(\prod_{j=1}^{6}(y_n^j, z_n^j)\Big)x_n', \quad x_n' - 1 \in r_n\mathcal{A}r_n, \quad \|x_n' - 1\| < \frac{1}{51200(n+1)^2},$$

$$T(\mathrm{Log}(x_n')) = 0, \quad y_n^j - 1, z_n^j - 1 \in r_n\mathcal{A}r_n, \quad \|y_n^j - 1\|, \|z_n^j - 1\| < \frac{2}{n}.$$

Apply Lemma 5.18 of [15] to p_{n+1}, q_{n+1} and $x_n' + r_n - 1 \in U^0(r_n\mathcal{A}r_n)$. (Recall that since $\mathcal{A}$ is TAI, $\mathcal{A}$ is K_1-injective; hence, $x_n' \in U^0(\mathcal{A})$ implies that $x_n' + r_n - 1 \in U^0(r_n\mathcal{A}r_n)$.) We then get $x_n'', y_n^j, z_n^j \in U^0(\mathcal{A})$ $(j = 7, 8)$ such that, for $j = 7, 8$, the following hold:

$$x_n = \Big(\prod_{j=1}^{8}(y_n^j, z_n^j)\Big)x_n'', \quad x_n'' - 1 \in q_{n+1}\mathcal{A}q_{n+1}, \quad \|x_n'' - 1\| < \frac{1}{6400(n+1)^2},$$

$$T(\mathrm{Log}(x_n'')) = 0, \quad y_n^j - 1, z_n^j - 1 \in r_n\mathcal{A}r_n, \quad \|y_n^j - 1\|, \|z_n^j - 1\| < \frac{2}{n}.$$

Now apply Lemma 2.15 to q_{n+1}, r_{n+1} and $x_n'' + q_{n+1} + r_{n+1} - 1 \in (q_{n+1} + r_{n+1})\mathcal{A}(q_{n+1} + r_{n+1})$ to get $x_{n+1}, y_n^9, z_n^9 \in U^0(\mathcal{A})$ such that the following hold:

$$x_n = \Big(\prod_{j=1}^{9}(y_n^j, z_n^j)\Big)x_{n+1}, \quad x_{n+1} - 1 \in r_{n+1}\mathcal{A}r_{n+1}, \quad \|x_{n+1} - 1\| < \frac{1}{6400(n+1)^2},$$

$$y_n^9 - 1, z_n^9 - 1 \in (q_{n+1} + r_{n+1})\mathcal{A}(q_{n+1} + r_{n+1}), \quad \|y_n^9 - 1\|, \|z_n^9 - 1\| < \frac{2}{n}.$$

This completes the inductive construction of the sequences in (2.43).

Observe that since $x_n'' - 1 \in q_{n+1}\mathcal{A}q_{n+1}$ and $x_{n+1} - 1 \in r_{n+1}\mathcal{A}r_{n+1}$ (and $x_n'' = (y_n^9, z_n^9)x_{n+1}$), we must have that $(y_n^9, z_n^9) - 1 \in r_n\mathcal{A}r_n + r_{n+1}\mathcal{A}r_{n+1}$.

We now modify the sequences in (2.43).

Let $y_0^9 =_{\mathrm{df}} z_0^9 =_{\mathrm{df}} 1$ and for $n \geqslant 1$,

$$\widetilde{y}_n^j =_{\mathrm{df}} (y_{n-1}^9, z_{n-1}^9)y_n^j(y_{n-1}^9, z_{n-1}^9)^*, \quad \widetilde{z}_n^j =_{\mathrm{df}} (y_{n-1}^9, z_{n-1}^9)z_n^j(y_{n-1}^9, z_{n-1}^9)^*$$

for $1 \leqslant j \leqslant 9$.

From the observation above, we have that $\widetilde{y}_n^j - 1, \widetilde{z}_n^j - 1 \in r_n\mathcal{A}r_n$ for $1 \leqslant j \leqslant 8$ and $\widetilde{y}_n^9 - 1, \widetilde{z}_n^9 - 1 \in (r_n + r_{n+1})\mathcal{A}(r_n + r_{n+1})$.

As a consequence, for $1 \leqslant j \leqslant 8$, the unitaries $y_n^j, z_n^j, \widetilde{y}_n^j, \widetilde{z}_n^j$ commute with the unitaries $y_m^k, z_m^k, \widetilde{y}_m^k, \widetilde{z}_m^k$ for $1 \leqslant k \leqslant 8$ and $m \neq n$, and also for $k = 9$ and $m \notin \{n-1, n, n+1\}$. One then can prove (by induction) the following two relations:

$$x_1 = \Big[\Big(\prod_{k=1}^{n}\widetilde{y}_k^1, \prod_{k=1}^{n}\widetilde{z}_k^1\Big)\cdots\Big(\prod_{k=1}^{n}\widetilde{y}_k^8, \prod_{k=1}^{n}\widetilde{z}_k^8\Big)\prod_{k=1}^{n}(y_k^9, z_k^9)\Big]x_{n+1}, \quad \text{and}$$

$$\prod_{k=1}^{2n}(y_k^9, z_k^9) = \Big(\prod_{k=1}^{n}\widetilde{y}_{2k-1}^9, \prod_{k=1}^{n}\widetilde{z}_{2k-1}^9\Big)\Big(\prod_{k=1}^{n}y_{2k}^9, \prod_{k=1}^{n}z_{2k}^9\Big).$$

For all $n \geqslant 1$, let $\overline{y}^j_n =_{\mathrm{df}} \prod_{k=1}^{2n} \widetilde{y}^j_k$ and $\overline{z}^j_n =_{\mathrm{df}} \prod_{k=1}^{2n} \widetilde{z}^j_k$ $(1 \leqslant j \leqslant 8)$, $\overline{y}^9_n =_{\mathrm{df}} \prod_{k=1}^{n} \widetilde{y}^9_{2k-1}$, $\overline{z}^9_n =_{\mathrm{df}} \prod_{k=1}^{n} \widetilde{z}^9_{2k-1}$, $\overline{y}^{10}_n =_{\mathrm{df}} \prod_{k=1}^{n} y^9_{2k}$, and $\overline{z}^{10}_n =_{\mathrm{df}} \prod_{k=1}^{n} z^9_{2k}$.

Clearly, as $n \to \infty$, the sequences $\{\overline{y}^j_n\}$, $\{\overline{z}^j_n\}$ converge in $\mathcal{A}$ to, say, $\overline{y}^j_\infty, \overline{z}^j_\infty$ respectively $(1 \leqslant j \leqslant 10)$. Also, $x_{n+1} \to 1$ as $n \to \infty$. Hence, we have that $x_1 = \prod_{j=1}^{10} (\overline{y}^j_\infty, \overline{z}^j_\infty)$. Combining this with the above, we have that

$$u = \Big(\prod_{k=1}^{24} C_k\Big)\Big(\prod_{j=1}^{10} (\overline{y}^j_\infty, \overline{z}^j_\infty)\Big);$$

i.e., u is the product of 34 commutators in $U^0(\mathcal{A})$.

The proof of (ii) is the same as the proof of (i), except that Lemma 2.13(i), Lemma 2.14(i), Lemma 2.15, and Lemma 5.18 of [15] are replaced with Lemma 2.13(ii), Lemma 2.14(ii), Lemma 5.11 of [15] and Lemma 5.1 of [15] respectively. ∎

We note that the above argument is an improvement on the (nonetheless important and interesting) argument of [30] in that there are uniform upper bounds (namely 34 and 46 for the two cases) for the number of commutators. (The proof in [30] itself does not give any upper bound and, conceivably, the number of commutators (in the argument) could get arbitrarily large depending on the unitary or invertible chosen.) The argument in [15] gives an upper bound (iv) for invertibles, but no explicit upper bound for unitaries — though the proof should lead to one.

It is an open question whether the number of commutators can be reduced.

In the next section, we will show that for the invertible case, the number (presently 34) of multiplicative commutators can be reduced to 8.

3. REDUCING THE NUMBER OF COMMUTATORS

LEMMA 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra and $p, q \in \mathcal{A}$ projections with $p+q=1$. Say that $x \in GL^0(\mathcal{A})$ is such that pxp, qxq are invertible and*

$$\|qxp(pxp)^{-1}pxq\| < \frac{1}{\|(qxq)^{-1}\|}.$$

Then there exist

$$s = \begin{bmatrix} p & 0 \\ qsp & q \end{bmatrix}, \quad t = \begin{bmatrix} p & ptq \\ 0 & q \end{bmatrix}, \quad d = \begin{bmatrix} pdp & 0 \\ 0 & qdq \end{bmatrix}$$

in $GL^0(\mathcal{A})$ such that $x = std$. Moreover, we have the following:

(i) *If x is a positive invertible, then pdp and qdq are positive invertibles.*

(ii) $\mathrm{dist}(pdp, U^0(p\mathcal{A}p)) = \mathrm{dist}(pxp, U^0(p\mathcal{A}p))$ *and* $\mathrm{dist}(qdq, U^0(q\mathcal{A}q)) \leqslant \mathrm{dist}(qxq, U^0(q\mathcal{A}q)) + \|qxp(pxp)^{-1}pxq\|$.

The proof is exactly the same as that of Lemma 5.8 in [15].

LEMMA 3.2. *Let $\mathcal{A}$ be a unital simple separable C^*-algebra such that either*
(i) *$\mathcal{A}$ is a* TAI*-algebra, or*
(ii) *$\mathcal{A}$ has real rank zero, strict comparison and cancellation of projections.*
Let $x \in \mathcal{A}$ be either a positive invertible or $\mathrm{dist}(x, U^0(\mathcal{A})) < 1/10$. Then for every nonzero projection $r \in \mathcal{A}$ with $r \neq 1$, there exists a projection $p \in \mathcal{A}$ with $p \sim r$ such that pxp and $(1-p)x(1-p)$ are invertible and

$$\|(1-p)xp(pxp)^{-1}px(1-p)\| < \frac{1}{\|((1-p)x(1-p))^{-1}\|}.$$

Moreover, in the case where $\mathrm{dist}(x, U^0(\mathcal{A})) < 1/10$, for every $\varepsilon > 0$, we can choose p so that

$$\|(1-p)xp(pxp)^{-1}px(1-p)\| \leqslant \frac{(\mathrm{dist}(x, U^0(\mathcal{A})) + \varepsilon)^2}{\sqrt{1 - (21/10)(\mathrm{dist}(x, U^0(\mathcal{A})) + \varepsilon)}}$$

and $\mathrm{dist}(pxp, U^0(p\mathcal{A}p)), \mathrm{dist}((1-p)x(1-p), U^0((1-p)\mathcal{A}(1-p))) \leqslant \mathrm{dist}(x, U^0(\mathcal{A})) + \varepsilon$. *(Note that the last quantity is bounded above by $1/10$, when ε is small enough.)*

Proof. Let us first assume that $\mathcal{A}$ is unital simple infinite-dimensional and TAI. We will prove the case where $\mathrm{dist}(x, U^0(\mathcal{A})) < 1/10$.

Let $u \in U^0(\mathcal{A})$ be such that $\|x - u\| < 1/10$. We may assume that $\varepsilon < 1/10 - \|x - u\|$.

Firstly, multiplying u (and also x) by a scalar in $\mathbb{T}$ if necessary, we may assume that $1 \in \mathrm{sp}(u)$. (Note that all relevant statements and inequalities are preserved under such a multiplication.)

Choose $\delta > 0$ such that if $c, d \in \mathcal{A}_{sa}$ with $\|c - d\| < \delta$ then $\|e^{i2\pi c} - e^{i2\pi d}\| < \varepsilon/100$. We may assume that $\delta < \varepsilon/100$ and that if $\alpha \in \mathbb{R}$ and $|\alpha| < \delta$ then $|e^{i2\pi\alpha} - 1| < \varepsilon/100 < 1/100$.

By Theorem 3.3 of [22], let $a \in \mathcal{A}$ be a self-adjoint element such that $\|u - e^{i2\pi a}\| < \varepsilon/100$. Since $1 \in \mathrm{sp}(u)$, we may assume that $0 \in \mathrm{sp}(a)$.

Let $f \in (-\infty, \infty) \to [0,1]$ be a continuous function such that

$$f(\alpha) \begin{cases} > 0 & \alpha \in (-\delta/10, \delta/10), \\ = 0 & \alpha \notin (-\delta/10, \delta/10). \end{cases}$$

Since $0 \in \mathrm{sp}(a)$, $f(a) \neq 0$. Hence, since $\mathcal{A}$ has (SP) (see Theorem 3.2 of [20]), let $e \in \mathcal{A}$ be a nonzero projection such that $e \in \mathrm{Her}(f(a))$. Moreover, since $\mathcal{A}$ is simple TAI and $r \neq 1$, we may choose e so that $e \prec 1 - r$. (Note that $\mathcal{A}$ is weakly divisible (see the argument for the existence of $\{p_n, q_n, r_n\}$ in the proof of Theorem 2.18) and has strict comparison (see Theorem 4.7 of [20]).) Hence, let $r' \in \mathrm{Her}(1 - e)$ be a projection such that $r' \sim r$.

Also, $\|(1-e)a(1-e)-a\| = \|-ea-ae+eae\| \leqslant \|e\chi_{(-\delta/10,\delta/10)}(a)a\| + \|a\chi_{(-\delta/10,\delta/10)}(a)e\| + \|e\chi_{(-\delta/10,\delta/10)}(a)ae\| < 3\delta/10$. Hence,

$$\|u-\mathrm{e}^{\mathrm{i}2\pi(1-e)a(1-e)}\| \leqslant \|u-\mathrm{e}^{\mathrm{i}2\pi a}\| + \|\mathrm{e}^{\mathrm{i}2\pi a}-\mathrm{e}^{\mathrm{i}2\pi(1-e)a(1-e)}\| < \frac{\varepsilon}{100}+\frac{\varepsilon}{100} = \frac{\varepsilon}{50}.$$

Since $\mathcal{A}$ is TAI, $\mathrm{Her}(1-r)$ is TAI, and there exist a projection $p' \in \mathrm{Her}(1-e)$ and a C^*-subalgebra $\mathcal{B} \subset \mathrm{Her}(1-e)$ with $\mathcal{B} \in \mathbb{INT}$ such that $p' \preceq e$, $1_{\mathcal{B}} = 1-e-p'$, unitaries $u_1 \in U^0(p'\mathcal{A}p')$, $u_2 \in U^0(\mathcal{B})$ and projectons $r'' \in \mathrm{Her}(p')$, $r''' \in \mathcal{B}$ such that

$$\|(1-e)\mathrm{e}^{\mathrm{i}2\pi(1-e)a(1-e)}(1-e)-(u_1\oplus u_2)\| < \frac{\varepsilon}{100} \quad \text{and} \quad \|r'-(r''\oplus r''')\| < \frac{\varepsilon}{100}.$$

Note that $\|x-(e\oplus u_1\oplus u_2)\| \leqslant \|x-u\| + \|u-\mathrm{e}^{\mathrm{i}2\pi(1-e)a(1-e)}\| + \|\mathrm{e}^{\mathrm{i}2\pi(1-e)a(1-e)} -(e\oplus u_1\oplus u_2)\| < \|x-u\|+\varepsilon/50+\varepsilon/100 = \|x-u\|+3\varepsilon/100 < 1/10$. We denote this computation by "$(*)$".

For simplicity, let us assume that $\mathcal{B} \cong \mathbb{M}_N(C[0,1])$.

Since $r'' \leqslant p' \preceq e$, let $p'' \leqslant e$ be a projection such that $p'' \sim r''$. Also, by Lemma 1.9 of [29], there exist pairwise orthogonal minimal projections $p_1, p_2, \ldots, p_N \in \mathbb{M}_N(C[0,1])$ and continuous functions $g_1, g_2, \ldots, g_N : [0,1] \to \mathbb{T}$ such that $\left\|u_2 - \sum_{j=1}^{N} g_j p_j\right\| < \varepsilon/100$.

Suppose that r''' is the sum of M minimal projections in $\mathcal{B}$ (where $M \leqslant N$). Then take $p =_{\mathrm{df}} p'' + \sum_{j=1}^{M} p_j$. Clearly, $p \sim r$.

By $(*)$, $\left\|pxp - \left(p''+\sum_{j=1}^{M} g_j p_j\right)\right\| \leqslant \|pxp - p(e\oplus u_1\oplus u_2)p\| + \left\|p(e\oplus u_1 \oplus u_2)p - \left(p''+\sum_{j=1}^{M} g_j p_j\right)\right\| < \|x-u\| + 3\varepsilon/100 + 0 < 1/10$. We note in particular that since $p''+\sum_{j=1}^{M} g_j p_j$ is in $U^0(p\mathcal{A}p)$, pxp is invertible and $\mathrm{dist}(pxp, U^0(p\mathcal{A}p)) < 1/10$.

By $(*)$, $\left\|(1-p)x(1-p) - \left((e-p'')\oplus u_1 \oplus \sum_{j=M+1}^{N} g_j p_j\right)\right\| \leqslant \|(1-p)x(1-p) - (1-p)(e\oplus u_1\oplus u_2)(1-p)\| + \left\|(1-p)(e\oplus u_1\oplus u_2)(1-p) - \left((e-p'')\oplus u_1 \oplus \sum_{j=M+1}^{N} g_j p_j\right)\right\| < \|x-u\|+3\varepsilon/100+0 < 1/10$. Note in particular that since $(e-p'')\oplus u_1\oplus \sum_{j=M+1}^{N} g_j p_j$ is in $U^0((1-p)\mathcal{A}(1-p))$, $(1-p)x(1-p)$ is invertible and $\mathrm{dist}((1-p)x(1-p), U^0((1-p)\mathcal{A}(1-p))) < 1/10$.

Note that since u, ε are arbitrary, the computations in the previous two paragraphs actually show that for every $\varepsilon > 0$, we can choose p so that $\mathrm{dist}(pxp, U^0(p\mathcal{A}p)), \mathrm{dist}((1-p)x(1-p), U^0((1-p)\mathcal{A}(1-p))) \leqslant \mathrm{dist}(x, U^0(\mathcal{A})) + \varepsilon$.

To simplify notation, let $u_3 =_{\mathrm{df}} p'' + \sum_{j=1}^{M} g_j p_j$. Then $\|(pxp)^* pxp - 1\| = \|(pxp)^* pxp - (pxp)^* u_3\| + \|(pxp)^* u_3 - u_3^* u_3\| \leqslant \|(pxp)^*\| \|pxp - u_3\| + \|(pxp)^* - u_3^*\| < (1 + 1/10)(1/10) + 1/10 = 21/100$. Hence, $\mathrm{sp}((pxp)^* pxp) \subset (1 - 21/100, 1 + 21/100)$. Hence, $\|((pxp)^* pxp)^{-1}\| < 1/(1 - 21/100) = 100/79$. Hence, $\|(pxp)^{-1}\| \leqslant 10/\sqrt{79}$. By a similar argument, $\|((1-p)x(1-p))^{-1}\| \leqslant 10/\sqrt{79}$.

Note that $(*)$ and the computation in the previous paragraph actually shows that $\|(pxp)^* pxp - 1\| \leqslant (21/10)\|pxp - pu_3 p\| \leqslant (21/10)(\|x - u\| + 3\varepsilon/100)$. Since u was arbitrary, for every $\varepsilon > 0$, we can choose p so that $\mathrm{sp}((pxp)^* pxp) \subset (1 - (21/10)(\mathrm{dist}(x, U^0(\mathcal{A})) + \varepsilon), 1 + (21/10)(\mathrm{dist}(x, U^0(\mathcal{A})) + \varepsilon))$. Hence, for every $\varepsilon > 0$, we can choose p so that $\|(pxp)^{-1}\| \leqslant 1/\sqrt{1 - (21/10)(\mathrm{dist}(x, U^0(\mathcal{A})) + \varepsilon)}$.

Next, $\|px(1-p)\| \leqslant \|px(1-p) - pu(1-p)\| + \|pu(1-p) - p(e \oplus u_1 \oplus u_2)(1-p)\| + \|p(e \oplus u_1 \oplus u_2)(1-p)\| < 1/10 + 3\varepsilon/100 + 0 < 1/10 + 3/1000 = 103/1000$. Similarly, $\|(1-p)xp\| < 103/1000$. Hence, $\|(1-p)xp\| \|px(1-p)\| < 10609/1000000 < 79/100 \leqslant \frac{1}{\|(pxp)^{-1}\| \|((1-p)x(1-p))^{-1}\|}$. Hence,

$$\|(1-p)xp(pxp)^{-1}px(1-p)\| < \frac{1}{\|((1-p)x(1-p))^{-1}\|}.$$

Since u, ε are arbitrary, the computation of the previous paragraph also yields that for every $\varepsilon > 0$, we can choose p so that $\|px(1-p)\|, \|(1-p)xp\| \leqslant \mathrm{dist}(x, U^0(\mathcal{A})) + \varepsilon$. Hence, for every $\varepsilon > 0$, we can choose p so that

$$\|(1-p)xp(pxp)^{-1}px(1-p)\| \leqslant \frac{(\mathrm{dist}(x, U^0(\mathcal{A})) + \varepsilon)^2}{\sqrt{1 - (21/10)(\mathrm{dist}(x, U^0(\mathcal{A})) + \varepsilon)}}.$$

The proof for the case where x is a positive invertible is similar (and easier).

For the case where $\mathcal{A}$ has real rank zero, strict comparison and cancellation, one uses that if y is positive invertible or unitary in $U^0(\mathcal{A})$ then y can be approximated (arbitrarily close in norm) by positive invertibles with finite spectrum or unitaries with finite spectrum, respectively. (See, for example, [17].) One also uses that $\mathcal{A}$ has strict comparison and the Riesz property. ∎

LEMMA 3.3. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra such that either*

(i) $\mathcal{A}$ *is* TAI, *or*

(ii) $\mathcal{A}$ *has real rank zero, strict comparison and cancellation of projections.*

Let $x \in \mathcal{A}$ be a unitary in $U^0(\mathcal{A})$ or a positive invertible. Then there exist pairwise orthogonal projections $p_1, p_2, \ldots, p_{93} \in \mathcal{A}$ with $\sum_{j=1}^{93} p_j = 1_{\mathcal{A}}$ and $p_j \sim p_k$ for $1 \leqslant j, k \leqslant 47$ or $48 \leqslant j, k \leqslant 93$, and elements $s, t, d \in GL(\mathcal{A})$ such that the following hold:

(a) *s is lower triangular: $s = 1 + \sum_{j>k} p_j s p_k$.*

(b) *t is upper triangular: $t = 1 + \sum_{j<k} p_j t p_k$.*

(c) *d is diagonal:* $d = \sum_j p_j d p_j$.

(d) $x = std$.

Moreover, (if $x \in U^0(\mathcal{A})$) we can choose the projections p_j so that for $1 \leqslant j \leqslant 93$, $p_j d p_j \in GL^0(p_j \mathcal{A} p_j)$ (in $U^0(p_j \mathcal{A} p_j)$ respectively).

Proof. Since $\mathcal{A}$ is weakly divisible (see the second paragraph in the proof of Theorem 2.18), there exist nonzero projections $p, q \in \mathcal{A}$ such that $47[p] + 46[q] = [1_{\mathcal{A}}]$.

The rest of the proof is similar to Lemma 6.4 of [15], except that we use (this paper) Lemma 3.1 and Lemma 3.2 instead of Lemma 6.3 in [15]. For the case where x is a unitary, in order to make the induction work, we additionally need to use the norm estimates in Lemmas 3.1 and 3.2, which require ε to be sufficiently small (at each step of the induction). By inspection, taking $\varepsilon = 1/10^{1000}$ (for all the steps) will suffice. ∎

LEMMA 3.4. *Let $\mathcal{A}$ be a unital simple separable C^*-algebra such that either*

(i) *$\mathcal{A}$ is a* TAI-*algebra, or*

(ii) *$\mathcal{A}$ has real rank zero, strict comparison and cancellation of projections.*

Let $x \in \mathcal{A}$ be either a unitary in $U^0(\mathcal{A})$ or a positive invertible element. Then there exist pairwise orthogonal and pairwise (Murray–von Neumann) equivalent projections $q_1, q_2, \ldots, q_{46} \in \mathcal{A}$ and elements $x_1, y_1, x_2, y_2, z \in GL(\mathcal{A})$ with

$$x = (x_1, y_1)(x_2, y_2)z \quad \text{and} \quad z - 1 \in q_1 \mathcal{A} q_1.$$

For the proof, the argument is exactly the same as Lemma 6.5 of [15], but where we use Lemma 3.3 instead of Lemma 6.4 in [15].

THEOREM 3.5. *Let $\mathcal{A}$ be a unital simple separable* TAI-*algebra. Let $x \in GL^0(\mathcal{A})$ be such that $\Delta_T(x) = 0$. Then there exist $x_j, y_j \in GL^0(\mathcal{A})$, $1 \leqslant j \leqslant 8$, such that*

$$x = \prod_{j=1}^{8} (x_j, y_j).$$

If, in addition, x is a unitary (in $U^0(\mathcal{A})$) or a positive invertible, then there exist $x_j, y_j \in GL^0(\mathcal{A})$ (not necessarily unitary or positive), $1 \leqslant j \leqslant 4$, such that

$$x = \prod_{j=1}^{4} (x_j, y_j).$$

Proof. In the case where x is either a unitary (in the connected component of the identity) or a positive invertible, the proof is exactly the same as Theorem 6.6 of [15], except that Lemma 6.5 of [15] is replaced with Lemma 3.4; and also, Proposition 6.1 of [15] is replaced with Theorem 2.18.

Now for the general case. If $x \in GL^0(\mathcal{A})$ is arbitrary, let $x = u|x|$ be the polar decomposition of x. Then by Lemma 2.11, $\Delta_T(u) = \Delta_T(|x|) = 0$. Then, by the cases for unitaries and positive invertibles, u and $|x|$ are both the product of 4 multiplicative commutators. Hence, x is the product of 8 multiplicative commutators, as required. ∎

4. THE REAL RANK ZERO CASE

THEOREM 4.1. *Let $\mathcal{A}$ be a unital simple separable C^*-algebra with real rank zero, strict comparison, and cancellation of projections.*

(i) *Suppose that $u \in U^0(\mathcal{A})$ is a unitary such that $\Delta_T(u) = 0$. Then there exist unitaries $x_j, y_j \in U^0(\mathcal{A})$, $1 \leqslant j \leqslant 34$, such that*

$$u = \prod_{j=1}^{34}(x_j, y_j).$$

(ii) *Suppose that $x \in GL^0(\mathcal{A})$ is an invertible such that $\Delta_T(x) = 0$. Then there exist invertibles $x_j, y_j \in GL^0(\mathcal{A})$, $1 \leqslant j \leqslant 46$, such that*

$$x = \prod_{j=1}^{46}(x_j, y_j).$$

Proof. The proof of this theorem is very similar to the proof of Theorem 2.18.

Firstly, by [19], there exists a unital simple AH-algebra $\mathcal{C}$ with bounded dimension growth and real rank zero and a unital $*$-homomorphism $\Phi : \mathcal{C} \to \mathcal{A}$ such that Φ is an isomorphism at the level of the K-theory invariant; i.e., we have the following:

(i) The following induced map is an isomorphism of ordered groups with unit:

$$K_*(\Phi) : (K_0(\mathcal{C}), K_0(\mathcal{C})_+, K_1(\mathcal{C}), [1_\mathcal{C}]) \to (K_0(\mathcal{A}), K_0(\mathcal{A})_+, K_1(\mathcal{A}), [1_\mathcal{A}]).$$

(ii) The induced map $T(\Phi) : T(\mathcal{A}) \to T(\mathcal{C})$ is an affine homeomorphism.

Replacing $\mathcal{C}$ with $\Phi(\mathcal{C})$ if necessary, we may assume that $\mathcal{C}$ is a unital C^*-subalgebra of $\mathcal{A}$. We denote the above statements by "(+)".

The proof (both parts (i) and (ii)) is exactly the same as the argument leading up to Theorem 2.18. In particular, one needs prove analogues to Lemma 2.12, Lemma 2.13 and Lemma 2.14 as well as the argument of Theorem 2.18 itself.

Here are the main additional ingredients:

(iii) Since $\mathcal{A}$ has real rank zero, if $u \in U^0(\mathcal{A})$, then u can be approximated by unitaries with finite spectrum ([17]). More precisely, for every $\delta_2 > 0$, there exists a self-adjoint element $a \in \mathcal{A}$, with finite spectrum, such that

$$\|u - \mathrm{e}^{\mathrm{i}2\pi a}\| < \delta_2.$$

(E.g., the above statement replaces the statement (2.2) from Lemma 2.12.) Note also that by (+) statement (i) (and since $\mathcal{A}$ has cancellation of projections), there exists a unitary $z \in \mathcal{A}$ such that $zaz^* \in \mathcal{C}$ and hence, $z\mathrm{e}^{\mathrm{i}2\pi a}z^* = \mathrm{e}^{i2\pi zaz^*} \in \mathcal{C}$. We then work with $\mathrm{e}^{\mathrm{i}2\pi zaz^*}$ inside $\mathcal{C}$, which is TAI.

(iv) If $x \in GL^0(\mathcal{A})$ is a *positive* invertible, then, since $\mathcal{A}$ has real rank zero, x can be approximated arbitrarily close by positive invertibles with finite spectrum. Once more, by (+) statement (i) (and since $\mathcal{A}$ has cancellation), these positive invertibles are unitarily equivalent to positive invertibles in $\mathcal{C}$, and we work in $\mathcal{C}$, which is TAI. ∎

The reduction of commutators argument goes through with essentially no change.

THEOREM 4.2. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra with real rank zero, strict comparison and cancellation of projections. Let $x \in GL^0(\mathcal{A})$ such that $\Delta_T(x) = 0$. Then there exist $x_j, y_j \in GL^0(\mathcal{A})$, $1 \leqslant j \leqslant 8$, such that*

$$x = \prod_{j=1}^{8}(x_j, y_j).$$

If, in addition, x is a unitary (in $U^0(\mathcal{A})$) or a positive invertible, then there exist $x_j, y_j \in GL^0(\mathcal{A})$ (not necessarily unitary or positive), $1 \leqslant j \leqslant 4$, such that

$$x = \prod_{j=1}^{4}(x_j, y_j).$$

Proof. The proof is exactly the same as the proof of Theorem 3.5, except that Theorem 2.18 is replaced with Theorem 4.1. Note that all the preliminary lemmas leading up to Theorem 3.5 include the real rank zero case. ∎

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