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WEAK TYPE ESTIMATES FOR THE ABSOLUTE VALUE MAPPING

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ABSTRACT. We prove that if A and B are bounded self-adjoint operators such that $A - B$ belongs to the trace class, then $|A| - |B|$ belongs to the principal ideal $\mathcal{L}_{1,\infty}$ in the algebra $\mathcal{L}(H)$ of all bounded operators on an infinite-dimensional Hilbert space generated by an operator whose sequence of eigenvalues is $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. Moreover, $\mu(j; |A| - |B|) \leqslant \mathrm{const}(1 + j)^{-1}\|A - B\|_1$. We also obtain a semifinite version of this result, as well as the corresponding commutator estimates.

KEYWORDS: *Operator ideals, commutator estimates, Lipschitz estimates, Schatten classes, weak type inequalities, absolute value mapping.*

MSC (2010): 47B10, 47L20, 47A30.

1. INTRODUCTION

Let H be a complex separable Hilbert space, let $\mathcal{K}(H)$ be the $*$-algebra of all compact operators on H and let $\mathcal{L}_p$, $1 \leqslant p < \infty$, be the p-th Schatten–von Neumann class (that is the class of all operators A from $\mathcal{K}(H)$ such that $\|A\|_p := \Big(\sum\limits_{k=0}^{\infty} \mu(k; A)^p \Big)^{1/p} < \infty$, where $\{\mu(k; A)\}_{k=0}^{\infty}$ is the sequence of singular numbers of the operator A [15], [19]). The following result was proved by E.B. Davies ([8], Theorem 8; for its extension to semifinite von Neumann algebras, we refer to [11]).

THEOREM 1.1. *If A, B are self-adjoint bounded operators on H and if $A - B \in \mathcal{L}_p, 1 < p < \infty$, then $|A| - |B| \in \mathcal{L}_p$ and*

$$\||A| - |B|\|_p \leqslant c_p \|A - B\|_p.$$

Here, c_p depends only on p and $c_p = O(p)$ as $p \to \infty$ and $c_p = O((p-1)^{-1})$ as $p \to 1$.

For various extensions and generalizations of Theorem 1.1, we refer to the papers [4], [11], [12], [18], [20] studying the Lipschitz continuity of the absolute value mapping $A \to |A|$ in the setting of symmetrically-normed ideals (and more general symmetric operator spaces). Here, we contribute to an interesting open question concerning the optimal form of Theorem 1.1 in the crucial case $p = 1$. It is well known (see Section 3 of [8]) that the absolute value mapping is not Lipschitz continuous in the trace class $(\mathcal{L}_1, \|\cdot\|_1)$. It was proved by H. Kosaki ([18], Theorem 12; see also Corollary 3.4 of [12]) that the absolute value mapping is Lipschitz continuous from $(\mathcal{L}_1, \|\cdot\|_1)$ into Banach ideal $(\mathcal{M}_{1,\infty}, \|\cdot\|_{\mathcal{M}_{1,\infty}})$, where

$$\mathcal{M}_{1,\infty} := \Big\{ A \in \mathcal{K}(H) : \|A\|_{\mathcal{M}_{1,\infty}} := \sup_{N \geqslant 0} \frac{1}{\log(N+2)} \sum_{k=0}^{N} \mu(k; A) < \infty \Big\}.$$

The main objective of this paper is to show that the latter result holds if we replace $(\mathcal{M}_{1,\infty}, \|\cdot\|_{\mathcal{M}_{1,\infty}})$ with a smaller (quasi-Banach) ideal $(\mathcal{L}_{1,\infty}, \|\cdot\|_{1,\infty})$, where

$$\mathcal{L}_{1,\infty} := \Big\{ A \in \mathcal{K}(H) : \|A\|_{\mathcal{L}_{1,\infty}} := \sup_{k \geqslant 0} (k+1)\mu(k; A) < \infty \Big\}. \tag{1.1}$$

THEOREM 1.2. *If A, B are self-adjoint bounded operators on H and if $A - B \in \mathcal{L}_1$, then $|A| - |B| \in \mathcal{L}_{1,\infty}$ and*

$$\||A| - |B|\|_{1,\infty} \leqslant \Big(34 + \frac{2560e}{\pi}\Big) \|A - B\|_1. \tag{1.2}$$

The strength of Theorem 1.2 is seen from the fact that it implies the result of Theorem 1.1 via a combination of methods used in [8], [11] linking Lipschitz continuity and commutator estimates with a noncommutative version of the Boyd interpolation theorem (see e.g. Theorem 5.8 of [9]). We refer to Remark 6.2 for more details. Such an implication is of course not available from the results of Theorem 12 in [18] and Corollary 3.4 of [12]. The result of Theorem 1.2 is also sharp in the sense that the quasi-norm $\|\cdot\|_{1,\infty}$ is the largest symmetric quasi-norm on the ideal of finite rank operators for which (1.2) holds (the latter follows from the proof of Lemma 10 in [8]).

From a certain perspective, the result of Theorem 1.2 is not unexpected. Indeed, the proof of Theorem 1.1 in [8], as well as the proofs of its analogues and extensions from [4], [11], [12], [18] are ultimately based on the famous results due to V.I. Macaev, I.C. Gohberg and M.G. Krein (see [7], [15]), describing the behavior of (generalized) triangular truncation operators in Schatten–von Neumann classes $\mathcal{L}_p$. In the case when $p = 1$, these results yield the fact that the latter operator acts boundedly from the Banach space $(\mathcal{L}_1, \|\cdot\|_1)$ into a quasi-Banach space $(\mathcal{L}_{1,\infty}, \|\cdot\|_{1,\infty})$. However, (and here lies the major difficulty) all the proofs in the just listed papers involve certain integration processes, which render them inapplicable in the quasi-normed setting. Exactly the same obstacle also manifested itself in Theorem 2.5(i) of [20]. Indeed, that theorem yields the result of Theorem 1.2 under the restrictive assumption that $\mathrm{rank(A - B)} = 1$ and the methods used in [20] do not seem applicable to treat the general case. To circumvent this difficulty, we employ a completely different approach coming back to a celebrated theorem of I. Schur concerning positive semidefiniteness of a Schur (or Hadamard) product of two semidefinite matrices.

Section 3 contains the proof of Theorem 1.2. In Section 4 we find a sharper result assuming the extra condition that A and B are compact operators. In this case it turns out that $|A| - |B|$ in fact lands in the separable part of $\mathcal{L}_{1,\infty}$, see Theorem 4.3. Section 5 contains the extension of Theorem 1.2 to the setting of semifinite von Neumann algebras. This theme has been explored already in [12], however, methods employed there (again due to the obstacle explained above) were not sufficiently strong to obtain the weak type estimate similar to (1.2). Furthermore, the setting used in [12] was restricted to the case of semifinite factors. The approach used in this paper allows us to dispense with the latter condition. In Section 6 we treat the consequences of Theorem 1.2 for commutator estimates. In the final Section we give a treatment of the consequences of Theorem 1.2 for certain Lipschitz functions f belonging to a subclass of the Davies class (the case $f = |\cdot|$ being Theorem 1.2). Note that for general Lipschitz functions f outside of that subclass the question whether the weak $(1,1)$ estimate holds remains open.

2. PRELIMINARIES

2.1. SINGULAR VALUES. Let $\mathcal{L}(H)$ be the $*$-algebra of all bounded operators on the Hilbert space H equipped with a uniform norm $\|\cdot\|_\infty$.

Every proper ideal in $\mathcal{L}(H)$ consists of compact operators. For brevity, we set $\mu(A) := \{\mu(k, A)\}_{k\geqslant 0}$.

If $B \in \mathcal{L}(H)$ and $A \in \mathcal{K}(H)$, then it is well known that

$$\mu(AB) \leqslant \|B\|_\infty \mu(A), \quad \mu(BA) \leqslant \|B\|_\infty \mu(A), \quad \mu(A^*) = \mu(A). \tag{2.1}$$

If $A, B \in \mathcal{K}(H)$, then (see e.g. Corollary 2.3.16 in [19]) we have

$$\mu(A + B) \leqslant \sigma_2(\mu(A) + \mu(B)). \tag{2.2}$$

Here, the dilation operator $\sigma_2 : l_\infty \to l_\infty$ (acting on the space l_∞ of all complex bounded sequences) is defined as follows

$$\sigma_2(a_0, a_1, \ldots) = (a_0, a_0, a_1, a_1, \ldots).$$

Two self-adjoint operators $A, B \in \mathcal{K}(H)$ are called *identically distributed* if we have $\mu(A_+) = \mu(B_+)$ and $\mu(A_-) = \mu(B_-)$. Here, $A = A_+ - A_-$ is the orthogonal decomposition of a self-adjoint operator A (see e.g. p. 36 of [5]). A self-adjoint operator $A \in \mathcal{K}(H)$ is called *symmetrically distributed* if $\mu(A_+) = \mu(A_-)$.

In what follows, the symbol $\mathrm{supp}(A)$ stands for the support projection of a self-adjoint operator $A \in \mathcal{L}(H)$ (that is, the spectral projection of A corresponding to the set $\mathbb{R}\backslash\{0\}$).

2.2. IDEAL $\mathcal{L}_{1,\infty}$. Let $A_0 \in \mathcal{K}(H)$ be such that $\mu(A_0) = \{1/(k+1)\}_{k\geqslant 0}$. The principal ideal generated by A_0 is frequently called weak-$\mathcal{L}_1$ and coincides with $\mathcal{L}_{1,\infty}$. The mapping $\|\cdot\|_{1,\infty}$ on $\mathcal{L}_{1,\infty}$ (see (1.1)) is a quasi-norm. Indeed, it follows from (2.2) that

$$\|A+B\|_{1,\infty} \leqslant 2\|A\|_{1,\infty} + 2\|B\|_{1,\infty}. \tag{2.3}$$

It can be shown (see e.g. Theorem 2.11.32 of [22]) that the quasi-normed space given by $(\mathcal{L}_{1,\infty}, \|\cdot\|_{1,\infty})$ is complete and is, therefore, a quasi-Banach ideal. It is important to note that the quasi-norm $\|\cdot\|_{1,\infty}$ is not equivalent to any norm. In particular, the weak form of triangle inequality in (2.3) is the best possible.

The ideals $\mathcal{L}_p$ and $\mathcal{L}_{1,\infty}$ both have the Fatou property. That is, if $A_n \in \mathcal{L}_{1,\infty}$, $\|A_n\|_{1,\infty} \leqslant 1$ and $A_n \to A$ in measure, then $A \in \mathcal{L}_{1,\infty}$ and $\|A\|_{1,\infty} \leqslant 1$. Exactly the same assertion holds for $\mathcal{L}_p$.

2.3. SCHUR MULTIPLICATION. Let

$$M_n = \{A = \{A_{k,l}\}_{k,l=0}^{n-1}\}$$

be the $*$-algebra of all complex $n \times n$ matrices. The algebra M_n is $*$-isomorphic to a subalgebra $P_n\mathcal{L}(H)P_n$ in $\mathcal{L}(H)$, where P_n is a projection in $\mathcal{L}(H)$ such that $\mathrm{Tr}(P_n) = n$, where Tr is the standard trace on $\mathcal{L}(H)$. We also frequently identify M_n with $\mathcal{L}(H)$ when $\dim(H) = n$. In the latter case, all previously introduced notations (e.g. $\mu(A), \|\cdot\|_p, \|\cdot\|_{1,\infty}, \mathrm{Tr}, \mathrm{supp}(A)$) and terminology (e.g. identically distributed) remain unambiguously defined.

For every $A, B \in M_n$, we define their Schur product (also called Hadamard product) $A \circ B$ by setting

$$(A \circ B)_{k,l} = A_{k,l}B_{k,l}, \quad 0 \leqslant k,l \leqslant n-1.$$

Fix $B \in M_n$. The Schur multiplication operator $M_B : M_n \to M_n$ is defined by setting $M_B(A) = A \circ B$. If $B \geqslant 0$, then, according to the Schur theorem, we have that $M_B(A) \geqslant 0$ for every $A \geqslant 0$. For a beautiful exposition of the latter theorem and other relevant properties of Schur multiplication, we refer the reader to [2]. For the next lemma, see [1]. We include a short proof for completeness.

LEMMA 2.1. *If $B \geqslant 0$, then*

$$\|M_B(A)\|_1 \leqslant 4\|\mathrm{diag}(B)\|_\infty \cdot \|A\|_1, \quad A \in M_n.$$

Proof. Suppose first that $A \geqslant 0$. It follows that $M_B(A) \geqslant 0$. Hence,

$$\begin{aligned}\|M_B(A)\|_1 &= \mathrm{Tr}(M_B(A)) = \mathrm{Tr}(A \circ B) = \sum_{k=0}^{n-1} A_{k,k}B_{k,k} \\ &\leqslant \Big(\max_{0\leqslant k<n} B_{k,k}\Big) \cdot \Big(\sum_{k=0}^{n-1} A_{k,k}\Big) = \|\mathrm{diag}(B)\|_\infty \mathrm{Tr}(A) = \|\mathrm{diag}(B)\|_\infty \|A\|_1.\end{aligned}$$

Consider now the general case of an arbitrary $A \in M_n$. Using Jordan decomposition (see e.g. p. 216 of [5]), we write

$$A = A_1 - A_2 + \mathrm{i}A_3 - \mathrm{i}A_4, \quad A_m \geqslant 0,\ \|A_m\|_1 \leqslant \|A\|_1, \quad 1 \leqslant m \leqslant 4.$$

Therefore,

$$\|M_B(A)\|_1 \leqslant \sum_{m=1}^{4} \|M_B(A_m)\|_1 \leqslant \|\mathrm{diag}(B)\|_\infty \cdot \Big(\sum_{m=1}^{4} \|A_m\|_1 \Big). \quad \blacksquare$$

3. PROOF OF THEOREM 1.2

The following lemma can be found in [3]. Its short proof is included for convenience of the reader.

LEMMA 3.1. *If $\alpha_k > 0$, $0 \leqslant k < n$, are decreasing, then the matrix*

$$\Phi = \Big\{ \frac{\alpha_{\max\{k,l\}}}{\alpha_k + \alpha_l} \Big\}_{k,l=0}^{n-1}$$

is positive semidefinite.

Proof. Set

$$\Phi_1 = \Big\{ \frac{1}{\alpha_k + \alpha_l} \Big\}_{k,l=0}^{n-1}.$$

Consider (rank one) projections p_k, $0 \leqslant k < n$, given by diagonal matrix units. That is,

$$(p_k)_{i,j} = \begin{cases} 0 & i \neq k, \\ 0 & j \neq k, \\ 1 & i = j = k, \end{cases}$$

and set

$$P_k := \sum_{j=0}^{k-1} p_j, \quad 0 \leqslant k < n.$$

The following equality can be verified directly:

$$\Phi = \Big(\sum_{k=0}^{n-2} (\alpha_k - \alpha_{k+1}) P_k \Phi_1 P_k \Big) + \alpha_{n-1}\Phi_1. \tag{3.1}$$

(For $n = 2$, we have

$$\begin{pmatrix} \frac{\alpha_0}{\alpha_0+\alpha_0} & \frac{\alpha_1}{\alpha_0+\alpha_1} \\ \frac{\alpha_1}{\alpha_0+\alpha_1} & \frac{\alpha_1}{\alpha_1+\alpha_1} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_0-\alpha_1}{\alpha_0+\alpha_0} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\alpha_1}{\alpha_0+\alpha_0} & \frac{\alpha_1}{\alpha_0+\alpha_1} \\ \frac{\alpha_1}{\alpha_0+\alpha_1} & \frac{\alpha_1}{\alpha_1+\alpha_1} \end{pmatrix}.$$

For $n = 3$, we have

$$\begin{pmatrix} \frac{\alpha_0}{\alpha_0+\alpha_0} & \frac{\alpha_1}{\alpha_0+\alpha_1} & \frac{\alpha_2}{\alpha_0+\alpha_2} \\ \frac{\alpha_1}{\alpha_0+\alpha_1} & \frac{\alpha_1}{\alpha_1+\alpha_1} & \frac{\alpha_2}{\alpha_1+\alpha_2} \\ \frac{\alpha_2}{\alpha_0+\alpha_2} & \frac{\alpha_2}{\alpha_1+\alpha_2} & \frac{\alpha_2}{\alpha_2+\alpha_2} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\alpha_0-\alpha_1}{\alpha_0+\alpha_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\alpha_1-\alpha_2}{\alpha_0+\alpha_0} & \frac{\alpha_1-\alpha_2}{\alpha_0+\alpha_1} & 0 \\ \frac{\alpha_1-\alpha_2}{\alpha_0+\alpha_1} & \frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\alpha_2}{\alpha_0+\alpha_0} & \frac{\alpha_2}{\alpha_0+\alpha_1} & \frac{\alpha_2}{\alpha_0+\alpha_2} \\ \frac{\alpha_2}{\alpha_0+\alpha_1} & \frac{\alpha_2}{\alpha_1+\alpha_1} & \frac{\alpha_2}{\alpha_1+\alpha_2} \\ \frac{\alpha_2}{\alpha_0+\alpha_2} & \frac{\alpha_2}{\alpha_1+\alpha_2} & \frac{\alpha_2}{\alpha_2+\alpha_2} \end{pmatrix}.$$

For larger n, the decomposition follows exactly the same way.)

It is well known (see e.g. [2]) that the Cauchy matrix Φ_1 is positive semidefinite. It is now immediate from (3.1) that the matrix Φ is also positive semidefinite. ∎

LEMMA 3.2. *Let $\alpha_k > 0, 0 \leqslant k < n$, be decreasing. Define an operator $S : M_n \to M_n$ by setting*

$$S(A) = \sum_{k,l=0}^{n-1} \frac{\alpha_k - \alpha_l}{\alpha_k + \alpha_l} p_k A p_l,$$

where p_k, $0 \leqslant k < n$, are the pairwise orthogonal rank one projections in M_n. We have

$$\|S(A)\|_{1,\infty} \leqslant \frac{80e}{\pi} \|A\|_1$$

for every $A \in M_n$.

Proof. It is sufficient to prove the assertion for the special case of projections p_k defined in the proof of the preceding lemma. Let T be the triangular truncation operator defined by setting

$$(T(A))_{i,j} = \begin{cases} A_{i,j} & i \geqslant j, \\ 0 & i < j, \end{cases}$$

and let M_Φ be the Schur multiplication operator with respect to Φ from Lemma 3.1. We have

$$S = (2T - 1)(2M_\Phi - 1).$$

Indeed,

$$(S(A))_{k,l} = \frac{\alpha_k - \alpha_l}{\alpha_k + \alpha_l} A_{k,l} = \Big(\frac{2\alpha_k}{\alpha_k + \alpha_l} - 1\Big) A_{k,l} = ((2M_\Phi - 1)(A))_{k,l}, \quad k \geqslant l$$

and

$$(S(A))_{k,l} = \frac{\alpha_k - \alpha_l}{\alpha_k + \alpha_l} A_{k,l} = \Big(1 - \frac{2\alpha_l}{\alpha_k + \alpha_l}\Big) A_{k,l} = -((2M_\Phi - 1)(A))_{k,l}, \quad k < l.$$

It is known (see Theorem IV.8.2 of [15]) that

$$\|(2T - 1)(X)\|_{1,\infty} \leqslant \frac{4e}{\pi} \|X\|_1, \quad X = X^* \in M_n, \ \mathrm{diag}(X) = 0.$$

Thus,

$$\|(2T-1)(X)\|_{1,\infty} \leqslant \frac{16e}{\pi}\|X\|_1, \quad X \in M_n,\ \mathrm{diag}(X)=0.$$

By Lemma 2.1 and Lemma 3.1, we have

$$\|2M_\Phi - 1\|_{\mathcal{L}_1 \to \mathcal{L}_1} \leqslant 1 + 8\|\mathrm{diag}(\Phi)\|_\infty = 5.$$

Therefore,

$$\|S(A)\|_{1,\infty} \leqslant \frac{16e}{\pi}\|(2M_\Phi - 1)(A)\|_1 \leqslant \frac{80e}{\pi}\|A\|_1. \quad \blacksquare$$

LEMMA 3.3. *Let $A, B \in M_{2n}$ be identically and symmetrically distributed matrices. We have*

$$\||A|-|B|\|_{1,\infty} \leqslant \Big(8 + \frac{640e}{\pi}\Big)\|A-B\|_1.$$

Proof. We have

$$A = \sum_{k=0}^{n-1}\mu(k,A_+)p_{1k} - \sum_{k=0}^{n-1}\mu(k,A_+)p_{2k}, \quad |A| = \sum_{k=0}^{n-1}\mu(k,A_+)p_{1k} + \sum_{k=0}^{n-1}\mu(k,A_+)p_{2k},$$

$$B = \sum_{l=0}^{n-1}\mu(l,A_+)q_{1l} - \sum_{l=0}^{n-1}\mu(l,A_+)q_{2l}, \quad |B| = \sum_{l=0}^{n-1}\mu(l,A_+)q_{1l} + \sum_{l=0}^{n-1}\mu(l,A_+)q_{2l},$$

where all the projections p_{1k}, p_{2k}, $0 \leqslant k < n$ are pairwise orthogonal and have rank 1 (and the same holds for the projections $q_{1l}, q_{2l}, 0 \leqslant l < n$). Hence, we have

$$\begin{aligned}
|A|-|B| &= \sum_{k,l=0}^{n-1}(\mu(k,A_+)-\mu(l,A_+))p_{1k}q_{1l} + \sum_{k,l=0}^{n-1}(\mu(k,A_+)-\mu(l,A_+))p_{2k}q_{2l} \\
&\quad + \sum_{k,l=0}^{n-1}(\mu(k,A_+)-\mu(l,A_+))p_{1k}q_{2l} + \sum_{k,l=0}^{n-1}(\mu(k,A_+)-\mu(l,A_+))p_{2k}q_{1l} \\
&= \sum_{k,l=0}^{n-1} p_{1k}(A-B)q_{1l} - \sum_{k,l=0}^{n-1} p_{2k}(A-B)q_{2l} \\
&\quad + \sum_{k,l=0}^{n-1} \frac{\mu(k,A_+)-\mu(l,A_+)}{\mu(k,A_+)+\mu(l,A_+)} p_{1k}(A-B)q_{2l} \\
&\quad - \sum_{k,l=0}^{n-1} \frac{\mu(k,A_+)-\mu(l,A_+)}{\mu(k,A_+)+\mu(l,A_+)} p_{2k}(A-B)q_{1l}.
\end{aligned}$$

Take unitary matrices $U, V \in M_{2n}$ such that $q_{2l} = Up_{1l}U^{-1}$ and $q_{1l} = Vp_{2l}V^{-1}$ for all $0 \leqslant l < n$. It is clear that

$$p_{1k}(A-B)q_{2l} = p_{1k}(\mathrm{supp}(A_+)(A-B)U\mathrm{supp}(A_+))p_{1l} \cdot U^{-1} \quad 0 \leqslant k,l < n;$$

$$p_{2k}(A-B)q_{1l} = p_{2k}(\mathrm{supp}(A_-)(A-B)V\mathrm{supp}(A_-))p_{2l} \cdot V^{-1} \quad 0 \leqslant k,l < n.$$

Consider $S_1 : \mathrm{supp}(A_+)M_{2n}\mathrm{supp}(A_+) \to \mathrm{supp}(A_+)M_{2n}\mathrm{supp}(A_+)$

$$S_1(X) := \sum_{k,l=0}^{n-1} \frac{\mu(k,A_+) - \mu(l,A_+)}{\mu(k,A_+) + \mu(l,A_+)} p_{1k}Xp_{1l}, \quad X \in \mathrm{supp}(A_+)M_{2n}\mathrm{supp}(A_+)$$

and $S_2 : \mathrm{supp}(A_-)M_{2n}\mathrm{supp}(A_-) \to \mathrm{supp}(A_-)M_{2n}\mathrm{supp}(A_-)$

$$S_2(X) := \sum_{k,l=0}^{n-1} \frac{\mu(k,A_+) - \mu(l,A_+)}{\mu(k,A_+) + \mu(l,A_+)} p_{2k}Xp_{2l}, \quad X \in \mathrm{supp}(A_-)M_{2n}\mathrm{supp}(A_-).$$

Employing these notations, we obtain

$$\begin{aligned} |A| - |B| = {} & \mathrm{supp}(A_+)(A-B)\mathrm{supp}(B_+) - \mathrm{supp}(A_-)(A-B)\mathrm{supp}(B_-) \\ & + S_1(\mathrm{supp}(A_+)(A-B)U\mathrm{supp}(A_+)) \cdot U^{-1} \\ & - S_2(\mathrm{supp}(A_-)(A-B)V\mathrm{supp}(A_-)) \cdot V^{-1}. \end{aligned}$$

Since the algebras $\mathrm{supp}(A_+)M_{2n}\mathrm{supp}(A_+)$ and $\mathrm{supp}(A_-)M_{2n}\mathrm{supp}(A_-)$ are $*$-isomorphic to the algebra M_n, it follows that the operators S_1 and S_2 satisfy the assumptions of Lemma 3.2. Applying Lemma 3.2, we obtain

$$\begin{aligned} \frac{1}{4}\||A| - |B|\|_{1,\infty} \leqslant {} & \|\mathrm{supp}(A_+)(A-B)\mathrm{supp}(B_+)\|_{1,\infty} \\ & + \|\mathrm{supp}(A_-)(A-B)\mathrm{supp}(B_-)\|_{1,\infty} \\ & + \|S_1(\mathrm{supp}(A_+)(A-B)U\mathrm{supp}(A_+))\|_{1,\infty} \\ & + \|S_2(\mathrm{supp}(A_-)(A-B)V\mathrm{supp}(A_-))\|_{1,\infty}, \end{aligned}$$

and

$$\||A| - |B|\|_{1,\infty} \leqslant \Big(4 + 4 + 4 \cdot \frac{80e}{\pi} + 4 \cdot \frac{80e}{\pi}\Big)\|A - B\|_1 = \Big(8 + \frac{640e}{\pi}\Big)\|A - B\|_1. \quad ∎$$

In the following lemma, we get rid of the auxiliary conditions on A and B imposed in Lemma 3.3.

LEMMA 3.4. *For all self-adjoint matrices $A, B \in M_{2n}$, we have*

$$\||A| - |B|\|_{1,\infty} \leqslant \Big(34 + \frac{2560e}{\pi}\Big) \cdot \|A - B\|_1.$$

Proof. Let matrices A, B be symmetrically (but not necessarily identically) distributed,

$$A = \sum_{k=0}^{n-1} \mu(k,A_+)p_{1k} - \sum_{k=0}^{n-1} \mu(k,A_+)p_{2k}, \quad B = \sum_{k=0}^{n-1} \mu(k,B_+)q_{1k} - \sum_{k=0}^{n-1} \mu(k,B_+)q_{2k},$$

where all the projections $p_{1k}, p_{2k}, q_{1k}, q_{2k}$, $0 \leqslant k < n$ are pairwise orthogonal and have rank 1. We introduce an auxiliary matrix

$$C := \sum_{k=0}^{n-1} \mu(k,B_+)p_{1k} - \sum_{k=0}^{n-1} \mu(k,B_+)p_{2k}.$$

Clearly, B and C are identically and symmetrically distributed matrices (in particular, we have $\mu(B) = \mu(C)$). Thus, we have

$$\||B|-|A|\|_{1,\infty} = \|(|B|-|C|)+(|C|-|A|)\|_{1,\infty} \leqslant 2\||B|-|C|\|_{1,\infty} + 2\||A|-|C|\|_{1,\infty}.$$

By Lemma 3.3, we have

$$\begin{aligned}\||B|-|C|\|_{1,\infty} &\leqslant \Big(8+\frac{640e}{\pi}\Big)\cdot\|B-C\|_1 \leqslant \Big(8+\frac{640e}{\pi}\Big)\cdot(\|A-B\|_1+\|A-C\|_1)\\ &= \Big(8+\frac{640e}{\pi}\Big)\cdot(\|A-B\|_1+\|\mu(A)-\mu(B)\|_1)\\ &\leqslant \Big(16+\frac{1280e}{\pi}\Big)\|A-B\|_1.\end{aligned}$$

Here, we used the fact (guaranteed by our definition of C) that $\|A-C\|_1 = \|\mu(A)-\mu(B)\|_1$. Since the matrices A and C commute, it follows that

$$\||A|-|C|\|_{1,\infty} \leqslant \|A-C\|_{1,\infty} \leqslant \|A-C\|_1 = \|\mu(A)-\mu(B)\|_1 \leqslant \|A-B\|_1,$$

where in the last step we used the classical fact (1.22) of [25]. Combining the above inequalities we complete the proof for the case of symmetrically distributed matrices.

Let now A and B be arbitrary self-adjoint matrices from M_{2n}. Consider an element $F \in M_2$ given by

$$F := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

and observe that

$$|A\otimes F| - |B\otimes F| = (|A|-|B|)\otimes 1,$$

where 1 is the identity in M_2. Note that

$$\|X\otimes 1\|_{1,\infty} = 2\|X\|_{1,\infty}, \quad \|X\otimes 1\|_1 = 2\|X\|_1.$$

Now, observing that $A\otimes F$ and $B\otimes F$ are symmetrically distributed matrices, we infer from the first part of the proof that

$$\begin{aligned}\||A|-|B|\|_{1,\infty} &= \frac{1}{2}\||A\otimes F|-|B\otimes F|\|_{1,\infty} \leqslant \frac{1}{2}\cdot\Big(34+\frac{2560e}{\pi}\Big)\|A\otimes F - B\otimes F\|_1\\ &= \Big(34+\frac{2560e}{\pi}\Big)\cdot\|A-B\|_1. \quad \blacksquare\end{aligned}$$

Proof of Theorem 1.2. Let p_n, $n \geqslant 0$, be a sequence of finite rank projections in $\mathcal{L}(H)$ such that $p_n \uparrow 1$. By Corollary 1.5 of [11],

$$(|p_nAp_n| - |p_nBp_n|)E \to (|A|-|B|)E$$

uniformly for every finite rank projection E. By Lemma 3.4, we have

$$\||p_nAp_n|-|p_nBp_n|\|_{1,\infty} \leqslant \Big(34+\frac{2560e}{\pi}\Big)\|p_n(A-B)p_n\|_1 \leqslant \Big(34+\frac{2560e}{\pi}\Big)\|A-B\|_1.$$

Thus,

$$\mu(k,(|p_nAp_n|-|p_nBp_n|)E)\leqslant \mu(k,|p_nAp_n|-|p_nBp_n|)$$
$$\leqslant\Big(34+\frac{2560e}{\pi}\Big)\|A-B\|_1\cdot\frac{1}{k+1},\quad k\geqslant 0,$$

for every finite rank projection E. Since uniform convergence implies the convergence of singular values, it follows that

$$\mu(k,(|A|-|B|)E)\leqslant\Big(34+\frac{2560e}{\pi}\Big)\|A-B\|_1\cdot\frac{1}{k+1},\quad k\geqslant 0,$$

for every finite rank projection E. Hence, using judicious choice of projection E (which is possible since our proof yields that $|A|-|B|$ is compact and hence E may be taken to be a suitable spectral projection of this operator), we have

$$\mu(k,|A|-|B|)\leqslant\Big(34+\frac{2560e}{\pi}\Big)\|A-B\|_1\cdot\frac{1}{k+1},\quad k\geqslant 0.\quad\blacksquare$$

4. THEOREM 1.2 FOR COMPACT OPERATORS

Define an ideal $(\mathcal{L}_{1,\infty})_0$ in $\mathcal{L}(H)$ by setting

$$(\mathcal{L}_{1,\infty})_0=\{A\in\mathcal{K}(H): k\mu(k,A)\to 0 \text{ as } k\to\infty\}.$$

This ideal coincides with the closure of the ideal of all finite rank operators in $\mathcal{L}_{1,\infty}$ and is commonly called the separable part of $\mathcal{L}_{1,\infty}$.

Define a (non-linear) functional θ on $\mathcal{L}_{1,\infty}$ by setting

$$\theta(A)=\limsup_{k\to\infty}k\mu(k,A).$$

LEMMA 4.1. *Let $A,B\in\mathcal{L}_{1,\infty}$. We have*

(i) *If $\frac{1}{\alpha}+\frac{1}{\beta}=1$, then*

$$\theta(A+B)\leqslant\alpha\theta(A)+\beta\theta(B).$$

(ii) *If $B\in\mathcal{L}_1$, then $\theta(B)=0$.*

(iii) *If $B\in\mathcal{L}_1$, then $\theta(A+B)=\theta(A)$.*

Proof. We have

$$\mu(k,A+B)\leqslant\mu([\tfrac{k}{\alpha}],A)+\mu([\tfrac{k}{\beta}],B).$$

Hence,

$$\theta(A+B)=\limsup_{k\to\infty}k\mu(k,A+B)\leqslant\limsup_{k\to\infty}k\mu([\tfrac{k}{\alpha}],A)+\limsup_{k\to\infty}k\mu([\tfrac{k}{\beta}],B).$$

It is obvious that

$$\limsup_{k\to\infty}k\mu([\tfrac{k}{\alpha}],A)=\alpha\theta(A),\quad \limsup_{k\to\infty}k\mu([\tfrac{k}{\beta}],B)=\beta\theta(B).$$

This proves (i).

If $B \in \mathcal{L}_1$ and if $p_n \uparrow 1$, then $\|B(1-p_n)\|_1 \to 0$ (see e.g. [6]). Let e_k be the eigenvector of $|B|$ corresponding to the eigenvalue $\mu(k,B)$ and set p_k to be the projection on the linear span of e_n, $0 \leqslant n < k$. We have

$$\sum_{m=k}^{\infty} \mu(m,B) = \|B(1-p_n)\|_1 \to 0.$$

Therefore,

$$\frac{k}{2}\mu(k,B) \leqslant \sum_{n=[k/2]}^{k} \mu(k,B) \leqslant \sum_{n=[k/2]}^{\infty} \mu(k,B) \to 0.$$

This proves (ii).

If $A \in \mathcal{L}_{1,\infty}$ and $B \in \mathcal{L}_1$, then it follows from (i) and (ii) that $\theta(A+B) \leqslant \alpha\theta(A)$. Since $\alpha > 1$ is arbitrary, it follows that $\theta(A+B) \leqslant \theta(A)$. Applying the same argument to the operators $A+B \in \mathcal{L}_{1,\infty}$ and $-B \in \mathcal{L}_1$, we infer that $\theta(A) \leqslant \theta(A+B)$. This proves (iii). ∎

LEMMA 4.2. *Let $\alpha_k > 0$, $k \geqslant 0$, be decreasing. Define an operator $S : \mathcal{L}_2 \to \mathcal{L}_2$ by setting*

$$S(A) = \sum_{k,l=0}^{\infty} \frac{\alpha_k - \alpha_l}{\alpha_k + \alpha_l} p_k A p_l,$$

where p_k, $k \geqslant 0$, are the pairwise orthogonal rank one projections in $\mathcal{L}(H)$. We have $S(A) \in (\mathcal{L}_{1,\infty})_0$ and

$$\|S(A)\|_{1,\infty} \leqslant \frac{80e}{\pi}\|A\|_1$$

for every $A \in \mathcal{L}_1$.

Proof. The norm estimate can be proved in exactly the same way as in Lemma 3.2. Set $P_n = \sum_{k=0}^{n-1} p_k$. We have

$$A = (1-P_n)A(1-P_n) + (AP_n + P_nA(1-P_n))$$

and, therefore,

$$S(A) = S((1-P_n)A(1-P_n)) + S(AP_n + P_nA(1-P_n)).$$

We have $S(AP_n + P_nA(1-P_n)) \in \mathcal{L}_1$ since this operator has finite rank (even though its norm may be quite large). It follows from Lemma 4.1 that

$$\begin{aligned}\theta(S(A)) &= \theta(S((1-P_n)A(1-P_n))) \\ &\leqslant \|S((1-P_n)A(1-P_n))\|_{1,\infty} \leqslant \text{const} \cdot \|(1-P_n)A(1-P_n)\|_1.\end{aligned}$$

However, $\|(1-P_n)A(1-P_n)\|_1 \to 0$ as $n \to \infty$. It follows that $\theta(S(A)) = 0$ and, therefore, $S(A) \in (\mathcal{L}_{1,\infty})_0$. ∎

THEOREM 4.3. *If $A, B \in \mathcal{L}(H)$ are compact operators such that $A - B \in \mathcal{L}_1$, then $|A| - |B| \in (\mathcal{L}_{1,\infty})_0$.*

The proof follows that in Lemma 3.3 and Lemma 3.4 *mutatis mutandi*.

5. GENERAL SEMIFINITE VERSION OF THEOREM 1.2

We begin by recalling a few relevant facts and notations from the theory of noncommutative integration on semifinite von Neumann algebras. For details on von Neumann algebra theory, the reader is referred to e.g. [10], [16], [17] or [28]. General facts concerning measurable operators may be found in [21], [24] (see also Chapter IX of [29]). For the convenience of the reader, some of the basic definitions are recalled.

In what follows, let $\mathcal{M}$ be a von Neumann algebra on a separable Hilbert space H. A linear operator $A : \mathrm{dom}(A) \to H$, where the domain $\mathrm{dom}(A)$ of A is a linear subspace of H, is said to be *affiliated* with $\mathcal{M}$ if it commutes with every element in $\mathcal{M}'$.

An operator A affiliated with $\mathcal{M}$ is called τ-measurable if there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of τ-finite projections in $\mathcal{M}$ such that $p_n \downarrow 0$ and $(1-p_n)(H) \subset \mathrm{dom}(A)$ for all n. The collection $\mathcal{S}(\mathcal{M},\tau)$ of all τ-measurable operators is a unital $*$-algebra with respect to the strong sum and strong multiplication. It is well known that a linear operator A affiliated with $\mathcal{M}$ belongs to $\mathcal{S}(\mathcal{M},\tau)$ if and only if there exists $\lambda > 0$ such that

$$\tau(E_{|A|}(\lambda,\infty)) < \infty.$$

Here, $E_{|A|}$ is the spectral family of the operator $|A|$. Alternatively, an unbounded operator A affiliated with $\mathcal{M}$ is τ-measurable (see [14]) if and only if

$$\tau(E_{|A|}(n,\infty)) = o(1), \quad n \to \infty.$$

Let a semifinite von Neumann algebra $\mathcal{M}$ be equipped with a faithful normal semi-finite trace τ. Let $A \in \mathcal{S}(\mathcal{M},\tau)$. The generalized singular value function $\mu(A) : t \to \mu(t;A)$ of the operator A is defined by setting

$$\mu(s;A) = \inf\{\|A(1-p)\|_\infty : p \in \mathcal{M} \text{ is a projection}, \tau(p) \leqslant s\}.$$

There exists an equivalent definition which involves the distribution function of the operator $|A|$. For every self-adjoint operator $A \in \mathcal{S}(\mathcal{M},\tau)$, setting

$$d_A(t) = \tau(E_A(t,\infty)), \quad t > 0,$$

we have (see e.g. [14])

$$\mu(t;A) = \inf\{s \geqslant 0 : d_{|A|}(s) \leqslant t\}.$$

If $\mathcal{M} = \mathcal{L}(H)$ and τ is the standard trace Tr, then it is not difficult to see that $\mathcal{S}(\mathcal{M},\tau) = \mathcal{M}$. In this case, for $A \in \mathcal{M}$, we have

$$\mu(n;A) = \mu(t;A), \quad t \in [n,n+1), \quad n \geqslant 0.$$

The sequence $\{\mu(n;A)\}_{n \geqslant 0}$ is just the sequence of singular values of the operator A.

For every $\varepsilon, \delta > 0$, we define the set

$$V(\varepsilon,\delta)=\{x\in\mathcal{S}(\mathcal{M},\tau):\exists p=p^2=p^*\in\mathcal{M} \text{ such that } \|x(1-p)\|\leqslant\varepsilon,\ \tau(p)\leqslant\delta\}.$$

The topology generated by the sets $V(\varepsilon,\delta)$, $\varepsilon,\delta > 0$, is called a measure topology.

Let $L_1(0,\infty)$ and $L_\infty(0,\infty)$ be Lebesgue spaces on $(0,\infty)$.

We define the space $\mathcal{L}_1(\mathcal{M},\tau) = \{A \in \mathcal{S}(\mathcal{M},\tau) : \mu(A) \in L_1(0,\infty)\}$. It is well-known that the functional

$$\|\cdot\|_1 : A \to \|\mu(A)\|_1$$

is a Banach norm on $\mathcal{L}_1(\mathcal{M},\tau)$. Similarly, we say that $A \in (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M},\tau)$ if and only if $\mu(A) \in (L_1 + L_\infty)(0,\infty)$. Here, we identify $\mathcal{M}$ with $\mathcal{L}_\infty(\mathcal{M},\tau)$. The space $(\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M},\tau)$ can be also viewed as a sum of Banach spaces $\mathcal{L}_1(\mathcal{M},\tau)$ and $\mathcal{L}_\infty(\mathcal{M},\tau)$ (the latter space is equipped with the uniform norm, which we denote simply by $\|\cdot\|_\infty$).

Define a linear space $(\mathcal{L}_1+\mathcal{L}_\infty)(\mathcal{M},\tau)=\{A\in\mathcal{S}(\mathcal{M},\tau):\mu(A)\in L_1+L_\infty\}$.

One can define the noncommutative weak L_1 space in a similar manner. Set

$$\mathcal{L}_{1,\infty}(\mathcal{M},\tau) = \Big\{A \in \mathcal{S}(\mathcal{M},\tau) : \sup_{t>0} t\mu(t,A) < \infty\Big\}.$$

The mapping

$$\|\cdot\|_{1,\infty} : A \to \sup_{t>0} t\mu(t,A), \quad A \in \mathcal{L}_{1,\infty}(\mathcal{M},\tau)$$

defines a quasi-norm on $\mathcal{L}_{1,\infty}(\mathcal{M},\tau)$. It can be easily seen that $\mathcal{L}_{1,\infty}(\mathcal{M},\tau)$ equipped with the latter quasi-norm becomes a quasi-Banach space. The quasi-Banach space $\mathcal{L}_{1,\infty}(\mathcal{M},\tau)$ has the Fatou property: if $A_n \in \mathcal{L}_{1,\infty}(\mathcal{M},\tau)$, $\|A_n\|_{1,\infty} \leqslant 1$ and $A_n \to A$ in measure, then $A \in \mathcal{L}_{1,\infty}(\mathcal{M},\tau)$ and $\|A\|_{1,\infty} \leqslant 1$.

LEMMA 5.1. *Let $\mathcal{M}$ be a semifinite von Neumann algebra. Let $\alpha_k > 0, 0 \leqslant k < n$, be decreasing. Define an operator $S : \mathcal{M} \to \mathcal{M}$ by setting*

$$S(A) = \sum_{k,l=0}^{n-1} \frac{\alpha_k - \alpha_l}{\alpha_k + \alpha_l} p_k A p_l,$$

where p_k, $0 \leqslant k < n$, are the pairwise orthogonal τ-finite projections in $\mathcal{M}$. We have

$$\|S(A)\|_{1,\infty} \leqslant \text{const} \cdot \|A\|_1, \quad A \in \mathcal{M}.$$

Proof. The proof follows that of Lemma 3.2 *mutatis mutandi*. The reference to [15] must be replaced with the reference to Theorem 1.4 in [12]. ∎

LEMMA 5.2. *Let $\mathcal{M}$ be a semifinite factor. Let $A, B \in \mathcal{M}$ be identically and symmetrically distributed finitely supported operators. We have*

$$\||A| - |B|\|_{1,\infty} \leqslant \text{const}\|A - B\|_1.$$

Proof. Since the type I factors were already treated in Theorem 1.2, we may assume without loss of generality that $\mathcal{M}$ is a type II factor. Suppose first that $\mu(A)$ (and, hence, $\mu(B)$) takes finitely many values. The proof in this case follows that of Lemma 3.3 *mutatis mutandi*. Observe that in the last argument we have used the assumption that $\mathcal{M}$ is a factor to find unitaries U and V as in the proof of Lemma 3.3 (recall: any two projections in a type II factor with equal finite trace are unitarily equivalent ([28], Theorem V.1.8)).

Let now $A, B \in \mathcal{M}$ be arbitrary identically and symmetrically distributed finitely supported operators. There exist projections $p_{k,s}$, $s > 0$, $1 \leqslant k \leqslant 4$, such that $\tau(p_{k,s}) = s$ and

$$A_+ = \int_0^\infty \mu(s, A_+) \mathrm{d}p_{1,s}, \quad A_- = \int_0^\infty \mu(s, A_+) \mathrm{d}p_{2,s},$$

$$B_+ = \int_0^\infty \mu(s, A_+) \mathrm{d}p_{3,s}, \quad B_- = \int_0^\infty \mu(s, A_+) \mathrm{d}p_{4,s}.$$

Fix ε such that

$$\int_0^\varepsilon \mu(s, A_+) \mathrm{d}s \leqslant \|A - B\|_1$$

and set

$$C_+ = \sum_{k=0}^\infty \mu((k+1)\varepsilon, A_+)(p_{1,(k+1)\varepsilon} - p_{1,k\varepsilon}),$$

$$C_- = \sum_{k=0}^\infty \mu((k+1)\varepsilon, A_+)(p_{2,(k+1)\varepsilon} - p_{2,k\varepsilon}),$$

$$D_+ = \sum_{k=0}^\infty \mu((k+1)\varepsilon, A_+)(p_{3,(k+1)\varepsilon} - p_{3,k\varepsilon}),$$

$$D_- = \sum_{k=0}^\infty \mu((k+1)\varepsilon, A_+)(p_{4,(k+1)\varepsilon} - p_{4,k\varepsilon}).$$

It is easy to see that

$$\begin{aligned}
\|A_+ - C_+\|_1 &= \sum_{k=0}^\infty \int_{k\varepsilon}^{(k+1)\varepsilon} (\mu(s, A_+) - \mu((k+1)\varepsilon, A_+)) \mathrm{d}s \\
&\leqslant \int_0^\varepsilon (\mu(s, A_+) - \mu(\varepsilon, A_+)) \mathrm{d}s + \sum_{k=1}^\infty \varepsilon(\mu(k\varepsilon, A_+) - \mu((k+1)\varepsilon, A_+)) \\
&= \int_0^\varepsilon (\mu(s, A_+) - \mu(\varepsilon, A_+)) \mathrm{d}s + \varepsilon\mu(\varepsilon, A_+) = \int_0^\varepsilon \mu(s, A_+) \mathrm{d}s \leqslant \|A - B\|_1.
\end{aligned}$$

We have

$$\begin{aligned}\||A|-|B|\|_{1,\infty} &= \|(|A|-|C|)+(|C|-|D|)-(|B|-|D|)\|_{1,\infty}\\ &\leqslant 4\||A|-|C|\|_{1,\infty}+4\||C|-|D|\|_{1,\infty}+4\||B|-|D|\|_{1,\infty}.\end{aligned}$$

Recall that C and D are symmetrically and identically distributed finitely supported operators. By construction, $\mu(C)$ (and, hence, $\mu(D)$) takes only finitely many values. We infer from the previous paragraph that

$$\||C|-|D|\|_{1,\infty} \leqslant \text{const}\cdot\|C-D\|_1.$$

Since A and C commute, it follows that

$$\||A|-|C|\|_{1,\infty} \leqslant \|A-C\|_{1,\infty} \leqslant \|A-C\|_1 \leqslant 2\|A-B\|_1.$$

Similarly,

$$\||B|-|D|\|_{1,\infty} \leqslant 2\|A-B\|_1.$$

Also, we have

$$\begin{aligned}\|C-D\|_1 &= \|(C-A)+(A-B)+(B-D)\|_1\\ &\leqslant \|C-A\|_1+\|A-B\|_1+\|B-D\|_1 \leqslant 5\|A-B\|_1.\end{aligned}$$

Combining these estimates, we conclude the proof. ∎

The following lemma should be compared to the results on positive Schur multipliers in Section 2.3.

LEMMA 5.3. *Let $\mathcal{M} \subseteq \mathcal{L}(H)$ be a von Neumann algebra. Let $B \in M_n$ and $B \geqslant 0$. Let $p_1, \ldots, p_n$ be mutually orthogonal projections in $\mathcal{M}$. Consider the operator valued Schur multiplier (or double operator integral) defined by,*

$$S_B : \mathcal{M} \to \mathcal{M} : x \mapsto \sum_{i,j=1}^{n} B_{i,j} p_i x p_j. \tag{5.1}$$

Then S_B preserves positive operators: $S_B(x) \geqslant 0$ whenever $x \geqslant 0$.

Proof. The Schur multiplier extends to a map $S_B : \mathcal{L}(H) \to \mathcal{L}(H)$ prescribed by the same formula (5.1) and hence it suffices to prove the statement for $\mathcal{M} = \mathcal{L}(H)$. In case each of the projections p_i are finite rank the statement is reduced to the matricial case and hence follows from Schur's theorem, see Section 2.3. Indeed, this is true in case each p_i is one dimensional. Else, write $p_i = \sum_m^{n_i} p_{i,m}$, a finite sum of mutually orthogonal rank 1 projections and apply the previous line to the set $\{p_{i,m} : i, 1 \leqslant m \leqslant n_i\}$ using that $(B_{i,j})_{(i,1\leqslant m\leqslant n_i),(j,1\leqslant k\leqslant n_j)}$ is again positive. The positivity of the latter matrix follows as this matrix is a corner of the Kronecker product $C_n \otimes B$ where C_n is the $n \times n$-matrix with entries equal to 1. In the general case of not necessarily finite rank projections p_i one can write each p_i as a strong limit of finite rank projections $p_{i,m} \to p_i$. Putting $P_m = \sum_i p_{i,m}$ we see that $x \mapsto P_m S_B(x) P_m$ preserves positive operators and $P_m S_B(x) P_m \to S_B(x)$ strongly. This concludes the lemma. ∎

LEMMA 5.4. *Let $\mathcal{M}$ be a semifinite factor. Let $A, B \in \mathcal{M}$ be self-adjoint finitely supported operators. We have*

$$\||A| - |B|\|_{1,\infty} \leqslant \mathrm{const}\|A - B\|_1.$$

Proof. The proof follows that of Lemma 3.4 *mutatis mutandi*. At the point that Schur's theorem is used, see the proof of Lemma 2.1, Lemma 5.3 can be invoked. ∎

LEMMA 5.5. *If $\mathcal{M}$ is a semifinite factor and if $A, B \in (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}, \tau)$ are such that $A - B \in \mathcal{L}_1(\mathcal{M}, \tau)$, then $|A| - |B| \in \mathcal{L}_{1,\infty}(\mathcal{M}, \tau)$ and*

$$\||A| - |B|\|_{1,\infty} \leqslant \mathrm{const} \cdot \|A - B\|_1.$$

Proof. Suppose first that $A, B \in \mathcal{M}$. Let p_n, $n \geqslant 0$, be a sequence of τ-finite projections in $\mathcal{M}$ such that $p_n \uparrow 1$. By Corollary 1.5 of [11],

$$(|p_n A p_n| - |p_n B p_n|)E \to (|A| - |B|)E$$

in measure for every τ-finite projection E. By Lemma 5.4, we have

$$\||p_n A p_n| - |p_n B p_n|\|_{1,\infty} \leqslant \mathrm{const} \cdot \|p_n(A - B)p_n\|_1 \leqslant \mathrm{const} \cdot \|A - B\|_1.$$

Therefore,

$$\mu(t, (|p_n A p_n| - |p_n B p_n|)E) \leqslant \mu(t, |p_n A p_n| - |p_n B p_n|) \leqslant \frac{\mathrm{const}}{t}\|A - B\|_1, \quad t > 0,$$

for every τ-finite projection E. Since convergence in measure implies the (almost everywhere) convergence of singular value functions (see e.g. Lemma 7 of [26]), it follows that

$$\mu(t, (|A| - |B|)E) \leqslant \frac{\mathrm{const}}{t}\|A - B\|_1, \quad t > 0,$$

for every τ-finite projection E. Hence, using a judicious choice of the projection E (namely a suitable spectral projection of $|A| - |B|$), we have

$$\mu(t, |A| - |B|) \leqslant \frac{\mathrm{const}}{t}\|A - B\|_1, \quad t > 0.$$

This proves the assertion for bounded A and B.

Let now $A, B \in (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}, \tau)$. Set

$$p_n = E_{|A|}[0, n] \wedge E_{|B|}[0, n].$$

Since A, B are τ-measurable, it follows from [30] (see also Theorem 1.1 of [11]) that

$$p_n A p_n \to A, \quad p_n B p_n \to B, \quad |p_n A p_n| \to |A|, \quad |p_n B p_n| \to |B|,$$

in measure. It follows from the above that

$$\||p_n A p_n| - |p_n B p_n|\|_{1,\infty} \leqslant \mathrm{const} \cdot \|p_n(A - B)p_n\|_1 \leqslant \mathrm{const} \cdot \|A - B\|_1.$$

Since the quasi-norm in $\mathcal{L}_{1,\infty}(\mathcal{M},\tau)$ has the Fatou property, it follows that

$$\||A|-|B|\|_{1,\infty} \leqslant \mathrm{const}\cdot \|A-B\|_1. \qquad \blacksquare$$

The following lemma shows the proper triangle inequality in $\mathcal{L}_{1,\infty}$ for pairwise orthogonal summands. Let $A_k \in \mathcal{L}_{1,\infty}(\mathcal{M},\tau)$, $k \geqslant 0$. We use the direct sum symbol $\bigoplus_{k=0}^{\infty} A_k$ to denote the operator on H formed with respect to some arbitrary Hilbert space isomorphism $\bigoplus_{k=0}^{\infty} H \simeq H$.

LEMMA 5.6. *If $A_k \in \mathcal{L}_{1,\infty}(\mathcal{M},\tau)$, $k \geqslant 0$, then*

$$\Big\|\bigoplus_{k=0}^{\infty} A_k\Big\|_{1,\infty} \leqslant \sum_{k=0}^{\infty} \|A_k\|_{1,\infty}.$$

Proof. Set $x(t)=1/t$, $t>0$. For simplicity of notations, denote $\|A_k\|_{1,\infty}$ by α_k. We have $\mu(t,A_k) \leqslant \alpha_k/t$, $t>0$. Using the notation $d_{\alpha_k x}(t)$ for the classical distribution, it is immediate that

$$d_{|\oplus_{k=0}^{\infty} A_k|}(t) = \sum_{k=0}^{\infty} d_{|A_k|}(t) \leqslant \sum_{k=0}^{\infty} d_{\alpha_k x}(t) = \sum_{k=0}^{\infty} \frac{\alpha_k}{t} = \frac{\sum_{k=0}^{\infty}\alpha_k}{t} = d_{(\sum_{k=0}^{\infty}\alpha_k)x}(t).$$

Hence,

$$\mu\big(\textstyle\bigoplus_{k=0}^{\infty} A_k\big) \leqslant \Big(\displaystyle\sum_{k=0}^{\infty}\alpha_k\Big)x$$

or, equivalently,

$$\Big\|\bigoplus_{k=0}^{\infty} A_k\Big\|_{1,\infty} \leqslant \Big(\sum_{k=0}^{\infty}\alpha_k\Big)\|x\|_{1,\infty} = \sum_{k=0}^{\infty}\|A_k\|_{1,\infty}. \qquad \blacksquare$$

The following lemma combines well-known facts from [28] and less known facts from [13].

LEMMA 5.7. *For every semifinite von Neumann algebra $(\mathcal{M},\tau)$, there exist semifinite factors $(\mathcal{M}_k,\tau_k)$, $k \geqslant 0$, and a trace preserving $*$-monomorphism of $(\mathcal{M},\tau)$ into $(\bigoplus_{k\geqslant 0}\mathcal{M}_k, \bigoplus_{k\geqslant 0}\tau_k)$.*

Proof. By Theorem V.1.19 in [28], we have $\mathcal{M} = \mathcal{M}^1 \oplus \mathcal{M}^2 \oplus \mathcal{M}^3$, where $\mathcal{M}^1$ is type I, $\mathcal{M}^2$ is type II_1 and $\mathcal{M}^3$ is type II_∞.

By Theorem V.1.27 in [28], there exist commutative algebras $\mathcal{A}_k$, $k \geqslant 0$, and Hilbert spaces H_k, $k \geqslant 0$, such that

$$\mathcal{M}^1 = \bigoplus_{k\geqslant 0} \mathcal{A}_k \bar{\otimes} \mathcal{L}(H_k).$$

Every $\mathcal{A}_k$ admits a trace preserving isomorphic embedding into $L_\infty(0,\infty)$ and then to the hyperfinite II_∞ factor $\mathcal{R}\bar{\otimes}\mathcal{L}(H)$. Every $\mathcal{L}(H_k)$ admits a trace preserving isomorphic embedding into $\mathcal{L}(H)$. Thus, $\mathcal{M}^1$ admits a trace preserving isomorphic embedding into

$$\bigoplus_{k\geqslant 0}\mathcal{R}\bar{\otimes}\mathcal{L}(H)\bar{\otimes}\mathcal{L}(H).$$

Equivalently, $\mathcal{M}^1$ admits a trace preserving isomorphic embedding into the von Neumann algebra $(\mathcal{R}\bar{\otimes}\mathcal{L}(H))^{\oplus\infty}$.

By Theorem 2.3 and Lemma 2.5 in [13], the algebra $\mathcal{M}^2$ admits a trace preserving isomorphic embedding into a II_1 factor.

By Theorem V.1.40 in [28], there exist type II_1-algebras $\mathcal{N}_k$, $k\geqslant 0$, such that

$$\mathcal{M}^3=\bigoplus_{k\geqslant 0}\mathcal{N}_k\bar{\otimes}\mathcal{L}(H).$$

Again applying Theorem 2.3 and Lemma 2.5 in [13], we embed the algebras $\mathcal{N}_k$, $k\geqslant 0$, into II_1 factors $\mathcal{O}_k$, $k\geqslant 0$. Since $\mathcal{O}_k\bar{\otimes}\mathcal{L}(H)$ is a type II_∞ factor, the assertion follows for $\mathcal{M}^3$ and, thus, for $\mathcal{M}$. ∎

THEOREM 5.8. *If $\mathcal{M}$ is a semifinite von Neumann algebra and whenever $A,B\in(\mathcal{L}_1+\mathcal{L}_\infty)(\mathcal{M},\tau)$ are such that $A-B\in\mathcal{L}_1(\mathcal{M},\tau)$, then $|A|-|B|\in\mathcal{L}_{1,\infty}(\mathcal{M},\tau)$ and*

$$\||A|-|B|\|_{1,\infty}\leqslant \mathrm{const}\cdot\|A-B\|_1.$$

Proof. According to the Lemma 5.7, we can embed $(\mathcal{M},\tau)$ into the von Neumann algebra $(\bigoplus_{k\geqslant 0}\mathcal{M}_k,\bigoplus_{k\geqslant 0}\tau_k)$, where $(\mathcal{M}_k,\tau_k)$, $k\geqslant 0$, are semifinite factors.

Thus, $A=\bigoplus\limits_{k\geqslant 0}A_k$ and $B=\bigoplus\limits_{k\geqslant 0}B_k$, where $A_k,B_k\in(\mathcal{L}_1+\mathcal{L}_\infty)(\mathcal{M}_k,\tau_k)$, $k\geqslant 0$. By Lemma 5.6, we have that

$$\||A|-|B|\|_{1,\infty}=\Big\|\bigoplus_{k\geqslant 0}|A_k|-|B_k|\Big\|_{1,\infty}\leqslant\sum_{k\geqslant 0}\||A_k|-|B_k|\|_{1,\infty}.$$

Since every $\mathcal{M}_k$ is a factor, it follows from Lemma 5.5 that

$$\||A_k|-|B_k|\|_{1,\infty}\leqslant\mathrm{const}\cdot\|A_k-B_k\|_1.$$

Therefore, we have

$$\begin{aligned}\||A|-|B|\|_{1,\infty}&\leqslant\mathrm{const}\cdot\sum_{k\geqslant 0}\|A_k-B_k\|_1=\mathrm{const}\cdot\Big\|\bigoplus_{k\geqslant 0}A_k-B_k\Big\|_1\\&=\mathrm{const}\cdot\|A-B\|_1.\end{aligned}$$ ∎

6. COMMUTATOR ESTIMATES

The proof of the following consequence is essentially the same as the implication (i) ⇒ (ii) of Theorem 2.2 in [11]. For completeness and the fact that Theorem 2.2 of [11] is not directly applicable since we are dealing with estimates between different spaces (and one of them only has a quasi-norm) we have included the proof.

THEOREM 6.1. *If $A, B \in (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}, \tau)$ are self-adjoint operators with $[A, B] \in \mathcal{L}_1(\mathcal{M}, \tau)$, then*

$$\|[|A|, B]\|_{1,\infty} \leqslant \text{const} \cdot \|[A, B]\|_1.$$

Proof. Suppose first that B is bounded. Setting $C = \mathrm{e}^{\mathrm{i}\varepsilon B} A \mathrm{e}^{-\mathrm{i}\varepsilon B}$, we have $|C| = \mathrm{e}^{\mathrm{i}\varepsilon B}|A|\mathrm{e}^{-\mathrm{i}\varepsilon B}$. We infer from Theorem 5.8 that

$$\|[\mathrm{e}^{\mathrm{i}\varepsilon B}, |A|]\|_{1,\infty} = \||C| - |A|\|_{1,\infty} \leqslant \text{const} \cdot \|C - A\|_1 = \text{const} \cdot \|[\mathrm{e}^{\mathrm{i}\varepsilon B}, A]\|_1.$$

However,

$$[\mathrm{e}^{\mathrm{i}\varepsilon B}, A] = \sum_{k=1}^{\infty} \frac{(i\varepsilon)^k}{k!}[B^k, A]$$

where the series on the right hand side converges in $\mathcal{L}_1(\mathcal{M}, \tau)$. Indeed,

$$[B^k, A] = \sum_{m=0}^{k-1} B^m [B, A] B^{k-1-m}, \quad k \geqslant 1$$

and therefore,

$$\begin{aligned}\|[\mathrm{e}^{\mathrm{i}\varepsilon B}, A]\|_1 &= \Big\| \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \frac{(\mathrm{i}\varepsilon)^k}{k!} B^m [B, A] B^{k-1-m} \Big\|_1 \leqslant \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \frac{\varepsilon^k}{k!} \|B^m [B, A] B^{k-1-m}\|_1 \\ &\leqslant \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \frac{\varepsilon^k}{k!} \|B\|_\infty^{k-1} \|[B, A]\|_1 = \Big(\sum_{k=1}^{\infty} \frac{k\varepsilon^k}{k!} \|B\|_\infty^{k-1} \Big) \|[A, B]\|_1 \\ &= \varepsilon \mathrm{e}^{\varepsilon \|B\|_\infty} \|[A, B]\|_1.\end{aligned}$$

Combining preceding estimates, we infer that

$$\|\varepsilon^{-1}[\mathrm{e}^{\mathrm{i}\varepsilon B}, |A|]\|_{1,\infty} \leqslant \text{const} \cdot \mathrm{e}^{\varepsilon\|B\|_\infty} \|[A, B]\|_1.$$

It follows from the spectral theorem and boundedness of B that

$$\varepsilon^{-1}(\mathrm{e}^{\mathrm{i}\varepsilon B} - 1) \to \mathrm{i}B$$

uniformly and, therefore,

$$\varepsilon^{-1}[\mathrm{e}^{\mathrm{i}\varepsilon B}, |A|] = [\varepsilon^{-1}(\mathrm{e}^{\mathrm{i}\varepsilon B} - 1), |A|] \to \mathrm{i}[B, |A|]$$

in the norm of the space $(\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}, \tau)$ and therefore in measure (see also [11]). Since the quasi-norm in $\mathcal{L}_{1,\infty}$ has the Fatou property, it follows that

$$\|[|A|, B]\|_{1,\infty} \leqslant \text{const} \cdot \|[A, B]\|_1.$$

This proves the assertion for the case of bounded B.

Consider now the general case of an arbitrary $B \in (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}, \tau)$. Set $p_n = E_{|B|}[0, n]$. We have that $p_n A p_n \to A$ and $p_n B p_n \to B$ in measure. Since p_n commutes with B, it follows that

$$[p_n A p_n, p_n B p_n] = p_n [A, B] p_n.$$

It follows from [30] (see also Theorem 1.1 of [11]) that $|p_nAp_n| \to |A|$ in measure. Thus,

$$[|p_nAp_n|, p_nBp_n] \to [|A|, B]$$

in measure. It is proved in the previous paragraph that

$$\begin{aligned}\|[|p_nAp_n|, p_nBp_n]\|_{1,\infty} &\leqslant \mathrm{const}\|[p_nAp_n, p_nBp_n]\|_1 \\ &= \mathrm{const}\cdot\|p_n[A,B]p_n\|_1 \leqslant \mathrm{const}\cdot\|[A,B]\|_1.\end{aligned}$$

Since the quasi-norm in $\mathcal{L}_{1,\infty}$ has the Fatou property, it follows that

$$\|[|A|, B]\|_{1,\infty} \leqslant \mathrm{const}\cdot\|[A,B]\|_1. \quad \blacksquare$$

REMARK 6.2. The result of Theorem 1.1 may be obtained from Theorem 6.1 as follows. Firstly, observe that we can interpolate between the weak $\mathcal{L}_1$-space, $\mathcal{L}_{1,\infty}$ and $\mathcal{L}_2$ using weak type interpolation (see e.g. [9] and references therein). This immediately implies the estimates for Schatten p-norms, $1 < p < 2$ analogous to that of Theorem 6.1, with the case $2 < p < \infty$ following by duality. The result of Theorem 1.2 follows now from Theorem 2.2 of [11].

7. FINAL COMMENTS

Davies introduced the class of functions representable in the form

$$f(t) = \int_{\mathbb{R}} |t-s|\mathrm{d}\nu_f(s),$$

where ν_f is a signed measure with finite support. He proved that

$$\|f(A) - f(B)\|_p \leqslant c_{p,f}\|A-B\|_p, \quad 1 < p < \infty.$$

Though we cannot fully extend this result to $p = 1$, the following is possible.

Define distorted variation $DV(\nu)$ as follows

$$DV(\nu) = \sup\Big\{\inf_{\pi}\sum_{k\geqslant 0} 2^{\pi(k)}|\nu(A_k)| : A_m \cap A_n = \varnothing \text{ for all } m \neq n, \bigcup_{k\geqslant 0} A_k = \mathbb{R}\Big\}.$$

Here, every A_n, $n \geqslant 0$, is an interval (or a semi-axis) and the infimum is taken over all permutations π of $\mathbb{Z}_+$.

LEMMA 7.1. *For every finitely supported measure ν with $DV(\nu) < \infty$, there exists a sequence ν_m, $m \geqslant 1$, of discrete measures such that*

$$\int_{\mathbb{R}} |t-s|\mathrm{d}\nu_m(s) \to \int_{\mathbb{R}} |t-s|\mathrm{d}\nu(s)$$

uniformly and $DV(\nu_m) \leqslant DV(\nu)$ for all $m \geqslant 1$.

Proof. Assume, for simplicity of notations, that ν is supported on $[0,1)$ and that $|\nu|([0,1)) = 1$. Define a measure ν_m by setting

$$\nu_m = \sum_{k=0}^{m-1} \nu([\tfrac{k}{m}, \tfrac{k+1}{m}))\delta_{\{\frac{k}{m}\}}.$$

It is immediate that $DV(\nu_m) \leqslant DV(\nu)$ (because every partition of the finite set $\{0, 1/m, \ldots, (m-1)/m\}$ extends to a partition of $[0,1)$).

Fix $t \in \mathbb{R}$ and for a given $m \in \mathbb{N}$, define the function g_m on $[0,1)$ by setting $g_m(s) = |k/m - t|$ for all $s \in [k/m, (k+1)/m)$, $0 \leqslant k < m$. Since g_m is a step function with steps at $\{0, 1/m, \ldots, (m-1)/m\}$, it follows that

$$\int_{\mathbb{R}} g_m(s) \mathrm{d}\nu_m(s) = \int_{\mathbb{R}} g_m(s) \mathrm{d}\nu(s).$$

It is clear that

$$\Big| \int_{\mathbb{R}} |t-s| \mathrm{d}\nu(s) - \int_{\mathbb{R}} g_m(s) \mathrm{d}\nu(s) \Big| \leqslant \frac{1}{m} \cdot |\nu|([0,1))$$

and

$$\Big| \int_{\mathbb{R}} |t-s| \mathrm{d}\nu_m(s) - \int_{\mathbb{R}} g_m(s) \mathrm{d}\nu_m(s) \Big| \leqslant \frac{1}{m} \cdot |\nu_m|([0,1)).$$

It follows that

$$\Big| \int_{\mathbb{R}} |t-s| \mathrm{d}\nu_m(s) - \int_{\mathbb{R}} |t-s| \mathrm{d}\nu(s) \Big| \leqslant \frac{2}{m}.$$

This proves the claim. ∎

The following lemma is a particular case of Lemma 17 in [27] (proved there for every quasi-Banach space and not just $\mathcal{L}_{1,\infty}$). It serves as a replacement for the triangle inequality.

LEMMA 7.2. *Let $A_k \in \mathcal{L}_{1,\infty}$, $k \geqslant 0$. We have*

$$\Big\| \sum_{k=0}^{\infty} A_k \Big\|_{1,\infty} \leqslant \mathrm{const} \cdot \sum_{k=0}^{\infty} 2^k \|A_k\|_{1,\infty}.$$

Here, the convergence of the series in the right hand side guarantees that the series in the left hand side converges in $\mathcal{L}_{1,\infty}$.

THEOREM 7.3. *If f is in the Davies class and if $DV(\nu_f) < \infty$, then for any two bounded self-adjoint operators A and B, such that $A - B \in \mathcal{L}_1$, we have*

$$\|f(A) - f(B)\|_{1,\infty} \leqslant \mathrm{const} \cdot DV(\nu_f) \cdot \|A - B\|_1.$$

Proof. By Lemma 7.1, we may approximate ν with $DV(\nu) \leqslant 1$ by the sequence of discrete measures ν_m with $DV(\nu_m) \leqslant 1$. It follows that we can assume without loss of generality that ν is discrete. Indeed, since A and B are bounded, we find that $f_m(A) \to f(A)$ and $f_m(B) \to f(B)$ uniformly, where

$$f_m(t) := \int_{\mathbb{R}} |t-s| \mathrm{d}\nu_m(s), \quad t \in \mathbb{R}.$$

Using the Fatou property of the $\mathcal{L}_{1,\infty}$, we infer that it is suffices to prove the assertion for discrete measures with finite distorted variation.

If the measure ν is discrete, then

$$f(t) = \sum_{k=0}^{\infty} \alpha_k |t - t_k|, \quad \sum_{k=0}^{\infty} 2^k |\alpha_k| < \infty.$$

We have

$$f(A) - f(B) = \sum_{k=0}^{\infty} \alpha_k (|A - t_k| - |B - t_k|).$$

By Theorem 1.2, we have

$$\| |A - t_k| - |B - t_k| \|_{1,\infty} \leqslant \mathrm{const} \cdot \|A - B\|_1.$$

It follows from Lemma 7.2 above that

$$\|f(A) - f(B)\|_{1,\infty} \leqslant \sum_{k=0}^{\infty} 2^k |\alpha_k| \|A - B\|_1. \quad \blacksquare$$

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