ON THE C^*-ALGEBRA GENERATED BY TOEPLITZ OPERATORS AND FOURIER MULTIPLIERS ON THE HARDY SPACE OF A LOCALLY COMPACT GROUP

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Dedicated to Prof. Aydın Aytuna on the occasion of his 65th birthday

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ABSTRACT. Let G be a locally compact abelian Hausdorff topological group which is non-compact and whose Pontryagin dual Γ is partially ordered. Let $\Gamma^+ \subset \Gamma$ be the semigroup of positive elements in Γ. The Hardy space $H^2(G)$ is the closed subspace of $L^2(G)$ consisting of functions whose Fourier transforms are supported on Γ^+. In this paper we consider the C^*-algebra $C^*(\mathcal{T}(G) \cup F(C(\Gamma^+)))$ generated by Toeplitz operators with continuous symbols on G which vanish at infinity and Fourier multipliers with symbols which are continuous on one point compactification of Γ^+ on the Hilbert–Hardy space $H^2(G)$. We characterize the character space of this C^*-algebra using a theorem of Power.

KEYWORDS: *C^*-algebras, Toeplitz operators, Hardy space of a locally compact group.*

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INTRODUCTION

For a locally compact abelian Hausdorff topological group G whose Pontryagin dual Γ is partially ordered, one can define the positive elements of Γ as $\Gamma^+ = \{\gamma \in \Gamma : \gamma \geqslant e\}$ where e is the identity of the group G and the Hardy space $H^2(G)$ as

$$H^2(G) = \{f \in L^2(G) : \widehat{f}(\gamma) = 0 \ \forall \gamma \notin \Gamma^+\}$$

where $\widehat{f}$ is the Fourier transform of f. It is not difficult to see that $H^2(G)$ is a closed subspace of $L^2(G)$ and since $L^2(G)$ is a Hilbert space there is a unique orthogonal projection $P : L^2(G) \to H^2(G)$ onto $H^2(G)$.

This definition of the Hardy space $H^2(G)$ is motivated by Riesz theorem in the classical cases when $G = \mathbb{T}$ i.e. when G is the unit circle, which characterizes the Hardy class functions among $f \in L^2(\mathbb{T})$ as the space of functions whose

negative Fourier coefficients vanish and by the Paley–Wiener theorem when $G = \mathbb{R}$, the real line, since the group Fourier transform is the Fourier series when $G = \mathbb{T}$ and coincides with the Euclidean Fourier transform when $G = \mathbb{R}$.

Actually the Paley–Wiener theorem is valid under more general conditions. We have the following Paley–Wiener type theorem due to Bochner (see p. 93 of [13]):

PALEY–WIENER THEOREM. *Let $\Pi = X \oplus i\Lambda$ be a tubular domain where $\Lambda \subseteq \mathbb{R}^n$ is a cone and $X \cong \mathbb{R}^n$. Then the Fourier transform*

$$(0.1) \qquad \mathcal{F}(f)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(t) e^{-ix \cdot t} dt$$

maps $H^2(\Pi)$ isometrically onto $L^2(\Lambda^*)$ where $\Lambda^* = \{y \in \mathbb{R}^n : x \cdot y > 0 \ \forall x \in \Lambda\}$ is the dual cone of Λ.

In this setup when $G = \mathbb{R}^2$ then $H^2(\Pi) = H^2(G)$ according to our Hardy space definition, with the help of the theorem above, where $\Pi = \mathbb{R}^2 \oplus i(\mathbb{R}^+)^2 = \mathbb{H}^2$. Here $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ is the upper half-plane. Note that $((\mathbb{R}^+)^2)^* = (\mathbb{R}^+)^2$. The Pontryagin dual Γ of the locally compact abelian topological group $\mathbb{R}^2$ with component-wise addition as the group operation is $\Gamma = \mathbb{R}^2$. When we put the partial order $(x_1, y_1) \geqslant (x_2, y_2) \Leftrightarrow x_1 - x_2 > 0$ and $y_1 - y_2 > 0$, then $\Gamma^+ = (\mathbb{R}^+)^2$. Note that this partial order is not a total order.

One can extend the theory of Toeplitz operators to this setting by defining a Toeplitz operator with symbol $\phi \in L^\infty(G)$ as $T_\phi = PM_\phi$ where M_ϕ is the multiplication by ϕ and P is the orthogonal projection of L^2 onto H^2. Such a definition was first considered by Coburn and Douglas in [2]. However the Toeplitz operators considered in [2] were more general since no partial order was assumed on the dual Γ whereas the Hardy space was defined as the space of functions whose Fourier transforms are supported on a fixed sub-semigroup Γ_0 of Γ. The definition of Hardy space of groups whose duals are partially ordered and their Toeplitz operators were introduced and studied by Murphy in [5] and [7]. However in these papers [5] and [7], Murphy studies the case where G is compact. In this paper we will study the case where G is not compact. One very important assumption that we will make is that Γ^+ separates the points of G, i.e. for any $t_1, t_2 \in G$ satisfying $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$.

The Toeplitz C^*-algebra of a locally compact group is defined as

$$\mathcal{T}(G) = C^*(\{T_\phi : \phi \in C_0(G)\} \cup \{I\})$$

where $C_0(G)$ is the space of continuous functions vanishing at infinity and I is the identity operator. In the study of this Toeplitz C^*-algebra, the most important notions are the commutator ideal $\mathrm{com}(G) = I^*(\{T_\phi T_\psi - T_\psi T_\phi : \phi, \psi \in C_0(G)\})$, the semi-commutator ideal $\mathrm{scom}(G) = I^*(\{T_{\phi\psi} - T_\psi T_\phi : \phi, \psi \in C_0(G)\})$ and the symbol map $\Sigma : C(\dot{G}) \to \mathcal{T}(G)/\mathrm{com}(G)$, $\Sigma(\phi) = [T_\phi]$ where $\dot{G}$ is the one point compactification of G and $[T_\phi]$ denotes the equivalence class of T_ϕ modulo

com(G). It is not difficult to see that com(G) $\subseteq$ scom(G). We start by proving the following important result whose proof is adapted from [8]:

LEMMA 0.1. *Let G be a locally compact abelian Hausdorff topological group whose Pontryagin dual Γ is partially ordered and let Γ^+ be the semigroup of positive elements of Γ. Suppose that Γ^+ separates the points of G i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Let com($G$) and scom($G$) be the commutator and the semi-commutator ideal of the Toeplitz C^*-algebra $\mathcal{T}(G)$ respectively. Then*

$$\text{com}(G) = \text{scom}(G).$$

It is shown in [2] and [7] that $\Sigma : C(\dot{G}) \to \mathcal{T}(G)/\text{com}(G)$ is an isometry but is not a homomorphism since it may not preserve the multiplication. However $\Sigma : C(\dot{G}) \to \mathcal{T}(G)/\text{scom}(G)$ is a homomorphism and combining this fact with Lemma 0.1 above we deduce that the symbol map $\Sigma : C(\dot{G}) \to \mathcal{T}(G)/\text{com}(G)$ is an isometric isomorphism which means that

$$M(\mathcal{T}(G)) = \dot{G}$$

where $M(A)$ is the character space of a C^*-algebra A.

We introduce another class of operators acting on $H^2(G)$ which are called *Fourier multipliers*. These operators in the classical case $G = \mathbb{R}$ were introduced in [4]. The space of Fourier multipliers is defined as

$$F(C(\dot{\Gamma^+})) = \{D_\theta = \mathcal{F}^{-1} M_\theta \mathcal{F}|_{H^2(G)} : \theta \in C(\dot{\Gamma^+})\}$$

where $\mathcal{F} : L^2(G) \to L^2(\Gamma)$ is the Fourier transform. By Plancherel theorem it is not difficult to see that the image $\mathcal{F}(H^2(G))$ of $H^2(G)$ under the Fourier transform is equal to $L^2(\Gamma^+)$. Again it is not difficult to see that $F(C(\dot{\Gamma^+}))$ is isometrically isomorphic to $C(\dot{\Gamma^+})$. This means that

$$M(F(C(\dot{\Gamma^+}))) = \dot{\Gamma^+}.$$

Lastly we consider the C^*-algebra generated by $\mathcal{T}(G)$ and $F(C(\dot{\Gamma^+}))$ which we denote by $\Psi(C_0(G), C(\dot{\Gamma^+}))$ i.e.

$$\Psi(C_0(G), C(\dot{\Gamma^+})) = C^*(\mathcal{T}(G) \cup F(C(\dot{\Gamma^+}))).$$

Using a theorem of Power [9], [10], which characterizes the character space of the C^*-algebra generated by two C^*-algebras as a certain subset of the cartesian product of character spaces of these two C^*-algebras, we prove the following theorem:

MAIN THEOREM. *Let G be a non-compact, locally compact abelian Hausdorff topological group whose Pontryagin dual Γ is partially ordered and let Γ^+ be the semigroup of positive elements of Γ. Suppose that Γ^+ separates the points of G i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Let*

$$\Psi(C_0(G), C(\dot{\Gamma^+})) = C^*(\mathcal{T}(G) \cup F(C(\dot{\Gamma^+})))$$

be the C^*-algebra generated by Toeplitz operators and Fourier multipliers on $H^2(G)$. Then for the character space $M(\Psi)$ of $\Psi(C_0(G), C(\dot{\Gamma}^+))$ we have

$$M(\Psi) \cong (\dot{G} \times \{\infty\}) \cup (\{\infty\} \times \dot{\Gamma}^+).$$

Very similar kinds of the above theorem were proven by Power in [9] and [10] when $G = \mathbb{R}^n$, however those theorems considered $L^2(\mathbb{R}^n)$ instead of $H^2(\mathbb{R}^n)$ and took multiplication operators instead of Toeplitz operators. Indeed Power proved his theorems with more general class of symbols like $CVO(\mathbb{R}^n)$ in place of $C_0(\mathbb{R}^n)$.

1. PRELIMINARIES

In this section we fix the notation that we will use throughout and recall some preliminary facts that will be used in the sequel.

Let S be a compact Hausdorff topological space. The space of all complex valued continuous functions on S will be denoted by $C(S)$. For any $f \in C(S)$, $\|f\|_\infty$ will denote the sup-norm of f, i.e.

$$\|f\|_\infty = \sup\{|f(s)| : s \in S\}.$$

If S is a locally compact Hausdorff topological space, $C_0(S)$ will denote the space of continuous functions f which vanish at infinity i.e. for any $\varepsilon > 0$ there is a compact subset $K \subset S$ such that $|f(x)| < \varepsilon$ for all $x \notin K$. For a Banach space X, $K(X)$ will denote the space of all compact operators on X and $\mathcal{B}(X)$ will denote the space of all bounded linear operators on X. The real line will be denoted by $\mathbb{R}$, the complex plane will be denoted by $\mathbb{C}$ and the unit circle group will be denoted by $\mathbb{T}$. The one point compactification of a locally compact Hausdorff topological space S will be denoted by $\dot{S}$. For any subset $S \subset \mathcal{B}(H)$, where H is a Hilbert space, the C^*-algebra generated by S will be denoted by $C^*(S)$ and for any subset $S \subset A$ where A is a C^*-algebra, the closed two-sided ideal generated by S will be denoted by $I^*(S)$.

For any $\phi \in L^\infty(G)$ where G is a Borel space (a topological space with a regular measure on it), M_ϕ will be the multiplication operator on $L^2(G)$ defined as

$$M_\phi(f)(t) = \phi(t)f(t).$$

For convenience, we remind the reader of the rudiments of theory of Banach algebras, some basic abstract harmonic analysis and Toeplitz operators.

Let A be a Banach algebra. Then its character space $M(A)$ is defined as

$$M(A) = \{x \in A^* : x(ab) = x(a)x(b) \ \forall a, b \in A\}$$

where A^* is the dual space of A. If A has identity then $M(A)$ is a compact Hausdorff topological space with the weak* topology. When A is commutative $M(A)$

is called the maximal ideal space of A. For a commutative Banach algebra A the Gelfand transform $\Gamma : A \to C(M(A))$ is defined as

$$\Gamma(a)(x) = x(a).$$

If A is a commutative C^*-algebra with identity, then Γ is an isometric $*$-isomorphism between A and $C(M(A))$. If A is a C^*-algebra and I is a two-sided closed ideal of A, then the quotient algebra A/I is also a C^*-algebra (see [6]). For a Banach algebra A, we denote by $\mathrm{com}(A)$ the closed ideal in A generated by the commutators $\{a_1 a_2 - a_2 a_1 : a_1, a_2 \in A\}$. It is an algebraic fact that the quotient algebra $A/\mathrm{com}(A)$ is a commutative Banach algebra. The reader can find detailed information about Banach and C^*-algebras in [6] and [12] related to what we have reviewed so far.

On a locally compact abelian Hausdorff topological group G there is a unique (up to multiplication by a constant) translation invariant measure λ on G i.e. for any Borel subset $E \subset G$ and for any $x \in G$,

$$\lambda(xE) = \lambda(E)$$

where $xE = \{xy : y \in E\}$ is the translate of E by x. This measure is called the Haar measure of G. Let $L^1(G)$ be the space of integrable functions with respect to this measure. Then $L^1(G)$ becomes a commutative Banach algebra with multiplication as the convolution defined as

$$(f * g)(t) = \int_G f(ts^{-1}) g(s) \mathrm{d}\lambda(s).$$

The Pontryagin dual Γ of G is defined to be the set of all continuous homomorphisms from G to the circle group $\mathbb{T}$:

$$\Gamma = \{\gamma : G \to \mathbb{T} : \gamma(st) = \gamma(s)\gamma(t) \text{ and } \gamma \text{ is continuous}\}.$$

It is a well known fact that Γ is in one to one correspondence with the maximal ideal space $M(L^1(G))$ of $L^1(G)$ via the Fourier transform:

$$\langle \gamma, f \rangle = \widehat{f}(\gamma) = \int_G \overline{\gamma(t)} f(t) \mathrm{d}\lambda(t).$$

When Γ is topologized by the weak* topology coming from $M(L^1(G))$, Γ becomes a locally compact abelian Hausdorff topological group with point-wise multiplication as the group operation:

$$(\gamma_1 \gamma_2)(t) = \gamma_1(t)\gamma_2(t).$$

Let $\widetilde{\lambda}$ be a fixed Haar measure on Γ. Plancherel theorem asserts that the Fourier transform $\mathcal{F}$ is an isometric isomorphism of $L^2(G)$ onto $L^2(\Gamma)$:

$$\mathcal{F}(f)(\gamma) = \widehat{f}(\gamma) = \int_G \overline{\gamma(t)} f(t) \mathrm{d}\lambda(t)$$

with inverse $\mathcal{F}^{-1}$ defined as

$$\mathcal{F}^{-1}(f)(t) = \check{f}(t) = \int_{\Gamma} \gamma(t)f(\gamma)\mathrm{d}\widetilde{\lambda}(\gamma).$$

Here we note that $\widetilde{\lambda}$ is normalized so that the above formula for the inverse Fourier transform holds. For detailed information on abstract harmonic analysis consult [11].

A partially ordered group Γ is a group with partial order $\geqslant$ on it satisfying $\gamma_1 \geqslant \gamma_2$ implies $\gamma\gamma_1 \geqslant \gamma\gamma_2 \,\forall\gamma \in \Gamma$. This definition of the ordered group was given in [5]. Let $\Gamma^+ = \{\gamma \in \Gamma : \gamma \geqslant e\}$ be the semi-group of positive elements of Γ where e is the unit of the group Γ. Let G be a locally compact abelian Hausdorff topological group and let Γ be the Pontryagin dual of G. Then the Hardy space $H^2(G)$ is defined as

$$H^2(G) = \{f \in L^2(G) : \widehat{f}(\gamma) = 0 \,\forall\gamma \notin \Gamma^+\}.$$

The Hardy space $H^2(G)$ is a closed subspace of $L^2(G)$ and since $L^2(G)$ is a Hilbert space, there is a unique orthogonal projection $P : L^2(G) \to H^2(G)$. For any $\phi \in L^\infty(G)$ the Toeplitz operator $T_\phi : H^2(G) \to H^2(G)$ is defined as

$$T_\phi = PM_\phi.$$

Toeplitz operators satisfy the following algebraic properties:

(i) $T_{c\phi+\psi} = cT_\phi + T_\psi \,\forall c \in \mathbb{C}, \,\forall\phi, \psi \in C(\dot{G})$,
(ii) $T_\phi^* = T_{\overline{\phi}} \,\forall\phi \in C(\dot{G})$.

The proofs of these properties are the same as in the classical case where $G = \mathbb{T}$ (or $G = \mathbb{R}$) and can be found in [3].

The Toeplitz C^*-algebra $\mathcal{T}(G)$ is defined to be the C^*-algebra generated by continuous symbols on G:

$$\mathcal{T}(G) = C^*(\{T_\phi : \phi \in C_0(G)\} \cup \{I\})$$

where I is the identity operator and $C_0(G)$ is the space of continuous functions which vanish at infinity:

$$C_0(G) = \{f : G \to \mathbb{C} : f \text{ is continuous and } \forall\varepsilon > 0 \,\exists K \subset\subset G | f(t)| < \varepsilon \,\forall t \notin K\}$$

where $K \subset\subset G$ denotes a compact subset of G. Actually one has

$$\mathcal{T}(G) = C^*(\{T_\phi : \phi \in C(\dot{G})\})$$

where $\dot{G}$ is the one-point compactification of G. In the case where G is compact one has $G = \dot{G}$ and the most prototypical concrete example of this case is $G = \mathbb{T}$. This case was analyzed by Coburn in [1]. The famous result of Coburn asserts that for any $T \in \mathcal{T}(\mathbb{T})$ there are unique $K \in K(H^2(\mathbb{T}))$ and $\phi \in C(\mathbb{T})$ such that $T = T_\phi + K$. Hence the quotient algebra $\mathcal{T}(\mathbb{T})/K(H^2(\mathbb{T}))$ modulo the compact operators is isometrically isomorphic to $C(\mathbb{T})$. The two sided closed $*$-ideal

com(G) generated by the commutators is called the commutator ideal of $\mathcal{T}(G)$:

$$\mathrm{com}(G) = I^*(\{T_\phi T_\psi - T_\psi T_\phi : \phi, \psi \in C(\dot{G})\})$$

and the semi-commutator ideal scom(G) is defined as

$$\mathrm{scom}(G) = I^*(\{T_{\phi\psi} - T_\psi T_\phi : \phi, \psi \in C(\dot{G})\}).$$

The symbol map $\Sigma : C(\dot{G}) \to \mathcal{T}(G)/\mathrm{com}(G)$ is defined as

$$\Sigma(\phi) = [T_\phi]$$

where $[\cdot]$ denotes the equivalence class modulo com(G). In [2] and [5] it is shown that Σ is an isometry. The symbol map Σ also preserves the $*$-operation however is not a homomorphism i.e. does not preserve multiplication. But if com(G) = scom(G) then it is an isometric isomorphism. We will show under certain conditions that com(G) = scom(G).

We introduce another class of operators which we call the *Fourier multipliers*. This class of operators in the case $G = \mathbb{R}$ was introduced in [4] and proved to be useful in calculating the essential spectra of a class of composition operators. The Fourier multiplier $D_\theta : H^2(G) \to H^2(G)$ with symbol $\theta \in C(\dot{\Gamma^+})$ is defined as

$$D_\theta(f)(t) = (\mathcal{F}^{-1} M_\theta \mathcal{F}(f))(t).$$

The most prototypical example of a Fourier multiplier is a convolution operator with kernel $k \in L^1(G)$:

$$(T_k f)(t) = \int_G k(ts^{-1}) f(s) \mathrm{d}\lambda(s).$$

It is not difficult to see that actually $T_k = D_{\hat{k}}$ where $\hat{k}$ denotes the Fourier transform of k. The set of all Fourier multipliers $F(C(\dot{\Gamma^+}))$ defined as

$$F(C(\dot{\Gamma^+})) = \{D_\theta : \theta \in C(\dot{\Gamma^+})\}$$

is a commutative C^*-algebra since the map $D : C(\dot{\Gamma^+}) \to F(C(\dot{\Gamma^+}))$ defined as $D(\theta) = D_\theta$ is an isometric $*$-isomorphism.

Lastly we consider the C^*-algebra generated by Toeplitz operators and Fourier multipliers. Let $\Psi(C_0(G), C(\dot{\Gamma}))$ be the C^*-algebra

$$\Psi(C_0(G), C(\dot{\Gamma})) = C^*(\mathcal{T}(G) \cup F(C(\dot{\Gamma^+})))$$

generated by Toeplitz operators with continuous symbols and continuous Fourier multipliers. The main result of this paper is a characterization of the character space $M(\Psi)$ of $\Psi(C_0(G), C(\dot{\Gamma}))$. We know that

$$M(F(C(\dot{\Gamma^+}))) \cong \dot{\Gamma^+},$$

under certain conditions we have scom(G) = com(G) and this implies that

$$M(\mathcal{T}(G)) \cong \dot{G}.$$

We will use the following theorem due to Power [9], [10] in identifying the character space of $\Psi(C_0(G), C(\dot{\Gamma}^+))$:

POWER'S THEOREM. *Let C_1, C_2 be C*-subalgebras of $B(H)$ with identity, where H is a separable Hilbert space, such that $M(C_i) \neq \emptyset$, where $M(C_i)$ is the space of multiplicative linear functionals of C_i, $i = 1$, 2 and let C be the C*-algebra that they generate. Then for the commutative C*-algebra $\widetilde{C} = C/\mathrm{com}(C)$ we have $M(\widetilde{C}) = P(C_1, C_2) \subset M(C_1) \times M(C_2)$, where $P(C_1, C_2)$ is defined to be the set of points $(x_1, x_2) \in M(C_1) \times M(C_2)$ satisfying the condition: Given $0 \leqslant a_1 \leqslant 1$, $0 \leqslant a_2 \leqslant 1$, $a_1 \in C_1$, $a_2 \in C_2$,*

$$x_i(a_i) = 1 \quad \text{with } i = 1, 2 \quad \Rightarrow \quad \|a_1 a_2\| = 1.$$

The proof of this theorem can be found in [10]. Power's theorem will give the character space $M(\Psi)$ of $\Psi(C_0(G), C(\dot{\Gamma}))$ as a certain subset of the cartesian product $\dot{G} \times \dot{\Gamma}^+$.

2. THE CHARACTER SPACE OF $\Psi(C_0(G), C(\dot{\Gamma}))$

In this section we will concentrate on the C*-algebra $\Psi(C_0(G), C(\dot{\Gamma}))$. But before that we will identify the character space $M(\mathcal{T}(G))$ of $\mathcal{T}(G)$ under certain conditions. The condition that we will pose on G is that Γ^+ separates the points of G i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Under this condition we show that $\mathrm{scom}(G) = \mathrm{com}(G)$ and this implies that $M(\mathcal{T}(G)) \cong \dot{G}$. Hence we begin by proving the following lemma whose proof is adapted from the proof of Theorem 2.2 of [8]:

PROPOSITION 2.1. *Let G be a locally compact abelian Hausdorff topological group whose Pontryagin dual Γ is partially ordered and let Γ^+ be the semigroup of positive elements of Γ. Suppose that Γ^+ separates the points of G i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Let $\mathrm{com}(G)$ and $\mathrm{scom}(G)$ be the commutator and the semi-commutator ideal of the Toeplitz C*-algebra $\mathcal{T}(G)$ respectively. Then*

$$\mathrm{com}(G) = \mathrm{scom}(G).$$

Proof. It is trivial that $\mathrm{com}(G) \subseteq \mathrm{scom}(G)$ hence we need to show that $\mathrm{scom}(G) \subseteq \mathrm{com}(G)$:

Let $B = \{\phi \in C_0(G) : T_\phi T_\psi - T_{\phi\psi} \in \mathrm{com}(G)\}$ then B is a self-adjoint subalgebra of $C_0(G)$: Let $\psi \in B$ then since

$$T_\phi T_{\overline{\psi}} - T_{\phi\overline{\psi}} = (T_\psi T_{\overline{\phi}} - T_{\overline{\phi}} T_\psi)^* + (T_{\overline{\phi}} T_\psi - T_{\overline{\phi}\psi})^*$$

we have $(T_\psi T_{\overline{\phi}} - T_{\overline{\phi}} T_\psi)^* \in \mathrm{com}(G)$, $(T_{\overline{\phi}} T_\psi - T_{\overline{\phi}\psi})^* \in \mathrm{com}(G)$ and hence $T_\phi T_{\overline{\psi}} - T_{\phi\overline{\psi}} \in \mathrm{com}(G)$ $\forall \phi \in C(\dot{G})$. This implies that $\overline{\psi} \in B$. It is clear that $\psi_1, \psi_2 \in B$ implies that $\psi_1 + \psi_2 \in B$. Let us check that $\psi_1 \psi_2 \in B$: We have

$$T_\phi T_{\psi_1\psi_2} - T_{\phi\psi_1\psi_2} = T_\phi(T_{\psi_1\psi_2} - T_{\psi_1} T_{\psi_2}) + (T_\phi T_{\psi_1} - T_{\phi\psi_1})T_{\psi_2} + (T_{\phi\psi_1} T_{\psi_2} - T_{\phi\psi_1\psi_2}).$$

Since $\psi_1 \in B$ and com(G) is an ideal we have $T_\phi(T_{\psi_1\psi_2} - T_{\psi_1}T_{\psi_2}) \in$ com(G), $(T_\phi T_{\psi_1} - T_{\phi\psi_1})T_{\psi_2} \in$ com(G) and $(T_{\phi\psi_1}T_{\psi_2} - T_{\phi\psi_1\psi_2}) \in$ com(G) which implies that $T_\phi T_{\psi_1\psi_2} - T_{\phi\psi_1\psi_2} \in$ com(G) $\forall \phi \in C_0(G)$. So we have $\psi_1\psi_2 \in B$. Now we need to show that B separates the points of G to conclude the proof since in that case B is closed and by Stone–Weierstrass theorem we will have $B = C_0(G)$: Now let $A(G) = \{\psi \in C_0(G) : \psi f \in H^2(G) \ \forall f \in H^2(G)\}$. Clearly $A(G) \subseteq B$, hence if we show that $A(G)$ separates the points of G we are done. For any $k \in L^1(\Gamma^+)$ consider

$$\check{k}(t) = \int_{\Gamma^+} k(\gamma)\gamma(t)\mathrm{d}\widetilde{\lambda}(\gamma)$$

then since for any $f \in H^2(G)$ we have

$$\mathcal{F}(\check{k}f) = k * \widehat{f}$$

the Fourier transform of $\check{k}f$ will be supported in Γ^+. This implies that $\check{k} \in A(G)$. Since Γ^+ separates the points of G, $\{\check{k} : k \in L^1(\Gamma^+)\}$ also separates the points of G and hence $A(G)$ separates the points of G. This implies that B separates the points of G. This proves our lemma. ∎

We have the following corollary of Proposition 2.1:

COROLLARY 2.2. *Let G be a locally compact abelian Hausdorff topological group whose Pontryagin dual Γ is partially ordered and let Γ^+ be the semigroup of positive elements of Γ. Suppose that Γ^+ separates the points of G i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Let $\mathcal{T}(G)$ be the Toeplitz C^*-algebra with symbols in $C(\dot{G})$ acting on $H^2(G)$. Then we have*

$$M(\mathcal{T}(G)) \cong \dot{G}.$$

Proof. The symbol map $\Sigma : C(\dot{G}) \to \mathcal{T}(G)/\mathrm{com}(G)$, $\Sigma(\phi) = [T_\phi]$ is an isometry that preserves the $*$-operation and $\Sigma : C(\dot{G}) \to \mathcal{T}(G)/\mathrm{scom}(G)$ is multiplicative. Since com$(G) = $ scom(G), Σ is an isometric isomorphism. Since characters kill the commutators we have

$$M(\mathcal{T}(G)) = M(\mathcal{T}(G)/\mathrm{com}(G)) \cong M(C(\dot{G})) = \dot{G}. \quad ∎$$

In [9] Power calls a C^*-algebra $C \subset B(K)$ inner with respect to a closed subspace $H \subset K$ of a Hilbert space K if C is generated by $T \in B(K)$ which leave H invariant i.e. $C = C^*(\{T|_H : T \in A\})$ for some $A \subset B(K)$ such that $T(H) \subset H$ $\forall T \in A$. Power proves that if $C \subset B(K)$ is inner then com$(C) = $ scom(C) where com$(C) \subset \mathcal{T}(C)$ is defined as com$(C) = I^*(\{(PTPS - PSPT)|_H : T, S \in C\}$ and scom$(C) \subset \mathcal{T}(C)$ is defined as scom$(C) = I^*(\{(PTPS - PTS)|_H : T, S \in C\})$ where $P : K \to H$ is the orthogonal projection. Here $\mathcal{T}(C) = C^*(\{PT|_H : T \in C\}) \subset B(H)$. In the Toeplitz operator theory setting Power's result gives that if $C \subset L^\infty(\mathbb{T})$, which is also interpreted as $C \subset B(L^2(\mathbb{T}))$ via the identification $f \to M_f$ where $M_f(h)(\mathrm{e}^{i\theta}) = f(\mathrm{e}^{i\theta})h(\mathrm{e}^{i\theta})$ is the multiplication operator, is inner with respect to $H^2(\mathbb{T})$ then com$(C) = $ scom(C) where com$(C) \subset \mathcal{T}(C)$ is defined

as $\mathrm{com}(C) = I^*(\{T_{c_1}T_{c_2} - T_{c_2}T_{c_1} : c_1, c_2 \in C\})$ and $\mathrm{scom}(C) \subset \mathcal{T}(C)$ is defined as $\mathrm{scom}(C) = I^*(\{T_{c_1}T_{c_2} - T_{c_2c_1} : c_1, c_2 \in C\})$. It was proved earlier by Douglas [3] that $C(\mathbb{T})$ and $L^\infty(\mathbb{T})$ are inner with respect to $H^2(\mathbb{T})$ in the sense of Power. The question whether $C_0(G)$ and $L^\infty(G)$ are inner with respect to $H^2(G)$ in our abstract group context in the sense of Power is another issue that we will not take up here.

We will need the following small observation in proving our main theorem:

LEMMA 2.3. *Let G be a locally compact, non-compact, abelian Hausdorff topological group and let $K_1, K_2 \subset G$ be two non-empty compact subsets of G. Then there is $t_0 \in G$ such that $K_1 \cap (t_0 K_2) = \emptyset$ where $t_0 K_2 = \{t_0 t : t \in K_2\}$.*

Proof. Since G is non-compact and locally compact there is a one-point compactification $\dot{G}$ of G. Hence there is a point at infinity $\infty \in \dot{G}$ such that $\infty \notin G$. Assume that the lemma does not hold i.e. there are two non-empty compact subsets $K_1, K_2 \subset G$ such that $K_1 \cap (tK_2) \neq \emptyset$ for all $t \in G$. Now take a net $\{t_\alpha\}_{\alpha \in \mathcal{I}} \subset G$ such that $\lim_{\alpha \in \mathcal{I}} t_\alpha = \infty$. Since $\forall \alpha \in \mathcal{I}$ we have $K_1 \cap (t_\alpha K_2) \neq \emptyset$, there are $x_\alpha \in K_1$ and $y_\alpha \in K_2$ such that $x_\alpha = t_\alpha y_\alpha$. Since K_1 and K_2 are compact there are $x_0 \in K_1$, $y_0 \in K_2$ and sub-nets $x_{\alpha_1} \in K_1$, $y_{\alpha_2} \in K_2$ such that $\lim x_{\alpha_1} = x_0$ and $\lim y_{\alpha_2} = y_0$. One can further find a common sub-net index set $\mathcal{I}_0 \subset \mathcal{I}$ such that $\lim_{\beta \in \mathcal{I}_0} x_\beta = x_0$ and $\lim_{\beta \in \mathcal{I}_0} y_\beta = y_0$. Since $x_\beta = t_\beta y_\beta$ which implies that $t_\beta = x_\beta y_\beta^{-1}$, $\lim_{\beta \in \mathcal{I}_0} t_\beta = \infty$ and multiplication is continuous this implies that

$$x_0 y_0^{-1} = \lim_{\beta \in \mathcal{I}_0} x_\beta y_\beta^{-1} = \lim_{\beta \in \mathcal{I}_0} t_\beta = \infty.$$

Since $x_0 y_0^{-1} \in K_1 K_2^{-1}$ this implies that $\infty \in K_1 K_2^{-1}$. Since $K_1 K_2^{-1}$ is a compact subset of G and ∞ is outside any compact subset of G, this gives a contradiction. This contradiction proves the lemma. ∎

Now we will show the following lemma which will shorten the proof of our main theorem. The proof of the following lemma is adapted from an idea of Power in p. 482 of [9]:

LEMMA 2.4. *Let G be a locally compact abelian Hausdorff topological group with Pontryagin dual Γ. Let $\phi \in C_0(G)$ and $\theta \in C_0(\Gamma)$ each have compact supports. Then $D_\theta M_\phi$ is a compact operator on $L^2(G)$ where $D_\theta = \mathcal{F}^{-1}M_\theta\mathcal{F}$.*

Proof. Let $K_1 \subset G$ and $K_2 \subset \Gamma$ be compact supports of ϕ and θ respectively. Then for any $f \in L^2(G)$ we have

$$(D_\theta M_\phi f)(t) = \int\limits_{K_2} \gamma(t)\theta(\gamma)\left(\int\limits_{K_1} \overline{\gamma(\tau)}\phi(\tau)f(\tau)\mathrm{d}\lambda(\tau)\right)\mathrm{d}\widetilde{\lambda}(\gamma)$$

$$= \int\limits_{K_1} \left(\phi(\tau)\int\limits_{K_2} \gamma(t\tau^{-1})\theta(\gamma)\mathrm{d}\widetilde{\lambda}(\gamma)\right)f(\tau)\mathrm{d}\lambda(\tau) = \int\limits_{K_1} k(t,\tau)f(\tau)\mathrm{d}\lambda(\tau)$$

where

$$k(t, \tau) = \phi(\tau) \int\limits_{K_2} \gamma(t\tau^{-1})\theta(\gamma)\mathrm{d}\widetilde{\lambda}(\gamma).$$

Now consider

$$\int\limits_{G}\int\limits_{G} |k(t,\tau)|^2 \mathrm{d}\lambda(t)\mathrm{d}\lambda(\tau) = \int\limits_{G}\int\limits_{G} \left|\phi(\tau) \int\limits_{K_2} \gamma(t\tau^{-1})\theta(\gamma)\mathrm{d}\widetilde{\lambda}(\gamma)\right|^2 \mathrm{d}\lambda(t)\mathrm{d}\lambda(\tau)$$

$$\leqslant \|\phi\|_\infty^2 \int\limits_{K_1}\int\limits_{G} |\check{\theta}(t\tau^{-1})|^2 \mathrm{d}\lambda(t)\mathrm{d}\lambda(\tau) = \|\phi\|_\infty^2 \int\limits_{K_1} \|\check{\theta}\|_2 \mathrm{d}\lambda(\tau)$$

$$= \|\phi\|_\infty^2 \int\limits_{K_1} \|\theta\|_2 \mathrm{d}\lambda(\tau) = \|\phi\|_\infty^2 \|\theta\|_2 \lambda(K_1) < \infty.$$

This implies that $D_\theta M_\phi$ is Hilbert–Schmidt and hence compact. ∎

Now we are ready to prove our main theorem as follows:

MAIN THEOREM. *Let G be a non-compact, locally compact abelian Hausdorff topological group whose Pontryagin dual Γ is partially ordered and let Γ^+ be the semi-group of positive elements of Γ. Suppose that Γ^+ separates the points of G i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Let*

$$\Psi(C_0(G), C(\dot{\Gamma}^+)) = C^*(\mathcal{T}(G) \cup F(C(\dot{\Gamma}^+)))$$

be the C^*-algebra generated by Toeplitz operators and Fourier multipliers on $H^2(G)$. Then for the character space $M(\Psi)$ of $\Psi(C_0(G), C(\dot{\Gamma}^+))$ we have

$$M(\Psi) \cong (\dot{G} \times \{\infty\}) \cup (\{\infty\} \times \dot{\Gamma}^+).$$

Proof. We will use Power's theorem. In the setup of Power's theorem $C_1 = \mathcal{T}(G)$ and $C_2 = F(C(\dot{\Gamma}^+))$. By Corollary 2.2 we have $M(C_1) = \dot{G}$ and we have $M(C_2) = \dot{\Gamma}^+$. So we need to determine $(t, \gamma) \in \dot{G} \times \dot{\Gamma}^+$ satisfying for $0 \leqslant \phi, \theta \leqslant 1$, $\phi(t) = \theta(\gamma) = 1$ implies $\|T_\phi D_\theta\| = 1$.

Let $(t, \gamma) \in G \times \Gamma^+$. Let $\phi \in C(\dot{G})$, $\theta \in C(\dot{\Gamma}^+)$ such that $0 \leqslant \phi, \theta \leqslant 1$ and $\phi(t) = \theta(\gamma) = 1$. Let us also assume that θ and ϕ have compact supports. Let $\widetilde{\theta} \in C(\dot{\Gamma})$ such that $0 \leqslant \widetilde{\theta} \leqslant 1$, $\widetilde{\theta}$ has compact support and $\widetilde{\theta}|_{C(\dot{\Gamma}^+)} = \theta$. Since

$$\|T_\phi D_\theta\| \leqslant \|M_\phi D_{\widetilde{\theta}}\|_{L^2(G)}$$

it suffices to show that $\|M_\phi D_{\widetilde{\theta}}\|_{L^2(G)} < 1$. We will also assume that $\phi(s) < 1$ $\forall s \in G - \{t\}$. Since $(M_\phi D_{\widetilde{\theta}})^* = D_{\widetilde{\theta}} M_\phi$ and $D_{\widetilde{\theta}} M_\phi$ is compact by Lemma 2.4, $M_\phi D_{\widetilde{\theta}}$ is also compact. Hence $M_\phi D_{\widetilde{\theta}}(M_\phi D_{\widetilde{\theta}})^* = M_\phi D_{\widetilde{\theta}}^2 M_\phi$ is a compact self-adjoint operator on $L^2(G)$ and this implies that $\|M_\phi D_{\widetilde{\theta}}^2 M_\phi\| = \mu$ where μ is the largest eigenvalue of $M_\phi D_{\widetilde{\theta}}^2 M_\phi$. Let $f \in L^2(G)$ be the corresponding eigenvector such that $\|f\|_2 = 1$, then we have

$$\mu = \|\mu f\|_2 = \|(M_\phi D_{\widetilde{\theta}}^2 M_\phi f)\|_2 < \|D_{\widetilde{\theta}}^2 M_\phi f\|_2 \leqslant 1$$

since $\phi(s) < 1 \ \forall s \in G - \{t\}$. This implies that $\|D_{\widetilde{\theta}}M_\phi\|^2 = \|M_\phi D_{\widetilde{\theta}}^2 M_\phi\| < 1$. This means that $(t,\gamma) \notin M(\Psi) \ \forall(t,\gamma) \in G \times \Gamma^+$. So if $(t,\gamma) \in M(\Psi)$ then either $t = \infty$ or $\gamma = \infty$.

Now let $t \in \dot{G}$ and $\gamma = \infty$. Let $\phi \in C(\dot{G})$ and $\theta \in C(\dot{\Gamma}^+)$ such that $0 \leqslant \phi, \theta \leqslant 1$ and $\phi(t) = \theta(\infty) = 1$. Observe that $P = D_{\chi_{\Gamma^+}}$ where χ_{Γ^+} is the characteristic function of Γ^+. So we have $D_\theta T_\phi = D_\theta D_{\chi_{\Gamma^+}} M_\phi = D_{\chi_{\Gamma^+}} D_\theta M_\phi = D_\theta M_\phi$. Since $\mathcal{F}$ is unitary we have

$$\|D_\theta M_\phi\|_{H^2(G)} = \|\mathcal{F}D_\theta M_\phi \mathcal{F}^{-1}\|_{L^2(\Gamma^+)} = \|M_\theta \mathcal{F} M_\phi \mathcal{F}^{-1}\|_{L^2(\Gamma^+)}.$$

Since $\theta(\infty) = 1$ we have $\forall \varepsilon > 0, \exists \gamma_0 \in \Gamma^+$ such that $1 - \varepsilon \leqslant \theta(\gamma) \leqslant 1 \ \forall \gamma \geqslant \gamma_0$. Consider the operator $S_{\gamma_0} : L^2(\Gamma^+) \to L^2(\Gamma^+)$ defined as $(S_{\gamma_0}f)(\gamma) = f(\gamma_0^{-1}\gamma)$, then S_{γ_0} is an isometry. Observe that

$$(\mathcal{F}^{-1}S_{\gamma_0}f)(t) = (\check{S}_{\gamma_0}f)(t) = \int\limits_{\Gamma^+} \gamma(t) f(\gamma_0^{-1}\gamma) d\widetilde{\lambda}(\gamma)$$

$$= \int\limits_{\Gamma^+} \gamma_0(t) u(t) f(u) d\widetilde{\lambda}(u) = \gamma_0(t)\check{f}(t) = (M_{\gamma_0}\check{f})(t).$$

Hence we have $S_{\gamma_0} = \mathcal{F}M_{\gamma_0}\mathcal{F}^{-1}$ which implies that

$$S_{\gamma_0}(\mathcal{F}M_\phi \mathcal{F}^{-1}) = (\mathcal{F}M_\phi \mathcal{F}^{-1})S_{\gamma_0}.$$

Now let $f \in L^2(\Gamma^+)$ such that $\|(\mathcal{F}M_\phi \mathcal{F}^{-1})f\|_2 > 1 - \varepsilon$ and $\|f\|_2 = 1$ then for $g = \mathcal{F}M_\phi \mathcal{F}^{-1}f$ we have

$$\|M_\theta S_{\gamma_0}g\|_2 \geqslant (1-\varepsilon)^2$$

since $S_{\gamma_0}g$ is supported on $\{\gamma : \gamma \geqslant \gamma_0\}$, $\theta(\gamma) \geqslant 1 - \varepsilon \ \forall \gamma \geqslant \gamma_0$ and $\|S_{\gamma_0}g\|_2 \geqslant 1 - \varepsilon$. Since $S_{\gamma_0}g = (\mathcal{F}M_\phi \mathcal{F}^{-1})S_{\gamma_0}f$ we have

$$\|(M_\theta \mathcal{F}M_{\gamma_0}\mathcal{F}^{-1})(S_{\gamma_0}f)\|_2 \geqslant (1-\varepsilon)^2.$$

Since S_{γ_0} is an isometry we have $\|S_{\gamma_0}f\|_2 = 1$ and this implies that

$$\|M_\theta \mathcal{F}M_\phi \mathcal{F}^{-1}\| \geqslant (1-\varepsilon)^2$$

$\forall \varepsilon > 0$. Therefore we have $\|M_\theta \mathcal{F}M_\phi \mathcal{F}^{-1}\| = \|D_\theta T_\phi\| = 1$. Hence $(t,\infty) \in M(\Psi)$ $\forall t \in \dot{G}$.

Now let $\gamma \in \dot{\Gamma}^+$ and $t = \infty$. Let $\phi \in C(\dot{G})$ and $\theta \in C(\dot{\Gamma}^+)$ such that $0 \leqslant \phi, \theta \leqslant 1$ and $\phi(\infty) = \theta(\gamma) = 1$. Since $\phi(\infty) = 1$, for any $\varepsilon > 0$ there is a compact subset $K_1 \subset G$ such that $1 - \varepsilon \leqslant \phi(t) \leqslant 1 \ \forall t \notin K_1$. Let $\widetilde{\theta} = \chi_{\Gamma^+}\theta$. Then we have

$$D_\theta T_\phi = D_\theta D_{\chi_{\Gamma^+}} M_\phi = D_{\chi_{\Gamma^+}\theta} M_\phi = D_{\widetilde{\theta}} M_\phi.$$

Let $\varepsilon > 0$ be given. Let $g \in H^2(G)$ so that $\|g\|_2 = 1$ and $\|D_{\widetilde{\theta}}g\|_2 \geqslant 1 - \varepsilon$. Let $K_2 \subset G$ be a compact subset of G so that

$$\left(\int\limits_{K_2} |g(t)|^2 d\lambda(t)\right)^{1/2} \geqslant 1 - \varepsilon.$$

By Lemma 2.3 we have $t_0 \in G$ such that $K_1 \cap (t_0 K_2) = \varnothing$. Let $(S_{t_0} g)(t) = g(t t_0^{-1})$ then $\left(\int\limits_{t_0 K_2} |S_{t_0} g(t)|^2 \mathrm{d}\lambda(t) \right)^{1/2} = \left(\int\limits_{K_2} |g(t)|^2 \mathrm{d}\lambda(t) \right)^{1/2} \geqslant 1 - \varepsilon$ and this implies that

$$\|S_{t_0} g - M_\phi S_{t_0} g\|_2 \leqslant 2\varepsilon.$$

We observe that $S_{t_0} = \mathcal{F}^{-1} M_{\widehat{t_0}} \mathcal{F}$ where $\widehat{t_0} : \Gamma \to \mathbb{C}$ defined as $\widehat{t_0}(\gamma) = \gamma(t_0)$. This implies that S_{t_0} is unitary and we have $D_{\widetilde{\theta}} S_{t_0} = S_{t_0} D_{\widetilde{\theta}}$. Since $\|D_{\widetilde{\theta}}\| = 1$ we have

$$\|D_{\widetilde{\theta}} S_{t_0} g - D_{\widetilde{\theta}} M_\phi S_{t_0} g\|_2 \leqslant 2\varepsilon.$$

Since S_{t_0} is unitary for $f = S_{t_0} g$ we have $\|f\|_2 = 1$ and $D_{\widetilde{\theta}} S_{t_0} = S_{t_0} D_{\widetilde{\theta}}$ together with $\|D_{\widetilde{\theta}} g\|_2 \geqslant 1 - \varepsilon$ implies that

$$\|D_{\widetilde{\theta}} M_\phi f\|_2 \geqslant 1 - 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have $\|D_{\widetilde{\theta}} M_\phi\| = \|D_\theta T_\phi\| = 1$. Therefore we have $(\infty, \gamma) \in M(\Psi), \forall \gamma \in \dot{\Gamma}^+$. Our theorem is thus proven. ∎

3. FURTHER PROBLEMS

At this stage it is natural to think about extending our result to Toeplitz operators which are of a more general class, for instance to $CVO(G)$ where

$$CVO(G) = \left\{ f \in BC(G) : \lim_{g \to \infty} (\sup\{|f(gh) - f(g)| : h \in K\}) = 0 \right\}$$

is the algebra of functions of vanishing oscillation on G where $K \subset G$ is a compact neighbourhood of $e \in G$ such that $\lambda(K) = 1$ and $BC(G)$ is the algebra of all bounded continuous functions on G.

REFERENCES

[1] L.A. COBURN, The C^*-algebra generated by an isometry, *Bull. Amer. Math. Soc.* **73**(1967), 722–726.

[2] L.A. COBURN, R.G. DOUGLAS, C^*-algebras of operators on a half space. I, *Publ. Math. Inst. Hautes Études Sci.* **40**(1971), 59–68.

[3] R.G. DOUGLAS, *Banach Algebra Techniques in Operator Theory*, Second Edition, Graduate Texts in Math., vol. 179. Springer, New York 1998.

[4] U. GÜL, Essential spectra of quasi-parabolic composition operators on Hardy spaces of analytic functions, *J. Math. Anal. Appl.* **377**(2011), 771–791.

[5] G.J. MURPHY, Ordered groups and Toeplitz algebras, *J. Operator Theory* **18**(1987), 303–326.

[6] G.J. MURPHY, *C*-Algebras and Operator Theory*, Academic Press, Boston, MA 1990.

[7] G.J. MURPHY, An index theorem for Toeplitz operators, *J. Operator Theory* **29**(1993), 97–114.

[8] G.J. MURPHY, The *C**-algebra of a function algebra, *Integral Equations Operator Theory* **47**(2003), 361–374.

[9] S.C. POWER, Commutator ideals and pseudo-differential *C**-algebras, *Quart. J. Math. Oxford (2)* **31**(1980), 467–489.

[10] S.C. POWER, Characters on *C**-algebras, the joint normal spectrum and a pseudo-differential *C**-algebra, *Proc. Edinburgh Math. Soc. (2)* **24**(1981), 47–53.

[11] W. RUDIN, *Fourier Analysis on Groups*, Intersci. Tracts Pure Appl. Math., vol. 12, Interscience Publ., New York-London 1962.

[12] W. RUDIN, *Functional Analysis*, McGraw-Hill Ser. Higher Math., McGraw Hill Inc., New York-Düsseldorf-Johannesburg 1973.

[13] H. UPMEIER, *Toeplitz Operators and Index Theory in Several Complex Variables*, Oper. Theory Adv. Appl., vol. 81, Birkhäuser, Basel 1996.

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