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FINITE-DIMENSIONAL TOEPLITZ KERNELS AND NEARLY-INVARIANT SUBSPACES

M.C. CÂMARA and J.R. PARTINGTON

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ABSTRACT. A systematic analysis of the structure of finite-dimensional nearly-invariant subspaces of the Hardy space on the half-plane of index p (with $1 < p < \infty$) is made, and a criterion given by which they may be recognised. As a consequence, a new approach to Hitt's theorem on nearly-invariant subspaces is developed. Moreover, an analogue is given of Hayashi's theorem for finite-dimensional Toeplitz kernels; this is used to establish a necessary and sufficient condition for a Toeplitz kernel to be non-trivial and of dimension n, in terms of a factorisation of its symbol, analogous to Nakazi's work for the disc.

KEYWORDS: *Toeplitz operator, Toeplitz kernel, model space, nearly-invariant subspace, inner–outer factorization, Riemann–Hilbert problem.*

MSC (2010): 47B35, 30H10.

1. INTRODUCTION

Let $H^p(\mathbb{D})$ denote the Hardy space on the unit disc $\mathbb{D}$, and let S^* denote the backward shift operator defined by

$$(S^* f)(z) = \frac{f(z) - f(0)}{z}.$$

Nearly S^*-invariant (abbreviated to n. S^*-invariant) subspaces of $H^2(\mathbb{D})$, i.e., the closed subspaces $\mathcal{E}$ of $H^2(\mathbb{D})$ such that $z^{-1}\mathcal{E} \cap H^2(\mathbb{D}) \subset \mathcal{E}$, were introduced by Hitt in [17] and have since been the subject of various works ([8], [9], [15], [16], [21], [22]). They are defined analogously in the H^p setting, for any $p \in (1, \infty)$, whether we consider the setting of the disc or the upper half-plane. The kernel of a Toeplitz operator T_g with an essentially bounded symbol g is a n. S^*-invariant subspace in each of these settings.

By Hitt's theorem, any n. S^*-invariant subspace of $H^2(\mathbb{D})$ can be described as a product $\mathcal{E} = fK_\theta$ where f is the function of unit norm which is orthogonal to

$\mathcal{E} \cap zH^2(\mathbb{D})$ in $H^2(\mathbb{D})$ and satisfies $f(0) > 0$, and θ is an inner function vanishing at the origin. Here K_θ denotes $H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$.

In [15] Hayashi addressed the question of which nontrivial closed subspaces of $H^2(\mathbb{D})$ are Toeplitz kernels (i.e., kernels of Toeplitz operators) and showed that they were precisely those that could be represented as a product $\mathcal{E} = fK_\theta$ where $f \in H^2(\mathbb{D})$ with f^2 rigid in $H^1(\mathbb{D})$ and θ is an inner function vanishing at the origin.

These results were further developed by Sarason in [21] and [22]. Their generalisation to the vectorial case in $H^2(\mathbb{D})$ was studied in [9], where a nice introduction to these problems and the corresponding known results can be found.

Few attempts have been made to extend them to the H^p setting, possibly due to the lack of a Hilbert space structure allowing one to use a similar line of reasoning. In [13] Dyakonov addressed the question of which closed subspaces of $H^p(\mathbb{D}), 1 \leqslant p \leqslant \infty$, are Toeplitz kernels and proposed an alternative parametrisation of the kernel of a Toeplitz operator T_g based on Bourgain's factorisation ([1], [3]) for its symbol, trying to cover the whole range $p \in [1, \infty]$ and avoid the use of rigid functions. One disadvantage of this approach is that not only is such a representation highly non-unique, but also it does not indicate the dimension of the Toeplitz kernel and, in particular, whether it is trivial or not.

The question of whether the natural extension of Hitt's and Hayashi's results to the H^p setting holds, where naturally f and θ should depend on p, remains open.

In this paper we study the case of finite-dimensional subspaces of H_p^+ (using the upper half-plane instead of the disk as in [7]) by taking an approach that may provide some useful lines of reasoning to study the unsolved problem of whether Hitt's and Hayashi's theorems can be extended to all nontrivial subspaces of H_p^+ (or $H^p(\mathbb{D})$). After some preliminary results and notation, the main results are contained in Sections 2 and 3. In Section 2 we establish an analogue of Hitt's theorem and we moreover present a simple criterion to recognise a n. S^*-invariant subspace in H_p^+, $p \in (1, \infty)$, by studying the quotients of any two non-zero functions in the subspace. In Section 3 we present an analogue of Hayashi's result for finite-dimensional Toeplitz kernels and use it to establish a necessary and sufficient condition for the kernel of a Toeplitz operator T_g to be non-trivial and of dimension n, in terms of a factorisation of its symbol. Some of these results provide analogues of the work of Nakazi [20] in the disc, although they are equivalent to Nakazi's results only for $p = 2$.

We take $0 < p \leqslant \infty$ and H_p^+, H_p^- to be the Hardy spaces of the upper and lower half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively ([12]). We write L_p to denote $L^p(\mathbb{R})$. By $\mathcal{G}H_\infty^\pm$ we denote the class of invertible elements in $H_\infty^\pm$, and similarly for $\mathcal{G}L_\infty$.

For $G \in L_\infty$ and $1 < p < \infty$, the *Toeplitz operator* $T_G : H_p^+ \to H_p^+$ is defined by

$$T_G f_+ = P^+(G f_+), \quad f_+ \in H_p^+,$$

where P^+ denotes the projection of L_p onto H_p^+ parallel to H_p^-. A *Toeplitz kernel*, or *T-kernel*, is a subspace of H_p^+, $1 < p < \infty$, which is the kernel of some Toeplitz operator.

2. NEARLY S^*-INVARIANT SUBSPACES AND MODEL SPACES

For $1 < p < \infty$, and $\theta \in H_\infty^+$ inner, we define the model space $K_\theta^p \subset H_p^+$ by

$$K_\theta^p = H_p^+ \cap \theta H_p^-.$$

In particular $K_\theta^2 = H_2^+ \ominus \theta H_2^+$. We shall write K_θ rather than K_θ^p when no confusion is likely.

We shall require the functions

$$\lambda_\pm(\xi) = \xi \pm \mathrm{i} \quad \text{and} \quad r(\xi) = \frac{\xi - \mathrm{i}}{\xi + \mathrm{i}},$$

and write S for the operator T_r on H_p^+ of multiplication by r, with S^* the operator $T_{\bar{r}}$.

DEFINITION 2.1 ([7]). Let $\mathcal{E}$ be a proper closed subspace of H_p^+, $1 < p < \infty$, and η a complex-valued function defined a.e. on $\mathbb{R}$. We say that $\mathcal{E}$ is *nearly η-invariant* if, for every $f_+ \in \mathcal{E}$ such that $\eta f_+ \in H_p^+$, we have $\eta f_+ \in \mathcal{E}$; that is, $\eta\mathcal{E} \cap H_p^+ \subset \mathcal{E}$. If $\mathcal{E}$ is nearly η-invariant with $\eta \in L_\infty$, then we also say that $\mathcal{E}$ is *nearly T_η-invariant.*

We abbreviate “nearly η-invariant” to “n. η-invariant".

Throughout this section $\mathcal{E}$ will denote a n. S^*-invariant subspace of H_p^+, that is, a nearly η-invariant subspace for $\eta = \bar{r}$. It is clear that $\mathcal{E}$ is one-dimensional if and only if $\mathcal{E} = \operatorname{span}\{f_+\}$ where $f_+ \in H_p^+$ and $f_+(\mathrm{i}) \neq 0$. More generally, we have:

PROPOSITION 2.2. *If* $\dim \mathcal{E} \geqslant N, N \in \mathbb{N}$, *then there are (at least) N linearly independent elements of $\mathcal{E}$ which do not vanish at* i.

Proof. If $\mathcal{E}$ is a n. S^*-invariant subspace of H_p^+, then there exists $f_1^+ \in \mathcal{E}$ such that $f_1^+(\mathrm{i}) \neq 0$. Let $\{f_1^+, f_2^+, \dots, f_N^+\}$ be a set of linearly independent elements of $\mathcal{E}$ and define $\widetilde{f}_{j+} := a_j f_{1+} + b_j f_{j+}, j = 2, \dots, N$, where $b_j \neq 0$ are such that $\widetilde{f}_{j+}(\mathrm{i}) \neq 0$. Then $f_{1+}, \widetilde{f}_{2+}, \dots, \widetilde{f}_{N+}$ are linearly independent elements of $\mathcal{E}$ that do not vanish at i. ∎

Thus, if $\dim \mathcal{E} = N$, we can choose a basis for $\mathcal{E}$ such that none of its elements vanishes at i.

We also have the following.

PROPOSITION 2.3. *If* $\dim \mathcal{E} \geqslant N$, *with* $N \in \mathbb{N}$, *then there exists at least one element* $\psi_+ \in \mathcal{E}$ *with a zero of order* $N - 1$ *at* i.

Proof. In any subspace of H_p^+ with N linearly independent elements there exists an element ψ_+ with a zero of order $m \geqslant N-1$ at i. Since $\mathcal{E}$ is n. S^*-invariant, $r^{-m+N-1}\psi_+$ belongs to $\mathcal{E}$ and has a zero of order $N-1$ at i. ∎

Remark that if $\dim \mathcal{E} = N$ then the function ψ_+ of Proposition 2.3 is unique up to a constant factor.

As a consequence, we can represent a finite dimensional n. S^*-invariant subspace in terms of a model space as follows.

THEOREM 2.4. *Let $\mathcal{E}$ be a n. S^*-invariant subspace of H_p^+ with $\dim \mathcal{E} = N$ (where $N \in \mathbb{N}$) and let $\psi_+ \in \mathcal{E}$ admit a zero of order $N-1$ at* i. *Defining $\widetilde{\psi}_+ := r^{-N+1}\psi_+$, we have*

$$\mathcal{E} = \lambda_+ \widetilde{\psi}_+ K_{r^N}. \tag{2.1}$$

Proof. It is clear that $\widetilde{\psi}_+, r\widetilde{\psi}_+, \ldots, r^{N-1}\widetilde{\psi}_+$ are linearly independent elements of $\mathcal{E}$, so that $\mathcal{E} = \operatorname{span}\{\widetilde{\psi}_+, r\widetilde{\psi}_+, \ldots, r^{N-1}\widetilde{\psi}_+\}$. Thus $\varphi_+ \in \mathcal{E}$ if and only if, for some $A_1, \ldots, A_N \in \mathbb{C}$, we have

$$\begin{aligned} \varphi_+ &= (A_1 + A_2 r + \cdots + A_N r^{N-1})\widetilde{\psi}_+ \\ &= \widetilde{\psi}_+ \lambda_+ \Big(\frac{A_1}{\lambda_+} + A_2 r \frac{1}{\lambda_+} + \cdots + A_N r^{N-1} \frac{1}{\lambda_+} \Big), \end{aligned}$$

and $\frac{A_1}{\lambda_+} + A_2 r \frac{1}{\lambda_+} + \cdots + A_N r^{N-1}\frac{1}{\lambda_+}$ is the general form of an element in K_{r^N}. ∎

The representation (2.1) is unique modulo rational functions belonging to $\mathcal{G}H_\infty^+$ and equivalence of inner functions. To state this more precisely, we introduce here some notation.

DEFINITION 2.5 ([6]). If θ_1 and θ_2 are inner functions, we say that $\theta_1 \sim \theta_2$ if and only if there are functions $h_\pm \in \mathcal{G}H_\infty^\pm$ such that

$$\theta_1 = h_- \theta_2 h_+ \tag{2.2}$$

and we say that $K_{\theta_1} \sim K_{\theta_2}$ if and only if

$$K_{\theta_1} = h_+ K_{\theta_2} \quad \text{with } h_+ \in \mathcal{G}H_\infty^+. \tag{2.3}$$

It is clear that $\theta_1 \sim \theta_2 \Rightarrow K_{\theta_1} \sim K_{\theta_2}$ ([6]). We will use the notation

$$\frac{K_{\theta_1}}{K_{\theta_2}} \simeq h_+ \tag{2.4}$$

whenever (2.3) holds.

THEOREM 2.6. *Let $\mathcal{E} \subset H_p^+$ be a n. S^*-invariant subspace with dimension $N \in \mathbb{N}$. If, for some function $F_+ \in H_p^+$ and an inner function θ, we have*

$$\mathcal{E} = \lambda_+ F_+ K_\theta, \tag{2.5}$$

then $F_+(\mathrm{i}) \neq 0, \theta \sim r^N$ *and, for any function* $\widetilde{\psi}_+$ *satisfying* (2.1),

$$\frac{K_\theta}{K_{r^N}} \simeq \frac{\widetilde{\psi}_+}{F_+} \in \mathcal{R} \cap \mathcal{G}H_\infty^+ \tag{2.6}$$

where $\mathcal{R}$ *denotes the set of all rational functions in* L_∞.

Proof. It is clear that $F^+(\mathrm{i}) \neq 0$, by Proposition 2.2. On the other hand, since $\dim K_\theta = \dim \mathcal{E} = N$, θ is a rational inner function and therefore $\theta = h_- r^N h_+$ where $h_\pm \in \mathcal{G}H_\infty^\pm$ are rational functions. It follows that $K_\theta = h_+ K_{r^N}$ and from (2.1) and (2.5) we have

$$\mathcal{E} = \lambda_+ \widetilde{\psi}_+ K_{r^N} = \lambda_+ F_+ h_+ K_{r^N}. \tag{2.7}$$

Since the second equality in (2.7) implies that $F_+ h_+ \in \mathcal{E}$, it follows from the first equality in (2.7) that, for some constants $A_1, A_2, \ldots, A_N \in \mathbb{C}$, with $A_1 \neq 0$,

$$F_+ h_+ = \widetilde{\psi}_+ (A_1 + A_2 r + \cdots + A_N r^{N-1}). \tag{2.8}$$

For $N = 1$ this means that $F_+ h_+ = A_1 \widetilde{\psi}_+$ with $A_1 \neq 0$ and therefore

$$\frac{\widetilde{\psi}_+}{F_+} = \frac{h_+}{A_1}, \tag{2.9}$$

so that (2.6) holds. If $N > 1$ it follows from (2.7) that we also have

$$r^{j+1} F_+ h_+ \in \mathcal{E} \quad \text{for all } j = 0, 1, \ldots, N-2 \tag{2.10}$$

and, taking (2.8) into account,

$$r^{j+1} F_+ h_+ = \widetilde{\psi}_+ (A_1 r^{j+1} + \cdots + A_{N-j-1} r^{N-1}) + \widetilde{\psi}_+ (A_{N-j} r^N + \cdots + A_N r^{N+j}). \tag{2.11}$$

Since, by (2.10), the left hand side of this equality represents a function in $\mathcal{E}$ and $\widetilde{\psi}_+ (A_1 r^{j+1} + \cdots + A_{N-j-1} r^{N-1}) \in \lambda_+ \widetilde{\psi}_+ K_{r^N} = \mathcal{E}$, we see that (2.11) implies that

$$\eta_j := \widetilde{\psi}_+ (A_{N-j} r^N + \cdots + A_N r^{N+j}) \in \mathcal{E}$$

and therefore, since η_j has a zero of order greater or equal to N at i, $\eta_j = 0$ for all $j = 0, 1, \ldots, N-2$. We have thus

$$A_{N-j} r^N + \cdots + A_N r^{N+j} = 0 \quad \text{for all } j = 0, 1, \ldots, N-2$$

and it follows that $A_N = A_{N-1} = \cdots = A_2 = 0$. From (2.8) we see therefore that (2.9) and, consequently, (2.6) hold. ∎

Defining $\varphi_+^* = \frac{\widetilde{\psi}_+}{\lambda_+^{N-1}}$, and noting that the set of functions

$$\{\lambda_+^{N-1}, \lambda_+^{N-2} \lambda_-, \ldots, \lambda_+ \lambda_-^{N-2}, \lambda_-^{N-1}\}$$

forms a basis for the space $\mathcal{P}_{N-1}$ of all polynomials of degree at most $N-1$, we arrive at the following result.

THEOREM 2.7. *Let $\mathcal{E}$ be a n. S^*-invariant subspace of H_p^+ with* $\dim \mathcal{E} = N$. *Then there is a function $\varphi_+^* \in \mathcal{E} \setminus rH_p^+$ such that*

$$\mathcal{E} = \{\varphi_+^* p_+ : p_+ \in \mathcal{P}_{N-1}\}. \tag{2.12}$$

In the case that $\mathcal{E}$ is a Toeplitz kernel, and hence nearly-invariant under division by every inner function ([7]), we may also conclude that φ_+^* is outer. In fact, a nontrivial Toeplitz kernel cannot be contained in θH_p^+ if θ is a non constant inner function ([6], Theorem 2.4).

We may ask then if, conversely, any set of the form (2.12) is a n. S^*-invariant subspace of H_p^+ and, in case φ_+^* is outer, if it is a Toeplitz kernel. While the latter question will be dealt with in the next section, the answer to the former is given in the following theorem, which moreover provides a simple criterion to recognise a finite-dimensional n. S^*-invariant subspace of H_p^+, $p \in (1, \infty)$.

THEOREM 2.8. *Suppose that $\mathcal{E} \subset H_p^+$ and* $\dim E = N$. *Then $\mathcal{E}$ is n. S^*-invariant if and only if:*

(i) *$\mathcal{E}$ contains at least one function that does not vanish at* i, *and*

(ii) *the quotient of any two functions in $\mathcal{E}$ is equal to a quotient of two polynomials of degrees at most $N-1$.*

Proof. The conditions are obviously necessary, by Theorem 2.7.

To show their sufficiency, we may clearly take $N > 1$. Pick a basis $v_1^+, \dots, v_N^+$ of $\mathcal{E}$, and assume that $v_1^+(\mathrm{i}) \neq 0$, as we can by Proposition 2.2.

For each $k = 1, \dots, N$ we write

$$v_k^+ = \frac{p_k}{q_k} v_1^+,$$

where $p_1 = q_1 = 1$, and in general p_k, q_k are polynomials of degree at most $N-1$ with no common factors. Since $v_1^+(\mathrm{i}) \neq 0$, we have $q_k(\mathrm{i}) \neq 0$ for all $k = 2, \dots, N$.

Let d denote the least common multiple of the polynomials $q_k, k = 2, \dots, N$. Clearly the set of zeroes of d is finite and does not contain i. We claim that there exists a complex linear combination

$$r = \frac{c_2 v_2^+ + \cdots + c_N v_N^+}{v_1^+},$$

such that every zero of multiplicity m in d is a pole of order m in r, which implies that d has at most $N-1$ zeroes, counting multiplicity.

For suppose that z_0 is a zero of d with multiplicity m. Then, for some $k_0 \in \{2, \dots, N\}$, z_0 must be a zero of q_{k_0} with multiplicity m and, in a neighbourhood of z_0, we have

$$\frac{v_k^+(\xi)}{v_1^+(\xi)} = \frac{1}{(\xi - z_0)^m}(b_k + O(\xi - z_0)), \quad k = 2, \dots, N,$$

where at least one b_k is non-zero. The point z_0 will be a pole of order m in r if and only if $\sum_{k=2}^{N} c_k b_k \neq 0$. Repeating the same reasoning for all zeroes of d, we see that it is sufficient to choose a point $(c_2, \dots, c_N)$ in $\mathbb{C}^{N-1}$ which does not belong to finitely-many hyperplanes of the form $\sum_{k=2}^{N} c_k b_k = 0$.

Our conclusion is that all the $\frac{v_k^+}{v_1^+}$ can be written over a common denominator $d \in \mathcal{P}_{N-1}$. Thus

$$\mathcal{E} = \left\{ v_1^+ \sum_{k=1}^{N} \frac{\alpha_k \widetilde{p}_k}{d} : \alpha_1, \dots, \alpha_N \in \mathbb{C} \right\}$$

for some polynomials $\widetilde{p}_1, \dots, \widetilde{p}_N$ belonging to $\mathcal{P}_{N-1}$, and indeed we may take $\widetilde{p}_1 = d$. Since $\dim \mathcal{E} = N$, we see that

$$\mathcal{E} = \left\{ v_1^+ \frac{Q}{d} : Q \in \mathcal{P}_{N-1} \right\}.$$

Now near-invariance is clear: since $\mathrm{d(i)} \neq 0$, if a function $f \in \mathcal{E}$ vanishes at i, then so does the corresponding polynomial Q, and thus $\frac{f}{r} \in \mathcal{E}$. ∎

The results of Theorems 2.4 and 2.6 naturally lead to the question of how they relate to Hitt's characterisation of n. S^*-invariant subspaces [17], [21] in the case when $p = 2$ and $n \neq 1$, since in general $\widetilde{\psi}_+$ is not orthogonal to the space $\mathcal{E}_0$ given by

$$\mathcal{E}_0 := \mathcal{E} \cap rH_2^+ = \operatorname{span}\{\psi_+, r^{-1}\psi_+, \dots, r^{-(N-2)}\psi_+\}, \tag{2.13}$$

using the notation of Theorem 2.4.

We can nevertheless obtain an element $\varphi_+ \in \mathcal{E} \cap \mathcal{E}_0^\perp$ by Gram–Schmidt orthogonalisation of the basis

$$\{\psi_+, r^{-1}\psi_+, \dots, r^{-(N-2)}\psi_+, r^{-(N-1)}\psi_+\},$$

where $r^{-(N-1)}\psi_+ = \widetilde{\psi}_+$ as in Theorem 2.4, which yields an orthogonal basis $\{\varphi_{1+}, \dots, \varphi_{N+}\}$.

In particular we have

$$\psi_+ = \varphi_{1+} \quad \text{and} \quad \varphi_+ := \varphi_{N+} \in \mathcal{E} \ominus \mathcal{E}_0. \tag{2.14}$$

Since $\mathcal{E}_0$ has codimension 1 in $\mathcal{E}$, the orthogonal element φ_+ is unique apart from a constant factor. From (2.1) it follows that, for some constants $A_0, A_1, \dots, A_{N-1} \in \mathbb{C}$, with $A := A_{N-1} \neq 0$, and $C_1, C_2, \dots, C_{N-1} \in \mathbb{C}$ we have

$$\begin{aligned}
\varphi_+ &= A_0\psi_+ + A_1 r^{-1}\psi_+ + \cdots + A_{N-1} r^{-(N-1)}\psi_+ \\
&= A(r^{-1} - C_1)(r^{-1} - C_2) \cdots (r^{-1} - C_{N-1})\psi_+ \\
&= A(r^{-1} - C_j) \prod_{s=1, s\neq j}^{N-1} (r^{-1} - C_s)\psi_+
\end{aligned}$$

for any $j = 1, 2, \dots, N-1$. So, defining

$$\psi_{j+}^* = \prod_{s=1, s\neq j}^{N-1} (r^{-1} - C_s)\psi_+,$$

we then have

$$\varphi_+ = Ar^{-1}\psi_{j+}^* - AC_j\psi_{j+}^*. \tag{2.15}$$

Now $\varphi_+ \in \mathcal{E}_0^\perp$ and $\psi_{j+}^* \in \mathcal{E}_0$, so $\langle \varphi_+, \psi_{j+}^* \rangle = 0$ and thus the constant C_j in (2.15) is given by

$$C_j = \frac{\langle r^{-1}\psi_{j+}^*, \psi_{j+}^* \rangle}{\langle \psi_{j+}^*, \psi_{j+}^* \rangle}.$$

Since $\|r^{-1}\psi_{j+}^*\|_2 = \|\psi_{j+}^*\|_2$, it follows from the fact that the Cauchy–Schwarz inequality is strict unless the vectors involved are linearly dependent that $|C_j| < 1$.

Now we may write

$$r^{-1}(\xi) - C_j = \frac{\xi + \mathrm{i}}{\xi - \mathrm{i}} - C_j = \frac{(1 - C_j)(\xi - \xi_j)}{\xi - \mathrm{i}}, \tag{2.16}$$

where $\xi_j = -\mathrm{i}\dfrac{1 + C_j}{1 - C_j} \in \mathbb{C}^-$ since $|C_j| < 1$. We have thus proved the following.

THEOREM 2.9. *Let $\varphi_+ \in \mathcal{E} \ominus \mathcal{E}_0$ with $\varphi_+ \neq 0$. Then*

$$\varphi_+(\xi) = A\frac{(\xi - \xi_1)\cdots(\xi - \xi_{N-1})}{(\xi - \mathrm{i})^{N-1}}\psi_+(\xi), \tag{2.17}$$

where $A \in \mathbb{C} \setminus \{0\}$, $\xi_1, \dots, \xi_{N-1} \in \mathbb{C}^-$ and $\psi_+ \in \mathcal{E}$ has a zero of order $N-1$ at i.

It follows from Theorems 2.4 and 2.9 that $\mathcal{E} = \lambda_+\widetilde{\psi}_+K_{r^N} = \lambda_+\varphi_+h_+K_{r^N}$, where $h_+ \in \mathcal{G}H_\infty^+$ is given by

$$h_+(\xi) = \frac{(\xi + \mathrm{i})^{N-1}}{(\xi - \xi_1)\cdots(\xi - \xi_{N-1})}.$$

Now

$$h_+K_{r^N} = h_+ \ker T_{r^{-N}} = \ker T_{h_+^{-1}r^{-N}} = \ker T_{g_-\overline{B}},$$

where $g_- \in \mathcal{G}H_\infty^-$ is given by

$$g_-(\xi) = \frac{(\xi - \overline{\xi}_1)\cdots(\xi - \overline{\xi}_{N-1})}{(\xi - \mathrm{i})^{N-1}}$$

and B is the Blaschke product given by

$$B = \frac{\xi - \mathrm{i}}{\xi + \mathrm{i}}\frac{\xi - \overline{\xi}_1}{\xi - \xi_1}\cdots\frac{\xi - \overline{\xi}_{N-1}}{\xi - \xi_{N-1}}. \tag{2.18}$$

Since $\ker T_{g_-\overline{B}} = \ker T_{\overline{B}} = K_B$, we have established the following theorem.

THEOREM 2.10. *Let $\mathcal{E}$ be a n. S^*-invariant subspace of H_2^+ with* $\dim \mathcal{E} = N$. *Let ψ_+ be the (unique, up to a constant factor) element of $\mathcal{E}$ admitting a zero of order $N-1$ at* i, *and let $\varphi_+ \in \mathcal{E} \ominus \mathcal{E}_0$. Then*

$$\mathcal{E} = \lambda_+ \varphi_+ K_B,$$

where B is the finite Blaschke product given by (2.18), *where $\xi_1, \dots, \xi_{N-1}$ are the zeroes of the rational function $\frac{\varphi_+}{\psi_+}$.*

Besides establishing a clear relation between Theorem 2.4 and Hitt's theorem, this result moreover defines explicitly the model space associated with Hitt's representation.

3. ON FINITE-DIMENSIONAL TOEPLITZ KERNELS

Next we address two closely related questions: *when does a Toeplitz operator have a nontrivial kernel of finite dimension, and when is a finite-dimensional subspace of H_p^+ a T-kernel?*

Here we need the theory of rigid functions.

DEFINITION 3.1. A function $f_+ \in H_q^+ \setminus \{0\}$, with $0 < q < \infty$, is called *rigid* if and only if, for any $g_+ \in H_q^+$ such that $\frac{g_+}{f_+} > 0$ a.e. on $\mathbb{R}$, we have $g_+ = \lambda f_+$ for some $\lambda \in \mathbb{R}^+$.

A rigid function is outer (in H_q^+), and every rigid function in H_q^+ is the square of an outer function in H_p^+, with $p = 2q$ ([7], [22]). If $f_+ \in H_p^+$ and f_+^2 is rigid in $H_{p/2}^+$, we say that f_+ is *square-rigid* in H_p^+.

It was shown in [22] that a one-dimensional subspace of $H^2(\mathbb{D})$ is a T-kernel if and only if it is spanned by a function that is square-rigid in $H^2(\mathbb{D})$. An analogous result holds for one-dimensional subspaces of H_p^+, for $1 < p < \infty$, as follows.

THEOREM 3.2 ([7]). *Let $f_+ \in H_p^+$, $1 < p < \infty$. Then* $\operatorname{span}\{f_+\}$ *is a T-kernel in H_p^+ if and only if f_+ is outer and square-rigid in H_p^+. In that case* $\operatorname{span}\{f_+\} = \ker T_{\bar{f}_+/f_+}$.

As a consequence, we also have:

COROLLARY 3.3. *If $O_+ \in H_p^+$ is outer and square-rigid then, for every $k \in \mathbb{N}$,*

$$\ker T_{r^{-k}\overline{O}_+/O_+} = \operatorname{span}\{O_+, rO_+, \dots, r^k O_+\} = \lambda_+ O_+ K_{r^{k+1}}. \tag{3.1}$$

Proof. It is clear that $O_+, rO_+, \dots, r^k O_+$ are linearly independent and belong to $\ker T_{r^{-k}\overline{O}_+/O_+}$. If the dimension of the latter was greater than $k+1$ we would have ([2])

$$\ker T_{\overline{O}_+/O_+} = \ker T_{r^k(r^{-k}\overline{O}_+/O_+)} > 1,$$

which is false since $\ker T_{\overline{O}_+/O_+} = \mathrm{span}\{O_+\}$ by Theorem 3.2. ∎

The following theorem is an analogue of Hayashi's result ([15]) for finite-dimensional subspaces of H_p^+.

THEOREM 3.4. *Suppose that $\mathcal{E} \subset H_p^+$ and* $\dim E = N$. *Then $\mathcal{E}$ is a T-kernel if and only if $\mathcal{E} = \lambda_+ O_+ K_{r^N}$, with O_+ a square-rigid outer function in H_p^+.*

Proof. Let $\mathcal{E} = \lambda_+ \varphi_+ K_{r^N}$, according to Theorem 2.4. If $\mathcal{E}$ is a T-kernel, then φ_+ must be outer, as remarked in Section 2, so we may write $\varphi_+ = O_+$, with O_+ outer. Since $\mathcal{E}$ is a T-kernel containing O_+, it must contain the minimal kernel $\mathcal{K}_{\min}(O_+) = \ker T_{\overline{O}_+/O_+}$.

Let $\psi_+ \in \ker T_{\overline{O}_+/O_+} \subset \mathcal{E}$; then $\psi_+ = \lambda_+ O_+ k_+$, where $k_+ \in K_{r^N}$ and, for some $\psi_- \in H_p^-$, we have $\frac{\overline{O}_+}{O_+}\psi_+ = \psi_-$. On the other hand,

$$\frac{\overline{O}_+}{O_+}\psi_+ = \psi_- \iff \overline{O}_+ \lambda_+ k_+ = \psi_- \iff \frac{\lambda_+}{\lambda_-} k_+ = \frac{\lambda_-^{-1}\psi_-}{\overline{O}_+}$$

and, since $\frac{\lambda_-^{-1}\psi_-}{\overline{O}_+} \in L_p \cap \overline{\mathcal{N}}_+$, we have $\frac{\lambda_-^{-1}\psi_-}{\overline{O}_+} \in H_p^-$, i.e., $\frac{\psi_-}{\overline{O}_+} \in \lambda_- H_p^-$.

From

$$\underbrace{\lambda_+ k_+}_{\in \lambda_+ H_p^+} = \underbrace{\frac{\psi_-}{\overline{O}_+}}_{\in \lambda_- H_p^-},$$

it follows that both sides are constant ([23]), so that $\psi_- = c\overline{O}_+$ and $\psi_+ = cO_+$ with $c \in \mathbb{C}$. Thus $\ker T_{\overline{O}_+/O_+} = \mathrm{span}\{O_+\}$, which is the same as saying that O_+^2 is rigid in $H_{p/2}^+$.

Conversely, assume that $\mathcal{E} = O_+ \lambda_+ K_{r^N}$ with O_+^2 rigid in $H_{p/2}^+$. Then, by Corollary 3.3,

$$\mathcal{E} = \lambda_+ O_+ K_{r^N} = \ker T_{r^{-(N-1)}(\overline{O}_+/O_+)},$$

so $\mathcal{E}$ is a T-kernel. ∎

Using the result of Theorem 3.4 we can also characterise any non zero finite-dimensional T-kernel and establish conditions for a T-kernel to be trivial, in terms of a factorisation of the symbol of the corresponding Toeplitz operator.

We will need the following results.

THEOREM 3.5 ([7]). *For every $\varphi_+ \in H_p^+ \setminus \{0\}$ there exists a T-kernel containing φ_+, denoted by $\mathcal{K}_{\min}(\varphi_+)$, such that for any $g \in L_\infty$ we have*

$$\varphi_+ \in \ker T_g \Rightarrow \mathcal{K}_{\min}(\varphi_+) \subset \ker T_g \tag{3.2}$$

and, if $\varphi_+ = I_+ O_+$ is an inner-outer factorisation of φ_+,

$$\mathcal{K}_{\min}(\varphi_+) = \ker T_{\overline{I}_+ \overline{O}_+/O_+}. \tag{3.3}$$

$\mathcal{K}_{\min}(\varphi_+)$ is called the *minimal kernel* for φ_+. It can be shown moreover that a nontrivial, proper, n. S^*-invariant subspace $\mathcal{E}$ of H_p^+ $(1 < p < \infty)$ is a T-kernel if and only if there exists $\varphi_+ \in H_p^+$ such that $\mathcal{E} = \mathcal{K}_{\min}(\varphi_+)$ ([7]).

DEFINITION 3.6 ([6]). If $K = \mathcal{K}_{\min}(\varphi_+)$, we say that φ_+ is a *maximal function* for K.

Clearly, if φ_+ is a maximal function for $\ker T_g$, then we have $g\varphi_+ = \varphi_-$ where $\varphi_- \in H_p^-$ is outer ([6]).

THEOREM 3.7. *For $g \in L_\infty$,* $\ker T_g$ *is nontrivial and of finite dimension if and only if, for some $N \in \mathbb{N}$, g admits a factorisation*

$$g = g_- r^{-N} g_+^{-1}, \tag{3.4}$$

where $\frac{g_-}{\lambda_-} \in H_p^-$ is outer and $\frac{g_+}{\lambda_+} \in H_p^+$ is outer and square-rigid. In that case $\ker T_g = \ker T_{r^{-N}(\overline{g}_+/g_+)}$ *and* $\dim \ker T_g = N$.

Proof. Step 1. Let $O_+ = \frac{g_+}{\lambda_+}, O_- = \frac{g_-}{\lambda_-}$. If g admits a representation of the form (3.4) then $\{O_+, rO_+, \dots, r^{N-1}O_+\} \subset \ker T_g$ and so, by Corollary 3.3, $\ker T_g \supset \ker T_{r^{-N+1}(\overline{O}_+/O_+)} \neq \{0\}$. On the other hand, if $\varphi_+ \in \ker T_g$, we have $g\varphi_+ = \varphi_-$ with $\varphi_- \in H_p^-$ which, taking (3.4) into account, is equivalent to

$$r^{-N+1}\frac{\overline{O}_+}{O_+}\varphi_+ = \frac{\overline{O}_+\varphi_-}{O_-}. \tag{3.5}$$

Since the right hand side of (3.5) represents a function whose conjugate belongs to the Smirnov class $\mathcal{N}_+$ as well as to L_p, it is a function in H_p^-; it follows that $\varphi_+ \in \ker T_{r^{-N+1}(\overline{O}_+/O_+)}$. Therefore (3.4) implies that $\ker T_g \subset \ker T_{r^{-N+1}(\overline{O}_+/O_+)}$. It follows that $\ker T_g = \ker T_{r^{-N+1}(\overline{O}_+/O_+)} = \lambda_+ O_+ K_{r^N}$ and $\dim \ker T_g = N$.

Step 2. Conversely, let now $\ker T_g$ be a (n. S^*-invariant) subspace of H_p^+ with dimension N. Let φ_+ be a maximal function for $\ker T_g$. By Theorem 3.4, $\ker T_g = O_+\lambda_+K_{r^N}$, where $O_+ \in \ker T_g$ is outer and square-rigid, so we have

$$r^{-N}O_+^{-1}\lambda_+^{-1}\varphi_+ = \psi_-, \tag{3.6}$$

where $\psi_- \in H_p^-$ is outer by Lemma 3.8 below. On the other hand,

$$g\varphi_+ = O_- \tag{3.7}$$

where O_- is outer since φ_+ is maximal in $\ker T_g$. From (3.6) and (3.7) we obtain

$$g = \frac{O_-}{\psi_-} r^{-N} \frac{O_+^{-1}}{\lambda_+}$$

where $\frac{O_-}{\psi_-\lambda_-} = gr^{-N+1}O_+ \in H_p^-$ is outer and $\left(\frac{O_+^{-1}}{\lambda_+}\right)^{-1}\lambda_+^{-1} = O_+$ is outer and square-rigid. ∎

LEMMA 3.8. *For $g \in L_\infty$ let φ_+ be a maximal function in* $\ker T_g = O_+\lambda_+K_{r^N}$, $N \in \mathbb{N}$, *as in Theorem* 3.4. *Then* $\psi_- = r^{-N}O_+^{-1}\lambda_+^{-1}\varphi_+$ *is outer in* H_p^-.

Proof. Since $\varphi_+ \in \ker T_g$, we have $\varphi_+ = O_+\lambda_+h_+$ with $h_+ \in \ker T_{r^{-N}} = K_{r^N}$ and thus

$$r^{-N}O_+^{-1}\lambda_+^{-1}\varphi_+ = \psi_- \in H_p^-. \tag{3.8}$$

Remark that $O_+^{-1}\lambda_+^{-1}\varphi_+ \in \mathcal{N}_+ \cap L_p$, so that $O_+^{-1}\lambda_+^{-1}\varphi_+ \in H_p^+$. If $\psi_- = I_-O_-$ is an inner-outer factorisation of ψ_- in H_p^-, then it follows from (3.8) that we have $\overline{I}_-O_+^{-1}\lambda_+^{-1}\varphi_+ \in \ker T_{r^{-N}}$. Therefore $\overline{I}_-\varphi+ \in O_+\lambda_+ \ker T_{r^{-N}} = \ker T_g$ and it follows that $\varphi_+ \in \ker T_{\overline{I}_-g}$. We conclude that I_- must be constant since $\ker T_g$ is the minimal kernel for φ_+, and $\ker T_{\overline{I}_-g} \subsetneqq \ker T_g$ if I_- is not constant ([7]). ∎

A similar characterisation of non-trivial finite-dimensional Toeplitz kernels in $H^p(\mathbb{D})$ was obtained by Nakazi ([20], Theorem 7). The result can be stated as follows (here $\mathbb{T}$ denotes the unit circle):

THEOREM 3.9 ([20]). *Let $1 < p < \infty$, and $\widetilde{g} \in L_\infty(\mathbb{T})$, with $n \in \mathbb{N}$ and let $T_{\widetilde{g}}$ be the associated Toeplitz operator on $H^p(\mathbb{D})$. Then the following conditions are equivalent:*

(i) $\dim\ker T_{\widetilde{g}} = n$;

(ii) *there is a square-rigid outer function $f_+ \in H^p(\mathbb{D})$ such that* $\ker T_{\widetilde{g}} = \{pf_+ : p \in P_{n-1}\}$;

(iii) *there is an outer function $h \in H^\infty(\mathbb{D})$ with $|\widetilde{g}| = |h|$ on $\mathbb{T}$, and a square-rigid outer function $f_+ \in H^p(\mathbb{D})$ such that*

$$\frac{\widetilde{g}}{|\widetilde{g}|}\frac{h}{|h|} = \overline{z}^n\frac{\overline{f}_+}{f_+} \quad \text{on } \mathbb{T}. \tag{3.9}$$

To compare (3.9) with (3.4), we shall suppose for simplicity that $|g| = 1$. Let $m : \mathbb{D} \to \mathbb{C}_+$ denote the conformal bijection given by $m(z) = \mathrm{i}\dfrac{1+z}{1-z}$ for $z \in \mathbb{D}$, and extending this to $\mathbb{T}$, let $\widetilde{g} = g \circ m$. There is a well-known isometric isomorphism $V : L_p(\mathbb{T}) \to L_p(\mathbb{R})$, whose restriction maps $H^p(\mathbb{D})$ onto H_p^+; it can be defined by

$$(V\psi)(\xi) = \frac{1}{\pi^{1/p}}\frac{1}{(\xi+\mathrm{i})^{2/p}}\psi(m^{-1}(\xi)), \quad \xi \in \mathbb{R}.$$

However, V does not map the complementary space $\overline{zH^p(\mathbb{D})}$ onto H_p^- unless $p = 2$, so that Toeplitz operators on the disc and half-plane are no longer equivalent. It is therefore not surprising to note that the condition $f_+ \in H^p(\mathbb{D})$ in Nakazi's Theorem 3.9 is not equivalent to the condition $\frac{g_+}{\lambda_+} \in H_p^+$ in Theorem 3.7 unless $p = 2$.

Nevertheless, if we define $w(t) = |t-1|^{1-2/p}$, then we have that the Toeplitz operator T_g defined on H_p^+ is equivalent to a Toeplitz operator on a weighted

Hardy space $H^+_{p,w}$; to define this, let $L_{p,w}(\mathbb{T}) = w^{-1}L_p(\mathbb{T})$ and note that $B : L_p(\mathbb{R}) \to L_{p,w}(\mathbb{T})$, given by

$$(B\varphi)(t) = \frac{1}{1-t}\varphi\Big(\mathrm{i}\frac{1+t}{1-t}\Big), \quad t \in \mathbb{T},$$

is an isomorphism between $L_p(\mathbb{R})$ and $L_{p,w}(\mathbb{T})$ (see [19]). Now let $S_{\mathbb{T}}$ denote the singular integral operator on $L_{p,w}(\mathbb{T})$ defined by

$$(S_{\mathbb{T}}\psi)(t) = \frac{1}{\pi\mathrm{i}}\int_{\mathbb{T}} \frac{\psi(\tau)}{\tau - t}\mathrm{d}\tau, \quad t \in \mathbb{T},$$

and $H^{\pm}_{p,w}$ the images of the projections $\frac{1}{2}(I \pm S_{\mathbb{T}})$. Then $H^{\pm}_p = B^{-1}H^{\pm}_{p,w}$.

Thus the Toeplitz operator T_g on H^+_p is equivalent to a Toeplitz operator $T_{\widetilde{g}}$ on $H^+_{p,w}$ by $T_g = B^{-1}T_{\widetilde{g}}B$.

For $p = 2$ we have $w = 1$ and there is a unitary equivalence between T_g (on H^+_2) and $T_{\widetilde{g}}$ (on $H^2(\mathbb{D})$), so that we recover a case of Theorem 3.9 from our work.

For $p \neq 2$, Theorem 3.7 can be used to extend Nakazi's result to a weighted Hardy space on $\mathbb{D}$; alternatively, Nakazi's result can be used to provide versions of Theorems 3.4 and 3.7 for weighted Hardy spaces of the upper half-plane.

Returning to the half-plane, we have the following.

THEOREM 3.10. *If $g \in L_\infty$ admits a factorisation*

$$g = g_-\theta g_+^{-1} \tag{3.10}$$

where $\frac{g_-}{\lambda_-} \in H^-_p$ is outer, $\frac{g_+}{\lambda_+} \in H^+_p$ is outer and square-rigid and $\theta \in H^+_\infty$ is an inner function, or if

$$g = g_- r^N g_+^{-1}, \tag{3.11}$$

where $\frac{g_-}{\lambda_-} \in H^-_p$ is outer and square-rigid, $\frac{g_+}{\lambda_+} \in H^+_p$ is outer and $N \in \mathbb{N}$, then $\ker T_g = \{0\}$.

Proof. Let $O_+ = \frac{g_+^{-1}}{\lambda_+}$ and $O_- = \frac{g_-}{\lambda_-}$. We have $\varphi_+ \in \ker T_g$ if and only if $\varphi_+ \in H^+_p$ and $g\varphi_+ = \varphi_-$ with $\varphi_- \in H^-_p$ which, from (3.10), is equivalent to

$$O_-\lambda_-\theta O_+^{-1}\lambda_+^{-1}\varphi_+ = \varphi_-.$$

Thus if $\varphi_+ \in \ker T_g$ we have

$$\frac{\overline{O}_+}{O_+}\frac{\varphi_+}{\lambda_+} = \frac{\overline{O}_+\varphi_-}{O_-}\lambda_-^{-1}\overline{\theta}$$

where the right hand side represents a function in H^-_p since its conjugate is in L_p and in the Smirnov class $\mathcal{N}_+$. Therefore $\lambda_+^{-1}\varphi_+ \in \ker T_{\overline{O}_+/O_+}$ and, by near invariance with respect to λ_+ ([7]), we also have $\varphi_+ \in \ker T_{\overline{O}_+/O_+}$. We conclude that $\varphi_+ = 0$ since $\ker T_{\overline{O}_+/O_+}$ is one-dimensional.

If g admits a representation (3.11) then $\overline{g}$ admits a factorisation (3.4) and, by Theorem 3.7, $\ker T_g^* = \ker T_{\overline{g}} \neq \{0\}$. By Coburn's lemma, it follows that $\ker T_g = \{0\}$. ∎

It is well known that various properties of a Toeplitz operator can be described in terms of an appropriate factorisation of its symbol. The representations (3.4) and (3.11) generalise the so called L_p factorisation, which is a representation of g as a product

$$g = g_- d g_+^{-1} \tag{3.12}$$

where

$$\frac{g_\pm}{\lambda_\pm} \in H_p^\pm, \quad \frac{g_\pm^{-1}}{\lambda_\pm} \in H_q^\pm, \quad \frac{1}{p} + \frac{1}{q} = 1 \tag{3.13}$$

and

$$d = r^k, k \in \mathbb{Z}. \tag{3.14}$$

The Toeplitz operator T_g is Fredholm if and only if g admits a factorisation (3.12) satisfying the conditions (3.13), (3.14) and such that $g_- P^+ g_-^{-1} I$ is a densely defined bounded operator in L_p. In that case (3.12) is called a *generalised p-factorisation*, or *Wiener–Hopf factorisation relative to L_p*, and we have $\dim \ker T_g = k$ if $k \geqslant 0$, $\dim \ker T_g^* = -k$ if $k \leqslant 0$ ([5], [18], [19]).

As an illustration, we consider $g(\xi) = r^\alpha, \alpha \in [-\frac{1}{2}, \frac{1}{2}]$, where we assume the discontinuity to be at ∞. A Wiener–Hopf factorisation relative to L_2 exists for all $\alpha \neq \pm\frac{1}{2}$ and T_g is invertible in H_2^+ ([11]). If $\alpha = \pm\frac{1}{2}$ then g does not admit a Wiener–Hopf factorisation relative to L_2. However we can write $g = g_- r^N g_+^{-1}$ with $g_- = (\xi - \mathrm{i})^{-1/2}, g_+^{-1} = (\xi + \mathrm{i})^{1/2}$ and $N = 0$ if $\alpha = -\frac{1}{2}$, $N = 1$ if $\alpha = \frac{1}{2}$. It is clear that $g_\pm \lambda_\pm^{-1} \in H_2^\pm$ are outer and, since we have

$$\mathcal{K}_{\min}(g_+ \lambda_+^{-1}) = \ker T_{r^{-3/2}} = \operatorname{span}\{g_+ \lambda_+^{-1}\},$$

the function $g_+ \lambda_+^{-1}$ is square-rigid. We have thus $\ker T_g = \ker T_g^* = \{0\}$, in accordance with Theorem 3.10.

The representation (3.12) is called a *bounded* factorisation if $g_-^{\pm 1} \in H_\infty^-$ and $g_+^{\pm 1} \in H_\infty^+$ ([4]). In various subalgebras of L_∞, every invertible element admits a bounded factorisation (3.12) where d is an inner function. This is the case for the Wiener algebra and the algebra of all Hölder continuous functions with exponent $\mu \in (0,1)$, with $d = r^k, k \in \mathbb{Z}$ ([18], [19]), and the algebra AP of almost periodic functions, with $d(\xi) = \exp(-\mathrm{i}\lambda\xi), \lambda \in \mathbb{R}$ ([10], [14]).

In the latter case, we easily see moreover that, for every $g \in AP$ which is invertible in L_∞ (and thus also in AP), $\ker T_g$ is either trivial or isomorphic to an infinite dimensional model space K_θ with $\theta(\xi) = \exp(\mathrm{i}\lambda\xi)$, depending on whether $\lambda \leqslant 0$ or $\lambda \geqslant 0$.

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M.C. CÂMARA, CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY AND DYNAMICAL SYSTEMS, MATHEMATICS DEPARTMENT, INSTITUTO SUPERIOR TÉCNICO, UNIVERSIDADE DE LISBOA, AV. ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL

E-mail address: ccamara@math.tecnico.ulisboa.pt

J.R. PARTINGTON, SCHOOL OF MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, U.K.

E-mail address: j.r.partington@leeds.ac.uk

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