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KMS STATES FOR QUASI-FREE ACTIONS ON FINITE-GRAPH ALGEBRAS

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ABSTRACT. Given a graph E and a labeling map ω, we consider the quasi-free action α^{ω} of $\mathbb{R}$ on the graph algebra $C^*(E)$. For a finite graph E, we give a complete characterization of all KMS_β states of a graph algebra in terms of a polyhedral set in $\mathbb{R}^{E^0}$. This characterization allows us to generalize the results of an Huef, Laca, Raeburn, and Sims. We make an explicit construction of all KMS_β states for β above a critical inverse temperature β_{c}, as well as a precise description of the KMS states for graphs with a certain strongly connected subgraph. In addition, we find a correspondence between the KMS states of a graph algebra and its dual-graph algebra when E is a row-finite graph with no sinks.

KEYWORDS: *KMS states, graph algebras, quasi-free actions, C^*-dynamical systems.*

MSC (2010): 46L30, 46L55.

1. INTRODUCTION

Given a graph $E = (E^0, E^1, r, s)$ and a labeling map $\omega : E^1 \to \mathbb{R}$, we consider the quasi-free action $\alpha^{\omega} : \mathbb{R} \curvearrowright C^*(E)$ that satisfies $\alpha_t^{\omega}(s_e) = \mathrm{e}^{\mathrm{i}\omega(e)t} s_e$ for all $e \in E^1$ and $\alpha_t^{\omega}(p_v) = p_v$ for all $v \in E^0$, which reduces to the gauge action of $\mathbb{R}$, when $\omega(e) = 1$ for all edges $e \in E^1$. For the gauge action γ of $\mathbb{R}$, Enomoto, Fujii and Watatani [6] gave a description of the KMS states of the Cuntz–Krieger algebra O_A, in terms of the eigenvalues of A. In particular, they showed that when A is an irreducible matrix, there exists a unique KMS state that has inverse temperature $\ln \rho(A)$, where $\rho(A)$ is the spectral radius of A (or, equivalently, the Perron–Frobenius eigenvalue of A). Exel and Laca [7] extended the results in [6] for quasi-free actions, where the labels are all positive and A is a finite matrix with no zero rows or columns. When A is an irreducible matrix, they gave a complete description of the KMS states for the Toeplitz–Cuntz–Krieger algebra $\mathcal{T}_A$. Among their results, they showed that at a critical inverse temperature $\beta_{\mathrm{c}} > 0$, there exists a unique $\mathrm{KMS}_{\beta_{\mathrm{c}}}$ state of $\mathcal{T}_A$. In addition, this state factors through O_A and is the only KMS state for the C^*-dynamical system consisting of quasi-fee actions on O_A. Zacharias [18] also showed that there exists a unique KMS state of O_A and is the only KMS state for the C^*-dynamical system consisting of quasi-free actions on O_A but without the use of the Toeplitz–Cuntz–Krieger algebra. The unique β_{c} satisfies $\rho(D_{\beta_{\mathrm{c}}}A) = 1$, where $D_{\beta_{\mathrm{c}}}$ is a diagonal matrix and each diagonal entry is of the form $\mathrm{e}^{-\beta_{\mathrm{c}}\lambda}$ for some label $\lambda > 0$.

More recently, there has been interest in the investigation of KMS states of C^*-algebras that are constructed from directed graphs. In [9], [10], [11], finite graphs were analyzed and, in [4], [5], emphasis was towards infinite graphs. In [10] and [11], an Huef, Laca, Raeburn and Sims gave considerable insight into KMS states on the C^*-algebras of finite graphs for the gauge action of $\mathbb{R}$; their papers consisted of studying KMS states on the Toeplitz algebra $\mathcal{T}C^*(E)$ for a finite graph E. In [10], they gave an explicit description of all KMS_β states, when β is above the critical inverse temperature $\ln\rho(A)$, where A is the vertex matrix of the corresponding graph. It was also shown that if E is a strongly connected graph, then there is a unique $\mathrm{KMS}_{\ln\rho(A)}$ state of $\mathcal{T}C^*(E)$ that factors through $C^*(E)$. In [11], they continued their analysis, with emphasis on graph algebras having reducible vertex matrices, by looking at the strongly connected components of a finite graph E and their interactions.

In this paper, we extend the results of the theorems in [10] to quasi-free actions (also known as generalized gauge actions). The characterization in Theorem 3.3 of this paper allows us to focus our attention on the graph algebra $C^*(E)$ directly. Then, as a consequence, we use Proposition 2.5 to recover the results for the Toeplitz algebra $\mathcal{T}C^*(E)$. We believe that the results in [11] can be extended to quasi-free actions, using the same techniques of this paper, but we leave that for future work.

The notation and preliminaries needed for this paper are in Section 2.

In Section 3, we characterize the simplex of all KMS states in terms of a polyhedral set in $\mathbb{R}^{E^0}$, which allows us to readily compute its extreme points and we illustrate this through examples in Section 9.

In Section 4, we find KMS states at a critical inverse temperature β_{c}; when $\omega(e) > 0$ for all $e \in E^1$, we show that this β_{c} exists and is unique.

In Section 5, we give a precise description of the KMS states above a critical inverse temperature and extend Theorem 3.1 of [10]. If H is the set of sinks and $E\backslash H$ is a strongly connected subgraph of E, then we give a precise description of all KMS states of $C^*(E)$; in particular, there is a unique $\mathrm{KMS}_{\beta_{\mathrm{c}}}$ state and it factors through a unique $\mathrm{KMS}_{\beta_{\mathrm{c}}}$ state of $C^*(E\backslash H)$. As a consequence, we extend Theorem 4.3 in [10] (see Section 6).

In Section 7, we analyze the connection between the KMS states of a graph algebra and its dual-graph algebra, when E is a row-finite graph with no sinks.

In Section 8, we extend Proposition 5.1 of [10].

2. NOTATION AND PRELIMINARIES

2.1. GRAPH ALGEBRAS. A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable sets E^0 and E^1 of vertices and edges, respectively, with range and source maps $r, s : E^1 \to E^0$. A directed graph $E = (E^0, E^1, r, s)$ is called *finite* if both E^0 and E^1 are finite and is called *row-finite* if $|s^{-1}(v)| < \infty$ for all $v \in E^0$. A *path* of length $n \geqslant 1$ is a finite sequence of edges $\mu := \mu_1\mu_2 \cdots \mu_n$ with $r(\mu_i) = s(\mu_{i+1})$ for $1 \leqslant i \leqslant n-1$. We regard vertices as paths of length 0. For $n \geqslant 0$, we let E^n denote the set of all paths of length n and define $E^* := \bigcup\limits_{n \geqslant 0} E^n$. The range and source maps extend to E^* in a natural way. For vertices v and w, we define $vE^n w$ to be the set $\{\mu \in E^n : s(\mu) = v \text{ and } r(\mu) = w\}$. A *cycle* is a path with its range and source equal; namely, a path $\mu := \mu_1\mu_2 \cdots \mu_n$ is a cycle provided that $r(\mu_n) = s(\mu_1)$. A vertex that does not emit an edge is called a *sink* and we denote E^0_{sinks} to be the set of all sinks in E^0. A vertex that emits at least one edge but not infinitely many edges is called a *regular vertex* and we denote E^0_{reg} to be the set of all regular vertices in E^0.

If E is a graph, a Cuntz–Krieger E-family in a C^*-algebra is a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges that satisfy the following Cuntz–Krieger relations:

(CK1) $s_e^* s_e = p_{r(e)}$

(CK2) $p_v = \sum\limits_{\{e \in E^1 : s(e) = v\}} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$, and

(CK3) $s_e s_e^* \leqslant p_{s(e)}$.

The *graph C^*-algebra* (or, simply, the graph algebra) of E is the C^*-algebra generated by the universal Cuntz–Krieger E-family and is denoted by $C^*(E)$.

2.2. STRONGLY CONNECTED GRAPH AND ITS DUAL GRAPH. Let E be a graph. Define the dual graph $\widehat{E}$ by $\widehat{E}^0 = E^1$ and $\widehat{E}^1 = E^2$ where $r_{\widehat{E}}(ef) = f$ and $s_{\widehat{E}}(ef) = e$. We note that if E is row-finite, then so is $\widehat{E}$. The vertex matrix of the dual graph corresponds to the edge matrix of the original graph:

$$A_{\widehat{E}}(e, f) = \begin{cases} 1 & \text{if } ef \text{ is a path,} \\ 0 & \text{if } ef \text{ is a not path,} \end{cases} = B_E(e, f).$$

We say non-empty graph E is *strongly connected* if for every pair of vertices $v, w \in E^0$, there is a path $|\mu| \geqslant 1$ such that $s(\mu) = v$ and $r(\mu) = w$.

PROPOSITION 2.1. *If E is a strongly connected directed graph, then so is $\widehat{E}$.*

Proof. Suppose E is strongly connected and $e, f \in \widehat{E}^0$. Let $r(e) = x$ and $s(f) = y$. Since E is strongly connected, there is a path $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ from x to y. Thus, $e\alpha_1, \alpha_2\alpha_3, \ldots, \alpha_n f$ are paths of length two in E, so they correspond to edges in $\widehat{E}$. Hence, we have that $(e\alpha_1)(\alpha_1\alpha_2) \cdots (\alpha_{n-1}\alpha_n)(\alpha_n f)$ is a path from e to f in $\widehat{E}$, as required. ∎

2.3. THE TOEPLITZ ALGEBRA. The Toeplitz algebra $\mathcal{T}C^*(E)$ is isomorphic to the graph algebra $C^*(E_{\mathcal{T}})$, where the associated graph $E_{\mathcal{T}}$ comes from E and is defined below. We refer the reader to Theorem 4.1 of [8] for the definition of the Toeplitz algebra $\mathcal{T}C^*(E)$.

DEFINITION 2.2. Let $E = (E^0, E^1, r, s)$ be a graph and set $R(E) := \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\}$. Define a new graph $E_{\mathcal{T}}$ by letting

$$E^0_{\mathcal{T}} := E^0 \cup \{v' : v \in R(E)\},$$
$$E^1_{\mathcal{T}} := E^1 \cup \{e' : e \in E^1 \text{ and } r(e) \in R(E)\},$$

with range and source maps extended to $E^1_{\mathcal{T}}$ by $s(e') = s(e)$, and $r(e') = r(e)'$.

PROPOSITION 2.3 ([14]). *Let E be a graph and let $\{s_e, p_v\}$ be a generating Toeplitz–Cuntz–Krieger E-family in $\mathcal{T}C^*(E)$. Then the Toeplitz algebra $\mathcal{T}C^*(E)$ is canonically isomorphic to the graph algebra $C^*(E_{\mathcal{T}})$. Furthermore, if we define*

$$q_w := \begin{cases} p_w & \text{if } w \notin R(E), \\ \sum_{\{e \in E^1 : s(e) = w\}} s_e s_e^* & \text{if } w \in R(E), \\ p_v - \sum_{\{e \in E^1 : s(e) = v\}} s_e s_e^* & \text{if } w = v' \text{ for some } v \in R(E), \end{cases}$$

$$t_f := \begin{cases} s_f q_{r(f)} & \text{if } f \in E^1, \\ s_e q_{r(e)'} & \text{if } f = e' \text{ for some } e \in E^1, \end{cases}$$

then $\{t_f, q_w\}$ generates a Cuntz–Krieger $E_{\mathcal{T}}$-family in $\mathcal{T}C^*(E)$.

REMARK 2.4. The above proposition is just a specific example of a more general result: it was shown that every relative graph algebra $C^*(E, V)$, where $V \subseteq R(E)$, is canonically isomorphic to the graph algebra $C^*(E_V)$ (see Theorem 3.7 of [14]). Since the Toeplitz algebra $\mathcal{T}C^*(E)$ is the relative graph algebra $C^*(E, \emptyset)$, we adopted the notation $C^*(E_{\mathcal{T}})$ instead of $C^*(E_{\emptyset})$.

PROPOSITION 2.5. *Let $E = (E^0, E^1, r, s)$ be a graph. Let ω be a labeling map on E^1 and extend ω to $E^1_{\mathcal{T}}$ by $\omega(e') = \omega(e)$. Then $(C^*(E_{\mathcal{T}}), \alpha^\omega)$ is covariantly isomorphic to $(\mathcal{T}C^*(E), \alpha^\omega)$.*

The proof follows immediately from Proposition 2.3.

2.4. IDEAL STRUCTURE. A set H of E^0 is *hereditary* if, for any $e \in E^1$, we have $s(e) \in H$ implies $r(e) \in H$. A hereditary set H is *saturated* if, whenever $v \in E^0$ is a regular vertex with $r(vE^1) \subseteq H$, then $v \in H$. If $H \subseteq E^0$ is a hereditary set, the *saturation* of H is the smallest saturated subset $\overline{H}$ of E^0 containing H. It was shown that there is a bijective correspondence between the gauge-invariant ideals in $C^*(E)$ and the saturated hereditary subsets of E^0 (see [16] and the references therein).

2.5. KMS STATES AND GROUND STATES. Given a C^*-algebra A and a homomorphism (dynamics) $\sigma : \mathbb{R} \to \mathrm{Aut}(A)$, an element $a \in A$ is called *analytic* if $t \to \sigma_t(a)$ extends to an entire function on $\mathbb{C}$. For $\beta \in (0, \infty)$, a KMS$_\beta$ state of (A, σ) is a state ϕ of A which satisfies the KMS$_\beta$ condition

$$\phi(ab) = \phi(b\sigma_{\mathrm{i}\beta}(a)) \tag{2.1}$$

for all a, b analytic in A. A KMS$_0$ state of (A, σ) is a state ϕ of A that is invariant, with respect to σ and that satisfies the trace condition $\phi(ab) = \phi(ba)$ for all $a, b \in A$. A KMS$_\infty$ state is a weak* limit of a sequence of KMS$_{\beta_n}$ states as $\beta_n \to \infty$ and a ground state is a state ϕ such that the functions $\phi_{a,b} : z \mapsto \phi(a\alpha_z(b))$ are bounded in the upper-half plane for every a, b analytic in A. Standard references for KMS states and ground states can be found in [3] and [15].

Throughout this paper, we consider the quasi-free action α^ω of $\mathbb{R}$ on $C^*(E)$ that corresponds to a labeling map ω on E^1. This labeling map has an extention to E^*, which we also denote by ω, and is defined below.

DEFINITION 2.6. Let $\omega : E^1 \to \mathbb{R}$ be a labeling map on E^1. We say ω is a *labeling map* on E^* if we extend ω to E^* by $\omega(\mu) = \omega(\mu_1) + \cdots + \omega(\mu_n)$ for $\mu = \mu_1 \cdots \mu_n \in E^* \backslash E^0$ and $\omega(v) = 0$ for $v \in E^0$.

In [5], it was shown that if $\omega(\mu) \neq 0$ for all $\mu \in E^* \backslash E^0$, then σ is a KMS$_\beta$ state of $(C^*(E), \alpha^\omega)$ if and only if

$$\sigma(s_\mu s_\nu) = \delta_{\mu,\nu} \mathrm{e}^{-\beta\omega(\mu)} \sigma(p_{r(\mu)}). \tag{2.2}$$

In Theorem 3.10 of [5], it was shown that there is a bijective correspondence between the KMS$_\beta$ states of $(C^*(E), \alpha^\omega)$ and a certain class of tracial states on $C_0(E^0)$:

DEFINITION 2.7. Let ω be a labeling map on E^1 that is bounded below and let $\beta \geqslant 0$. Given a tracial state τ on $C_0(E^0) \cong \overline{\mathrm{span}}\{p_v\}_{v \in E^0}$, we can define a trace on $C_0(E^0)$ by

$$\mathcal{F}_{\omega,\beta}(\tau)(p_v) = \lim_{D \to s^{-1}(v)} \sum_{e \in D} \mathrm{e}^{-\beta\omega(e)} \tau(p_{r(e)}),$$

where the limit is taken on finite subsets D of $s^{-1}(v)$ and $\mathcal{F}_{\omega,\beta}(\tau)(p_v) = 0$ if $s^{-1}(v) = \emptyset$ [5].

THEOREM 2.8 ([5]). *Let γ be the standard gauge action of $\mathbb{T}$ on $C^*(E)$ and $C^*(E)^\gamma$ the fixed-point subalgebra of $C^*(E)$. Let ω be a labeling map on E^1 that is bounded below and let $\beta \geqslant 0$. If σ is a state on $C^*(E)^\gamma$ satisfying* (2.2), *then its restriction τ to $C_0(E^0)$ satisfies:*

(K1) $\mathcal{F}_{\omega,\beta}(\tau)(a) = \tau(a)$ *for all* $a \in \overline{\mathrm{span}}\{p_v : 0 < |s^{-1}(v)| < \infty\}$;
(K2) $\mathcal{F}_{\omega,\beta}(\tau)(a) \leqslant \tau(a)$ *for all* $a \in C_0(E^0)^+$.

Conversely, if τ is a tracial state on $C_0(E^0)$ satisfying (K1) *and* (K2), *then there is a unique state σ on $C^*(E)^\gamma$ satisfying* (2.2) *with $\sigma|_{C_0(E^0)} = \tau$. This correspondence preserves convex combinations.*

3. CHARACTERIZING KMS STATES FOR QUASI-FREE ACTIONS

In this section, we characterize the KMS_β states of $C^*(E)$ in terms of vectors that satisfy a certain *Property* P_β defined below. When E is a finite graph, the simplex of all KMS_β states of $C^*(E)$ can be viewed as a polyhedral set in $\mathbb{R}^{E^0}$, and, in turn, we can readily compute its extreme points.

DEFINITION 3.1. Let $E = (E^0, E^1, r, s)$ be a row-finite graph and $\alpha^\omega : \mathbb{R} \curvearrowright C^*(E)$ be the quasi-free action that corresponds to the labeling map $\omega : E^1 \to \mathbb{R}$. Let $\beta \in \mathbb{R}$ and $C_\beta \in M_{E^0}(\mathbb{R})$ the matrix defined by $C_\beta(v, w) := \sum_{e \in vE^1 w} \mathrm{e}^{-\beta\omega(e)}$. Note that, if $vE^1w = \emptyset$, then $C_\beta(v, w) = 0$, by standard convention. (The matrix C_β may also be written as $C_{\beta,E}$ or $C_{\beta,E,\omega}$, if we need to be more specific). We say that a vector $m := (m_v)_{v\in E^0}$ satisfies *Property* P_β on E^0 if m is a probability measure with $(C_\beta m)_v = m_v$, whenever v is a regular vertex.

REMARK 3.2. If we reduce to the gauge action, we note that $C_\beta = \mathrm{e}^{-\beta} A$, where A is the vertex matrix of E. Note that, if m satisfies Property P_β on E^0, then m satisfies the subinvariance relation $C_\beta m \leqslant m$. That is, $(C_\beta m)_v \leqslant m_v$ for each $v \in E^0$. Also, if E is a strongly connected graph, then C_β is an irreducible matrix.

THEOREM 3.3. *Let E be a row-finite graph, ω a labeling map on E^1 that is bounded below, and $\beta \geqslant 0$. Let K_{β,α^ω} be the set of all* KMS_β *states for the quasi-free action α^ω on $C^*(E)$ and let $L_{\beta,\alpha^\omega} := \{m = (m_v)_{v\in E^0} : m$ satisfies Property P_β on $E^0\}$. Suppose that ω satisfies $\omega(\mu) \neq 0$ for all $\mu \in E^*\backslash E^0$. Then K_{β,α^ω} is affine-isomorphic to L_{β,α^ω}. More specifically, for each $m \in L_{\beta,\alpha^\omega}$, the corresponding* KMS_β *state ϕ_m satisfies*

$$\phi_m(s_\mu s_\nu^*) = \delta_{\mu,\nu} \mathrm{e}^{-\beta\omega(\mu)} m_{r(\mu)}. \tag{3.1}$$

Proof. Let $\{s_e, p_v\}$ be the canonical Cuntz–Krieger E-family that generates $C^*(E)$. Let $\pi : C^*(E) \to B(\mathcal{H})$ be a faithful nondegenerate representation and define $P_v := \pi(p_v)$ and $S_e := \pi(s_e)$. Define a map $\Psi : K_{\beta,\gamma} \to \mathbb{R}^{E^0}$ by $\Psi(\phi) = m^\phi$, where $m^\phi := (\phi(p_v))_{v\in E^0}$. Clearly, Ψ is an affine map that is weak*-continuous. Since ϕ is a state, we have that m^ϕ is a probability measure. Also, whenever v is a regular vertex, we have

$$\begin{aligned} m_v^\phi = \phi(p_v) &= \sum_{e\in vE^1} \phi(s_e s_e^*) = \sum_{e\in vE^1} \mathrm{e}^{-\beta\omega(e)} \phi(p_{r(e)}) \\ &= \sum_{w\in E^0} \sum_{e\in vE^1 w} \mathrm{e}^{-\beta\omega(e)} \phi(p_w) = \sum_{w\in E^0} C_\beta(v, w)\phi(p_w) = (C_\beta m^\phi)_v. \end{aligned}$$

Thus, $(C_\beta m^\phi)_v = m_v^\phi$ and $m^\phi \in L_{\beta,\alpha^\omega}$.

To show the image of Ψ is L_{β,α^ω}, choose an $x \in L_{\beta,\alpha^\omega}$. Define a tracial state τ on $C_0(E^0)$ by $\tau(a) = \sum_{v\in E^0} x_v(\pi|_{C_0(E^0)}(a)P_v, P_v)$. Indeed, $\tau(1) = \sum_{v\in E^0} x_v = 1$. If v

is a regular vertex, then

$$\begin{aligned}
\mathcal{F}_{\omega,\beta}(\tau)(p_v) &= \sum_{e\in vE^1} \mathrm{e}^{-\beta\omega(e)}\tau(p_{r(e)}) = \sum_{w\in E^0}\sum_{e\in vE^1 w} \mathrm{e}^{-\beta\omega(e)} x_w \\
&= \sum_{w\in E^0} C_\beta(v,w)x_w \qquad (3.2)\\
&= x_v \qquad (3.3)\\
&= \tau(p_v),
\end{aligned}$$

where (3.2) equals (3.3) since $x \in L_{\beta,\alpha^\omega}$. By Theorem 2.8, we have a unique state σ on the core of $C^*(E)$ that satisfies (2.2) with $\sigma|_{C_0(E^0)} = \tau$. Hence, $\phi = \sigma \circ \Phi$ is a KMS$_\beta$ state on $C^*(E)$ by Theorem 3.3 of [5]. So, $\Psi(\phi)(v) = \phi(p_v) = \sigma(p_v) = x_v$ and $\Psi(\phi) = x$.

To prove injectivity, suppose $\Psi(\phi_1) = \Psi(\phi_2)$. Then $\phi_1(p_v) = \phi_2(p_v)$ for all $v \in E^0$. Hence, by Proposition 3.2 of [5], both KMS$_\beta$ states coincide on its core. Since $\omega(\mu) \neq 0$ for all $\mu \in E^*$, the KMS$_\beta$ states are equal. ∎

REMARK 3.4. We note that if $x = (x_v)_{v\in E^0}$ is in L_{β,α^ω}, it will satisfy the following equations:

$$x_v - \sum_{w\in E^0} C_\beta(v,w)x_w = 0 \quad \text{for each } v \in E^0_{\mathrm{reg}},$$

$$\sum_{w\in E^0} x_w = 1.$$

Let R_β be the coefficient matrix of the linear system above and $d = (0\ 0 \cdots 1)^{\mathrm{tr}}$. Then we have that

$$L_{\beta,\alpha^\omega} = \{x = (x_v)_{v\in E^0} : R_\beta x = d, x \geqslant 0\}. \tag{3.4}$$

For the gauge action γ of $\mathbb{R}$, we can multiply each row (except the row of ones) of R_β by e^β to allow for simpler calculations (see Example 9.1). We denote this by $\widetilde{R}_\beta$.

When E is a finite graph, L_{β,α^ω} is a polyhedral set. Thus, we can easily calculate the extreme points of the set K_{β,α^ω} of all KMS$_\beta$ states of $(C^*(E), \alpha^\omega)$ (see Section 9).

4. KMS STATES AT A CRITICAL INVERSE TEMPERATURE

In this section, we show that there exists a KMS state at a critical inverse temperature $\beta_{\mathrm{c}} \geqslant 0$, where β_{c} satisfies $\rho(C_{\beta_{\mathrm{c}}}) = 1$ and $C_{\beta_{\mathrm{c}}}$ is the matrix defined in Definition 3.1. First, we will prove the existence and uniqueness of β_{c}. We recall that the edge matrix of E is the matrix $B \in M_{E^1}(\mathbb{N})$ defined by

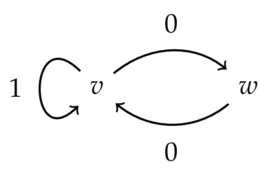
$$B(e,f) = \begin{cases} 1 & \text{if } ef \text{ is a path,} \\ 0 & \text{if } ef \text{ is not a path.} \end{cases}$$

LEMMA 4.1. *Let E be a strongly connected finite graph with $\omega(e) > 0$ for all $e \in E^1$ (or $\omega(e) < 0$ for all $e \in E^1$). For each β, let D_β be the diagonal matrix in $M_{E^1}(\mathbb{R})$ having diagonal entries $\mathrm{e}^{-\beta\omega(e)}$ and $B \in M_{E^1}(\mathbb{N})$ the edge matrix of E. Then there exists a unique $\beta_{\mathrm{c}} \geqslant 0$ ($\beta_{\mathrm{c}} \leqslant 0$) with $\rho(D_{\beta_{\mathrm{c}}}B) = 1$. Furthermore, if E is a strongly connected graph that consists of a single cycle, then $\beta_{\mathrm{c}} = 0$. Otherwise, if $\omega(e) > 0$ for all $e \in E^1$ (or $\omega(e) < 0$ for all $e \in E^1$), then $\beta_{\mathrm{c}} > 0$ ($\beta_{\mathrm{c}} < 0$).*

Proof. Assume $\omega(e) > 0$ for all $e \in E^1$. Since E is strongly connected, so is $\widehat{E}$ by Proposition 2.1. Thus, B is irreducible and this implies that $D_\beta B$ is irreducible for each β. By Proposition 18.3 of [7], there exists a β_{c} that satisfies $\rho(D_{\beta_{\mathrm{c}}}B) = 1$. By the Perron–Frobenius theorem, $\rho(D_{\beta_{\mathrm{c}}}B) = 1$ if and only if we have an eigenvector $(y_f)_f$ with $y_f > 0$ and $\sum_{f \in E^1} y_f = 1$ such that $D_{\beta_{\mathrm{c}}}By = y$. Let $a_e := \sum_{f \in E^1} B(e,f)y_f$ and $\varphi(x) = \sum_{e \in E^1} a_e x^{\omega(e)} - 1$. Since E is strongly connected, $|r^{-1}(s(e))| \geqslant 1$ for every $e \in E^1$ and thus $\sum_{e \in E^1} a_e = \sum_{e \in E^1} |r^{-1}(s(e))| y_e \geqslant 1$. Hence, φ is a real valued function that has a unique positive real root $\xi \in (0, 1]$. Since $D_{\beta_{\mathrm{c}}}By = y$, we have that $\xi = \mathrm{e}^{-\beta_{\mathrm{c}}}$ and therefore, $\beta_{\mathrm{c}} \geqslant 0$.

If E is a cycle, then $|r^{-1}(s(e))| = 1$ for all $e \in E^1$ and hence $\beta_{\mathrm{c}} = 0$. Otherwise, there is a vertex $v \in E^0$ that receives two edges, say e and f. Since E is strongly connected, v is not a sink and so it emits some edge $g \in E^1$. So $e, f \in r^{-1}(s(g))$ and we get that $|r^{-1}(s(g))| \geqslant 2$. Thus, $\sum_{e \in E^1} a_e > 1$ and therefore, $\beta_{\mathrm{c}} > 0$. ∎

REMARK 4.2. If E is strongly connected and $\omega(e) = 0$ for all $e \in E^1$, then $\rho(D_\beta B) = \rho(B) = \rho(A)$ by Proposition 4.1 in [12]. Hence, if E is a cycle, then $\rho(D_\beta B) = 1$ for all β by Lemma A.1 in [10]. Otherwise, $\rho(A) > 1$ and there is no such β. Also note that if $\omega(e) = 0$ for some, but not all edges $e \in E^1$, then there need not exist a β that satisfies $\rho(D_\beta B) = 1$. For example, let E be the graph below having labels 0 and 1.



Then $\rho(D_\beta B) \neq 1$ for all $\beta \in \mathbb{R}$.

PROPOSITION 4.3. *Let E be a strongly connected finite graph with $\omega(e) > 0$ for all $e \in E^1$ (or $\omega(e) < 0$ for all $e \in E^1$). Then there exists a unique $\beta_{\mathrm{c}} \geqslant 0$ ($\beta_{\mathrm{c}} \leqslant 0$) with $\rho(C_{\beta_{\mathrm{c}}}) = 1$. Furthermore, if E is a strongly connected graph that consists of a single cycle, then $\beta_{\mathrm{c}} = 0$. Otherwise, if $\omega(e) > 0$ for all $e \in E^1$ (or $\omega(e) < 0$ for all $e \in E^1$), then $\beta_{\mathrm{c}} > 0$ ($\beta_{\mathrm{c}} < 0$).*

Proof. Let S_β be the $E^0 \times E^1$ matrix defined by

$$S_\beta(v,e) = \begin{cases} e^{-\beta\omega(e)} & \text{if } s(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

and let R be the $E^1 \times E^0$ matrix defined by

$$R(e,v) = \begin{cases} 1 & \text{if } r(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

Then $RS_\beta = BD_\beta$ and $S_\beta R = C_\beta$. Since $\rho(C_\beta) = \rho(BD_\beta) = \rho(D_\beta B)$, we have that the rest follows from Lemma 4.1. ∎

PROPOSITION 4.4. *Let E be a finite graph with $\omega(e) > 0$ for all $e \in E^1$ (or $\omega(e) < 0$ for all $e \in E^1$). Then there exists a unique $\beta_c \geqslant 0$ ($\beta_c \leqslant 0$) with $\rho(C_{\beta_c}) = 1$.*

Proof. Let $F_1, F_2, \ldots, F_n$ be the strongly connected components of E. From the Seneta decomposition of C_β, we have that $\rho(C_\beta) = \max\{\rho(C_{\beta,F_k}) : k = 1,2,\ldots,n\}$, where each C_{β,F_k} is an irreducible matrix (see [17] and [11]). For each $k = 1,2,\ldots n$, there is a unique $\beta_k \geqslant 0$ that satisfies $\rho(C_{\beta_k,F_k}) = 1$ by Proposition 4.3. Let $\beta_c := \max\{\beta_k : k = 1,2,\ldots,n\}$. Then $\beta_c \geqslant \beta_k$ implies that $\rho(C_{\beta_c,F_k}) \leqslant \rho(C_{\beta_k,F_k}) = 1$ and hence, $\rho(C_{\beta_c}) = 1$. Suppose that there exist a $\widetilde{\beta}_c > 0$ with $\widetilde{\beta}_c \neq \beta_c$ and $\rho(C_{\widetilde{\beta}_c}) = 1$. Suppose without loss of generality that $\widetilde{\beta}_c > \beta_c > 0$. For each $k = 1,2,\ldots n$, we have that $1 \geqslant \rho(C_{\beta_c,F_k}) > \rho(C_{\widetilde{\beta}_c,F_k})$ by the min-max version of the Collatz-Wielandt formula. This is a contradiction and therefore, β_c uniquely satisfies $\rho(C_{\beta_c}) = 1$. ∎

From this point on, if $\omega(e) > 0$ for all $e \in E^1$, then the critical inverse temperature is the unique β that satisfies $\rho(C_\beta) = 1$ and this is denoted by β_c.

PROPOSITION 4.5. *Let E be a finite graph and $\beta \geqslant 0$ be such that $\rho(C_\beta) = 1$. Then there exists a* KMS$_\beta$ *state.*

Proof. Let H be the set of sinks and decompose E^0 as $E^0\backslash H \cup H$. Then we can write the matrix C_β as a block matrix

$$C_\beta = \begin{pmatrix} C_{\beta,E\backslash H} & F \\ 0 & 0 \end{pmatrix}, \tag{4.1}$$

and so, C_β is an upper triangular block matrix with $\rho(C_\beta) = \rho(C_{\beta,E\backslash H})$. Since $C_\beta \geqslant 0$, there exists a $z = (z_v)_{v\in E^0\backslash H}$ with $z > 0$ and $\|z\|_1 = 1$ so that $C_{\beta,E\backslash H} z = z$ (see, for example, 8.3 of [13]). Let $x = (z \ \ 0)^{\mathrm{tr}}$. Then $\|x\|_1 = 1$ and $C_\beta x = x$. Hence, by Theorem 3.3, there exists a KMS$_\beta$ state that satisfies

$$\phi_x(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta\omega(\mu)} x_{r(\mu)}. \quad ∎$$

The following corollary shows the existence of a KMS state for quasi-free actions acting on finite graph algebras. This extends Corollary 4.2 of [10] to quasi-free actions; in addition, the results show the existence of a KMS state for not only the Toeplitz algebra, but the graph algebra as well. Thus, there exists a KMS state of $(\mathcal{T}C^*(E), \alpha^\omega)$ that will always factor through a KMS state of $(C^*(E), \alpha^\omega)$. It is noteworthy that there are no restrictions on the structure of the graph E, such as strong connectivity.

COROLLARY 4.6. *Let E be a finite graph and $\omega(e) > 0$ for all $e \in E^1$. Then there exists a* KMS_{β_c} *state.*

This is a consequence of Proposition 4.4 and Proposition 4.5.

As a consequence of Corollary 4.6, we get the existence of a $\mathrm{KMS}_{\ln \rho(A)^{1/k}}$ state, when all the edges have label $k > 0$. In particular, when $k = 1$, the action reduces to the gauge action of the reals and we have the existence of a $\mathrm{KMS}_{\ln \rho(A)}$ state. This was exactly the critical inverse temperature described in [10].

COROLLARY 4.7. *Let E be a finite graph with at least one cycle and $\omega(e) = k > 0$ for all $e \in E^1$. Then there exists a* $\mathrm{KMS}_{\ln \rho(A)^{1/k}}$ *state.*

Proof. We note that $C_\beta = \mathrm{e}^{-\beta k} A$ and $\beta_c = \ln \rho(A)^{1/k}$. Since E has at least one cycle, $\rho(A) \geqslant 1$ (see Appendix A of [10]). Thus, we have a $\mathrm{KMS}_{\ln \rho(A)^{1/k}}$ state by Corollary 4.6 above. ∎

5. KMS STATES ABOVE THE CRITICAL INVERSE TEMPERATURE

In this section, we study the KMS states above a critical inverse temperature and extend the results of Theorem 3.1 in [10].

THEOREM 5.1. *Let E be a finite directed graph and $C_\beta \in M_{E^0}(\mathbb{R})$ be the matrix defined by $C_\beta(v, w) = \sum\limits_{e \in vE^1 w} \mathrm{e}^{-\beta\omega(e)}$. Let α^ω be the quasi-free action corresponding to a labeling map ω, where $\omega(\mu) \neq 0$ for all $\mu \in E^* \backslash E^0$. Assume β is such that $\rho(C_\beta) < 1$.*

(i) *For $v \in E^0$, the series $\sum\limits_{\mu \in E^* v} \mathrm{e}^{-\beta\omega(\mu)}$ either converges or is finite with sum $y_v \geqslant 1$. Set $y := (y_v) \in [1, \infty)^{E^0}$ and consider $\epsilon \in [0, \infty)^{E^0}$. Then $m := (I - C_\beta)^{-1}\epsilon$ is a probability measure on E^0 if and only if $\epsilon \cdot y = 1$.*

(ii) *Suppose $\epsilon \in [0, \infty)^{E^0}$ satisfies $\epsilon \cdot y = 1$ and $\epsilon_v = 0$ whenever v is a regular vertex. Then there is a* KMS_β *state ϕ_ϵ on $(C^*(E), \alpha^\omega)$ satisfying*

$$\phi_\epsilon(s_\mu s_\nu^*) = \delta_{\mu,\nu} \mathrm{e}^{-\beta\omega(\mu)} m_{r(\mu)}. \tag{5.1}$$

(iii) *The map $\epsilon \longmapsto \phi_\epsilon$ is an affine isomorphism of*

$$\sum_\beta := \{\epsilon \in [0, \infty)^{E^0} : \epsilon \cdot y = 1 \text{ and } \epsilon_v = 0 \text{ for } v \in E^0_{\mathrm{reg}}\}$$

onto the simplex of KMS_β *states of* $(C^*(E), \alpha^\omega)$. *The inverse of this isomorphism takes the* KMS_β *state* ϕ *to* $(I - C_\beta)m^\phi$, *where* $m^\phi := (\phi(p_v))_v$.

Proof. (i) Let $v \in E^0$. Note that $C^n_\beta(w,v) = \sum\limits_{\mu \in wE^n v} \mathrm{e}^{-\beta\omega(\mu)}$. Then

$$\sum_{\mu \in E^* v} \mathrm{e}^{-\beta\omega(\mu)} = \sum_{n=0}^{\infty} \sum_{\mu \in E^n v} \mathrm{e}^{-\beta\omega(\mu)} = \sum_{n=0}^{\infty} \sum_{w \in E^0} \sum_{\mu \in wE^n v} \mathrm{e}^{-\beta\omega(\mu)} \tag{5.2}$$

$$= \sum_{n=0}^{\infty} \sum_{w \in E^0} C^n_\beta(w,v). \tag{5.3}$$

Since $\rho(C_\beta) < 1$, the series $\sum\limits_{n=0}^{\infty} C^n_\beta$ converges in the operator norm. Thus, for every fixed $w \in E^0$, the series $\sum\limits_{n=0}^{\infty} C^n_\beta(w,v)$ converges and hence, the last sum in (5.3) converges. Also, since $C^0_\beta(v,v) = 1$, we have $y_v \geqslant 1$.

The expansion $m = \sum\limits_{n=0}^{\infty} C^n_\beta \epsilon$ shows that $m \geqslant 0$ and

$$\begin{aligned}
m(E^0) &= \sum_{v \in E^0} m_v = \sum_{v \in E^0} ((I - C_\beta)^{-1}\epsilon)_v \\
&= \sum_{v \in E^0} \Big(\Big(\sum_{n=0}^{\infty} C^n_\beta\Big)\epsilon\Big)_v = \sum_{v \in E^0} \sum_{n=0}^{\infty} \sum_{w \in E^0} C^n_\beta(v,w)\epsilon_w \\
&= \sum_{w \in E^0} \epsilon_w \Big(\sum_{v \in E^0} \sum_{n=0}^{\infty} C^n_\beta(v,w)\Big) = \sum_{w \in E^0} \epsilon_w \Big(\sum_{\mu \in E^* w} \mathrm{e}^{-\beta\omega(e)}\Big) = \epsilon \cdot y.
\end{aligned}$$

(ii) By (i) we have a probability measure $m := (I - C_\beta)^{-1}\epsilon$ on E^0. Let v be a regular vertex. Then $\epsilon_v = 0$ and we get that $m_v = (\sum_{n=0}^{\infty} C^{n+1}_\beta \epsilon)_v$ and

$$(C_\beta m)_v = (C_\beta(I - C_\beta)^{-1}\epsilon)_v = \Big(\Big(\sum_{n=0}^{\infty} C^{n+1}_\beta\Big)\epsilon\Big)_v = m_v.$$

Hence, by Theorem 3.3, there exists a KMS_β state that satisfies (3.1).

(iii) To see that every KMS_β state ϕ has the form ϕ_ϵ, note that $m^\phi = (\phi(p_v))_{v \in E^0}$ satisfies Property P_β on E^0 and take $\epsilon := (I - C_\beta)m^\phi$. Then $m := (I - C_\beta)^{-1}\epsilon = m^\phi$ shows that $\phi = \phi_\epsilon$. The formula (5.1) also shows that the map $F : \epsilon \longmapsto \phi_\epsilon$ is injective and that F is weak*-continuous from $\sum\limits_\beta \subset \mathbb{R}^{E^0}$ to the state space of $C^*(E)$. To show that F is affine, let $\lambda \in (0,1)$ and $\epsilon_1, \epsilon_2 \in \sum\limits_\beta$ and let $\epsilon := \lambda\epsilon_1 + (1-\lambda)\epsilon_2$. Let $m := (I - C_\beta)^{-1}\epsilon$, $m_1 := (I - C_\beta)^{-1}\epsilon_1$, and $m_2 := (I - C_\beta)^{-1}\epsilon_2$. Then $m = \lambda m_1 + (1-\lambda)m_2$ and $\phi_\epsilon = \lambda\phi_{\epsilon_1} + (1-\lambda)\phi_{\epsilon_2}$. ∎

REMARK 5.2. In part (ii) of Theorem 5.1, we could have, instead, let $\{s_e, p_v\}$ be the canonical Cuntz–Krieger E-family that generates $C^*(E)$, $\pi : C^*(E) \to$

$B(\mathcal{H})$ be a faithful nondegenerate representation and defined $P_v := \pi(p_v)$ and $S_e := \pi(s_e)$. For $\mu \in E^*$ we set $\Delta_\mu := \mathrm{e}^{-\beta\omega(\mu)}\epsilon_{r(\mu)}$. Define

$$\phi_\epsilon(a) = \sum_{\mu \in E^*} \Delta_\mu(\pi(a)S_\mu | S_\mu) \quad \text{for } a \in C^*(E).$$

The rest follows from the argument in the proof of Theorem 3.1(b) in [10].

COROLLARY 5.3. *Suppose β is such that $\rho(C_\beta) < 1$. Then there exists a* KMS_β *state if and only if E has a sink. If $|E^0_{\text{sinks}}| \neq 0$, then $\sum_\beta$ is a simplex of dimension* $|E^0_{\text{sinks}}| - 1$.

Proof. Suppose E is a graph with no sinks. Then $\epsilon_v = 0$ for all $v \in E^0$. Thus, $\epsilon \cdot y \neq 1$ and by Theorem 5.1(iii) $\sum_\beta = \emptyset$ and there are no KMS_β states of $(C^*(E), \alpha^\omega)$. Suppose now that E has a sink v and define $\epsilon^v := (\epsilon^v_w) = (\delta_{w,v} y_v^{-1})$. Then $\epsilon^v \cdot y = 1$, so there is a KMS_β state ϕ_{ϵ^v}. We can also note that $\{\phi_{\epsilon^v}\}_{v \in E^0_{\text{sinks}}}$ are the set of all extreme points of the simplex of KMS_β states of $(C^*(E), \alpha^\omega)$. To see this note the set $\sum_\beta$ is a polyhedral set in $\mathbb{R}^{E^0}$ with basic feasible solutions $\{\epsilon^v\}_{v \in E^0_{\text{sinks}}}$ (see Theorem 2.6.4 in [2]). ∎

REMARK 5.4. The graph $E_\mathcal{T}$ is a graph with $|E^0|$ sinks. Hence, by Proposition 2.3, the simplex of KMS_β states of $(\mathcal{T}C^*(E), \alpha^\omega)$ is of dimension $|E^0| - 1$ (compare with Remark 3.2 in [10]).

6. KMS STATES OF GRAPHS WITH A STRONGLY CONNECTED SUBGRAPH

Below we give a complete description of the KMS states of $(C^*(E), \alpha^\omega)$, where E is a finite graph, H is the set of sinks in E^0 and $E \backslash H$ is strongly connected. As a consequence of Theorem 6.1, we extend the results of Theorem 4.3 in [10].

THEOREM 6.1. *Let E be a finite graph with no sources and H be the set of sinks. Let $\omega(e) > 0$ for all $e \in E^1$. Suppose that $E \backslash H := (E^0 \backslash H, E^1 \backslash r^{-1}(H), r, s)$ is strongly connected and let $x = (y \ \ 0)^{\mathrm{tr}}$, where $y = (y_v)_{v \in E^0 \backslash H}$ is the unimodular Perron–Frobenius eigenvector of the matrix $C_{\beta_c, E \backslash H}$.*

(i) *If $\beta > \beta_c$, then the set of all* KMS_β *states of $(C^*(E), \alpha^\omega)$ is a simplex of dimension $|H| - 1$ and is affine-isomorphic to*

$$\sum_\beta := \{\epsilon \in [0, \infty)^{E^0} : \epsilon \cdot y = 1 \text{ and } \epsilon_v = 0 \text{ for } v \in E^0_{\text{reg}}\}.$$

(ii) *The system $(C^*(E), \alpha^\omega)$ has a unique* KMS_{β_c} *state ϕ. This state satisfies*

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} \mathrm{e}^{-\beta\omega(\mu)} x_{r(\mu)} \tag{6.1}$$

and factors through a KMS_{β_c} *state* $\overline{\phi}$ *of* $(C^*(E\backslash H), \alpha^\omega)$.

(iii) *The state* $\overline{\phi}$ *is the only* KMS *state of* $(C^*(E\backslash H), \alpha^\omega)$.

(iv) *If* $\beta < \beta_c$, *then* $(C^*(E), \alpha^\omega)$ *has no* KMS_β *states.*

Proof. (i) Follows from Theorem 5.1 and Corollary 5.3.

(ii) By Corollary 4.6, there exists a KMS_{β_c} state ϕ that satisfies (6.1). Suppose there exists another KMS_{β_c} state $\widetilde{\phi}$. Then, by Theorem 3.3, there exists an $\widetilde{x} \in \mathbb{R}^{E^0}$ that satisfies Property β on E^0, where C_{β_c} is of the form (4.1) with $\widetilde{x} = (\widetilde{y} \;\; z)^{\mathrm{tr}}$ and $\widetilde{y} \in \mathbb{R}^{E^0\backslash H}$. Then we have that $C_{\beta_c, E\backslash H}\widetilde{y} \leqslant C_{\beta_c, E\backslash H}\widetilde{y} + Fz = \widetilde{y}$. Since $\rho(C_{\beta_c}) = \rho(C_{\beta_c, E\backslash H})$, we have $C_{\beta_c, E\backslash H}\widetilde{y} = \widetilde{y}$ (see Theorem 1.6 of [17]). Thus, $Fz = 0$. Since E has no sources, we get that F has no zero columns and thus $z = 0$. Hence, $\widetilde{y}$ is the unimodular Perron–Frobenius eigenvector of the vertex matrix $C_{\beta_c, E\backslash H}$ and thus $\widetilde{\phi} = \phi$.

Suppose $H \neq \emptyset$. Since $E\backslash H$ is strongly connected, it contains a cycle and thus, $|E^0| \geqslant 2$. Let w be the basepoint of a cycle in E. Then, for every regular vertex $v \in E^0_{\mathrm{reg}} = E^0\backslash H$, there is a path from v to w since $E\backslash H$ is strongly connected. This implies that $r(s^{-1}(v)) \nsubseteq H$. Hence, $H = \overline{H}$ and so H is a saturated hereditary subset of E^0. Since $C^*(E)/I_H \cong C^*(E\backslash H)$ and $\phi(p_v) = 0$ for all $v \in H$, ϕ factors through a KMS_{β_c} state $\overline{\phi}$ of $(C^*(E\backslash H), \alpha^\omega)$ (see Lemma 2.2 of [11]).

(iii) Follows immediately from Perron–Frobenius theory and Theorem 3.3.

(iv) Suppose ϕ is a KMS_β state of $(C^*(E), \alpha^\omega)$. Then, by Theorem 3.3, there exists a $y \in \mathbb{R}^{E^0\backslash H}$ such that $C_{\beta, E\backslash H}y \leqslant C_{\beta, E\backslash H}y + Fy = y$. Since $y \geqslant 0$, we have that $\rho(C_{\beta_c}) = 1 \geqslant \rho(C_\beta)$ by Theorem 1.6 of [17]. Hence, $\beta \geqslant \beta_c$. ∎

REMARK 6.2. Suppose E is a strongly connected graph. If E consists of a single cycle, then there is a unique KMS_0 state by Proposition 4.3 and Theorem 6.1 above. Otherwise, $(C^*(E), \alpha^\omega)$ has no KMS_0 states.

REMARK 6.3. If E is strongly connected, then $E_{\mathcal{T}}$ is a graph with no sources and $E_{\mathcal{T}}\backslash H = E$. Thus, Theorem 6.1 holds for $(\mathcal{T}C^*(E), \alpha^\omega)$ when $\omega(e) > 0$ for all $e \in E^1$, by Proposition 2.3. In particular, Theorem 4.3 of [10] follows immediately as a consequence of Theorem 6.1, above.

7. KMS STATES ON THE DUAL-GRAPH ALGEBRA

In this section, we study the KMS states on the dual-graph algebra $C^*(\widehat{E})$ and find a correspondence to the KMS states on the graph algebra $C^*(E)$. Given a KMS state of one of the algebras, we are able to construct the corresponding KMS state of the other.

DEFINITION 7.1. Let $E = (E^0, E^1, r, s)$ be a row-finite graph, H the set of sinks in E, and $\alpha^\omega : \mathbb{R} \curvearrowright C^*(E)$ be a quasi-free action, where ω is a labeling map on the edges E^1. Let $\beta \in \mathbb{R}$, $D_\beta = \mathrm{diag}(\mathrm{e}^{-\beta\omega(e)})_{e\in E^1}$ and $B \in M_{E^1}(\mathbb{N})$ the edge matrix of E. We say that a vector $y := (y_e)_{e\in E^1}$ satisfies Property P_β on E^1 if y is a probability measure on E^1 and $(D_\beta B y)_e = y_e$ whenever $e \in E^1\backslash r^{-1}(H)$.

REMARK 7.2. We note that if a vector y satisfies Property P_β on E^1, then y satisfies the subinvariance relation $D_\beta B y \leqslant y$.

PROPOSITION 7.3. *Let $E = (E^0, E^1, r, s)$ be a row-finite graph. Let $\omega : E^1 \to \mathbb{R}$ be a labeling map on E^1 that is bounded below. Define a labeling $\widehat{\omega} : E^2 \to \mathbb{R}$ on the edges of the dual graph by $\widehat{\omega}(ef) = \omega(e)$ for all $ef \in E^2$ and let η be the corresponding quasi-free action on $C^*(\widehat{E})$. Suppose that $\widehat{\omega}(\widehat{\mu}) \neq 0$ for all $\widehat{\mu} \in \widehat{E}^*\backslash E^1$. For each $\beta \geqslant 0$, let $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$ be the set of all* KMS$_\beta$ *states of $(C^*(\widehat{E}), \eta^{\widehat{\omega}})$ and let $L_{\beta,\eta^{\widehat{\omega}}} := \{y = (y_e)_{e\in E^1} :$ y satisfies Property P_β on $E^1\}$. Then $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$ is affine-isomorphic to $L_{\beta,\eta^{\widehat{\omega}}}$.*

Proof. We have that $C_{\beta,\widehat{E},\widehat{\omega}}$ is a matrix in $M_{E^1}(\mathbb{R})$ and $C_{\beta,\widehat{E},\widehat{\omega}} = D_\beta B$, where $B \in M_{E^1}(\mathbb{N})$ is the edge matrix of E and $D_\beta = \mathrm{diag}(\mathrm{e}^{-\beta\omega(e)})_{e\in E^1}$. The rest follows from Theorem 3.3. ∎

REMARK 7.4. If we instead define a labeling map $\widehat{\omega} : E^2 \to \mathbb{R}$ on the edges of the dual graph by $\widehat{\omega}(ef) = \omega(f)$ for all $ef \in E^2$, we get $C_{\beta,\widehat{E},\widehat{\omega}} = BD_\beta$. We could apply Theorem 3.3 and obtain a similar affine-isomorphism as in Proposition 7.3. However, we will need the labeling map defined in Proposition 7.3 to find a correspondence between the KMS states on the graph algebra and dual-graph algebra.

LEMMA 7.5. *Suppose E is a row-finite graph with no sinks and $\omega : E^1 \to \mathbb{R}$ a labeling of the edges of E. Define a labeling $\widehat{\omega} : E^2 \to \mathbb{R}$ on the edges of the dual graph by $\widehat{\omega}(ef) = \omega(e)$ for all $ef \in E^2$ and let $\eta^{\widehat{\omega}}$ be the corresponding quasi-free action on $C^*(\widehat{E})$. Then $(C^*(\widehat{E}), \eta^{\widehat{\omega}}, \mathbb{R})$ is covariantly isomorphic to $(C^*(E), \alpha^\omega, \mathbb{R})$.*

Proof. Let $\{q_e, r_{ef}\}$ be the universal Cuntz–Krieger family for $\widehat{E}$ and $\{p_v, s_e\}$ the universal Cuntz–Krieger family for E. By Corollary 2.6 in [16], there is an isomorphism $\Phi : C^*(\widehat{E}) \to C^*(E)$ with $\Phi(q_e) = s_e s_e^*$ and $\Phi(r_{ef}) = s_e s_f s_f^*$. Then

$$(\Phi \circ \eta_t)(q_e) = \Phi(q_e) = s_e s_e^* = (\alpha_t \circ \Phi)(q_e)$$

and

$$(\Phi \circ \eta_t)(r_{ef}) = \mathrm{e}^{\widehat{\omega}(ef)\mathrm{i}t}\Phi(r_{ef}) = \mathrm{e}^{\omega(e)\mathrm{i}t} s_e s_f s_f^* = (\alpha_t \circ \Phi)(r_{ef}). \quad ∎$$

THEOREM 7.6. *Suppose E is a row-finite graph without sinks and $\beta \geqslant 0$. Let $\omega : E^1 \to \mathbb{R}$ be bounded below and $\omega(\mu) \neq 0$ for all $\mu \in E^*\backslash E^0$. Then $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$ is affine-isomorphic to K_{β,α^ω}. Furthermore, if $x = (x_v)_{v\in E^0}$ is the corresponding vector for the* KMS *state ϕ_x in K_{β,α^ω}, then $y := (y_e)_{e\in E^1}$, where $y_e = \phi_x(s_e s_e^*)$ is the vector with corresponding* KMS *state $\widehat{\phi}_y$ in $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$. Conversely, if $y = (y_e)_{e\in E^1}$ is the vector with* *corresponding* KMS *state* $\widehat{\phi}_y$ *in* $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$, *then* $x = (x_v)_{v\in E^0}$, *where* $x_v = \sum_{e\in vE^1} y_e$ *is the corresponding vector for the* KMS *state* ϕ_x *in* K_{β,α^ω}.

Proof. By Lemma 7.5, the first part of the statement holds. For the second part, let $\widehat{\phi}_y$ be a KMS$_\beta$ state corresponding to $y \in L_{\beta,\eta^{\widehat{\omega}}}$. Let $\Phi : \widehat{K}_{\beta,\eta^{\widehat{\omega}}} \to K_{\beta,\alpha^\omega}$; $\widehat{\phi}_y \mapsto \phi_x$ by

$$\phi_x(s_\mu s_\nu^*) = \mathrm{e}^{-\beta\omega(\mu)} x_{r(\mu)},$$

where $x_v = \sum_{e\in vE^1} y_e$. We have that $x_v \geqslant 0$ for every $v \in E^0$ and $\sum_{v\in E^0} x_v = 1$, since E is a row-finite graph with no sinks. For each $v \in E^0$,

$$\begin{aligned}
(C_\beta x)_v - x_v &= \sum_{w\in E^0} C_\beta(v,w)x_w - x_v = \sum_{w\in E^0}\Big(\sum_{e\in vE^1 w} \mathrm{e}^{-\beta\omega(e)} x_w\Big) - x_v \\
&= \sum_{e\in vE^1} \mathrm{e}^{-\beta\omega(e)} x_{r(e)} - x_v = \sum_{e\in vE^1} \mathrm{e}^{-\beta\omega(e)}\Big(\sum_{s(f)=r(e)} y_f\Big) - x_v \\
&= \sum_{e\in vE^1}\Big(\sum_{f\in E^1} D_\beta B(e,f) y_f\Big) - x_v = \sum_{e\in vE^1} y_e - x_v = 0.
\end{aligned}$$

So, by Theorem 3.3, we have that ϕ_x is a KMS$_\beta$ state and thus, Φ is a well-defined map.

To show Φ is injective, suppose $\phi_x = \phi_{\widetilde{x}}$. Then $\phi_x(s_e s_e^*) = \mathrm{e}^{-\beta\omega(e)} x_{r(e)} = y_e$ and similarly, $\phi_{\widetilde{x}}(s_e s_e^*) = \widetilde{y}_e$. Since $\omega(\mu) \neq 0$ for all $\mu \in E^*\backslash E^0$, we have that $\widehat{\omega}(\widehat{\mu}) \neq 0$ for all $\widehat{\mu} \in \widehat{E}^*\backslash E^1$. By Proposition 7.3, $\widehat{\phi}_y = \widehat{\phi}_{\widetilde{y}}$ since $y_e = \widetilde{y}_e$ for all $e \in E^1$ and y is a probability measure on E^1.

To prove surjectivity, suppose ϕ is a KMS$_\beta$ of $(C^*(E), \alpha^\omega)$. Let $y_e := \phi(s_e s_e^*)$ and $y := (y_e)_{e\in E^1}$. Then y satisfies Property P_β on E^1. Hence, by Proposition 7.3, there exists KMS$_\beta$ state $\widehat{\phi}_y$ of $(C^*(\widehat{E}), \eta^{\widehat{\omega}})$. Since $\phi(p_v) = \sum_{s(e)=v} \phi(s_e s_e^*) = \sum_{s(e)=v} y_e$, we have that $\Phi(\widehat{\phi}_y) = \phi_x = \phi$. We have that Φ is an affine and weak*-continuous map from $\widehat{K}_{\beta,\eta^{\widehat{\omega}}}$ to K_{β,α^ω}. Therefore, Φ is the affine-isomorphism and hence, the correspondence follows as desired. ∎

8. GROUND STATES AND KMS$_\infty$ STATES

PROPOSITION 8.1. *Let E be a finite directed graph and let $\alpha^\omega : \mathbb{R} \curvearrowright C^*(E)$ be the quasi-free action corresponding to a labeling map ω, where $\omega(e) > 0$ for all $e \in E^1$. Suppose that ϵ is a probability measure on E^0. Then there is a* KMS$_\infty$ *state ϕ_ϵ satisfying*

$$\phi_\epsilon(s_\mu s_\nu^*) = \begin{cases} 0 & \textit{unless } |\mu| = |\nu| = 0 \textit{ and } \mu = \nu, \\ \epsilon_v & \textit{if } \mu = \nu = v \in E^0. \end{cases}$$

Every ground state of $(C^*(E), \alpha^\omega)$ is a KMS$_\infty$ *state and the map $\epsilon \longmapsto \phi_\epsilon$ is an affine isomorphism of the simplex of probability measures of E^0 onto the set of ground states of $(C^*(E), \alpha^\omega)$.*

Proof. Choose a sequence $\beta_j \to \infty$ as $j \to \infty$ with $\beta_j > \beta_c$. For each j, define (y_v^j) as in Theorem 5.1(i) by $y_v^j = \sum_{\mu \in E^* v} e^{-\beta_j \omega(\mu)}$. Set $\epsilon_v^j := \epsilon_v (y_v^j)^{-1}$, and let ϕ_j be the KMS$_{\beta_j}$ state ϕ_{ϵ^j} of $(C^*(E), \alpha^\omega)$, described in Theorem 5.1(ii). We note that if we choose an $M > 0$ so that $\beta_j > M > \beta_c$ for all j sufficiently large, we can apply the dominated convergence theorem and prove that $y_v^j \to 1$ as $j \to \infty$. The rest follows from the argument in Theorem 5.1 of [10]. ∎

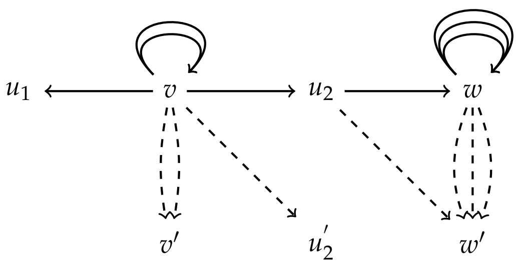
9. EXAMPLES

In the examples below, we calculate the KMS states of $(C^*(E), \alpha^\omega)$ and $(\mathcal{T}C^*(E), \alpha^\omega)$. By Proposition 2.3, finding the KMS states of $(\mathcal{T}C^*(E), \alpha^\omega)$ reduces to finding the KMS states of $(C^*(E_\mathcal{T}), \alpha^\omega)$.

By Theorem 3.3, we can compute the KMS states by finding all elements in the polyhedral set L_{β,α^ω} (see (3.4) in Remark 3.4). Calculating the extreme points in L_{β,α^ω} coincides with finding all basic feasible solutions (see Theorem 2.6.4 of [2]).

Below, the solid lines represent all of the edges in the graph E and the solid lines along with the dashed lines represent all of the edges of the graph $E_\mathcal{T}$.

EXAMPLE 9.1. In this example, we calculate the KMS state for the gauge action γ of $\mathbb{R}$, where the graph E has one sink and $E_\mathcal{T}$ has many more sinks. This example was computed in Example 6.4 of [11] using strongly connected components, but we calculate it here using Theorem 3.3.



(a) KMS$_\beta$ states of $(C^*(E), \gamma)$:

$$\widetilde{R}_\beta = \begin{pmatrix} e^\beta - 2 & -1 & 0 & -1 \\ 0 & e^\beta & -1 & 0 \\ 0 & 0 & e^\beta - 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(i) For $\beta > \ln 3$, we have a unique KMS$_\beta$ state that corresponds to the vector $m_1^\beta = \begin{pmatrix} \frac{1}{e^\beta-1} & 0 & 0 & \frac{e^\beta-2}{e^\beta-1} \end{pmatrix}^{\mathrm{tr}}$.

(ii) For $\beta = \ln 3$, we have a 2-dimensional simplex of KMS$_\beta$ states with extreme points that correspond to the vectors

$$m_1^\beta = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 \end{pmatrix}^{\mathrm{tr}} \quad \text{and} \quad m_2^\beta = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}^{\mathrm{tr}}.$$

(iii) For $\ln 2 < \beta < \ln 3$, we have a unique KMS$_\beta$ state that corresponds to the vector

$$m_1^\beta = \begin{pmatrix} \frac{1}{e^\beta-1} & 0 & 0 & \frac{e^\beta-2}{e^\beta-1} \end{pmatrix}^{\mathrm{tr}}.$$

(iv) For $\beta = \ln 2$, we have a unique KMS$_\beta$ state corresponding to the vector $m_1^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{tr}}$.

(v) For $\beta < \ln 2$, no solution exists and therefore, there are no KMS$_\beta$ states.

(b) KMS$_\beta$ states of $(C^*(E_{\mathcal{T}}), \gamma)$:

$$\widetilde{R}_\beta = \begin{pmatrix} e^\beta - 2 & -1 & 0 & -1 & -2 & -1 & 0 \\ 0 & e^\beta & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & e^\beta - 3 & 0 & 0 & 0 & -3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

(i) For $\beta > \ln 3$, we have a 3-dimensional simplex of KMS$_\beta$ states with extreme points that correspond to the vectors $m_1^\beta = \begin{pmatrix} \frac{1}{e^\beta-1} & 0 & 0 & \frac{e^\beta-2}{e^\beta-1} & 0 & 0 & 0 \end{pmatrix}^{\mathrm{tr}}$, $m_2^\beta = \begin{pmatrix} \frac{2}{e^\beta} & 0 & 0 & 0 & \frac{e^\beta-2}{e^\beta} & 0 & 0 \end{pmatrix}^{\mathrm{tr}}$, $m_3^\beta = \begin{pmatrix} \frac{1}{e^\beta-1} & 0 & 0 & 0 & 0 & \frac{e^\beta-2}{e^\beta-1} & 0 \end{pmatrix}^{\mathrm{tr}}$, and $m_4^\beta = \begin{pmatrix} \frac{1}{e^{2\beta}-e^\beta-1} & \frac{e^\beta-2}{e^{2\beta}-e^\beta-1} & \frac{3e^\beta-6}{e^{2\beta}-e^\beta-1} & 0 & 0 & 0 & \frac{e^{2\beta}-5e^\beta+6}{e^{2\beta}-e^\beta-1} \end{pmatrix}^{\mathrm{tr}}$.

(ii) For $\beta = \ln 3$, we have 3-dimensional simplex of KMS$_\beta$ states with extreme points that correspond to the vectors $m_1^\beta = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}^{\mathrm{tr}}$, $m_2^\beta = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \end{pmatrix}^{\mathrm{tr}}$, $m_3^\beta = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}^{\mathrm{tr}}$ and $m_4^\beta = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 \cdots 0 \end{pmatrix}^{\mathrm{tr}}$.

(iii) For $\ln 2 < \beta < \ln 3$, we have a 2-dimensional simplex of KMS$_\beta$ states with extreme points that correspond to the vectors

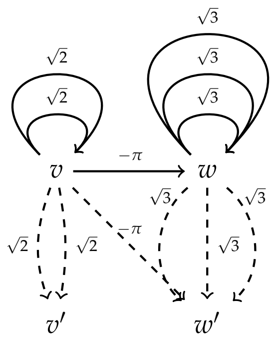
$$m_1^\beta = \begin{pmatrix} \frac{1}{e^\beta-1} & 0 & 0 & \frac{e^\beta-2}{e^\beta-1} & 0 & 0 & 0 \end{pmatrix}^{\mathrm{tr}}, \quad m_2^\beta = \begin{pmatrix} \frac{2}{e^\beta} & 0 & 0 & 0 & \frac{e^\beta-2}{e^\beta} & 0 & 0 \end{pmatrix}^{\mathrm{tr}},$$

$$\text{and} \quad m_3^\beta = \begin{pmatrix} \frac{1}{e^\beta-1} & 0 & 0 & 0 & 0 & \frac{e^\beta-2}{e^\beta-1} & 0 \end{pmatrix}^{\mathrm{tr}}.$$

(iv) For $\beta = \ln 2$, we have a unique KMS$_\beta$ state corresponding to the vector $m_1^\beta = \begin{pmatrix} 1 & 0 \cdots 0 \end{pmatrix}^{\mathrm{tr}}$.

(v) For $\beta < \ln 2$, no solution exists and therefore there are no KMS$_\beta$ states.

EXAMPLE 9.2. In this example, we find the KMS states for a quasi-free action that is not the gauge action, with labels that are both positive and negative.



(a) KMS_β states of $(C^*(E), \alpha^\omega)$:

$$R_\beta = \begin{pmatrix} 1-2\mathrm{e}^{-\beta\sqrt{2}} & -\mathrm{e}^{\beta\pi} \\ 0 & 1-3\mathrm{e}^{-\beta\sqrt{3}} \\ 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(i) For $\beta > \frac{\ln 3}{\sqrt{3}}$, no solution exists and therefore, there are no KMS_β states.

(ii) For $\beta = \frac{\ln 3}{\sqrt{3}}$, we have a unique KMS_β state corresponding to the vector $m_1^\beta = \begin{pmatrix} \frac{1}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} & \frac{\xi^\pi(1-2\xi^{\sqrt{2}})}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} \end{pmatrix}^{\mathrm{tr}}$, where $\xi = (\frac{1}{3})^{1/\sqrt{3}}$.

(iii) For $\frac{\ln 2}{\sqrt{2}} < \beta < \frac{\ln 3}{\sqrt{3}}$, no solution exists and therefore, there are no KMS_β states.

(iv) For $\beta = \frac{\ln 2}{\sqrt{2}}$, we have a unique KMS_β state corresponding to the vector $m_1^\beta = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{tr}}$.

(v) For $\beta < \frac{\ln 2}{\sqrt{2}}$, no solution exists and therefore, there are no KMS_β states.

(b) KMS_β states of $(C^*(E_{\mathcal{T}}), \alpha^\omega)$:

$$R_\beta = \begin{pmatrix} 1-2\mathrm{e}^{-\beta\sqrt{2}} & -\mathrm{e}^{\beta\pi} & -2\mathrm{e}^{-\beta\sqrt{2}} & -\mathrm{e}^{\beta\pi} \\ 0 & 1-3\mathrm{e}^{-\beta\sqrt{3}} & 0 & -3\mathrm{e}^{-\beta\sqrt{3}} \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(i) For $\beta > \frac{\ln 3}{\sqrt{3}}$, we have a 1-dimensional simplex of KMS_β states with extreme points that correspond to the vectors $m_1^\beta = \begin{pmatrix} 2\xi^{\sqrt{2}} & 0 & 1-2\xi^{\sqrt{2}} & 0 \end{pmatrix}^{\mathrm{tr}}$ and $m_2^\beta = \begin{pmatrix} \frac{1}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} & \frac{3\xi^{\sqrt{3}+\pi}(1-2\xi^{\sqrt{2}})}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} & 0 & \frac{\xi^\pi(1-2\xi^{\sqrt{2}})(1-3\xi^{\sqrt{3}})}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} \end{pmatrix}^{\mathrm{tr}}$, where $\xi = \mathrm{e}^{-\beta}$.

(ii) For $\beta = \frac{\ln 3}{\sqrt{3}}$, we have a 1-dimensional simplex of KMS_β states with extreme points that correspond to the vectors $m_1^\beta = \begin{pmatrix} \frac{1}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} & \frac{\xi^\pi(1-2\xi^{\sqrt{2}})}{1-2\xi^{\sqrt{2}+\pi}+\xi^\pi} & 0 & 0 \end{pmatrix}^{\mathrm{tr}}$ and $m_2^\beta = \begin{pmatrix} 2\xi^{\sqrt{2}} & 0 & 1-2\xi^{\sqrt{2}} & 0 \end{pmatrix}^{\mathrm{tr}}_{,}$ where $\xi = (\frac{1}{3})^{1/\sqrt{3}}$.

(iii) For $\frac{\ln 2}{\sqrt{2}} < \beta < \frac{\ln 3}{\sqrt{3}}$, we have a unique KMS_β state that corresponds to the vector $m_1^\beta = \begin{pmatrix} 2\xi^{\sqrt{2}} & 0 & 1-2\xi^{\sqrt{2}} & 0 \end{pmatrix}^{\mathrm{tr}}$, where $\xi = \mathrm{e}^{-\beta}$.

(iv) For $\beta = \frac{\ln 2}{\sqrt{2}}$, we have a unique KMS_β state corresponding to the vector $m_1^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{tr}}$.

(v) For $\beta < \frac{\ln 2}{\sqrt{2}}$, no solution exists and therefore, there are no KMS_β states.

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