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CONTINUOUS FAMILIES OF PROPERLY INFINITE C^*-ALGEBRAS

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ABSTRACT. Any unital separable continuous $C(X)$-algebra with properly infinite fibres is properly infinite as soon as the compact Hausdorff space X has finite topological dimension. We study conditions under which this is still the case if the compact space X has infinite topological dimension.

KEYWORDS: *C^*-algebra, classification, proper infiniteness.*

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1. INTRODUCTION

One of the basic C^*-algebras studied in the classification programme launched by G. Elliott [17] of nuclear C^*-algebras through K-theoretical invariants is the Cuntz C^*-algebra $\mathcal{O}_\infty$ generated by infinitely many isometries with pairwise orthogonal ranges [11]. This C^*-algebra is pretty rigid in so far as it is a strongly self-absorbing C^*-algebra [29]: Any separable unital continuous $C(X)$-algebra A the fibres of which are isomorphic to the same strongly self-absorbing C^*-algebra D is a trivial $C(X)$-algebra provided the compact Hausdorff base space X has finite topological dimension. Indeed, the strongly self-absorbing C^*-algebra D tensorially absorbs the Jiang–Su algebra $\mathcal{Z}$ [30]. Hence, this C^*-algebra D is K_1-injective [28] and the $C(X)$-algebra A satisfies $A \cong D \otimes C(X)$ [14].

But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous C^*-bundle over the infinite dimensional compact product $\prod\limits_{n=0}^{\infty} S^2$ such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type 2^∞ ([18], Example 4.7). More recently, M. Dădârlat has constructed in Section 3 of [13] for all pair (Γ_0, Γ_1) of countable abelian torsion groups a unital separable continuous $C(X)$-algebra A such that:

– the base space X is the compact Hilbert cube $X = B_\infty$ of infinite dimension,

– all the fibres A_x ($x \in B_\infty$) are isomorphic to the strongly self-absorbing Cuntz C^*-algebra $\mathcal{O}_2$ generated by two isometries s_1, s_2 satisfying $1_{\mathcal{O}_2} = s_1 s_1^* + s_2 s_2^*$,

– $K_i(A) \cong C(Y_0, \Gamma_i)$ for $i=0,1$, where $Y_0 \subset [0,1]$ is the canonical Cantor set.

These K-theoretical conditions imply that the $C(B_\infty)$-algebra A is not a trivial one. But these arguments do not work anymore when the strongly self-absorbing algebra D is the Cuntz algebra $\mathcal{O}_\infty$ [11], in so far as $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ is a torsion free group.

We study in this paper a more modest question: Assume that X is a second countable compact Hausdorff space, A is a separable unital continuous $C(X)$ whose fibres all unitally contains $\mathcal{O}_\infty$. Is there a unital embedding of C^*-algebra $\mathcal{O}_\infty \hookrightarrow A$? After fixing our notations in Section 2, we show in Section 3 that all separable unital continuous $C(X)$-algebras with properly infinite fibres are properly infinite C^*-algebras if and only if the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K_1-injective (Proposition 3.3). We describe the link between the different notions of proper infinite $C(X)$-algebras which appeared during the recent years [9], [10], [21], [24] in the next section. We eventually give in Section 5 conditions under which the Pimsner–Toeplitz algebra [23] of a Hilbert $C(X)$-modules with fibres of dimension greater than 2 is a properly infinite C^*-algebra.

2. A FEW NOTATIONS

We present in this section the main notations which are used in this article. We denote by $\mathbb{N} = \{0,1,2,\ldots\}$ the set of positive integers and we denote by $[S]$ the closed linear span of a subset S in a Banach space.

DEFINITION 2.1 ([3], [15], [20]). Let X be a compact Hausdorff space and let $C(X)$ be the C^*-algebra of continuous functions on X.

(i) A *unital* $C(X)$*-algebra* is a unital C^*-algebra A endowed with a unital morphism of C^*-algebra from $C(X)$ to the centre of A.

(ii) For all closed subset $F \subset X$ and all element $a \in A$, one denotes by $a_{|F}$ the *image* of a in the quotient $A_{|F} := A/C_0(X \setminus F) \cdot A$. If x is a point in X, one defines the *fibre* at x to be the quotient $A_x := A_{|\{x\}}$ and one writes a_x for $a_{|\{x\}}$.

(iii) The $C(X)$-algebra A is said to be *continuous* if the upper semicontinuous map $x \in X \mapsto \|a_x\| \in \mathbb{R}_+$ is continuous for all $a \in A$.

REMARKS 2.2. (i) ([9], [12]) For all integers $n \geqslant 2$, the C^*-algebra $\mathcal{T}_n := \mathcal{T}(\mathbb{C}^n)$ is the universal unital C^*-algebra generated by n isometries $s_1, \ldots, s_n$ satisfying the relation

$$s_1 s_1^* + \cdots + s_n s_n^* \leqslant 1 . \tag{2.1}$$

(ii) A unital C^*-algebra A is said to be *properly infinite* if and only if one of the following equivalent conditions holds true ([11], and Proposition 2.1 of [27]):

(a) the C^*-algebra A contains two isometries with mutually orthogonal range projections, i.e. A unitally contains a copy of $\mathcal{T}_2$;

(b) the C^*-algebra A contains a unital copy of the simple Cuntz C^*-algebra $\mathcal{O}_\infty$ generated by infinitely many isometries with pairwise orthogonal ranges.

(iii) If A is a C^*-algebra and E is a right Hilbert A-module, one denotes by $\mathcal{L}(E)$ the set of all adjointable A-linear operators acting on E and by $\mathcal{K}(E) \subset \mathcal{L}(\mathcal{E})$ the closed two sided ideal of compact operators generated by the rank 1 operators $\zeta \mapsto \theta_{\xi_1,\xi_2}\zeta := \xi_1 \cdot \langle \xi_2, \zeta \rangle$ [20]. The C^*-algebra $\mathcal{K}(\ell^2(\mathbb{N}))$ of compact operators on the Hilbert space $\ell^2(\mathbb{C})$ is often written $\mathcal{K}$.

(iv) If the C^*-algebra A is abelian, then $A \to \mathcal{L}(E)$ and $a\zeta = \zeta \cdot a$ for $a \in A$ and $\zeta \in E$.

3. GLOBAL PROPER INFINITENESS

The semiprojectivity of the C^*-algebra $\mathcal{T}_2$ ([2], Theorem 3.2) entails the following property of stable proper infiniteness for unital continuous $C(X)$-algebras with properly infinite fibres.

PROPOSITION 3.1. *Let X be a second countable perfect compact Hausdorff space, i.e. without any isolated point, and let A be a separable unital continuous $C(X)$-algebra with properly infinite fibres.*

(i) *There exist:*

(a) *a finite integer $n \geqslant 1$,*

(b) *a covering $X = \overset{o}{F_1} \cup \cdots \cup \overset{o}{F_n}$ by the interiors of closed balls $F_1, \ldots, F_n$,*

(c) *unital embeddings of C^*-algebra $\sigma_k : \mathcal{O}_\infty \hookrightarrow A_{|F_k}$ $(1 \leqslant k \leqslant n)$.*

(ii) *The tensor product $M_p(\mathbb{C}) \otimes A$ is properly infinite for all large enough integer p.*

REMARK 3.2. If X is a second countable compact Hausdorff space and A is a separable unital continuous $C(X)$-algebra, then the compact Hausdorff space $\widetilde{X} := X \times [0,1]$ is perfect, $\widetilde{A} := A \otimes C([0,1])$ is a unital continuous $C(\widetilde{X})$-algebra and every morphism of unital C^*-algebra $\mathcal{O}_\infty \to \widetilde{A}$ induces a unital $*$-homomorphism $\mathcal{O}_\infty \to A$ by composition with the projection map $\widetilde{A} \to A$ coming from the injection $x \in X \mapsto (x, 0) \in \widetilde{X}$.

Proof of Proposition 3.1. (i) For all points $x \in X$, the semiprojectivity of the unital C^*-subalgebra $\mathcal{O}_\infty \hookrightarrow A_x$ ([2], Theorem 3.2) entails that there are a closed neighbourhood $F \subset X$ of the point x and a unital embedding of the simple C^*-algebra $\mathcal{O}_\infty$ in $A_{|F}$, whence a morphism $\mathcal{O}_\infty \otimes C(F) \hookrightarrow A_{|F}$ of unital $C(F)$-algebra. The compactness of the topological space X implies the result.

(ii) Proposition 2.7 of [9] entails that the C^*-algebra $M_{2^{n-1}}(A)$ is properly infinite. Proposition 2.1 of [26] implies that the C^*-algebra $M_p(A)$ is properly infinite for all integer $p \geqslant 2^{n-1}$. ∎

The proper infiniteness of the tensor product $M_p(\mathbb{C}) \otimes A$ does not always imply that the C^*-algebra is properly infinite [19]. Indeed, there exists a unital C^*-algebra A which is not properly infinite, but such that the tensor product $M_2(\mathbb{C}) \otimes A$ is a properly infinite C^*-algebra ([27], Proposition 4.5). Thus, assertion (ii) of Proposition 3.1 does not imply that any unital continuous $C(X)$-algebra with properly infinite fibres is properly infinite. We were only able to prove the following.

PROPOSITION 3.3. *Let $\jmath_0, \jmath_1$ denote the two canonical unital embeddings of the Cuntz extension $\mathcal{T}_2$ in the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ and let $\widetilde{u} \in \mathcal{U}(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ be a K_1-trivial unitary such that $\jmath_1(s_1) = \jmath_1(s_1)\jmath_0(s_1)^*\jmath_0(s_1) = \widetilde{u} \cdot \jmath_0(s_1)$ ([9], Lemma 2.4).*

The following assertions are equivalent:

(i) *The full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ is K_1-injective.*

(ii) *The unitary $\widetilde{u}$ belongs to the connected component $\mathcal{U}^0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$ of $1_{\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2}$.*

(iii) *Every separable unital continuous $C(X)$-algebra A with properly infinite fibres is a properly infinite C^*-algebra.*

Proof. (i)⇒(ii) A unital C^*-algebra A is called K_1-injective if and only if all K_1-trivial unitaries $v \in \mathcal{U}(A)$ are homotopic to the unit 1_A in $\mathcal{U}(A)$ (see e.g. [25]). Thus, (ii) is a special case of (i) since $K_1(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) = \{1\}$ (see e.g. Lemma 4.4 in [5]).

(ii)⇒(iii) Let A be a separable unital continuous $C(X)$-algebra with properly infinite fibres. Take a finite covering $X = \overset{o}{F_1} \cup \cdots \cup \overset{o}{F_n}$ such that there exist unital embeddings $\sigma_k : \mathcal{T}_2 \to A_{|F_k}$ for all $1 \leqslant k \leqslant n$. Set $G_k := F_1 \cup \cdots \cup F_k \subset X$ and let us construct by induction isometries $w_k \in A_{|G_k}$ such that the two projections $w_k w_k^*$ and $1_{|G_k} - w_k w_k^*$ are properly infinite and full in the restriction $A_{|G_k}$:

– If $k = 1$, the isometry $w_1 := \sigma_1(s_1)$ has the requested properties.

– If $k \in \{1, \ldots, n-1\}$ and the isometry $w_k \in A_{|G_k}$ is already constructed, then Lemma 2.4 of [9] implies that there exists a morphism of unital C^*-algebra $\pi_k : \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \to A_{|G_k \cap F_{k+1}}$ satisfying

$$(3.1)\quad \pi_k(\jmath_0(s_1)) = w_{k|G_k\cap F_{k+1}}, \quad \pi_k(\jmath_1(s_1)) = \sigma_{k+1}(s_1)_{|G_k\cap F_{k+1}} = \pi_k(\widetilde{u}) \cdot w_{k|G_k\cap F_{k+1}}.$$

If the unitary $\widetilde{u}$ belongs to the connected component $\mathcal{U}^0(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2)$, then $\pi_k(\widetilde{u})$ is homotopic to $1_{A_{|G_k\cap F_{k+1}}} = \pi_k(1_{\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2})$ in $\mathcal{U}(A_{|G_k\cap F_{k+1}})$, so that $\pi_k(\widetilde{u})$ admits a unitary lifting z_{k+1} in $\mathcal{U}^0(A_{|F_{k+1}})$ (see e.g. Lemma 2.1.7 in [22]). The unique isometry $w_{k+1} \in A_{|G_{k+1}}$ satisfying the two constraints

$$(3.2)\qquad w_{k+1|G_k} = w_k, \quad w_{k+1|F_{k+1}} = (z_{k+1})^* \cdot \sigma_{k+1}(s_1)$$

verifies that the two projections $w_{k+1}w_{k+1}^*$ and $1_{|G_{k+1}} - w_{k+1}w_{k+1}^*$ are properly infinite and full in $A_{|G_{k+1}}$.

The proper infiniteness of the projection $w_n w_n^*$ in $A_{|G_n} = A$ implies that the unit $1_A = w_n^* w_n = w_n^* \cdot w_n w_n^* \cdot w_n$ is also a properly infinite projection in A, i.e. the C^*-algebra A is properly infinite.

(iii)⇒(i) The C^*-algebra $\mathcal{D} := \{ f \in C([0,1], \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2) : f(0) \in \jmath_0(\mathcal{T}_2)$ and $f(1) \in \jmath_1(\mathcal{T}_2) \}$ is a unital continuous $C([0,1])$-algebra all the fibres of which are properly infinite. Thus, condition (iii) implies that the C^*-algebra $\mathcal{D}$ is properly infinite, a statement which is equivalent to the K_1-injectivity of $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ ([5], Proposition 4.2). ∎

REMARK 3.4. The sum $\widetilde{u} \oplus 1$ belongs to $\mathcal{U}^0(M_2(\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2))$ ([5]).

4. A QUESTION OF PROPER INFINITENESS

We describe in this section the different notions of proper infiniteness which have been introduced during the last decades.

The first one has been introduced by J. Cuntz in [11] where he defines the properly infinite unital C^*-algebras as those which unitally contain a copy of the C^*-algebra $\mathcal{T}_2$ generated by two isometries with orthogonal ranges (see Remark 2.2). E. Kirchberg extended this notion by defining what are the properly infinite positive elements in a C^*-algebra (see e.g. Proposition 3.2 in [21]). More recently, K.T. Coward, G. Elliott and C. Ivanescu defined in [10] a separable Hilbert module E over a separable C^*-algebra A to be properly infinite if there is an embedding of Hilbert A-module $\ell^2(\mathbb{N}) \otimes A \hookrightarrow E$. This is another way of speaking about strictly positive compact elements acting on a Hilbert module E which are properly infinite in $\mathcal{K}(E)$. Indeed, the following holds true.

PROPOSITION 4.1. *Let A be a separable C^*-algebra and let $a \in \mathcal{K} \otimes A$ be a positive compact operator. The following assertions are equivalent:*

(i) *a is properly infinite in $\mathcal{K} \otimes A$, i.e. $a \oplus a \precsim a$ in $\mathcal{K} \otimes A$ (*[21], *Definition* 3.2*).*

(ii) *There is an embedding of Hilbert A-module $\ell^2(\mathbb{N}) \otimes A \hookrightarrow [a \cdot \ell^2(\mathbb{N}) \otimes A]$* [10].

Proof. (i) ⇒ (ii) If $\{d_i\}$ is an infinite sequence in $\mathcal{K} \otimes A$ such that $a = d_i^* d_i \geqslant \sum\limits_{j \in \mathbb{N}} d_j d_j^*$ for all $i \in \mathbb{N}$, then we have an inclusion of Hilbert modules

$$[a \cdot \ell^2(\mathbb{N}) \otimes A] \supset \sum_j [d_j A] \cong \ell^2(\mathbb{N}) \otimes A\,.$$

(ii)⇒(i) One has embeddings of Hilbert A-modules

$$[a \cdot \ell^2(\mathbb{N}) \otimes A] \oplus [a \cdot \ell^2(\mathbb{N}) \otimes A] \subset \ell_2(\mathbb{N}) \otimes A \subset [a \cdot \ell^2(\mathbb{N}) \otimes A]\,. \quad ∎$$

REMARK 4.2. A separable Hilbert A-module E is *properly infinite* if and only if one (hence all) strictly positive operator $a \in \mathcal{K}(E)$ is properly infinite in $\mathcal{K}(E)$.

These different notions of proper infiniteness imply the following result for continuous fields of properly infinite C^*-algebras.

PROPOSITION 4.3. *Let X be a second countable compact Hausdorff space, let A be a separable continuous $C(X)$-algebra with non-zero fibres and let $a \in A_+$ be a strictly positive contraction. Consider the following assertions:*

(i) *All the operators a_x are properly infinite in A_x $(x \in X)$.*

(ii) *The operator a is properly infinite in A.*

(iii) *The multiplier C^*-algebra $\mathcal{M}(A)$ is a unital properly infinite C^*-algebra.*

Then (iii)$\Rightarrow$(ii)$\Rightarrow$(i). *But* (i)$\not\Rightarrow$(ii) *and* (ii)$\not\Rightarrow$(iii).

Proof. (iii)$\Rightarrow$(ii) If $\sigma : \mathcal{T}_2 = C^*\langle s_1, s_2\rangle \to \mathcal{M}(A)$ is a unital $*$-homomorphism and $\delta_{i,j}$ denotes the Kronecker delta, then the two elements $d_1 = \sigma(s_1)\cdot a^{1/2}$ and $d_2 = \sigma(s_2)\cdot a^{1/2}$ satisfy $d_i^* d_j = \delta_{i,j}\cdot a$ in A.

(ii)$\Rightarrow$(i) The relations $c_i^* c_j = \delta_{i=j}\cdot a$ between 2 operators c_1, c_2 in a $C(X)$-algebra A entail that $(c_i)_x^*(c_j)_x = \delta_{i,j}\cdot a_x$ in the quotient $A_x = A/C_0(X\setminus\{x\})\cdot A$ for all $x \in X$.

(i)$\not\Rightarrow$(ii) Let $B_3 = \{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2+x_2^2+x_3^2 \leqslant 1\}$ be the unit ball of dimension 3, let B_3^+, B_3^- be the two open semi-disks $B_3^+ = \{(x_1,x_2,x_3)\in B_3 : x_3 > -\frac{1}{2}\}$, $B_3^- = \{(x_1,x_2,x_3)\in B_3 : x_3 < \frac{1}{2}\}$ and let $S_2 = \{(x_1,x_2,x_3)\in B_3 : x_1^2+x_2^2+x_3^2 = 1\} \subset B_3$ be the unit sphere of dimension 2.

The self-adjoint operator $f \in C(B_3)\otimes M_2(\mathbb{C}) \cong C(B_3, M_2(\mathbb{C}))$ given by

$$f(x_1,x_2,x_3) = \frac{1}{2}\cdot\begin{pmatrix} 1+x_3 & x_1 - \mathrm{i}\,x_2 \\ x_1+\mathrm{i}\,x_2 & 1-x_3\end{pmatrix} \tag{4.1}$$

is a positive contraction since each self-adjoint matrix $f(x_1,x_2,x_3)\in M_2(\mathbb{C})$ satisfies

$$f(x_1,x_2,x_3)^2 = f(x_1,x_2,x_3) + \frac{1}{4}(x_1^2+x_2^2+x_3^2-1)\cdot 1_{M_2(\mathbb{C})}\,, \tag{4.2}$$

i.e. $(f(x_1,x_2,x_3) - \frac{1}{2}\cdot 1_{M_2(\mathbb{C})})^2 = \frac{x_1^2+x_2^2+x_3^2}{4}\cdot 1_{M_2(\mathbb{C})} \leqslant (\frac{1}{2}\cdot 1_{M_2(\mathbb{C})})^2$.

The non trivial Hilbert $C(B_3)$-module $F := \left[f\cdot\begin{pmatrix} C(B_3)\\ C(B_3)\end{pmatrix}\right]$ satisfies the two isomorphisms of Hilbert $C(B_3)$-modules:

$$\begin{aligned} F\cdot C_0(B_3^+) &\cong C_0(B_3^+)\oplus C_0(B_3^+\setminus S_2\cap B_3^+),\\ F\cdot C_0(B_3^-) &\cong C_0(B_3^-)\oplus C_0(B_3^-\setminus S_2\cap B_3^-)\,. \end{aligned} \tag{4.3}$$

The set $B_\infty := \left\{x\in\ell^2(\mathbb{N}); \sum_p |x_p|^2 \leqslant 1\right\}$ is a metric compact space called the *complex Hilbert cube* when equipped with the distance $d((x_p),(y_p)) = \sum_p 2^{-p-2}\,|x_p - y_p|$. Denote by E_{DD} the non-trivial Hilbert $C(B_\infty)$-module with fibres $\ell^2(\mathbb{N})$ constructed by J. Dixmier and A. Douady (Section 17 of [16] and Proposition 3.6 of [7]).

Finally, consider the product $X := B_\infty\times B_3$ and the Hilbert $C(X)$-module

$$H := E_{\mathrm{DD}}\otimes C(B_3)\ \oplus\ C(B_\infty)\otimes F\,. \tag{4.4}$$

The two Hilbert $C(X)$-submodules $H \cdot C_0(B_\infty \times B_3^+)$ and $H \cdot C_0(B_\infty \times B_3^-)$ are properly infinite, i.e. there exist embeddings of Hilbert $C(X)$-modules $\ell^2(\mathbb{N}) \otimes C_0(B_\infty \times B_3^+) \hookrightarrow H \cdot C_0(B_\infty \times B_3^+)$ and $\ell^2(\mathbb{N}) \otimes C_0(B_\infty \times B_3^-) \hookrightarrow H \cdot C_0(B_\infty \times B_3^-)$. Hence, all the fibres of the Hilbert $C(X)$-module H are properly infinite Hilbert spaces, i.e. $\ell^2(\mathbb{N}) \hookrightarrow H_x$ for all points x in the compact space $X = B_\infty \times B_3^+ \cup B_\infty \times B_3^-$ ([10]). But the Hilbert $C(X)$-module H is not properly infinite i.e. $\ell^2(\mathbb{N}) \otimes C(X) \not\hookrightarrow H$ ([24], Example 9.11). The equality $C(B_3) = C_0(B_3^+) + C_0(B_3^-)$ only implies that $\ell^2(\mathbb{N}) \otimes C(X) \hookrightarrow H \oplus H$.

(ii)$\not\Rightarrow$(iii) There exists a continuous field $\widetilde{H}$ of Hilbert spaces over the compact space $Y := B_\infty \times (B_3)^\infty$ such that $\widetilde{H} = [a \cdot \widetilde{H}]$ for some properly infinite contraction $a \in \mathcal{K}(\widetilde{H})$ and the C^*-algebra $\mathcal{L}(\widetilde{H})$ is not properly infinite. Indeed, let $\eta \in \ell^\infty(B_\infty, \ell^2(\mathbb{N}) \oplus \mathbb{C})$ be the section $x \mapsto x \oplus \sqrt{1 - \|x\|^2}$, let $\ddot{F}$ be the closed Hilbert $C(B_\infty)$-module $\ddot{F} := [C(B_\infty, \ell^2(\mathbb{N}) \oplus 0) + C(B_\infty) \cdot \eta]$, let $\theta_{\eta,\eta} \in \mathcal{L}(\ddot{F})$ be the projection $\zeta \mapsto \eta\langle \eta, \zeta\rangle$ and let $E_{\mathrm{DD}} = (1 - \theta_{\eta,\eta}) \cdot \ddot{F}$ be the Hilbert $C(B_\infty)$-submodule built in [16]. Define also the sequence of norm 1 contractions $\widetilde{f} = (\widetilde{f}_n)$ in $\ell^\infty(\mathbb{N}, M_2(C((B_3)^\infty)))$ by

$$x_n = (x_{n,k}) \in (B_3)^\infty \longmapsto \widetilde{f}_n(x_n) := f(x_{n,n}) \in M_2(\mathbb{C})\,. \tag{4.5}$$

The Hilbert $C(Y)$-module

$$\widetilde{H} := C(Y) \,\oplus\, E_{\mathrm{DD}} \otimes C((B_3)^\infty) \,\oplus\, C(B_\infty) \otimes \Big[\widetilde{f} \cdot \ell^2\Big(\mathbb{N}, \begin{pmatrix} C((B_3)^\infty) \\ C((B_3)^\infty) \end{pmatrix}\Big)\Big] \tag{4.6}$$

has the desired properties ([24], Example 9.13). ∎

REMARK 4.4. If the strictly positive contraction $a \in A$ in Proposition 4.3 is a projection, then this projection must be the unit of the C^*-algebra A and so (ii)$\Leftrightarrow$ (iii) in that case.

QUESTION 4.5. Is the full unital free product $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$ a properly infinite C^*-algebra which is not K_1-injective? (see the equivalence (i)$\Leftrightarrow$(iii) in Proposition 3.3).

5. THE PIMSNER–TOEPLITZ ALGEBRA OF A HILBERT $C(X)$-MODULE

We look in this section at the proper infiniteness question for the unital continuous $C(X)$-algebras with fibres $\mathcal{O}_\infty$ corresponding to the Pimsner–Toeplitz $C(X)$-algebras of Hilbert $C(X)$-modules with infinite dimension fibres.

DEFINITION 5.1 ([23]). Let X be a compact Hausdorff space and let E be a full Hilbert $C(X)$-module, i.e. without any zero fibre.

(i) The *full Fock Hilbert $C(X)$-module* $\mathcal{F}(E)$ of E is the direct sum

$$\mathcal{F}(E) := \bigoplus_{m\in\mathbb{N}} E^{(\otimes_{C(X)})\, m}\,, \tag{5.1}$$

where $E^{(\otimes_{C(X)})\, m} := \begin{cases} C(X) & \text{if } m=0\,,\\ E\otimes_{C(X)}\cdots\otimes_{C(X)} E \ \ (m \text{ terms}) & \text{if } m\geqslant 1\,.\end{cases}$

(ii) The *Pimsner–Toeplitz $C(X)$-algebra* $\mathcal{T}(E)$ of E is the unital subalgebra of the $C(X)$-algebra $\mathcal{L}(\mathcal{F}(E)\,)$ of adjointable $C(X)$-linear operators acting on $\mathcal{F}(E)$ generated by the creation operators $\ell(\zeta)$ ($\zeta\in E$), where

$$\begin{aligned} \ell(\zeta)\,(f\cdot\hat{1}_{C(X)}) &:= f\cdot\zeta=\zeta\cdot f && \text{for } f\in C(X) \quad \text{and}\\ \ell(\zeta)\,(\zeta_1\otimes\cdots\otimes\zeta_k) &:= \zeta\otimes\zeta_1\otimes\cdots\otimes\zeta_k && \text{for } \zeta_1,\dots,\zeta_k\in E \quad \text{if } k\geqslant 1\,. \end{aligned} \tag{5.2}$$

(iii) Let $(C^*(\mathbb{Z}),\Delta)$ be the abelian compact quantum group generated by a unitary $\mathbf{u}$ with spectrum the unit circle and with coproduct $\Delta(\mathbf{u})=\mathbf{u}\otimes\mathbf{u}$. Then, there is a unique coaction α of the Hopf C^*-algebra $(C^*(\mathbb{Z}),\Delta)$ on the Pimsner–Toeplitz $C(X)$-algebra $\mathcal{T}(E)$ such that $\alpha(\ell(\zeta))=\ell(\zeta)\otimes\mathbf{u}$ for all $\zeta\in E$, i.e.

$$\begin{aligned} \alpha: \mathcal{T}(E) &\to \mathcal{T}(E)\otimes C^*(\mathbb{Z}) && = C(\mathbb{T},\mathcal{T}(E)),\\ \ell(\zeta) &\mapsto \ell(\zeta)\otimes\mathbf{u} && =(z\mapsto \ell(z\zeta)). \end{aligned} \tag{5.3}$$

The fixed point $C(X)$-subalgebra $\mathcal{T}(E)^\alpha=\{a\in\mathcal{T}(E) : \alpha(a)=a\otimes 1\}$ under this coaction is the closed linear span

$$\mathcal{T}(E)^\alpha=\Big[C(X)\cdot 1_{\mathcal{T}(E)}+\sum_{k\geqslant 1}\ell(E)^k\cdot(\ell(E)^k)^*\Big]. \tag{5.4}$$

Besides, the projection $P\in\mathcal{L}(\,\mathcal{F}(E))$ onto the submodule E induces a quotient morphism of $C(X)$-algebra $a\in\mathcal{T}(E)^\alpha\mapsto\overline{\mathbf{q}}(a):=P\cdot a\cdot P\in\mathcal{K}(E)+C(X)\cdot 1_{\mathcal{L}(E)}\subset\mathcal{L}(E)$.

PROPOSITION 5.2. *Let X be a second countable compact Hausdorff perfect space and let E be a separable Hilbert $C(X)$-module with infinite dimensional fibres.*

(i) *There exist a covering $X=\overset{o}{F_1}\cup\cdots\cup\overset{o}{F_m}$ by the interiors of closed subsets $F_1,\dots,F_m$ and m norm 1 sections $\zeta_1,\dots,\zeta_m$ in E such that*

$$\mathcal{T}(E)=C^*\langle\mathcal{T}(E)^\alpha,\ell(\zeta_1),\dots,\ell(\zeta_m)\rangle,\quad (\ell(\zeta_k)\ell(\zeta_k)^*)_{|F_k},\quad \text{and}\quad (1-\ell(\zeta_k)\ell(\zeta_k)^*)_{|F_k},$$

are properly infinite projections in $\mathcal{T}(E)_{|F_k}$ for all index $k\in\{1,\dots,m\}$.

(ii) *Set $G_k:=F_1\cup\cdots\cup F_k$ for all integer $k\in\{1,\dots,m\}$ and $\overline{G}_l:=G_l\cap F_{l+1}$ for all integer $l\in\{1,\dots,m-1\}$. If $\xi(l)\in E_{|G_l}$ is a section such that $\|\xi(l)_y\|=1$ for all $y\in\overline{G}_l$, then there is a unitary $w_l\in\mathcal{T}(E)_{|\overline{G}_l}$ such that:*

(a) *$w_l\cdot\ell(\xi(l))_{|\overline{G}_l}=\ell(\zeta_{l+1})_{|\overline{G}_l}$,*

(b) *$w_l\oplus 1_{|\overline{G}_l}$ is homotopic to $1_{|\overline{G}_l}\oplus 1_{|\overline{G}_l}$ among the unitaries in $M_2(\mathcal{T}(E)_{|\overline{G}_l})$.*

(iii) *If for all K_1-trivial unitary $w_k \in \mathcal{T}(E)_{|\overline{G}_k}$ there is a unitary $z_{k+1} \in \mathcal{T}(E)^{\alpha}{}_{|F_{k+1}}$ such that $(z_{k+1})_{|\overline{G}_k} = w_k$ $(1 \leqslant k \leqslant m-1)$, then there is a section $\xi \in E$ satisfying*

$$\forall\, x \in X, \quad \|\xi_x\| = 1, \tag{5.5}$$

so that Lemma 6.1 *of* [6] *implies that the C^*-algebra $\mathcal{T}(E)$ is properly infinite.*

Proof. (i) Given a point $x \in X$ and a unit vector $\zeta \in E_x$, let ξ_1, ξ_2, ξ_3 be three norm 1 sections in E such that $(\xi_1)_x = \zeta$ and the matrix $a := [\langle \xi_i, \xi_j \rangle] \in M_3(C(X))$ satisfies $a_x = 1_3 \in M_3(\mathbb{C})$. Let $F \subset X$ be a closed neighbourhood of x such that $\|a_y - 1_3\| \leqslant \frac{1}{2}$ for all $y \in F$. Define the sections ξ'_1, ξ'_2, ξ'_3 in $E_{|F}$ by

$$\xi'_1 \oplus \xi'_2 \oplus \xi'_3 = (\xi_1 \oplus \xi_2 \oplus \xi_3)_{|F} \cdot (a^* a)_{|F}{}^{-1/2}. \tag{5.6}$$

One has $\langle \xi'_1 \oplus \xi'_2 \oplus \xi'_3, \xi'_1 \oplus \xi'_2 \oplus \xi'_3 \rangle = (a^*_{|F}\, a_{|F})^{-1/2} \cdot (a^*_{|F}\, a_{|F}) \cdot (a^*_{|F}\, a_{|F})^{-1/2} = 1$ in $M_3(C(F))$. Hence, $\ell(\xi'_1)\ell(\xi'_1)^*$ and $q := 1_{|F} - \ell(\xi'_1)\ell(\xi'_1)^*$ are properly infinite projections in $\mathcal{T}(E)_{|F}$ since

$$\begin{aligned} 1_{|F} - q = \ell(\xi'_1)\ell(\xi'_1)^* &\geqslant \ell(\xi'_1)\ell(\xi'_2)\ell(\xi'_2)^*\ell(\xi'_1)^* + \ell(\xi'_1)\ell(\xi'_3)\ell(\xi'_3)^*\ell(\xi'_1)^*, \\ q \geqslant \ell(\xi'_2)\ell(\xi'_2)^* + \ell(\xi'_3)\ell(\xi'_3)^* &\geqslant \ell(\xi'_2)\, q^2\, \ell(\xi'_2)^* + \ell(\xi'_3)\, q^2\, \ell(\xi'_3)^*, \end{aligned} \tag{5.7}$$

so that there exist unital $*$-homomorphisms from $\mathcal{T}_2$ to $(1-q) \cdot \mathcal{T}(E)_{|F} \cdot (1-q)$ and $q \cdot \mathcal{T}(E)_{|F} \cdot q$ given by $s_i \mapsto \ell(\xi'_1)\ell(\xi'_{1+i})\,\ell(\xi'_1)^*$ and $s_i \mapsto \ell(\xi'_{1+i})\, q$ for $i = 1, 2$.

The compactness of the space X enables to end the proof of this first assertion.

(ii) Let $v_l \in \mathcal{T}(E)_{|\overline{G}_l}$ be the partial isometry $v_l := \ell(\zeta_{l+1})_{|\overline{G}_l} \cdot \ell(\xi(l))^*{}_{|\overline{G}_l}$. There exists by Lemma 2.4 of [9] a K_1-trivial unitary w_l in the properly infinite unital C^*-algebra $\mathcal{T}(E)_{|\overline{G}_l}$ which has the two requested properties (a) and (b).

(iii) One constructs inductively the restrictions $\xi_{|G_k}$ in $E_{|G_k}$. Set $\xi_{|G_1} := \zeta_1$ and assume $\xi_{|G_k}$ already constructed. As $\ell(\xi_{|G_k})_{|\overline{G}_k} = z^*_{k+1} \cdot \ell(\zeta_{k+1})_{|\overline{G}_k}$, the unique extension $\xi_{|G_{k+1}} \in E_{|G_{k+1}}$ such that $(\xi_{|G_{k+1}})_{|G_k} = \xi_{|G_k}$ and $(\xi_{|G_{k+1}})_{|F_{k+1}} = \overline{\mathbf{q}}(z_{k+1})^* \cdot (\zeta_{k+1})_{|F_{k+1}}$ satisfies $\|(\xi_{|G_{k+1}})_x\| = 1$ for all points $x \in G_{k+1}$. ∎

REMARKS 5.3. (i) The non trivial separable Hilbert $C(B_\infty)$-module E_{DD} constructed by J. Dixmier and A. Douady [16] has infinite dimensional fibres and every section $\zeta \in E_{\mathrm{DD}}$ satisfies $\zeta_x = 0$ for at least one point $x \in B_\infty$. Thus, it cannot satisfy the assumptions for the assertion (iii) of Proposition 5.2. There are some $k \in \{1, \ldots, m-1\}$ and a unitary $a_{k+1} \in \mathcal{U}^0(M_2(\mathcal{T}(E_{\mathrm{DD}})_{|F_{k+1}}))$ such that

$$(a_{k+1})_{|\overline{G}_k} = w_k \oplus 1_{|\overline{G}_k} \tag{5.8}$$

and either $a_{k+1} \notin \mathcal{T}(E_{\mathrm{DD}})_{|F_{k+1}} \oplus C(F_{k+1})$ or $\alpha(a_{k+1}) \neq a_{k+1} \otimes 1$.

(i′) If $\widetilde{H}$ is the Hilbert $C(Y)$-module constructed in equation (4.6), is the Pimsner–Toeplitz algebra $\mathcal{T}(\widetilde{H})$ properly infinite?

(ii) If each of the K_1-trivial unitaries w_l introduced in assertion (ii) of Proposition 5.2 satisfies $\alpha(w_l) = w_l \otimes 1$ and $w_l \sim_h 1_{|\overline{G}_l}$ in $\mathcal{U}(\mathcal{T}E)^{\alpha}{}_{|\overline{G}_l})$, then there exist by

Lemma 2.1.7 of [22] $m-1$ unitaries $z_{l+1} \in \mathcal{T}(E)^{\alpha}{}_{|F_{l+1}}$ such that $(z_{l+1})_{|\overline{G}_l} = w_l$, so that there exists a section $\xi \in E$ with $\xi_x \neq 0$ for all $x \in X$.

(iii) Let $\mathbb{D}$ be the unit ball $\mathbb{D} := \{z \in \mathbb{C} : |z| \leqslant 1\}$ and define the compact space S^2 by

$$C(S^2) := \{f \in C(\mathbb{D}) : f(z) = f(1) \text{ if } |z| = 1\}.$$

If Y' is the compact product $Y' := \prod_{n=1}^{\infty} S^2$ and $Y_0 \subset [0,1]$ is the canonical Cantor set, then M. Dădârlat has constructed in Section 3 of [13] for every pair of countable abelian torsion groups (Γ_0, Γ_1) a separable unital continuous $C(Y')$-algebra A with fibres isomorphic to $\mathcal{O}_2$ such that $K_i(A) \cong C(Y_0, \Gamma_i)$ for $i = 0, 1$. Take a continuous field of faithful states φ on A. The $C(Y')$-algebra $A' \subset \mathcal{L}(L^2(A, \varphi))$ generated by $\pi_\varphi(A)$ and the algebra of compact operators $\mathcal{K}(L^2(A, \varphi))$ is a continuous $C(Y')$-algebra since both the ideal $\mathcal{K}(L^2(A, \varphi))$ and the quotient $A \cong A'/\mathcal{K}(L^2(A, \varphi))$ are continuous (see e.g. Lemma 4.2 of [4]). All the fibres of A' are isomorphic to the Cuntz extension $\mathcal{T}_2$. But A' is not a trivial $C(Y')$-algebra since $K_0(A') = C(Y_0, \Gamma_0) \oplus \mathbb{Z}$ and $K_1(A') = C(Y_0, \Gamma_1)$.

(iv) Let Q be the universal UHF algebra with $K_0(Q) = \mathbb{Q}$ and $[1_Q] = 1$ in $K_0(Q)$. Then the unital continuous $C(Y')$-algebra D constructed by M. Dădârlat in [13] satisfies $K_0(D) = C(Y_0, \mathbb{Z})$ and all its fibres are isomorphic to the C^*-algebra Q. The tensor product $D \otimes \mathcal{O}_\infty$ is a strongly purely infinite C^*-algebra ([21]) since $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ [12]. But this unital continuous $C(Y')$-algebra with fibres $Q \otimes \mathcal{O}_\infty$ is a non trivial $C(Y')$-algebra since $K_0(D \otimes \mathcal{O}_\infty) = C(Y_0, \mathbb{Z})$ whereas $(K_0(C(Y')), K_1(C(Y'))) = (\mathbb{Z}^2, 0)$, $(K_0(Q), K_1(Q)) = (\mathbb{Q}, 0)$ and so $K_0(C(Y', Q \otimes \mathcal{O}_\infty)) = \mathbb{Q} \oplus \mathbb{Q}$ by the Künneth formula [2].

QUESTION 5.4. The Pimsner–Toeplitz algebra $\mathcal{T}(E_{\mathrm{DD}})$ is locally purely infinite ([8], Definition 1.3) since all its simple quotients are isomorphic to the Cuntz algebra $\mathcal{O}_\infty$ ([8], Proposition 5.1). But is $\mathcal{T}(E_{\mathrm{DD}})$ properly infinite?

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