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REGULAR REPRESENTATIONS OF LATTICE ORDERED SEMIGROUPS

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ABSTRACT. We establish a necessary and sufficient condition for a representation of a lattice ordered semigroup to be regular, in the sense that certain extensions are completely positive definite. This result generalizes a theorem due to Brehmer where the lattice ordered group was taken to be $\mathbb{Z}_+^\Omega$. As an immediate consequence, we prove that contractive Nica-covariant representations are regular. We also introduce an analog of commuting row contractions on a lattice ordered group and show that such a representation is regular.

KEYWORDS: *Nica-covariant, regular dilation, positive definite, lattice ordered group.*

MSC (2010): 43A35, 47A20, 47D03.

INTRODUCTION

A contractive map of a group has a unitary dilation if and only if it is completely positive definite, in the sense that certain operator matrices are positive. Consequently, for a semigroup P contained in a group G, a contractive representation of P has a unitary dilation if and only if it can be extended to a completely positive definite map on G. Introduced in [6], such representations of a semigroup are called completely positive definite. In particular, when the group is lattice ordered, a representation is called regular if a certain natural extension to the group is completely positive definite.

Nica [13] introduced the study of isometric representations of quasi-lattice ordered semigroups. This generalized the notion of doubly commuting representations of semigroups with nice generators. Laca and Raeburn [10] developed the theory, and showed there is a universal C^*-algebra for isometric Nica-covariant representations. This field has also been explored in [15].

Davidson, Fuller, and Kakariadis [6], [8] defined and studied the contractive Nica-covariant representation on lattice ordered semigroups. The regularity of such representations was seen as a critical property in describing the C^*-envelope of semicrossed products. They posed a question ([6], Question 2.5.11) of whether regularity is automatic for Nica-covariant representations. Fuller [8] established this for certain abelian semigroups.

This paper answers this question affirmatively by establishing a necessary and sufficient condition for a representation of a lattice ordered semigroup to be regular. This condition generalizes a result of Brehmer [3], where he gave a necessary and sufficient condition for a representation of $\mathbb{Z}_+^\Omega$ to be regular. As an immediate consequence of Brehmer's condition, it is known that doubly commuting representations and commuting column contractions are both regular ([17], Proposition I.9.2). This paper generalizes both results in the lattice ordered group settings. We first show that a Nica-covariant representation, which is an analog of a doubly commuting representation, is regular. We then introduce an analog of commuting column contractions, which is shown to be regular as well.

1. PRELIMINARIES

Let G be a group. A unital semigroup $P \subseteq G$ is called a cone. A cone P is *spanning* if $PP^{-1} = G$, and is *positive* when $P \cap P^{-1} = \{e\}$. A positive cone P defines a partial order on G via $x \leqslant y$ when $x^{-1}y \in P$. (G, P) is called *totally ordered* if $G = P \cup P^{-1}$, in which case the partial order on G is a total order. If any finite subset of G with a upper bound in P also has a least upper bound in P, the pair (G, P) is called a *quasi-lattice ordered group*. We call this partial order *compatible with the group* if for any $x \leqslant y$ and $g \in G$, we always have $gx \leqslant gy$ and $xg \leqslant yg$. Equivalently, the corresponding positive cone satisfies a normality condition that $gPg^{-1} \subseteq P$ for any $g \in G$, and thus $x \leqslant y$ whenever $yx^{-1} \in P$ as well. When P is a positive spanning cone of G whose partial order is compatible with the group, if every two elements $x, y \in G$ have a least upper bound (denoted by $x \vee y$) and a greatest lower bound (denoted by $x \wedge y$), the pair (G, P) is called a *lattice ordered group*. It is immediate that a lattice ordered group is also a quasi-lattice ordered group.

EXAMPLE 1.1 (Examples of lattice ordered groups). (i) $(\mathbb{Z}, \mathbb{Z}_{\geqslant 0})$ is a lattice ordered group. In fact, this partial order is also a total order. More generally, any totally ordered group (G, P) is also a lattice ordered group.

(ii) Let $(G_i, P_i)_{i \in I}$ be a family of lattice ordered group. Then, their direct product $(\prod G_i, \prod P_i)$ is also a lattice ordered group.

(iii) Let $G = C_{\mathbb{R}}[0, 1]$, the set of all continuous functions on $[0, 1]$. Let P be the set of all non-negative functions in G. Then (G, P) is a lattice ordered group.

(iv) Let $\mathcal{T}$ be a totally ordered set. A permutation α on $\mathcal{T}$ is called order preserving if for any $p, q \in \mathcal{T}$, $p \leqslant q$, we also have $\alpha(p) \leqslant \alpha(q)$. Let G be the set of all order preserving permutations, which is clearly a group under composition. Let $P = \{\alpha \in G : \alpha(t) \geqslant t, \text{ for all } t \in \mathcal{T}\}$. Then (G, P) is a non-abelian lattice ordered group [1].

(v) Let $\mathbb{F}_n$ be the free group of n generators, and $\mathbb{F}_n^+$ be the semigroup generated by the n generators. Then $(\mathbb{F}_n, \mathbb{F}_n^+)$ defines a quasi-lattice ordered group ([13], Examples 2.3). However, the induced partial order is not compatible with the group and the pair is not a lattice ordered group.

For any element $g \in G$ of a lattice ordered group (G, P), g can be written uniquely as $g = g_+ g_-^{-1}$ where $g_+, g_- \in P$, and $g_+ \wedge g_- = e$. In fact, $g_+ = g \vee e$ and $g_- = g^{-1} \vee e$. Here are some important properties of a lattice ordered group:

LEMMA 1.2. *Let (G, P) be a lattice order group, and $a, b, c \in G$.*

(i) *$a(b \vee c) = (ab) \vee (ac)$ and $(b \vee c)a = (ba) \vee (ca)$. A similar distributive law holds for $\wedge$.*

(ii) *$(a \wedge b)^{-1} = a^{-1} \vee b^{-1}$ and similarly $(a \vee b)^{-1} = a^{-1} \wedge b^{-1}$.*

(iii) *$a \geqslant b$ if and only if $a^{-1} \leqslant b^{-1}$.*

(iv) *$a(a \wedge b)^{-1} b = a \vee b$. In particular, when $a \wedge b = e$, $ab = ba = a \vee b$.*

(v) *If $a, b, c \in P$, then $a \wedge (bc) \leqslant (a \wedge b)(a \wedge c)$.*

One may refer to [4] for a detailed discussion of this subject. Notice by statement (iv) of Lemma 1.2 g_+, g_- commute and thus $g = g_+ g_-^{-1} = g_-^{-1} g_+$.

For a group G, a unital map $S : G \to \mathcal{B}(\mathcal{H})$ is called *completely positive definite* if for any $g_1, g_2, \ldots, g_n \in G$

$$[S(g_i^{-1} g_j)]_{1 \leqslant i,j \leqslant n} \geqslant 0.$$

Here, i denotes the row index and j the column index, and we shall follow this convention throughout this paper. A well-known result ([12], see also Proposition I.7.1 of [17]) stated that a completely positive definite map of G has a unitary dilation. The converse is elementary.

THEOREM 1.3. *If $S : G \to \mathcal{B}(\mathcal{H})$ is a unital completely positive definite map. Then there exists a unitary representation $U : G \to \mathcal{B}(\mathcal{K})$ where $\mathcal{H}$ is a subspace of $\mathcal{K}$, and that $P_{\mathcal{H}} U(g)|_{\mathcal{H}} = S(g)$. Moreover, this unitary representation can be chosen to be minimal in the sense of $\mathcal{K} = \bigvee_{g \in G} U(g)\mathcal{H}$.*

When (G, P) is a lattice ordered group, we may simultaneously increase or decrease g_i so that it would suffices to take $g_i \in P$:

LEMMA 1.4. *Let $S : G \to \mathcal{B}(\mathcal{H})$ be a map, then the following are equivalent:*

(i) *$[S(g_i^{-1} g_j)]_{1 \leqslant i,j \leqslant n} \geqslant 0$ for any $g_1, g_2, \ldots, g_n \in G$;*

(ii) *$[S(g_i g_j^{-1})]_{1 \leqslant i,j \leqslant n} \geqslant 0$ for any $g_1, g_2, \ldots, g_n \in G$;*

(iii) *$[S(p_i^{-1} p_j)]_{1 \leqslant i,j \leqslant n} \geqslant 0$ for any $p_1, p_2, \ldots, p_n \in P$;*

(iv) *$[S(p_i p_j^{-1})]_{1 \leqslant i,j \leqslant n} \geqslant 0$ for any $p_1, p_2, \ldots, p_n \in P$.*

Proof. Since G is a group, by considering g_i and g_i^{-1}, it is clear that (i) and (ii) are equivalent. Statement (i) clearly implies statement (iii), and conversely when statement (iii) holds true, for any $g_1, \dots, g_n \in G$, take $g = \bigvee_{i=1}^{n} (g_i)_-$. Denote $p_i = g \cdot g_i$ and notice that from our choice of g, $g \geqslant (g_i)_-$. Hence,

$$p_i = g \cdot (g_i)_-^{-1}(g_i)_+ \in P.$$

But notice that for each i, j, $p_i^{-1}p_j = g_i^{-1}g^{-1}gg_j = g_i^{-1}g_j$. Therefore,

$$[S(g_i^{-1}g_j)]_{1\leqslant i,j\leqslant n} = [S(p_i^{-1}p_j)]_{1\leqslant i,j\leqslant n} \geqslant 0.$$

Similarly, statements (ii) and (iv) are equivalent. ∎

For the convenience of computation, when (G, P) is a lattice ordered group, $S : G \to \mathcal{B}(\mathcal{H})$ is called completely positive definite when

$$[S(p_i p_j^{-1})]_{1\leqslant i,j\leqslant n} \geqslant 0.$$

For a spanning cone $P \subset G$, a contractive representation $T : P \to \mathcal{B}(\mathcal{H})$ is called *completely positive definite* when it can be extended to some completely positive definite map on G. There is a well-known result due to Sz.-Nagy that every contraction has a unitary dilation, and therefore, every contractive representation of $\mathbb{Z}_+$ is completely positive definite. Ando [2] further showed that every contractive representation of $\mathbb{Z}_+^2$ is completely positive definite. However, Parrott [14] provided an counterexample where a contractive representations on $\mathbb{Z}_+^3$ is not completely positive definite.

For a completely positive definite representation T on a lattice ordered semigroup, one might wonder what its extension looks like. In a lattice ordered group (G, P), any element $g \in G$ can be uniquely written as $g = g_+ g_-^{-1}$ where $g_\pm \in P$ and $g_+ \wedge g_- = e$. Suppose $U : G \to \mathcal{B}(\mathcal{K})$ is a unitary dilation of T, we can make the following observation:

$$\widetilde{T}(g) = P_{\mathcal{H}} U(g)|_{\mathcal{H}} = P_{\mathcal{H}} U(g_-)^* U(g_+)|_{\mathcal{H}}.$$

This motivates the question of whether the extension $\widetilde{T}(g) = T(g_-)^*T(g_+)$ is completely positive definite. We call a contractive representation T *right regular* whenever $\widetilde{T}$ defined in such way is completely positive definite. There is a dual definition that calls T *left regular* (or *$*$-regular*) if $\overline{T}(g) = T(g_+)T(g_-)^*$ is completely positive definite.

When (G, P) is a lattice ordered group, (G, P^{-1}) is also a lattice ordered group. A representation $T : P \to \mathcal{B}(\mathcal{H})$ gives raise to a dual representation $T^* : P^{-1} \to \mathcal{B}(\mathcal{H})$ where $T^*(p^{-1}) = T(p)^*$. Consider $g = g_+g_-^{-1} = g_-^{-1}(g_+^{-1})^{-1}$, we have

$$\widetilde{T}(g) = T(g_-)^*T(g_+) = T^*(g_-^{-1})T^*(g_+^{-1})^* = \overline{T}^*(g).$$

Hence, $\widetilde{T}$ agrees with $\overline{T}^*$ on G. Therefore, we obtain the following proposition.

PROPOSITION 1.5. *Let (G, P) be a lattice ordered group, and $T : P \to \mathcal{B}(\mathcal{H})$ be a representation and T^* defined as above. Then the following are equivalent:*

(i) *T is right regular.*

(ii) T^* *is left regular.*

(iii) *For any* $p_1, \dots, p_n \in P$, $[\widetilde{T}(p_i p_j^{-1})] \geqslant 0$ *(equivalently,* $[\overline{T}^*(p_i p_j^{-1})] \geqslant 0$*).*

Due to this equivalence, we shall focus on the right regularity and call a representation *regular* when it is right regular. Regular dilations were first studied by Brehmer [3], and they were also studied in [9], [16]. A necessary and sufficient condition for regularity for the abelian group $\mathbb{Z}^\Omega$ was proven by Brehmer ([17], Theorem I.9.1).

THEOREM 1.6 (Brehmer). *Let Ω be a set, and denote $\mathbb{Z}^\Omega$ to be the set of $(t_\omega)_{\omega\in\Omega}$ where $t_\omega \in \mathbb{Z}$ and $t_\omega = 0$ except for finitely many ω. Also, for a finite set $V \subset \Omega$, denote $e_V \in \mathbb{Z}^\Omega$ to be 1 at those $\omega \in V$ and 0 elsewhere. If $\{T_\omega\}_{\omega\in\Omega}$ is a family of commuting contractions, we may define a contractive representation $T : \mathbb{Z}_+^\Omega \to \mathcal{B}(\mathcal{H})$ by*

$$T(t_\omega)_{\omega\in\Omega} = \prod_{\omega\in\Omega} T_\omega^{t_\omega}.$$

Then T is right regular if and only if for any finite $U \subseteq \Omega$, the operator

$$\sum_{V\subseteq U} (-1)^{|V|} T(e_V)^* T(e_V) \geqslant 0.$$

It turns out that not all completely positive definite representations are regular.

EXAMPLE 1.7. It follows from Brehmer's theorem that a representation T on $\mathbb{Z}_+^2$ is regular if and only if $T_1 = T(e_1), T_2 = T(e_2)$ are contractions that satisfy

$$I - T_1^* T_1 - T_2^* T_2 + (T_1 T_2)^* T_1 T_2 \geqslant 0.$$

Take $T_1 = T_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and notice,

$$I - T_1^* T_1 - T_2^* T_2 + (T_1 T_2)^* T_1 T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Brehmer's result implies that T is not regular. However, from Ando's theorem [2], any contractive representation on $\mathbb{Z}_+^2$ has a unitary dilation and thus is completely definite.

Isometric Nica-covariant representations on quasi-lattice ordered groups were first introduced by Nica [13]: an isometric representation $W : G \to \mathcal{B}(\mathcal{H})$ is Nica-covariant if for any x, y with an upper bound, $W_x W_x^* W_y W_y^* = W_{x\vee y} W_{x\vee y}^*$. When the order is a lattice order, this is equivalent to the property that W_s, W_t^* commute whenever $s \wedge t = e$. Therefore, the notion of Nica-covariant is extended to abelian lattice ordered groups in [6], and we shall further extend such definition to non-abelian lattice ordered groups and call a representation $T : P \to \mathcal{B}(\mathcal{H})$ *Nica-covariant* if $T(s)T(t)^* = T(t)^* T(s)$ whenever $s \wedge t = e$. For a Nica-covariant representation T, since $T(g^+)$ commutes with $T(g^-)^*$ for any $g \in G$, there is no difference between left and right regularity. It was observed in [6] that Nica-covariant representations are regular in many cases.

EXAMPLE 1.8 (Examples of Nica-covariant representations). (i) On $(\mathbb{Z}, \mathbb{Z}_+)$, a contractive representation T on $\mathbb{Z}_+$ only depends on $T_1 = T(1)$ since $T(n) = T_1^n$. This representation is always Nica-covariant since for any $s, t \geqslant 0$, $s \wedge t = 0$ if and only if one of s, t is 0. A well-known result due to Sz.-Nagy shows that its extension to $\mathbb{Z}$ by $\widetilde{T}(-n) = T^{*n}$ is completely positive definite and thus T is regular.

(ii) Similarly, any contractive representation of a totally ordered group (G, P) is Nica-covariant. A theorem of Mlak [11] shows that such representations are regular.

(iii) $(\mathbb{Z}^n, \mathbb{Z}_+^n)$, the finite Cartesian product of $(\mathbb{Z}, \mathbb{Z}_+)$ is a lattice ordered group. A representation T on $\mathbb{Z}_+^n$ depends on n contractions $T_1 = T(1, 0, \dots, 0)$, $T_2 = T(0, 1, 0, \dots, 0), \dots$, $T_n = T(0, \dots, 0, 1)$. Notice T is Nica-covariant if and only if T_i, T_j $*$-commute whenever $i \neq j$. Hence Nica-covariant representations are equivalent to doubly commuting. It is known ([17], Section I.9) that doubly commuting contractive representations are regular.

(iv) For a lattice ordered group made from a direct product of totally ordered groups, Fuller [8] showed that their contractive Nica-covariant representations are regular.

A question posed in Question 2.5.11 of [6] asks whether contractive Nica-covariant representations on abelian lattice ordered groups are regular in general. For example, for $G = C_{\mathbb{R}}[0, 1]$ and P equal to the set of non-negative continuous functions, there are no known results on whether contractive Nica-covariant representations are regular on such semigroup. Little is known for the non-abelian lattice ordered groups as well. In this paper, we establish that all Nica-covariant representations of lattice ordered semigroups are regular.

Let (G, P) be a lattice ordered group, not necessarily abelian. Recall that the regularity conditions require a matrix involving entries in the form of $\widetilde{T}(pq^{-1})$ to be positive, where $p, q \in P$. We start by investigating this quantity of pq^{-1}.

LEMMA 1.9. *Let $p, q \in P$. Then,*

$$(pq^{-1})_+ = p(p \wedge q)^{-1} \quad \text{and} \quad (pq^{-1})_- = q(p \wedge q)^{-1}.$$

Proof. By property (i) and (ii) in Lemma 1.2,

$$(pq^{-1})_+ = (pq^{-1} \vee e) = p(q^{-1} \vee p^{-1}) = p(p \wedge q)^{-1}.$$

Similarly, $(pq^{-1})_- = q(p \wedge q)^{-1}$. ∎

LEMMA 1.10. *Let $p, q, g \in P$ such that $g \wedge q = e$. Then $(pg) \wedge q = p \wedge q$.*

Proof. By the property (v) of Lemma 1.2, we have that

$$(pg) \wedge q \leqslant (p \wedge q)(g \wedge q) = p \wedge q.$$

On the other hand, $p \wedge q$ is clearly a lower bound for both $p \leqslant pg$ and q, and hence $p \wedge q \leqslant (pg) \wedge q$. This proves the equality. ∎

LEMMA 1.11. *Let $p, q \in P$. If $g \in P$ is another element where $g \wedge q = 0$, then*

$$(pgq^{-1})_- = (pq^{-1})_- \quad \text{and} \quad (pgq^{-1})_+ = (pq^{-1})_+ g.$$

In particular, if $0 \leqslant g \leqslant p$, then

$$(pg^{-1}q^{-1})_- = (pq^{-1})_- \quad \text{and} \quad (pg^{-1}q^{-1})_+ = (pq^{-1})_+ g^{-1}.$$

Proof. By Lemma 1.9, we get $(pgq^{-1})_+ = pg(q \wedge pg)^{-1}$. Apply Lemma 1.10 to get

$$(q \wedge pg)^{-1} = (q \wedge p)^{-1}.$$

Now $g \wedge (p \wedge q) = e$ and thus g commutes with $p \wedge q$ by property (iv) of Lemma 1.2. Therefore,

$$(pgq^{-1})_+ = pg(q \wedge pg)^{-1} = p(q \wedge p)^{-1} g = (pq^{-1})_+ g.$$

The statement $(pgq^{-1})_- = (pq^{-1})_- g$ can be proven in a similar way.

Finally, for the case where $0 \leqslant g \leqslant p$, it follows immediately by considering $p' = pg^{-1}$ and thus $p = p'g$. ∎

LEMMA 1.12. *If $p_1, p_2, \ldots, p_n \in P$ and $g_1, \ldots, g_n \in P$ be such that $g_i \leqslant p_i$ for all $i = 1, 2, \ldots, n$. Then $\bigwedge_{i=1}^{n} p_i g_i^{-1} \leqslant \bigwedge_{i=1}^{n} p_i$. In particular, when $\bigwedge_{i=1}^{n} p_i = e$, we have $\bigwedge_{i=1}^{n} p_i g_i^{-1} = e$.*

Proof. It is clear that $e \leqslant p_i g_i^{-1} \leqslant p_i$, and thus

$$e \leqslant \bigwedge_{i=1}^{n} p_i g_i^{-1} \leqslant \bigwedge_{i=1}^{n} p_i.$$

Therefore, the equality holds when the later is e. ∎

2. A NECESSARY AND SUFFICIENT CONDITION FOR REGULARITY

When $T : P \to \mathcal{B}(\mathcal{H})$ is a representation of lattice ordered semigroup, we denote $\widetilde{T}(g) = T(g^-)^* T(g^+)$. Recall that T is *regular* if $\widetilde{T}$ is completely positive definite. The main result is the following necessary and sufficient condition for regularity:

THEOREM 2.1. *Let (G, P) be a lattice ordered group and $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation. Then T is regular if and only if for any $p_1, \ldots, p_n \in P$ and $g \in P$ where $g \wedge p_i = e$ for all $i = 1, 2, \ldots, n$, we have*

$$[T(g)^* \widetilde{T}(p_i p_j^{-1}) T(g)] \leqslant [\widetilde{T}(p_i p_j^{-1})]. \tag{2.1}$$

REMARK 2.2. If we denote

$$X = [\widetilde{T}(p_i p_j^{-1})]$$

and $D = \operatorname{diag}(T(g), T(g), \dots, T(g))$, condition (2.1) is equivalent to saying that $D^* X D \leqslant X$. Notice that we made no assumption on $X \geqslant 0$. Indeed, it follows from the main result that condition (2.1) is equivalent of saying the representation T is regular, which in turn implies $X \geqslant 0$. Therefore, when checking the condition (2.1), we may assume $X \geqslant 0$.

REMARK 2.3. By setting $p_1 = e$ and picking any $g \in P$, condition (2.1) implies that $T(g)^* T(g) \leqslant I$, and thus T must be contractive.

The following lemma is taken from Lemma 14.13 of [5].

LEMMA 2.4. *If A, X, D are operators in $\mathcal{B}(\mathcal{H})$ where $A \geqslant 0$. Then a matrix of the form $\begin{bmatrix} A & A^{1/2}X \\ X^* A^{1/2} & D \end{bmatrix}$ is positive if and only if $D \geqslant X^* X$.*

Condition (2.1) can thus be interpreted in the following way.

LEMMA 2.5. *Let $p_1, \dots, p_n \in P$ and $g \in P$ with $g \wedge p_i = e$ for all $1 \leqslant i \leqslant n$. Denote $q_1 = p_1 g, \dots, q_n = p_n g$ and $q_{n+1} = p_1, \dots, q_{2n} = p_n$. Then condition* (2.1) *is equivalent to $[\widetilde{T}(q_i q_j^{-1})] \geqslant 0$.*

Proof. Let $X = [\widetilde{T}(p_i p_j^{-1})] \geqslant 0$ and $D = \operatorname{diag}(T(g), T(g), \dots, T(g))$. Notice by Lemma 1.11 that

$$(p_i g p_j^{-1})_+ = (p_i p_j^{-1})_+ g, \quad (p_i g p_j^{-1})_- = (p_i p_j^{-1})_-,$$

and thus $\widetilde{T}(p_i g p_j^{-1}) = \widetilde{T}(p_i p_j^{-1}) T(g)$. Therefore,

$$[\widetilde{T}(q_i q_j^{-1})] = \begin{bmatrix} X & XD \\ D^* X & X \end{bmatrix}.$$

Lemma 2.4 implies that this matrix is positive if and only if $D^* X D \leqslant X$, which is condition (2.1). ∎

We shall first show that $[\widetilde{T}(p_i p_j^{-1})] \geqslant 0$ given $p_i \wedge p_j = e$ and condition (2.1). This will serve as a base case in the proof of the main result.

LEMMA 2.6. *Let (G, P) be a lattice ordered group, and T be a representation on P that satisfies condition* (2.1). *If $p_i \wedge p_j = e$ for all $i \neq j$, then $[\widetilde{T}(p_i p_j^{-1})] \geqslant 0$.*

Proof. Let $q_1 = e, q_2 = p_1$ and for each $1 < m \leqslant n$, recursively define $q_{2^{m-1}+k} = p_m q_k$ where $1 \leqslant k \leqslant 2^{m-1}$. Since T is contractive,

$$[\widetilde{T}(q_i q_j^{-1})]_{1 \leqslant i,j \leqslant 2} = \begin{bmatrix} I & \widetilde{T}(q_1 q_2^{-1}) \\ \widetilde{T}(q_2 q_1^{-1}) & I \end{bmatrix} \geqslant 0.$$

By Lemma 2.5, for each m, $[\widetilde{T}(q_iq_j^{-1})]_{1\leqslant i,j\leqslant 2^m} \geqslant 0$. Notice that $q_{2^{m-1}} = p_m$ for each $1 \leqslant m \leqslant n$. Therefore, $[\widetilde{T}(p_ip_j^{-1})]$ is a corner of $[\widetilde{T}(q_iq_j^{-1})] \geqslant 0$, and thus must be positive. ∎

For arbitrary choices of $p_1, \dots, p_n \in P$, the goal is to reduce it to the case where $p_i \wedge p_j = e$. The following lemma does the reduction.

LEMMA 2.7. *Let (G, P) be a lattice ordered group. Assume T is a representation that satisfies condition* (2.1).

Assume there exists $2 \leqslant k < n$ such that for each $J \subset \{1, 2, \dots, n\}$ with $|J| > k$, $\bigwedge_{j\in J} p_j = e$. Then let

$$g = \bigwedge_{j=1}^{k} p_j \quad \textit{and} \quad q_1 = p_1g^{-1}, \dots, q_k = p_kg^{-1}, \quad \textit{and} \quad q_{k+1} = p_{k+1}, \dots, q_n = p_n.$$

Then

$$[\widetilde{T}(p_ip_j^{-1})] \geqslant 0 \quad \textit{if } [\widetilde{T}(q_iq_j^{-1})] \geqslant 0.$$

Proof. Let us denote $X = [\widetilde{T}(q_jq_i^{-1})] \geqslant 0$ and its lower right $(n-k) \times (n-k)$ corner to be Y. Notice first of all, when $i, j \in \{1, 2, \dots, k\}$,

$$q_iq_j^{-1} = p_ig^{-1}gp_j^{-1} = p_ip_j^{-1}.$$

So the upper left $k \times k$ corner of $[\widetilde{T}(q_iq_j^{-1})]$ and the lower right $(n-k) \times (n-k)$ corner of X are both the same as those in $[\widetilde{T}(p_ip_j^{-1})]$.

Now consider $i \in \{1, 2, \dots, k\}$ and $j \in \{k+1, \dots, n\}$. It follows from the assumption that

$$g \wedge p_j = \Big(\bigwedge_{s=1}^{k} p_s\Big) \wedge p_j = e \quad \text{and} \quad g \leqslant p_i.$$

Therefore, we can apply Lemma 1.11 to get

$$(p_ig^{-1}p_j^{-1})_- = (p_ip_j^{-1})_-, \quad (p_ig^{-1}p_j^{-1})_+ = (p_ip_j^{-1})_+g^{-1}.$$

Now $g \in P$, so that

$$T((q_iq_j^{-1})_+)T(g) = T((p_ip_j^{-1})_+), \quad T((q_iq_j^{-1})_-) = T((p_ip_j^{-1})_-).$$

Hence,

$$\widetilde{T}(q_iq_j^{-1})T(g) = \widetilde{T}(p_ip_j^{-1}).$$

Similarly, for $i \in \{k+1, \dots, n\}$, $j \in \{1, 2, \dots, k\}$, we have

$$\widetilde{T}(p_ip_j^{-1}) = T(g)^*\widetilde{T}(q_jq_i^{-1}).$$

Now define $D = \operatorname{diag}(I, \dots, I, T(g), \cdots, T(g))$ to be the block diagonal matrix with k copies of I followed by $n-k$ copies of $T(g)$. Consider DXD^*: it follows immediately from the assumption that $D^*XD \geqslant 0$. We have

$$D^*[\widetilde{T}(q_iq_j^{-1})]D = \left[\begin{array}{ccc|c} \cdots & \cdots & \cdots & \vdots \\ \cdots & \widetilde{T}(p_ip_j^{-1}) & \cdots & \widetilde{T}(q_iq_j^{-1})T(g) \\ \cdots & \cdots & \cdots & \vdots \\ \hline \cdots & T(g)^*\widetilde{T}(q_iq_j^{-1}) & \cdots & [T(g)^*\widetilde{T}(p_ip_j^{-1})T(g)] \end{array}\right] \geqslant 0.$$

It follows from previous computation that each entry in the lower left $(n-k) \times k$ corner and upper right $k \times (n-k)$ corner is the same as those in $[\widetilde{T}(p_ip_j^{-1})]$. DXD^* only differs from $[\widetilde{T}(p_ip_j^{-1})]$ on the lower right $(n-k) \times (n-k)$ corner. It follows from condition (2.1) that

$$[T(g)^*\widetilde{T}(p_ip_j^{-1})T(g)] \leqslant [\widetilde{T}(p_ip_j^{-1})].$$

Therefore, the matrix remains positive when the lower right corner in D^*XD is changed from $[T(g)^*\widetilde{T}(p_ip_j^{-1})T(g)]$ to $[\widetilde{T}(p_ip_j^{-1})]$. The resulting matrix is exactly $[\widetilde{T}(p_ip_j^{-1})]$, which must be positive. ∎

Now the main result (Theorem 2.1) can be deduced inductively:

Proof. First assume that $T : P \to \mathcal{B}(\mathcal{H})$ is a representation that satisfies condition (2.1), which has to be contractive. The goal is to show for any n elements $p_1, p_2, \dots, p_n \in P$, the operator matrix $[\widetilde{T}(p_ip_j^{-1})] \geqslant 0$ and thus T is regular. We proceed by induction on n.

For $n = 1$, $\widetilde{T}(p_1p_1^{-1}) = I \geqslant 0$.

For $n = 2$, we have,

$$[\widetilde{T}(p_ip_j^{-1})] = \begin{bmatrix} I & \widetilde{T}(p_1p_2^{-1}) \\ \widetilde{T}(p_2p_1^{-1}) & I \end{bmatrix}.$$

Here, $\widetilde{T}(p_2p_1^{-1}) = \widetilde{T}(p_1p_2^{-1})^*$, and they are contractions since T is contractive. Therefore, this 2×2 operator matrix is positive.

Now assume that there is an N such that for any $n < N$, we have for any $p_1, p_2, \dots, p_n \in P$ $[\widetilde{T}(p_ip_j^{-1})] \geqslant 0$. Consider the case when $n = N$:

For arbitrary choices $p_1, \dots, p_N \in P$, let $g = \bigwedge_{i=1}^{N} p_i$, and replace p_i by p_ig^{-1}. By doing so, $p_ig^{-1}(p_jg^{-1})^{-1} = p_ip_j^{-1}$, and thus they give the same matrix $[\widetilde{T}(p_ip_j^{-1})]$. Moreover, $\bigwedge_{i=1}^{n} p_ig^{-1} = \left(\bigwedge_{i=1}^{N} p_i\right)g^{-1} = e$. Hence, without loss of generality, we may assume $\bigwedge_{i=1}^{N} p_i = e$.

Let m be the smallest integer such that for all $J \subseteq \{1, 2, \ldots, N\}$ and $|J| > m$, we have $\bigwedge_{j \in J} p_j = e$. It is clear that $m \leqslant N - 1$. Now do induction on m:

For the base case when $m = 1$, we have $p_i \wedge p_j = e$ for all $i \neq j$. Lemma 2.6 tells that condition (2.1) implies $[\widetilde{T}(p_i p_j^{-1})] \geqslant 0$.

Now assume $[\widetilde{T}(p_i p_j^{-1})] \geqslant 0$ whenever $m \leqslant M - 1 < N - 1$ and consider the case when $m = M$: For a subset $J \subseteq \{1, 2, \ldots, n\}$ with $|J| = M$, let $g = \bigwedge_{j \in J} p_j$ and set $q_j = p_j g^{-1}$ for all $j \in J$, and $q_j = p_j$ otherwise. Lemma 2.7 concluded that $[\widetilde{T}(p_i p_j^{-1})] \geqslant 0$ whenever $[\widetilde{T}(q_i q_j^{-1})] \geqslant 0$ and the sub-matrix $[\widetilde{T}(p_i p_j^{-1})]_{i,j \notin J} \geqslant 0$.

Since $|\{1, 2, \ldots, N\} \backslash J| = N - M < N$, the induction hypothesis on n implies that $[\widetilde{T}(p_i p_j^{-1})]_{i,j \notin J} \geqslant 0$. Therefore, $[\widetilde{T}(p_i p_j^{-1})] \geqslant 0$ whenever $[\widetilde{T}(q_i q_j^{-1})] \geqslant 0$, and by dropping from p_i to q_i, we may, without loss of generality, assume that $\bigwedge_{j \in J} p_j = e$. Repeat this process for all subsets $J \subset \{1, 2, \ldots, n\}$ where $|J| = M$, and with Lemma 1.12, we eventually reach a state when $\bigwedge_{j \in J} p_j = e$ for all $J \subseteq \{1, 2, \ldots, N\}$, $|J| = M$. But in such case, for all $|J| \geqslant M$, we have $\bigwedge_{j \in J} p_j = e$. Therefore, we are in a situation where $m \leqslant M - 1$. The result follows from the induction hypothesis on m.

Conversely, suppose that T is regular. Fix $g \in P$ and $p_1, p_2, \ldots, p_k \in P$ where $g \wedge p_i = e$ for all $i = 1, 2, \ldots, k$. Denote $q_1 = p_1 g, q_2 = p_2 g, \ldots, q_k = p_k g$, and $q_{k+1} = p_1, q_{k+2} = p_2, \ldots, q_{2k} = p_k$. It follows from regularity that $[\widetilde{T}(q_i q_j^{-1})] \geqslant 0$, which is equivalent to condition (2.1) by Lemma 2.5. ∎

As an immediate consequence of Theorem 2.1, we can show that isometric representations on any lattice ordered group must be regular.

COROLLARY 2.8. *Let $T : P \to \mathcal{B}(\mathcal{H})$ be an isometric representation of a lattice ordered semigroup. Then T is regular.*

Proof. Take $p_1, \ldots, p_n \in P$ and $g \in P$ with $g \wedge p_i = e$. It is clear that $g \wedge (p_i p_j^{-1})_\pm = e$ and therefore g commutes with each $(p_i p_j^{-1})_\pm$. Hence,

$$\begin{aligned} T(g)^* \widetilde{T}(p_i p_j^{-1}) T(g) &= T(g)^* T((p_i p_j^{-1})_-)^* T((p_i p_j^{-1})_+) T(g) \\ &= T((p_i p_j^{-1})_-)^* T(g)^* T(g) T((p_i p_j^{-1})_+) \\ &= T((p_i p_j^{-1})_-)^* T((p_i p_j^{-1})_+) = \widetilde{T}(p_i p_j^{-1}). \end{aligned}$$

Therefore, $[T(g)^* \widetilde{T}(p_i p_j^{-1}) T(g)] = [\widetilde{T}(p_i p_j^{-1})]$ and condition (2.1) is satisfied. ∎

For a contractive representation T, it would suffice to dilate it to an isometric representation. This provides an analog of Proposition 2.5.4 in [6] on non-abelian lattice ordered groups.

COROLLARY 2.9. *Let $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive representation. Then T is completely positive definite if and only if there exists an isometric representation $V : P \to \mathcal{B}(\mathcal{K})$ such that $P_{\mathcal{H}} V(p)|_{\mathcal{H}} = T(p)$ for all $p \in P$. Such V can be taken to be minimal in the sense that $\mathcal{K} = \bigvee_{p\in P} V(p)\mathcal{H}$.*

In particular, T is regular if and only if there exists such isometric dilation V and in addition, $P_{\mathcal{H}} V(p)^* V(q)|_{\mathcal{H}} = T(p)^* T(q)$ for all $p, q \in P$ with $p \wedge q = e$.

Proof. When $T : P \to \mathcal{B}(\mathcal{H})$ is completely positive definite and its extension S to G has minimal unitary dilation $U : G \to \mathcal{B}(\mathcal{L})$, let $\mathcal{K} = \bigvee_{p\in P} U(p)\mathcal{H}$. It is clear that $\mathcal{K}$ is invariant for any $U(p)$, $p \in P$. Define a map $V : P \to \mathcal{B}(\mathcal{K})$ via $V(p) = P_{\mathcal{K}} U(p)|_{\mathcal{K}}$, which must be isometric due to the invariance of $\mathcal{K}$. V is an isometric dilation of T that satisfies $P_{\mathcal{H}} V(p)|_{\mathcal{H}} = T(p)$, and $\mathcal{K} = \bigvee_{p\in P} V(p)\mathcal{H}$. In other words, V is a minimal isometric dilation of T. In particular, when T is regular, for any $p, q \in P$ with $p \wedge q = e$

$$T(p)^* T(q) = P_{\mathcal{H}} U(p)^* U(q)|_{\mathcal{H}} = P_{\mathcal{H}} P_{\mathcal{K}} U(p)^* U(q)|_{\mathcal{K}}|_{\mathcal{H}} = P_{\mathcal{H}} V(p)^* V(q)|_{\mathcal{H}}.$$

Conversely, when $V : P \to \mathcal{B}(\mathcal{K})$ is a minimal isometric dilation of T, Corollary 2.8 implies that V is regular and thus completely positive definite. There exists a unitary dilation $U : G \to \mathcal{B}(\mathcal{L})$ where $P_{\mathcal{K}} U(p)|_{\mathcal{K}} = V(p)$. Therefore,

$$P_{\mathcal{H}} U(p)|_{\mathcal{H}} = P_{\mathcal{H}} P_{\mathcal{K}} U(p)|_{\mathcal{H}} = P_{\mathcal{H}} V(p)|_{\mathcal{H}} = T(p).$$

Hence, U is also a unitary dilation of T and thus T is completely positive definite. Moreover, when $P_{\mathcal{H}} V(p)^* V(q)|_{\mathcal{H}} = T(p)^* T(q)$ for all $p, q \in P$ with $p \wedge q = e$, by the regularity of V,

$$P_{\mathcal{H}} U(p)^* U(q)|_{\mathcal{H}} = P_{\mathcal{H}} P_{\mathcal{K}} U(p)^* U(q)|_{\mathcal{K}}|_{\mathcal{H}} = T(p)^* T(q).$$

Therefore, $\widetilde{T}(g) = T(g_-)^* T(g_+)$ is completely positive definite and T is regular. ∎

3. NICA-COVARIANT REPRESENTATIONS

This section answers the question of whether contractive Nica-covariant representations are regular. It suffices to show that contractive Nica-covariant representations on lattice ordered groups satisfy condition (2.1).

THEOREM 3.1. *A contractive Nica-covariant representation on a lattice ordered group is regular.*

Proof. Let $p_1, \ldots, p_k \in P$ and $g \in P$ with $g \wedge p_i = e$ for all $i = 1, 2, \ldots, k$. $X = [\widetilde{T}(p_i p_j^{-1})]$ and $D = \mathrm{diag}(T(g), T(g), \ldots, T(g))$. By Remark 2.2, we may assume $X \geqslant 0$.

For each $p_i, p_j \in P$, $\widetilde{T}(p_i p_j^{-1}) = T(p_{i,j}^-)^* T(p_{i,j}^+)$ where $e \leqslant p_{i,j}^\pm \leqslant p_i, p_j$. Hence, $g \wedge p_{i,j}^\pm = e$ and thus g commutes with $p_{i,j}^\pm$. Therefore $T(g)$ commutes with $T(p_{i,j}^+)$ because T is a representation and it also commutes with $T(p_{i,j}^-)^*$ by the Nica-covariant condition. As a result, $T(g)$ commutes with each entry in X, and thus D commutes with X. Similarly, D^* commutes with X as well.

By continuous functional calculus, since $X \geqslant 0$, we know D, D^* also commutes with $X^{1/2}$. Hence, in such case,

$$D^* X D = D^* X^{1/2} X^{1/2} D = X^{1/2} D^* D X^{1/2} \leqslant X. \quad \blacksquare$$

It was shown in Proposition 2.5.10 of [6] that a contractive Nica-covariant representation on abelian lattice ordered groups can be dilated to an isometric Nica-covariant representation. Here, we shall extend this result to non-abelian case.

COROLLARY 3.2. *Any minimal isometric dilation $V : P \to \mathcal{B}(\mathcal{K})$ of a contractive Nica-covariant representation $T : P \to \mathcal{B}(\mathcal{H})$ is also Nica-covariant.*

Proof. Let $T : P \to \mathcal{B}(\mathcal{H})$ be a contractive Nica-covariant representation. Theorem 3.1 implies that T is regular, and thus by Theorem 1.3, it has a minimal unitary dilation $U : G \to \mathcal{B}(\mathcal{L})$, which gives rise to a minimal isometric dilation $V : P \to \mathcal{B}(\mathcal{K})$. Here $\mathcal{K} = \bigvee_{p \in P} V(p)\mathcal{H}$ and $V(p) = P_{\mathcal{K}} U(p)|_{\mathcal{K}}$. Notice that $\mathcal{K}$ is invariant for U and therefore, $P_{\mathcal{K}} U(p)^* U(q)|_{\mathcal{K}} = V(p)^* V(q)$ for any $p, q \in P$. In particular, if $p \wedge q = e$, $p, q \in P$, we have from the regularity that

$$T(p)^* T(q) = P_{\mathcal{H}} U(p)^* U(q)|_{\mathcal{H}} = P_{\mathcal{H}} (P_{\mathcal{K}} U(p)^* U(q)|_{\mathcal{K}})|_{\mathcal{H}} = P_{\mathcal{H}} V(p)^* V(q)|_{\mathcal{H}}.$$

Now let $s, t \in P$ be such that $s \wedge t = e$. First, we shall prove $V(s)^* V(t)|_{\mathcal{H}} = V(t) V(s)^*|_{\mathcal{H}}$: Since $\{V(p)h : p \in P, h \in \mathcal{H}\}$ is dense in $\mathcal{K}$, it suffices to show for any $h, k \in \mathcal{H}$ and $p \in P$,

$$\langle V(s)^* V(t) h, V(p) k \rangle = \langle V(t) V(s)^* h, V(p) k \rangle.$$

Start from the left,

$$\begin{aligned}
\langle V(s)^* V(t) h, V(p) k \rangle &= \langle V(p)^* V(s)^* V(t) h, k \rangle = \langle V(sp)^* V(t) h, k \rangle \\
&= \langle V((sp \wedge t)^{-1} sp)^* V(sp \wedge t)^* V(sp \wedge t) V((sp \wedge t)^{-1} t) h, k \rangle \\
&= \langle V((sp \wedge t)^{-1} sp)^* V((sp \wedge t)^{-1} t) h, k \rangle \\
&= \langle T((sp \wedge t)^{-1} sp)^* T((sp \wedge t)^{-1} t) h, k \rangle.
\end{aligned}$$

The last equality follows from $((sp \wedge t)^{-1} sp) \wedge ((sp \wedge t)^{-1} t) = e$ and thus,

$$T((sp \wedge t)^{-1} sp)^* T((sp \wedge t)^{-1} t) = P_{\mathcal{H}} V((sp \wedge t)^{-1} sp)^* V((sp \wedge t)^{-1} t)|_{\mathcal{H}}.$$

Since $s \wedge t = e$, Lemma 1.10 implies that $sp \wedge t = p \wedge t$. Notice $(p \wedge t) \wedge s \leqslant t \wedge s = e$, and thus by property (iv) of Lemma 1.2, s commutes with $p \wedge t$. By the

Nica-covariance of T, this also implies $T(s)^*$ commutes with $T((p\wedge t)^{-1}t)$. Put all these back to the equation:

$$\begin{aligned}\langle T((sp\wedge t)^{-1}sp)^*T((sp\wedge t)^{-1}t)h,k\rangle &= \langle T(s(p\wedge t)^{-1}p)^*T((p\wedge t)^{-1}t)h,k\rangle \\ &= \langle T((p\wedge t)^{-1}p)^*T(s)^*T((p\wedge t)^{-1}t)h,k\rangle \\ &= \langle T((p\wedge t)^{-1}p)^*T((p\wedge t)^{-1}t)(T(s)^*h),k\rangle \\ &= \langle V((p\wedge t)^{-1}p)^*V((p\wedge t)^{-1}t)(T(s)^*h),k\rangle \\ &= \langle V((p\wedge t)^{-1}p)^*V((p\wedge t)^{-1}t)(V(s)^*h),k\rangle \\ &= \langle V(p)^*V(t)(V(s)^*h),k\rangle \\ &= \langle V(t)V(s)^*h,V(p)k\rangle.\end{aligned}$$

Here we used the fact that $P_{\mathcal{H}}V(p)^*V(q)|_{\mathcal{H}} = T(p)^*T(q)$ whenever $p\wedge q = e$. Also, that $\mathcal{H}$ is invariant under $V(s)^*$, so that $T(s)^*h\in\mathcal{K}$ is the same as $V(s)^*h$.

Now to show $V(s)^*V(t) = V(t)V(s)^*$ in general, it suffices to show for every $p\in P$, $V(s)^*V(t)V(p)|_{\mathcal{H}} = V(t)V(s)^*V(p)|_{\mathcal{H}}$. Start with the left hand side and repeatedly use similar argument as above,

$$\begin{aligned}V(s)^*V(t)V(p)|_{\mathcal{H}} = V(s)^*V_{tp}|_{\mathcal{H}} &= V((s\wedge tp)^{-1}s)^*V((s\wedge tp)^{-1}tp)|_{\mathcal{H}} \\ &= V(t(s\wedge p)^{-1}p)V((s\wedge p)^{-1}s)^*|_{\mathcal{H}} \\ &= V(t(s\wedge p)^{-1}p)V((s\wedge p)^{-1}s)^*|_{\mathcal{H}} \\ &= V(t)V((s\wedge p)^{-1}s)^*V((s\wedge p)^{-1}p)|_{\mathcal{H}} = V(t)V(s)^*V(p)|_{\mathcal{H}}.\end{aligned}$$

This finishes the proof. ∎

4. ROW AND COLUMN CONTRACTIONS

A commuting n-tuple $(T_1,\dots,T_n)$ where each $T_i\in\mathcal{B}(\mathcal{H})$ is called a *row contraction* if $\sum_{i=1}^{n} T_iT_i^*\leqslant I$. Equivalently, the operator $[T_1,T_2,\dots,T_n]\in\mathcal{B}(\mathcal{H}^n,\mathcal{H})$ is contractive. It can be naturally associated with a contractive representation $T:\mathbb{Z}_+^n\to\mathcal{B}(\mathcal{H})$ that sends the i-th generator e_i to T_i. There is a dual definition called column contractions, when T_i satisfies $\sum_{i=1}^{n} T_i^*T_i\leqslant I$. It is clear that T is a row contraction if and only if T^* is a column contraction.

As an immediate corollary to Brehmer's theorem (Theorem 1.6), a column contraction T is always right regular ([17], Proposition I.9.2), and therefore a row contraction T is always left regular. This section generalizes the notion of row contraction to arbitrary lattice ordered groups and establishes a similar result.

DEFINITION 4.1. Let $T:P\to\mathcal{B}(\mathcal{H})$ be a contractive representation of a lattice ordered group (G,P). T is called *row contractive* if for any $p_1,\dots,p_n\in P$

where $p_i \neq e$ and $p_i \wedge p_j = e$ for all $i \neq j$,

$$\sum_{i=1}^{n} T(p_i)T(p_i)^* \leqslant I.$$

Dually, T is called *column contractive* if for such p_i,

$$\sum_{i=1}^{n} T(p_i)^*T(p_i) \leqslant I.$$

REMARK 4.2. Definition 4.1 indeed generalizes the notion of commuting row contractions: when the group is $(\mathbb{Z}^\Omega, \mathbb{Z}_+^\Omega)$ where Ω is countable, a representation $T : \mathbb{Z}_+^\Omega \to \mathcal{B}(\mathcal{H})$ is uniquely determined by its value on the generators $T_\omega = T(e_\omega)$. T is called commuting row contraction when $\sum_{\omega\in\Omega} T_\omega T_\omega^* \leqslant I$. For any $p_1, \ldots, p_k \in \mathbb{Z}_+^\Omega$ where $p_i \wedge p_j = 0$ for all $i \neq j$ and $p_i \neq 0$, each p_i can be seen as a function from Ω to $\mathbb{Z}_+$ with finite support. Let $S_i \subseteq \Omega$ be the support of p_i, which is non-empty since $p_i \neq 0$. We have $S_i \cap S_j = \emptyset$ since $p_i \wedge p_j = 0$. Therefore, pick any $\omega_i \in S_i$ and by T contractive, $T(\omega_i)T(\omega_i)^* \geqslant T(p_i)T(p_i)^*$. Since S_i are pairwise-disjoint, ω_i are distinct. Therefore, we get that

$$\sum_{i=1}^{n} T(p_i)T(p_i)^* \leqslant \sum_{i=1}^{n} T(\omega_i)T(\omega_i)^* \leqslant I,$$

and thus T satisfies the Definition 4.1. Hence, on $(\mathbb{Z}^\Omega, \mathbb{Z}_+^\Omega)$, two definitions coincides.

Our goal is to prove the following result:

THEOREM 4.3. *A column contractive representation is right regular. Therefore, a row contractive representation is left regular.*

We shall proceed with a method similar to the proof of Theorem 2.1.

LEMMA 4.4. *Let T be a column contractive representation. Let $p_1, \ldots, p_n \in P$ and $g_1, \ldots, g_k \in P$ where $p_i \wedge p_{i'} = p_i \wedge g_j = g_j \wedge g_{j'} = e$ for all $1 \leqslant i \neq i' \leqslant n$ and $1 \leqslant j \neq j' \leqslant k$. Moreover, assume that $g_i \neq e$. Denote $X = [\widetilde{T}(p_i p_j^{-1})]$ and $D_i = \operatorname{diag}(T(g_i), \ldots, T(g_i))$. Then,*

$$\sum_{i=1}^{k} D_i^* X D_i \leqslant X.$$

Proof. The statement is clearly true for all k when $n = 1$. Now assume it is true for all k whenever $n < N$, and consider the case when $n = N$:

It is clear that when all of the p_i are equal to e, $X - \sum_{i=1}^{k} D_i^* X D_i$ is a $n \times n$ matrix whose entries are all equal to $I - \sum_{i=1}^{k} T(g_i)^*T(g_i) \geqslant 0$, and thus the statement is true. Otherwise, we may assume without loss of generality that $p_1 \neq e$.

Let $q_1 = e$ and $q_2 = p_2, \dots, q_n = p_n$. Denote $X_0 = [\widetilde{T}(q_i q_j^{-1})]$ and let $E = \operatorname{diag}(I, T(p_1), \dots, T(p_1))$ be a $n \times n$ block diagonal matrix.

Denote $Y = [\widetilde{T}(p_i p_j^{-1})]_{2 \leqslant i,j \leqslant n}$ and set $E_i = \operatorname{diag}(T(g_i), \dots, T(g_i))$ to be a $(n-1) \times (n-1)$ block diagonal matrix. Finally, set $E_{k+1} = \operatorname{diag}(T(p_1), \dots, T(p_1))$ to be a $(n-1) \times (n-1)$ block diagonal matrix.

From the proof of Theorem 2.1,

$$X = E^* X_0 E + \begin{bmatrix} 0 & 0 \\ 0 & Y - E_{k+1}^* Y E_{k+1} \end{bmatrix}.$$

Now Y is a matrix of smaller size and thus by induction hypothesis, $\sum\limits_{i=1}^{k+1} E_i^* Y E_i \leqslant Y$. Hence,

$$Y - E_{k+1}^* Y E_{k+1} \geqslant \sum_{i=1}^{k} E_i^* Y E_i \geqslant \sum_{i=1}^{k} E_i^* (Y - E_{k+1}^* Y E_{k+1}) E_i.$$

Also notice that E commutes with D_i and therefore, if $\sum\limits_{i=1}^{k} D_i^* X_0 D_i \leqslant X_0$, we have

$$\begin{aligned} \sum_{i=1}^{k} D_i^* X D_i &= E^* \Big(\sum_{i=1}^{k} D_i^* X_0 D_i \Big) E + \begin{bmatrix} 0 & 0 \\ 0 & \sum_{i=1}^{k} E_i^* (Y - E_{k+1}^* Y E_{k+1}) E_i \end{bmatrix} \\ &\leqslant E^* X_0 E + \begin{bmatrix} 0 & 0 \\ 0 & Y - E_{k+1}^* Y E_{k+1} \end{bmatrix} = X. \end{aligned}$$

Hence, $\sum\limits_{i=1}^{k} D_i^* X D_i \leqslant X$ if $\sum\limits_{i=1}^{k} D_i^* X_0 D_i \leqslant X_0$. This reduction from X to X_0 changes one $p_i \neq e$ to e, and therefore by repeating this process, we eventually reach a state where all $p_i = e$. ∎

The main result can be deduced immediately from the following proposition:

PROPOSITION 4.5. *Let T be a column contractive representation on a lattice ordered semigroup P. Let $p_1, \dots, p_n \in P$ and $g_1, \dots, g_k \in P$ where $g_i \wedge p_j = e$ and $g_i \wedge g_l = e$ for all $i \leqslant l$. Assume $g_i \neq e$ and denote $X = [\widetilde{T}(p_i p_j^{-1})]$ and $D_i = \operatorname{diag}(T(g_i), \dots, T(g_i))$. Then*

$$\sum_{i=1}^{k} D_i^* X D_i \leqslant X.$$

In particular, condition (2.1) *is satisfied when $k = 1$.*

Proof. The statement is clear when $n = 1$. Assuming it is true for $n < N$, consider the case when $n = N$. Let m be the smallest integer such that for all $J \subseteq \{1, 2, \dots, N\}$ and $|J| > m$, $\bigwedge\limits_{j \in J} p_j = e$. It was observed in the proof of Theorem 2.1 that $m \leqslant N - 1$. Proceed by induction on m:

In the base case when $m = 1$, $p_i \wedge p_j = e$ for all $i \neq j$, the statement is shown in Lemma 4.4. Assume the statement is true for $m < M - 1 < N - 1$ and consider the case when $m = M$. For each $J \subseteq \{1, 2, \ldots, N\}$ with $|J| = M$ and $\bigwedge_{j=1}^{M} p_j = g \neq e$, denote $q_i = p_i$ when $i \notin J$ and $q_i = q_i g^{-1}$ when $i \in J$. Let $X_0 = [\widetilde{T}(q_i q_j^{-1})]$ and E be a block diagonal matrix whose i-th diagonal entry is I when $i \notin J$ and $T(g)$ otherwise. Denote $Y = [\widetilde{T}(q_i q_j^{-1})]_{i,j \notin J}$ and $E_i = \mathrm{diag}(T(g_i), \ldots, T(g_i))$ with $N - M$ copies of $T(g_i)$. Finally, let $E_{k+1} = \mathrm{diag}(T(g), \ldots, T(g))$ with $N - M$ copies of $T(g)$.

From the proof of Theorem 2.1, by assuming without loss of generality that $J = \{1, 2, \ldots, M\}$, we have

$$X = E^* X_0 E + \begin{bmatrix} 0 & 0 \\ 0 & Y - E_{k+1}^* Y E_{k+1} \end{bmatrix}.$$

Now Y has a smaller size and thus by induction hypothesis on n,

$$\sum_{i=1}^{k+1} E_i^* Y E_i \leqslant Y,$$

and thus

$$Y - E_{k+1}^* Y E_{k+1} \geqslant \sum_{i=1}^{k} E_i^* Y E_i \geqslant \sum_{i=1}^{k} E_i^* (Y - E_{k+1}^* Y E_{k+1}) E_i.$$

Therefore, if $\sum_{i=1}^{k} D_i^* X_0 D_i \leqslant X_0$,

$$\begin{aligned} \sum_{i=1}^{k} D_i^* X D_i &= E^* \Big(\sum_{i=1}^{k} D_i^* X_0 D_i \Big) E + \begin{bmatrix} 0 & 0 \\ 0 & \sum_{i=1}^{k} E_i^* (Y - E_{k+1}^* Y E_{k+1}) E_i \end{bmatrix} \\ &\leqslant E^* X_0 E + \begin{bmatrix} 0 & 0 \\ 0 & Y - E_{k+1}^* Y E_{k+1} \end{bmatrix} = X. \end{aligned}$$

Hence, the statement is true for p_i if it is true for q_i, where $\bigwedge_{j \in J} q_j = e$. Repeat the process until all such $|J| = M$ has $\bigwedge_{j \in J} p_j = e$, which reduces to a case where $m < M$. This finishes the induction. Notice condition (2.1) is clearly true when $g = e$, and when $g \neq e$, it is shown by the case when $m = 1$. This finishes the proof. ∎

5. BREHMER'S CONDITION

Brehmer [3] established a necessary and sufficient condition for a representation on $P = \mathbb{Z}_+^{\Omega}$ to be regular (see Theorem 1.6). This section explores how Brehmer's result relates to condition (2.1) without invoking their equivalence to regularity. In particular, we show that Brehmer's condition allows us to decompose certain $X = [\widetilde{T}(p_i - p_j)]$ as a product R^*R, where R is an upper triangular matrix.

Let $\{T_\omega\}_{\omega\in\Omega}$ be a family of commuting contractions, which leads to a contractive representation on $\mathbb{Z}_+^\Omega$ by sending each e_ω to T_ω. For each $U \subseteq \Omega$, denote

$$Z_U = \sum_{V\subseteq U} (-1)^{|V|} T(e_V)^* T(e_V).$$

For example,

$$\begin{aligned}
Z_\emptyset &= I \\
Z_{\{1\}} &= I - T_1^* T_1 \\
Z_{\{1,2\}} &= Z_{\{1\}} - T_2^* Z_{\{1\}} T_2 = I - T_1^* T_1 - T_2^* T_2 + T_2^* T_1^* T_1 T_2 \\
&\vdots
\end{aligned}$$

Brehmer's theorem stated that T is regular if and only if $Z_U \geqslant 0$ for any finite subset $U \subseteq \Omega$. We shall first transform Brehmer's condition into an equivalent form.

LEMMA 5.1. *$Z_U \geqslant 0$ for each finite subset $U \subseteq \Omega$ if and only if for any finite set $J \subseteq \Omega$ and $\omega \in \Omega$, $\omega \notin J$,*

$$T_\omega^* Z_J T_\omega \leqslant Z_J.$$

Proof. Take any finite subset $J \subseteq \Omega$ and $\omega \in \Omega$, $\omega \notin J$.

$$\begin{aligned}
Z_J - T_\omega^* Z_J T_\omega &= \sum_{V\subseteq J} (-1)^{|V|} T(e_V)^* T(e_V) + \sum_{V\subseteq J} (-1)^{|V|+1} T_\omega^* T(e_V)^* T(e_V) T_\omega \\
&= \sum_{V\subseteq\{\omega\}\cup J, \omega\notin V} (-1)^{|V|} T(e_V)^* T(e_V) + \sum_{V\subseteq\{\omega\}\cup J, \omega\in V} (-1)^{|V|} T(e_V)^* T(e_V) \\
&= Z_{\{\omega\}\cup J}.
\end{aligned}$$

Therefore, $T_\omega^* Z_J T_\omega \leqslant Z_J$ if and only if $Z_{\{\omega\}\cup J} \geqslant 0$. This finishes the proof. ∎

A major tool is the following version of Douglas lemma [7]:

LEMMA 5.2 (Douglas). *For $A, B \in \mathcal{B}(\mathcal{H})$, $A^*A \leqslant B^*B$ if and only if there exists a contraction C such that $A = CB$.*

As an immediate consequence of Lemma 5.2, $T_\omega^* Z_J T_\omega \leqslant Z_J$ is satisfied if and only if there is a contraction $W_{\omega,J}$ such that $Z_J^{1/2} T_\omega = W_{\omega,J} Z_J^{1/2}$. Therefore, it would suffice to find such contraction $W_{\omega,J}$ for each finite subset $J \subseteq \Omega$ and $\omega \in \Omega$, $\omega \notin J$. By symmetry, it would suffice to do so for each $J_n = \{1, 2, \ldots, n\}$ and $\omega_n = n+1$. Without loss of generality, we shall assume that $\Omega = \mathbb{N}$.

Consider $\mathcal{P}(J_n) = \{U \subseteq J_n\}$, and denote $p_U = \sum_{i\in U} e_i \in \mathbb{Z}_+^\Omega$. Denote $X_n = [\widetilde{T}(p_U - p_V)]$ where U is the row index and V is the column index.

LEMMA 5.3. *Assume $Z_J \geqslant 0$ for all $J \subseteq J_n$. Then for a fixed $F \subseteq J_n$, we have*

$$\sum_{U \subseteq F} T_U^* Z_{F \backslash U} T_U = I.$$

Proof. We first notice that by definition, $Z_J = \sum_{U \subseteq J} (-1)^{|U|} T_U^* T_U$. Therefore,

$$\sum_{U \subseteq F} T_U^* Z_{F \backslash U} T_U = \sum_{U \subseteq F} \sum_{V \subseteq F \backslash U} (-1)^{|V|} T_{U \cup V}^* T_{U \cup V}.$$

For a fixed set $W \subseteq F$, consider the coefficient of $T_W^* T_W$ in the double summation. It appears in the expansion of every $T_U^* Z_{F \backslash U} T_U$, where $U \subseteq W$, and its coefficient in the expansion of such term is equal to $(-1)^{|W \backslash U|}$. Therefore, the coefficient of $T_W^* T_W$ is equal to

$$\sum_{U \subseteq W} (-1)^{|W \backslash U|} = \sum_{i=0}^{|W|} \binom{|W|}{i} (-1)^i.$$

This evaluates to 0 when $|W| > 0$ and 1 when $|W| = 0$, in which case, $W = \varnothing$ and $T_W = I$. ∎

Now can now decompose $X_n = R_n^* R_n$ explicitly.

PROPOSITION 5.4. *Assuming $Z_J \geqslant 0$ for all $J \subseteq J_n$. Define a block matrix R_n, whose rows and columns are indexed by $\mathcal{P}(J_n)$, by $R_n(U, V) = Z_{J_n \backslash U}^{1/2} T_{U \backslash V}$ whenever $V \subseteq U$ and 0 otherwise. Then $X_n = R_n^* R_n$.*

Proof. Fix $U, V \subseteq J_n$, the (U, V)-entry in X_n is $\widetilde{T}(p_U - p_V) = T_{V \backslash U}^* T_{U \backslash V}$. Now the (U, V)-entry in $R_n^* R_n$ is equal to

$$\sum_{W \subseteq J_n} R_n(W, U)^* R_n(W, V).$$

It follows from the definition that $R_n(W, U)^* R_n(W, V) = 0$ unless $U, V \subseteq W$, and thus $U \cup V \subseteq W$. Hence,

$$\begin{aligned}
\sum_{W \in \mathcal{P}(J_n)} R_n(W, U)^* R_n(W, V) &= \sum_{U \cup V \subseteq W} T_{W \backslash U}^* Z_{J_n \backslash W} T_{W \backslash V} \\
&= \sum_{U \cup V \subseteq W} T_{V \backslash U}^* T_{W \backslash (U \cup V)}^* Z_{J_n \backslash W} T_{W \backslash (U \cup V)} T_{W \backslash U} \\
&= T_{V \backslash U}^* \Big(\sum_{U \cup V \subseteq W} T_{W \backslash (U \cup V)}^* Z_{J_n \backslash W} T_{W \backslash (U \cup V)} \Big) T_{W \backslash U}.
\end{aligned}$$

If we denote $F = J_n \backslash (U \cup V)$ and $W' = W \backslash (U \cup V)$, since $U \cup V \subseteq W$, we have $J_n \backslash W = F \backslash W'$. Hence the summation becomes

$$\sum_{U \cup V \subseteq W} T_{W \backslash (U \cup V)}^* Z_{J_n \backslash W} T_{W \backslash (U \cup V)} = \sum_{W' \subseteq F} T_{W'}^* Z_{F \backslash W'} T_{W'},$$

which by Lemma 5.3 is equal to I. Therefore, the (U, V)-entry in $R_n^* R_n$ is equal to $T_{V \backslash U}^* T_{W \backslash U}$ and $X_n = R_n^* R_n$. ∎

REMARK 5.5. If we order the subsets of J_n by cardinality and put larger sets first, then since $R_n(U,V) \neq 0$ only when $V \subseteq U$, R_n becomes a lower triangular matrix. In particular, the row of $\emptyset$ contains exactly one non-zero entry, which is $Z_{J_n}^{1/2}$ at $(\emptyset,\emptyset)$.

EXAMPLE 5.6. Let us consider the case when $n = 2$, and J_2 has 4 subsets $\{1,2\}$, $\{2\}$, $\{1\}$, $\emptyset$. Under this ordering,

$$X_n = \begin{bmatrix} I & T_1 & T_2 & T_1T_2 \\ T_1^* & I & T_1^*T_2 & T_2 \\ T_2^* & T_2^*T_1 & I & T_1 \\ T_1^*T_2^* & T_2^* & T_1^* & I \end{bmatrix}.$$

Proposition 5.4 gives that

$$R_n = \begin{bmatrix} I & T_1 & T_2 & T_1T_2 \\ 0 & Z_1^{1/2} & 0 & Z_1^{1/2}T_2 \\ 0 & 0 & Z_2^{1/2} & Z_2^{1/2}T_1 \\ 0 & 0 & 0 & Z_{1,2}^{1/2} \end{bmatrix}$$

satisfies $R_n^*R_n = X_n$.

We can now prove Brehmer's condition from condition (2.1) without invoking their equivalence to regularity.

PROPOSITION 5.7. *In the case of* $T : \mathbb{Z}_+^\Omega \to \mathcal{B}(\mathcal{H})$, *condition* (2.1) *implies the Brehmer's condition.*

Proof. Without loss of generality, we may assume $\Omega = \mathbb{N}$. We shall proceed by induction on the size of $J \subseteq \mathbb{N}$.

For $|J| = 1$ (i.e. $J = \{\omega\}$), condition (2.1) implies T is contractive. Hence, $Z_J = I - T_\omega^* T_\omega \geqslant 0$. Assuming $Z_J \geqslant 0$ for all $|J| \leqslant n$, and consider the case when $|J| = n+1$. By symmetry, it would suffice to show this for $J = J_{n+1} = \{1, 2, \ldots, n+1\}$.

By Proposition 5.4, $X_n = R_n^* R_n$ where the $(\emptyset,\emptyset)$-entry of R_n is equal to $Z_{J_n}^{1/2}$. Let D_n be a block diagonal matrix with 2^n copies of T_{n+1} along the diagonal. Condition (2.1) implies that

$$D_n^* X_n D_n = D_n^* R_n^* R_n D_n \leqslant X_n = R_n^* R_n.$$

Therefore, by Lemma 5.2, there exists a contraction W_n such that $W_n R_n = R_n D_n$. By comparing the $(\emptyset,\emptyset)$-entry on both sides, there exists C_n such that $C_n Z_{J_n}^{1/2} = Z_{J_n}^{1/2} T_{n+1}$, where C_n is the $(\emptyset,\emptyset)$-entry of W_n, which must be contractive as well. Hence, by Lemma 5.1 and 5.2,

$$Z_{J_{n+1}} = Z_{J_n} - T_{n+1}^* Z_{J_n} T_{n+1} \geqslant 0.$$

This finishes the proof. ∎

6. COVARIANT REPRESENTATIONS

The semicrossed products of a dynamical system by Nica-covariant representations were discussed in [6], [8], where its regularity is seen as a key to many results. Our result on the regularity of Nica-covariant representations (Theorem 3.1 and Corollary 3.2) allows us to generalize some of the results to arbitrary lattice ordered abelian groups.

DEFINITION 6.1. A C^*-dynamical system is a triple (A, α, P) where

(i) A is a C^*-algebra;
(ii) $\alpha : P \to \mathrm{End}(A)$ maps each $p \in P$ to a $*$-endomorphism on A;
(iii) P is a spanning cone of some group G.

DEFINITION 6.2. A pair (π, T) is called a *covariant pair* for a C^*-dynamical system if

(i) $\pi : A \to \mathcal{B}(\mathcal{H})$ is a $*$-representation;
(ii) $T : P \to \mathcal{B}(\mathcal{H})$ is a contractive representation of P;
(iii) $\pi(a)T(s) = T(s)\pi(\alpha_s(a))$ for all $s \in P$ and $a \in A$.

In particular, a covariant pair (π, T) is called Nica-covariant/isometric, if T is Nica-covariant/isometric.

The main goal is to prove that Nica-covariant pairs on C^*-dynamical systems can be lifted to isometric Nica-covariant pairs. This can be seen from Theorem 4.1.2 of [6] and Corollary 3.2. However, we shall present a slightly different approach by taking the advantage of the structure of lattice ordered abelian groups.

THEOREM 6.3. *Let (A, α, P) be a C^*-dynamical system over a positive cone P of a lattice ordered abelian group G. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ and $T : P \to \mathcal{B}(\mathcal{H})$ form a Nica-covariant pair (π, T) for this C^*-dynamical system. If $V : P \to \mathcal{K}$ is a minimal isometric dilation of T, then there is an isometric Nica-covariant pair (ρ, V) such that for all $a \in A$,*

$$P_{\mathcal{H}}\rho(a)|_{\mathcal{H}} = \pi(a).$$

Moreover, $\mathcal{H}$ is invariant for $\rho(a)$.

Proof. Fix a minimal dilation V of T and consider any $h \in \mathcal{H}$, $p \in P$, and $a \in A$; define

$$\rho(a)V(p)h = V(p)\pi(\alpha_p(a))h.$$

We shall first show that this is a well defined map. First of all, since V is a minimal isometric dilation, the set $\{V(p)h\}$ is dense in $\mathcal{K}$. Suppose $V(p)h_1 = V(s)h_2$ for some $p, s \in P$ and $h_1, h_2 \in \mathcal{H}$. It suffices to show that for any $t \in P$ and $h \in \mathcal{H}$, we have

$$\langle V(p)\pi(\alpha_p(a))h_1, V(t)h\rangle = \langle V(s)\pi(\alpha_s(a))h_2, V(t)h\rangle. \tag{6.1}$$

Since A is a C^*-dynamical system, it follows from the covariant condition that $T(s)^*\pi(a) = \pi(\alpha_s(a))T(s)^*$. Hence,

$$\begin{aligned}\langle V(p)\pi(\alpha_p(a))h_1, V(t)h\rangle &= \langle V(t)^*V(p)\pi(\alpha_p(a))h_1, h\rangle \\ &= \langle V(t-t\wedge p)^*V(p-t\wedge p)\pi(\alpha_p(a))h_1, h\rangle \\ &= \langle T(t-t\wedge p)^*T(p-t\wedge p)\pi(\alpha_p(a))h_1, h\rangle \\ &= \langle \pi(\alpha_{p-(p-t\wedge p)+(t-t\wedge p)}(a))T(t-t\wedge p)^*T(p-t\wedge p)h_1, h\rangle \\ &= \langle \pi(\alpha_t(a))T(t-t\wedge p)^*T(p-t\wedge p)h_1, h\rangle.\end{aligned}$$

Here we used that fact that V is regular and thus

$$P_{\mathcal{H}}V(t-t\wedge p)^*V(p-t\wedge p)|_{\mathcal{H}} = T(t-t\wedge p)^*T(p-t\wedge p).$$

Now notice that

$$T(t-t\wedge p)^*T(p-t\wedge p)h_1 = P_{\mathcal{H}}V(t-t\wedge p)^*V(p-t\wedge p)h_1 = P_{\mathcal{H}}V(t)^*V(p)h_1.$$

Similarly,

$$\langle V(s)\pi(\alpha_s(a))h_2, V(t)h\rangle = \langle \pi(\alpha_t(a))T(t-t\wedge s)^*T(s-t\wedge s)h_2, h\rangle,$$

where

$$T(t-t\wedge s)^*T(s-t\wedge s)h_2 = P_{\mathcal{H}}V(t)^*V(s)h_2 = P_{\mathcal{H}}V(t)^*V(p)h_1.$$

Therefore, ρ is well defined on the dense subset $\{V(p)h\}$.

Since $V(p)$ is isometric and π, α are completely contractive,

$$\|V(p)\pi(\alpha_p(a))h\| = \|\pi(\alpha_p(a))h\| \leqslant \|h\| = \|V(p)h\|,$$

and thus $\rho(a)$ is contractive on $\{V(p)h\}$. Hence, $\rho(a)$ can be extended to a contractive map on $\mathcal{K}$. Moreover, for any $h \in \mathcal{H}$ and $a \in A$, we have $\rho(a)h = \pi(a)h \in \mathcal{H}$, and thus $\mathcal{H}$ is invariant for ρ. For any $a, b \in A$, $p \in P$, and $h \in \mathcal{H}$,

$$\rho(a)\rho(b)V(p)h = V(p)\pi(\alpha_p(a))\pi(\alpha_p(b))h = V(p)\pi(\alpha_p(ab))h = \rho(ab)V(p)h.$$

Therefore, ρ is a contractive representation of A and thus a $*$-representation. Now for any $p, t \in P$ and $h \in \mathcal{H}$,

$$\begin{aligned}\rho(a)V(p)V(t)h &= V(p+t)\pi(\alpha_{p+t}(a))h = V(p)V(t)\rho(\alpha_{p+t}(a))h \\ &= V(p)\rho(\alpha_p(a))V(t)h.\end{aligned}$$

Hence, (ρ, V) is an isometric Nica-covariant pair. ∎

This lifting of contractive Nica-covariant pairs to isometric Nica-covariant pairs has significant implications in its associated semi-crossed product. A family of covariant pairs gives rise to a semi-crossed product algebra in the following way [6], [8]. For a C^*-dynamical system (A, α, P), let $\mathcal{P}(A, P)$ be the algebra of all formal polynomials q of the form

$$q = \sum_{i=1}^{n} e_{p_i} a_{p_i},$$

where $p_i \in P$ and $a_{p_i} \in A$. The multiplication on such polynomials follows the rule that $ae_s = e_s\alpha(a)$ and $e_pe_q = e_{pq}$. For a covariant pair (σ, T) on this dynamical system, define a representation of $\mathcal{P}(A, P)$ by

$$(\sigma \times T)\Big(\sum_{i=1}^{n} e_{p_i}a_{p_i}\Big) = \sum_{i=1}^{n} T(p_i)\sigma(a_{p_i}).$$

Now let $\mathcal{F}$ be a family of covariant pairs on this dynamical system. We may define a norm on $\mathcal{P}(A, S)$ by

$$\|p\|_{\mathcal{F}} = \sup\{(\sigma \times T)(p) : (\sigma, T) \in \mathcal{F}\},$$

and the semi-crossed product algebra is defined as

$$A \times_\alpha^{\mathcal{F}} P = \overline{\mathcal{P}(A, S)}^{\|\cdot\|_{\mathcal{F}}}.$$

In particular, $A \times_\alpha^{\mathrm{nc}} P$ is determined by the Nica-covariant representations, and $A \times_\alpha^{\mathrm{nc,iso}} P$ is determined by the isometric Nica-covariant representation. As an immediate corollary from Theorems 2.1 and 6.3,

COROLLARY 6.4. *For a C^*-dynamical system (A, α, P), the semi-crossed product algebra given by Nica-covariant pairs agrees with that given by isometric Nica-covariant pairs. In other words,*

$$A \times_\alpha^{\mathrm{nc}} P \cong A \times_\alpha^{\mathrm{nc,iso}} P.$$

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