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SIMILARITY TO THE BACKWARD SHIFT OPERATOR ON THE DIRICHLET SPACE

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ABSTRACT. We give a necessary condition for a Cowen–Douglas operator to be similar to the backward shift operator on the Dirichlet space. A model theorem for weighted shifts provides an eigenvector bundle structure for the operators involved and plays a fundamental role in this geometric description.

KEYWORDS: *Backward shift, Cowen–Douglas operator, Dirichlet space, eigenvector bundle, reproducing kernel, similarity.*

MSC (2010): 47A99, 47B32, 47B38, 30J99, 53C55.

INTRODUCTION

Previously, partial results to the similarity problem of Cowen–Douglas operators in terms of the curvatures of the corresponding eigenvector bundles were obtained by restricting attention to the similarity to the backward shift operator on the Hardy space or the weighted Bergman spaces [5], [8]. Recall that the importance of the backward shift operator, or the adjoint to the operator of multiplication by the independent variable, comes from the fact that it serves as a simple model for a large class of operators [1], [2], [10]. Unlike the case for unitary equivalence that was completely solved by M.J. Cowen and R.G. Douglas in [4], the characterization for similarity is still far from being complete. Hence, a continued investigation of the backward shift operator on various function spaces can serve as a basis for the general solution and in the present paper, we focus on the backward shift operator on the Dirichlet space, the remaining holomorphic function space of a single variable that has attracted much attention over the years.

In comparison to the Hardy and the Bergman spaces, $\frac{1}{K(z,\overline{w})}$, where $K(z,\overline{w})$ denotes the reproducing kernel function, is no longer a polynomial in z and $\overline{w}$ in the Dirichlet space. One can ask whether the similarity characterization in terms of the eigenvector bundles of the operators obtained in the previous spaces still holds in the Dirichlet space. By invoking a model theorem that applies to weighted shifts due to V. Müller [9], we acquire an eigenvector bundle structure of the backward shift operator on the Dirichlet space in the exact form as in these other spaces. This structure will then allow us to give a necessary condition for similarity.

1. PRELIMINARIES

The Dirichlet space $\mathcal{D}$ consists of all analytic functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ defined on the unit disk $\mathbb{D}$ of the complex plane satisfying

$$\|f\|^2 = \sum_{n=0}^{\infty} (n+1)|\widehat{f}(n)|^2 < \infty.$$

Just like the Hardy and weighted Bergman spaces, the Dirichlet space $\mathcal{D}$ is a reproducing kernel Hilbert space and its reproducing kernel is of the form

$$k_\lambda(z) = K(z,\overline{\lambda}) = \frac{1}{\overline{\lambda}z} \log \frac{1}{1-\overline{\lambda}z} = 1 + \frac{1}{2}\overline{\lambda}z + \frac{1}{3}\overline{\lambda}^2 z^2 + \cdots .$$

We can define $\mathcal{D}_{\mathcal{E}}$, the vector-valued analogue of $\mathcal{D}$ taking values in a Hilbert space $\mathcal{E}$, in an obvious way. We will write $\mathcal{D}_n$ when $\mathcal{E}$ is of dimension n.

The operator of multiplication by z on $\mathcal{D}$, denoted D, is a bounded linear operator. We denote its adjoint, called the backward shift operator, by D^* and can define $D^*_{\mathcal{E}}$ in an analogous way on the space $\mathcal{D}_{\mathcal{E}}$. A simple but a crucial observation that will be useful later is the following: since an orthonormal basis for $\mathcal{D}$ is given by $\left\{ \frac{z^n}{\sqrt{n+1}} \right\}_{n=0}^{\infty}$, $D^*_{\mathcal{E}}$ can be viewed as the weighted backward shift

$$S_\alpha(x_0, x_1, x_2, \ldots) = \Big(\frac{\sqrt{2}}{\sqrt{1}} x_1, \frac{\sqrt{3}}{\sqrt{2}} x_2, \frac{\sqrt{4}}{\sqrt{3}} x_3, \ldots \Big),$$

for $x_i \in \mathcal{E}$, corresponding to the weight sequence $\alpha = \{\alpha_n\}_{n=1}^{\infty} = \left\{ \frac{\sqrt{n+1}}{\sqrt{n}} \right\}_{n=1}^{\infty}$. Moreover,

$$D^* k_{\overline{\lambda}} = \lambda k_{\overline{\lambda}},$$

for all $\lambda \in \mathbb{D}$, so that the eigenvectors of $D^*_{\mathcal{E}}$ corresponding to the eigenvalue $\lambda \in \mathbb{D}$ are of the form $k_{\overline{\lambda}} e$ for $e \in \mathcal{E}$.

2. MAIN RESULT

Let $\mathcal{H}$ be a separable Hilbert space. We assume that an operator $T \in \mathcal{L}(\mathcal{H})$ satisfies the following assumptions:

(i) $\sum_{n=1}^{\infty} \frac{\|T^n\|^2}{n+1} \leqslant 1$;

(ii) $\bigvee_{\lambda\in\mathbb{D}} \ker(T-\lambda) = \mathcal{H}$; and

(iii) the subspaces $\ker(T-\lambda)$ depend analytically on the spectral parameter $\lambda \in \mathbb{D}$.

REMARK 2.1. Although assumption (i) looks technical at first sight, it can easily be shown that certain nilpotent operators and operators with $\|T\| < 1$ are among those satisfying (i). We will see in the next section the importance of placing this assumption in obtaining the main result.

By assumption (iii), we have for each $\lambda \in \mathbb{D}$, a neighborhood U_λ of λ and an operator-valued, bounded, analytic function $F_\lambda \in H^\infty_{\mathcal{H}\to\mathcal{H}}$ defined on U_λ with ran $F_\lambda(\omega) = \ker(T-\omega)$ for $\omega \in U_\lambda$. This analytic function F_λ has a left inverse in $L^\infty_{\mathcal{H}\to\mathcal{H}}$ and it can be easily shown that $\dim\ker(T-\lambda)$ is constant for all $\lambda \in \mathbb{D}$. Note that a Cowen–Douglas operator in $B_m(\mathbb{D})$, where m is a positive integer, satisfies assumptions (ii) and (iii) (cf. [4]). One of the main results in [4] states that the disjoint union

$$E_T = \coprod_{\lambda\in\mathbb{D}} \ker(T-\lambda) = \{(\lambda, v_\lambda) : \lambda \in \mathbb{D}, v_\lambda \in \ker(T-\lambda)\}$$

is a Hermitian holomorphic vector bundle over $\mathbb{D}$ with the metric inherited from $\mathcal{H}$ and the natural projection π, $\pi(\lambda, v_\lambda) = \lambda$.

Since we are interested in working with the vector bundle E_T for the study of the operator T, we define a $\mathcal{C}^\infty$ function Π defined on $\mathbb{D}$, with values that are orthogonal projections onto the fibers of E_T, that is,

$$\Pi(\lambda) = \mathcal{P}_{\ker(T-\lambda)},$$

for $\lambda \in \mathbb{D}$. The following theorem is the main result of the paper. Here Δ stands for the normalized Laplacian

$$\Delta = \partial\bar{\partial} = \bar{\partial}\partial = \frac{1}{4}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big),$$

and $\mathfrak{S}_2$ is the Hilbert–Schmidt class of operators.

THEOREM 2.2. *Let* $T \in \mathcal{L}(\mathcal{H})$ *satisfy assumptions* (i) *through* (iii). *Let*

$$\dim\ker(T-\lambda) = n < \infty$$

for every $\lambda \in \mathbb{D}$, *and let* $\Pi(\lambda)$ *denote the orthogonal projection onto* $\ker(T-\lambda)$. *If* T *is similar to the backward shift operator* D_n^* *on the vector-valued space* $\mathcal{D}_n$, *that is, if there exists a bounded, invertible operator* $A : \mathcal{D}_n \to \mathcal{H}$ *satisfying* $TA = AD_n^*$, *then*

$$\Delta\phi(\lambda) \geqslant \left\| \frac{\partial\Pi(\lambda)}{\partial\lambda} \right\|^2_{\mathfrak{S}_2} + l(\lambda) \quad \textit{for all } \lambda \in \mathbb{D},$$

for some bounded subharmonic function ϕ defined on $\mathbb{D}$, where

$$l(\lambda) = \begin{cases} \frac{n[\log(1-|\lambda|^2)+|\lambda|^2]}{[\log(1-|\lambda|^2)(1-|\lambda|^2)]^2} & \text{if } 0 \neq \lambda \in \mathbb{D}, \\ -\frac{n}{2} & \text{if } \lambda = 0. \end{cases}$$

The $-\frac{1}{2}$ appearing in the theorem is the limit of $\frac{\log(1-|\lambda|^2)+|\lambda|^2}{[\log(1-|\lambda|^2)(1-|\lambda|^2)]^2}$ as λ approaches 0. Taking note that $-\frac{\partial \Pi(\lambda)}{\partial \lambda}$ is the second fundamental form of the bundle E_T, we can see that $-\left\| \frac{\partial \Pi(\lambda)}{\partial \lambda} \right\|^2_{\mathfrak{S}_2}$ is its curvature [7]. The other term $l(\lambda)$ to the right of the inequality is the curvature for $E_{D_n^*}$ for $\lambda \neq 0$.

3. COMPUTATION OF E_T

Let us first consider the following model theorem by V. Müller [9] that generalizes and plays the role of the results by B. Sz-Nagy and C. Foias [10] and by J. Agler [1], [2] in our previous work. Recall that the backward shift operator D^* in the Dirichlet space $\mathcal{D}_{\mathcal{E}}$ is of the form

$$S_\alpha(x_1, x_2, \ldots) = \Big(\frac{\sqrt{2}}{\sqrt{1}}x_1, \frac{\sqrt{3}}{\sqrt{2}}x_2, \frac{\sqrt{4}}{\sqrt{3}}x_3, \ldots\Big),$$

with the weight sequence $\alpha = \{\alpha_n\}_{n=1}^{\infty} = \left\{\frac{\sqrt{n+1}}{\sqrt{n}}\right\}_{n=1}^{\infty}$.

THEOREM 3.1. *Let $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ be such that $\alpha_j \geqslant \alpha_{j+1} > 0$ for all $j \geqslant 1$. Then there exists a subspace $\mathcal{K} \subset \mathcal{D}_{\mathcal{E}}$, $S_\alpha \mathcal{K} \subset \mathcal{K}$, such that T is unitarily equivalent to $S_\alpha|_{\mathcal{K}}$ if and only if $\sum_{n=1}^{\infty} \|T^n\|^2 b_n \leqslant 1$, where for $n \geqslant 1$, $b_n = \alpha_n^{-2} \cdots \alpha_1^{-2}$.*

Theorem 3.1 provides the grounds for having assumption (i) about T; for D^*, we have $b_n = \frac{1}{n+1}$. It allows us to apply Theorem 3.1 to conclude that T is unitarily equivalent to $D^*_{\mathcal{E}}$ restricted to some invariant subspace.

We therefore have the eigenspace structure of $T = D^*_{\mathcal{E}}|_{\mathcal{K}}$ as

$$\ker(T-\lambda) = \{k_{\overline{\lambda}} e : e \in \mathcal{E}(\lambda)\},$$

where $\mathcal{E}(\lambda)$ is the subspace

$$\mathcal{E}(\lambda) = \{e \in \mathcal{E}; k_{\overline{\lambda}} e \in \mathcal{K}\}.$$

Note that $\ker(T-\lambda)$ is a holomorphic vector bundle due to assumption (iii) and this implies that the family of subspaces $\mathcal{E}(\lambda)$ is again a holomorphic vector bundle over $\mathbb{D}$.

The vector-valued Dirichlet space $\mathcal{D}_{\mathcal{E}}$ can be realized as the tensor product $\mathcal{D} \otimes \mathcal{E}$ and one can then express each fiber of E_T as

$$\ker(T-\lambda) = \bigvee \{k_{\overline{\lambda}}\} \otimes \mathcal{E}(\lambda).$$

Having this tensor product form of the eigenvector bundle, one can now represent $\Pi(\lambda)$, the orthogonal projection onto $\ker(T-\lambda)$, as a tensor product of the operators $\Pi_1(\lambda)$ and $\Pi_2(\lambda)$, which are the orthogonal projections from $\mathcal{D}$ onto $\bigvee\{k_{\overline{\lambda}}\}$ and from $\mathcal{E}$ onto $\mathcal{E}(\lambda)$, respectively:

$$\Pi(\lambda) = \Pi_1(\lambda) \otimes \Pi_2(\lambda).$$

It will be shown next from the theorems stated below that the proof of the main theorem really depends on the second part $\Pi_2(\lambda)$ of $\Pi(\lambda)$. Note first that

$$\operatorname{rank} \Pi_1(\lambda) = 1,$$

and that

$$\operatorname{rank} \Pi(\lambda) = \operatorname{rank} \Pi_2(\lambda) = n.$$

THEOREM 3.2. *For* $0 \neq \lambda \in \mathbb{D}$,

$$\begin{aligned} \left\| \frac{\partial \Pi(\lambda)}{\partial \lambda} \right\|^2_{\mathfrak{S}_2} &= n \left\| \frac{\partial \Pi_1(\lambda)}{\partial \lambda} \right\|^2_{\mathfrak{S}_2} + \left\| \frac{\partial \Pi_2(\lambda)}{\partial \lambda} \right\|^2_{\mathfrak{S}_2} \\ &= -\frac{n[\log(1-|\lambda|^2)+|\lambda|^2]}{[\log(1-|\lambda|^2)(1-|\lambda|^2)]^2} + \left\| \frac{\partial \Pi_2(\lambda)}{\partial \lambda} \right\|^2_{\mathfrak{S}_2}. \end{aligned}$$

Proof. For the first equality, we use the product rule, the fact that for an orthogonal projection P, we have $\| P \|^2_{\mathfrak{S}_2} = \operatorname{rank} P$, and the identities

$$\Pi_2(\lambda) \frac{\partial \Pi_2(\lambda)}{\partial \lambda} = 0,$$

and

$$(I - \Pi_2(\lambda)) \frac{\partial \Pi_2(\lambda)}{\partial \lambda} \Pi_2(\lambda) = \frac{\partial \Pi_2(\lambda)}{\partial \lambda},$$

that follow from assumption (iii). For details on these identities, we refer the reader to [5] or [8]. To complete the proof, it suffices to show the identity

$$\left\| \frac{\partial \Pi_1(\lambda)}{\partial \lambda} \right\|^2_{\mathfrak{S}_2} = -\frac{\log(1-|\lambda|^2)+|\lambda|^2}{[\log(1-|\lambda|^2)(1-|\lambda|^2)]^2}.$$

By [4], the curvature of E_{D^*} can be calculated via the simple formula

$$-\Delta \log \|k_\lambda\|^2,$$

where $k_\lambda(z) = \frac{1}{\overline{\lambda} z} \log(\frac{1}{1-\overline{\lambda} z})$ for $\lambda \neq 0$, and this will prove the identity. One can also give an alternative proof as follows:

First, the reproducing kernel property of k_λ implies that

$$\|k_\lambda\|_2^2 = -\frac{\log(1-|\lambda|^2)}{|\lambda|^2}.$$

Therefore for $f \in \mathcal{D}$,

$$\Pi_1(\lambda) f = -\frac{|\lambda|^2}{\log(1-|\lambda|^2)} f(\overline{\lambda}) k_{\overline{\lambda}}.$$

If we take $\frac{\partial}{\partial\lambda}$ and use the fact that $\frac{\partial f(\overline{\lambda})}{\partial\lambda} = 0$, $\frac{\partial\Pi_1(\lambda)}{\partial\lambda} f$ equals

$$-\frac{f(\overline{\lambda})\overline{\lambda}}{[\log(1-|\lambda|^2)]^2}\Big(\Big[\log(1-|\lambda|^2) + \frac{|\lambda|^2}{1-|\lambda|^2}\Big]k_{\overline{\lambda}} + \lambda\log(1-|\lambda|^2)\widetilde{k}_{\overline{\lambda}}\Big),$$

where

$$\widetilde{k}_{\overline{\lambda}}(z) = \frac{\partial}{\partial\lambda}k_{\overline{\lambda}}(z) = \frac{\lambda z^2 + z(1-\lambda z)\log(1-\lambda z)}{(1-\lambda z)(\lambda z)^2}.$$

Next, since

$$\langle f, \widetilde{k}_\lambda\rangle = f'(\lambda),$$

for all $f \in \mathcal{D}$,

$$\|\widetilde{k}_\lambda\|_2^2 = \|\widetilde{k}_{\overline{\lambda}}\|_2^2 = -\frac{(1-|\lambda|^2)^2\log(1-|\lambda|^2) + |\lambda|^2 - 2|\lambda|^4}{(1-|\lambda|^2)^2|\lambda|^4},$$

and using the reproducing property for k_λ one more time results in

$$\langle\widetilde{k}_{\overline{\lambda}}, k_{\overline{\lambda}}\rangle = \frac{|\lambda|^2\overline{\lambda} + \overline{\lambda}(1-|\lambda|^2)\log(1-|\lambda|^2)}{(1-|\lambda|^2)|\lambda|^4}.$$

All of the above calculations add up to help us conclude that

$$\begin{aligned}\Big\|\Big[\log(1-|\lambda|^2) + \frac{|\lambda|^2}{1-|\lambda|^2}\Big]k_{\overline{\lambda}} + \lambda\log(1-|\lambda|^2)\widetilde{k}_{\overline{\lambda}}\Big\|_2^2 \\ = \frac{[\log(1-|\lambda|^2)]^2 + |\lambda|^2\log(1-|\lambda|^2)}{(1-|\lambda|^2)^2}.\end{aligned}$$

Thus,

$$\Big\|\frac{\partial\Pi_1(\lambda)}{\partial\lambda}\Big\|^2 = -\frac{\log(1-|\lambda|^2) + |\lambda|^2}{[\log(1-|\lambda|^2)(1-|\lambda|^2)]^2},$$

and since rank $\frac{\partial\Pi_1(\lambda)}{\partial\lambda} = 1$, the operator and the Hilbert–Schmidt norms of $\frac{\partial\Pi_1(\lambda)}{\partial\lambda}$ are the same. ∎

THEOREM 3.3. *For $\lambda = 0$,*

$$\Big\|\frac{\partial\Pi(\lambda)}{\partial\lambda}\Big\|_{\mathfrak{S}_2}^2 = \Big\|\frac{\partial\Pi_2(\lambda)}{\partial\lambda}\Big\|_{\mathfrak{S}_2}^2.$$

Proof. For $\lambda = 0$, $k_0(z) = 1$ so that $\Big\|\frac{\partial\Pi_1(\lambda)}{\partial\lambda}\Big\|^2 = 0$. ∎

Therefore, in order to prove Theorem 2.2, it is enough to show the existence of a bounded, subharmonic function ϕ defined on $\mathbb{D}$ such that

$$\Delta\phi(\lambda) \geqslant \Big\|\frac{\partial\Pi_2(\lambda)}{\partial\lambda}\Big\|_{\mathfrak{S}_2}^2.$$

4. PROOF OF THE NECESSITY

It is obvious that similar operators induce a bundle map bijection between the corresponding eigenvector bundles. If Ψ is such a bundle map between E_T and E_{D^*}, then it is an analytic function of λ that linearly moves each fiber $\ker(D_n^* - \lambda)$ to $\ker(T - \lambda)$ for every $\lambda \in \mathbb{D}$. It should then be of the form

$$\Psi(k_{\overline{\lambda}}e) = k_{\overline{\lambda}} \cdot F(\lambda)e,$$

for a bounded, analytic, operator-valued function $F \in H^\infty_{\mathbb{C}^n \to \mathcal{E}}$ whose range equals $\mathcal{E}(\lambda)$, such that

$$c^{-1}I \leqslant F^*(\lambda)F(\lambda) \leqslant cI,$$

for all $\lambda \in \mathbb{D}$ and for all $e \in \mathbb{C}^n$. Hence, the orthogonal projection $\Pi_2(\lambda)$ from $\mathcal{E}$ onto $\mathcal{E}(\lambda)$ can be represented in terms of the operator $F(\lambda)$ as

$$\Pi_2(\lambda) = F(\lambda)(F^*(\lambda)F(\lambda))^{-1}F^*(\lambda).$$

Now differentiating, we get

$$\frac{\partial \Pi_2(\lambda)}{\partial \lambda} = (I - \Pi_2(\lambda))F'(\lambda)(F(\lambda)^*F(\lambda))^{-1}F(\lambda)^*,$$

and therefore,

$$\left\| \frac{\partial \Pi_2(z)}{\partial z} \right\|_{\mathfrak{S}_2} \leqslant C \left\| F'(z) \right\|_{\mathfrak{S}_2}.$$

We then take $\phi(\lambda) = \| F(\lambda) \|^2_{\mathfrak{S}_2}$ and note that

$$\Delta\phi(\lambda) = \| F'(\lambda) \|^2_{\mathfrak{S}_2}$$

to end the proof of Theorem 2.2.

5. REMARK ON THE SUFFICIENCY

Unlike in previous work, one can no longer use the theorem by S. Treil and B.D. Wick [11] to conclude the similarity of T to D_n^* from the existence of a function ϕ. The bounded operator that establishes the similarity between them in other spaces was a Toeplitz operator with symbol that is related to the outer part of the analytic projection formed from $\Pi_2(\lambda)$. However, there are issues with the boundedness of the Toeplitz operator thus formed in the Dirichlet space setting and the problem of whether it has dense range ([3], [6]) is still another obstacle.

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