J. OPERATOR THEORY
76:1(2016), 141–158
doi: 10.7900/jot.2015oct16.2084

HYPERCYCLIC CONVOLUTION OPERATORS ON SPACES OF ENTIRE FUNCTIONS

VINÍCIUS V. FÁVARO and JORGE MUJICA

Dedicated to the memory of Jorge Alberto Barroso (1928-2015)

Communicated by Stefaan Vaes

ABSTRACT. A classical result of Birkhoff states that every nontrivial translation operator on the space $\mathcal{H}(\mathbb{C})$ of entire functions of one complex variable is hypercyclic. Godefroy and Shapiro extended this result considerably by proving that every nontrivial convolution operator on the space $\mathcal{H}(\mathbb{C}^n)$ of entire functions of several complex variables is hypercyclic. In sharp contrast with these classical results we show that no convolution operator on the space $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ of entire functions of infinitely many complex variables is hypercyclic. On the positive side we obtain hypercyclicity results for convolution operators on spaces of entire functions on important locally convex spaces.

KEYWORDS: *Hypercyclicity, convolution operators, entire functions, Banach spaces, locally convex spaces.*

MSC (2010): 47A16, 46G20, 32A15, 30H50, 46A11.

INTRODUCTION

A classical result of Birkhoff [8] states that every nontrivial translation operator on $\mathcal{H}(\mathbb{C})$ is hypercyclic. We recall that if X is a topological vector space, then a continuous linear operator $T : X \to X$ is said to be *hypercyclic* if its *orbit* $\{x, T(x), T^2(x), \ldots\}$ is dense in X for some $x \in X$. In this case, x is said to be a *hypercyclic vector for T*. On the other hand a result of MacLane [36] asserts that every differentiation operator on $\mathcal{H}(\mathbb{C})$ is hypercyclic. These results were generalized by Godefroy and Shapiro [25], who proved that every nontrivial convolution operator on $\mathcal{H}(\mathbb{C}^n)$ is hypercyclic. We recall that a *convolution operator* on $\mathcal{H}(\mathbb{C}^n)$ is a continuous linear operator that commutes with translations. By a *nontrivial convolution operator* we mean a convolution operator which is not a scalar multiple of the identity.

There are several directions and ramifications of the study of hypercyclic operators. References [4], [27], [28] provide an overview of the theory. We remark that several results on the hypercyclicity of operators on spaces of entire functions of infinitely many complex variables have appeared in the last few decades. See for instance [2], [5], [6], [7], [14], [24], [27], [39], [44].

In this paper we are mainly interested in the hypercyclicity of convolution operators on spaces of entire functions of infinitely many complex variables. In sharp contrast with the aforementioned results of Birkhoff [8], MacLane [36] and Godefroy and Shapiro [25], we show that no convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is hypercyclic. At first sight this result may look surprising, since it is well known that every $f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ depends only of finitely many variables. But our result follows precisely from this fact.

On the positive side we show that, when E is the strong dual of a Fréchet nuclear space (for short a (DFN)-space), then every nontrivial convolution operator on $\mathcal{H}(E)$ is hypercyclic. More generally we obtain the same conclusion when $E = F'_c$, where F is a separable Fréchet space with the approximation property. We denote by F'_c the dual of a Fréchet space F with the topology of compact convergence (for short a (DFC)-space).

Our proofs combine results for the spaces $\mathcal{H}_{\Theta b}(E)$ of entire functions of a given bounded holomorphy type on a complex Banach space E obtained in [6], [22], with a factorization method introduced by Colombeau and Matos [16]. Besides that, our proofs rest on a classical hypercyclicity criterion, first obtained by Kitai [34], but never published and later on rediscovered by Gethner and Shapiro [24].

It is clear that our first positive result follows from the second one, but we have preferred to present both results separately to illustrate the usefulness of different holomorphy types. Besides that, we first obtained the result in the case of (DFN)-spaces, whose proof is simpler, and later on we extended the result to the case of (DFC)-spaces.

Nowadays it is known, by a result of Costakis and Sambarino [18], that the classical hypercyclicity criterion of Kitai ensures that the operator is mixing, a stronger property than hypercyclicity. We recall that if X is a topological vector space, then a continuous linear operator $T : X \to X$ is said to be *mixing* if for any two non-empty open sets $U, V \subset X$, there is $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$, for all $n \geqslant n_0$. Actually it is known that if X is a Fréchet space, then a continuous linear operator $T : X \to X$ is mixing if and only if it is *hereditarily hypercyclic*, that is, for any strictly increasing sequence $(n_j) \subset \mathbb{N}$ there exists $x \in X$ such that the sequence $(T^{n_j}(x))$ is dense in X (this is proved in [26] in the case of Banach spaces, but the proof works equally well in the case of Fréchet spaces).

Because of this, the aforementioned results on (DFN) and (DFC)-spaces ensure that every nontrivial convolution operator on $\mathcal{H}(E)$ is actually mixing, or equivalently hereditarily hypercyclic.

This paper is organized as follows. In Section 2 we collect some general results which are often used in subsequent sections. Sections 3 and 4 are devoted to the study of convolution operators on spaces of entire functions on (DFN) and

(DFC)-spaces, respectively. Finally, in Section 5 we prove the aforementioned result that no convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is hypercyclic.

In the original version of Theorem 4.1(ii) we had proved that no translation operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ is hypercyclic. By a clever refinement of our original proof the referee was able to prove that no convolution operator on $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$ is hypercyclic.

Throughout this paper $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$. All the locally convex spaces are assumed to be complex and Hausdorff. By Δ we mean the open unit disk in the complex field $\mathbb{C}$. If E is a locally convex space, then E'_{b} (respectively E'_{c}) denotes the dual E' of E with the topology of bounded convergence (respectively compact convergence). If E and F are normed spaces, with F complete, then the Banach space of all continuous m-homogeneous polynomials from E into F endowed with its usual sup norm is denoted by $\mathcal{P}(^mE; F)$. The subspace of $\mathcal{P}(^mE; F)$ of all polynomials of finite type is represented by $\mathcal{P}_f(^mE; F)$. For E and F locally convex spaces, with F complete, $\mathcal{H}(E; F)$ denotes the vector space of all holomorphic mappings from E into F. In all these cases, when $F = \mathbb{C}$ we write $\mathcal{P}(^mE)$, $\mathcal{P}_f(^mE)$ and $\mathcal{H}(E)$ instead of $\mathcal{P}(^mE; \mathbb{C})$, $\mathcal{P}_f(^mE; \mathbb{C})$ and $\mathcal{H}(E; \mathbb{C})$, respectively. The compact-open topology on the space $\mathcal{H}(E)$ is denoted by τ_0. For the general theory of homogeneous polynomials and holomorphic functions we refer to Dineen [20] and Mujica [38].

Finally, $cs(E)$ denotes the set of all continuous seminorms on the locally convex space E. If $p \in cs(E)$, then E_p denotes the normed space $(E, p)/p^{-1}(0)$, and π_p denotes the canonical surjective mapping $\pi_p : E \to E_p$. We say that $D \subset cs(E)$ is a *fundamental family* if D is a directed subset of $cs(E)$ and the seminorms $p \in D$ generate the topology of E.

1. PRELIMINAIRES

In this section we recall the concepts and results about holomorphic functions on normed spaces that we need and we introduce some similar concepts for holomorphic functions defined on locally convex spaces. It is important to say that all definitions of this section and all results of [6] and [22] that we will use during the paper were originally stated for E and F Banach spaces, with exception of Definitions 1.5, 1.9 and 1.11 that are introduced in this paper for the first time. But it is clear that they are still valid if we consider E only normed.

DEFINITION 1.1. Let E and F be normed spaces, with F complete, and U be an open subset of E. A mapping $f : U \longrightarrow F$ is said to be *holomorphic on U* if for every $a \in U$ there exists a sequence $(P_m)_{m=0}^{\infty}$, where each $P_m \in \mathcal{P}(^mE; F)$ ($\mathcal{P}(^0E; F) = F$), such that $f(x) = \sum_{m=0}^{\infty} P_m(x - a)$ uniformly on some open ball with center a. The m-homogeneous polynomial $m!P_m$ is called the *m-th derivative* *of f at a* and is denoted by $\widehat{d}^m f(a)$. In particular, if $P \in \mathcal{P}(^m E; F)$, $a \in E$ and $k \in \{0, 1, \ldots, m\}$, then

$$\widehat{d}^k P(a)(x) = \frac{m!}{(m-k)!} \check{P}(\underbrace{x, \ldots, x}_{k \text{ times}}, a, \ldots, a)$$

for every $x \in E$, where $\check{P}$ denotes the unique symmetric m-linear mapping associated to P.

A mapping $f : U \to F$ is holomorphic if and only if f is continuous and G-holomorphic, that is, $f(a + \lambda b)$ is a holomorphic function of the complex variable λ for each $a \in U$ and $b \in E$ (see Theorem 8.7 of [38]).

DEFINITION 1.2 (Nachbin [41]). Let E and F be normed spaces, with F complete. A *holomorphy type* Θ from E to F is a sequence of Banach spaces $(\mathcal{P}_\Theta(^m E; F))_{m=0}^\infty$, the norm on each of them being denoted by $\| \cdot \|_\Theta$, with the following properties:

(i) Each $\mathcal{P}_\Theta(^m E; F)$ is a linear subspace of $\mathcal{P}(^m E; F)$.

(ii) $\mathcal{P}_\Theta(^0 E; F)$ coincides with $\mathcal{P}(^0 E; F) = F$ as a normed vector space.

(iii) There is a real number $\sigma \geqslant 1$ such that, given any $k \in \mathbb{N}_0$, $m \in \mathbb{N}_0$, $k \leqslant m$, $a \in E$ and $P \in \mathcal{P}_\Theta(^m E; F)$, then

$$\widehat{d}^k P(a) \in \mathcal{P}_\Theta(^k E; F) \quad \text{and} \quad \left\| \frac{1}{k!} \widehat{d}^k P(a) \right\|_\Theta \leqslant \sigma^m \|P\|_\Theta \|a\|^{m-k}.$$

A holomorphy type from E to F shall be denoted by either Θ or $(\mathcal{P}_\Theta(^m E; F))_{m=0}^\infty$.

DEFINITION 1.3. Let $(\mathcal{P}_\Theta(^m E; F))_{m=0}^\infty$ be a holomorphy type from the normed space E to the Banach space F. A given function $f \in \mathcal{H}(E; F)$ is said to be of *Θ-bounded type* if

(i) $\frac{1}{m!} \widehat{d}^m f(0) \in \mathcal{P}_\Theta(^m E; F)$ for all $m \in \mathbb{N}_0$,

(ii) $\lim\limits_{m \to \infty} \left(\frac{1}{m!} \|\widehat{d}^m f(0)\|_\Theta \right)^{1/m} = 0.$

The linear subspace of $\mathcal{H}(E; F)$ of all functions f of Θ-bounded type is denoted by $\mathcal{H}_{\Theta b}(E; F)$.

For each $\rho > 0$, the correspondence

$$f \in \mathcal{H}_{\Theta b}(E; F) \mapsto \|f\|_{\Theta, \rho} = \sum_{m=0}^\infty \frac{\rho^m}{m!} \|\widehat{d}^m f(0)\|_\Theta < \infty$$

is a well-defined seminorm and $\mathcal{H}_{\Theta b}(E; F)$ becomes a Fréchet space when endowed with the locally convex topology generated by these seminorms (see, e.g, Proposition 2.3 of [22]).

When $F = \mathbb{C}$ we write $\mathcal{H}_{\Theta b}(E)$ instead of $\mathcal{H}_{\Theta b}(E; \mathbb{C})$ and when Θ is the *current holomorphy type*, that is when $\mathcal{P}_\Theta(^m E; F) = \mathcal{P}(^m E; F)$ for every $m \in \mathbb{N}_0$, we write $\mathcal{H}_b(E; F)$ instead of $\mathcal{H}_{\Theta b}(E; F)$ and $\| \cdot \|_\rho$ instead of $\| \cdot \|_{\Theta, \rho}$.

DEFINITION 1.4 ([20], Definition 3.6). Let E and F be locally convex spaces and let U be an open subset of E. A mapping $f : U \to F$ is said to be *holomorphic* if f is continuous and G-holomorphic.

When $F = \mathbb{C}$, then $f : U \to \mathbb{C}$ is holomorphic if and only if for each $a \in U$ there is a unique sequence $(P_m)_{m=0}^{\infty}$, with $P_m \in \mathcal{P}(^mE)$, $m = 0, 1, \ldots$, such that $f(x) = \sum_{m=0}^{\infty} P_m(x - a)$, where the series converges uniformly for x in a neighborhood of a (see for instance p. 23 of [19]).

Following Colombeau and Matos [16] we introduce a vector subspace of $\mathcal{H}(E)$, when E is a locally convex space, that will play a central role in this paper.

DEFINITION 1.5. Let E be a locally convex space and F a Banach space. A mapping $f \in \mathcal{H}(E; F)$ is said to be of *uniform Θ-bounded type* if there exist $p \in cs(E)$ and $f_p \in \mathcal{H}_{\Theta b}(E_p; F)$ such that $f = f_p \circ \pi_p$. Let $\mathcal{H}_{u\Theta b}(E; F)$ denote the vector space of all holomorphic functions of uniform Θ-bounded type from E to F. Let π_p^* denote the injective mapping

$$\pi_p^* : f_p \in \mathcal{H}_{\Theta b}(E_p; F) \to f_p \circ \pi_p \in \mathcal{H}_{u\Theta b}(E; F).$$

Then

$$\mathcal{H}_{u\Theta b}(E; F) = \bigcup_{p \in cs(E)} \pi_p^*(\mathcal{H}_{\Theta b}(E_p; F))$$

and we endow $\mathcal{H}_{u\Theta b}(E; F)$ with the locally convex inductive topology with respect to the mappings π_p^*. Thus $\mathcal{H}_{u\Theta b}(E; F) = \text{ind}\, \mathcal{H}_{\Theta b}(E_p; F)$ is an inductive limit of Fréchet spaces.

If $D \subset cs(E)$ is a fundamental family, then it is clear that

$$\mathcal{H}_{u\Theta b}(E; F) = \bigcup_{p \in D} \pi_p^*(\mathcal{H}_{\Theta b}(E_p; F)).$$

When Θ is the current holomorphy type, then we write $\mathcal{H}_{ub}(E; F)$ instead of $\mathcal{H}_{u\Theta b}(E; F)$, and when $F = \mathbb{C}$ we write $\mathcal{H}_{u\Theta b}(E)$ instead of $\mathcal{H}_{u\Theta b}(E; \mathbb{C})$

The next definition is a slight variation of the concept of π_1-holomorphy type (originally introduced in Definitions 2.3 of [22]) and can be found in Definition 2.5 of [6].

DEFINITION 1.6. Let E and F be normed spaces, with F complete. A holomorphy type $(\mathcal{P}_\Theta(^mE; F))_{m=0}^{\infty}$ from E to F is said to be a *π_1-holomorphy type* if the following conditions hold:

(i) Polynomials of finite type belong to $(\mathcal{P}_\Theta(^mE; F))_{m=0}^{\infty}$ and there exists $K > 0$ such that

$$\|\phi^m \cdot b\|_\Theta \leqslant K^m \|\phi\|^m \cdot \|b\|$$

for all $\phi \in E'_b$, $b \in F$ and $m \in \mathbb{N}$;

(ii) For each $m \in \mathbb{N}_0$, $\mathcal{P}_f(^mE; F)$ is dense in $(\mathcal{P}_\Theta(^mE; F), \| \cdot \|_\Theta)$.

The main examples that we are interested in are the following:

EXAMPLE 1.7. Let E and F be normed spaces, with F complete.

(i) It is clear that the sequence of nuclear polynomials $(\mathcal{P}_N(^mE;F))_{m=0}^{\infty}$ is a π_1-holomorphy type (see [22] or [30]), which defines the Fréchet space $\mathcal{H}_{Nb}(E;F)$ of entire functions of nuclear bounded type.

(ii) A polynomial $P \in \mathcal{P}(^mE;F)$ is said to be *approximable* if $P \in \overline{\mathcal{P}_f(^mE;F)}^{\|\cdot\|}$. Let $\mathcal{P}_A(^mE;F)$ denotes the subspace of all approximable members of $\mathcal{P}(^mE;F)$, endowed with the sup norm. Then it is clear that the sequence $(\mathcal{P}_A(^mE;F))_{m=0}^{\infty}$ is a π_1-holomorphy type, which defines the Fréchet space $\mathcal{H}_{Ab}(E;F)$ of entire functions of approximable bounded type.

DEFINITION 1.8. Let E and F be normed spaces, with F complete and let $(\mathcal{P}_\Theta(^mE;F))_{m=0}^{\infty}$ be a π_1-holomorphy type.

(i) We recall that the *polynomial Borel transform*

$$\mathcal{B}_m : \mathcal{P}_\Theta(^mE;F)' \to \mathcal{P}(^mE'_{\mathrm{b}};F'_{\mathrm{b}})$$

is defined by

$$(\mathcal{B}_mT)(\phi)(y) = T(\phi^m y) \quad \text{for every } T \in \mathcal{P}_\Theta(^mE;F)', \phi \in E'_{\mathrm{b}}, y \in F.$$

Then $\mathcal{B}_m$ is linear, continuous and injective. The image of $\mathcal{B}_m$ in $\mathcal{P}(^mE'_{\mathrm{b}};F'_{\mathrm{b}})$ is denoted by $\mathcal{P}_{\Theta'}(^mE'_{\mathrm{b}};F'_{\mathrm{b}})$, and the function

$$\mathcal{B}_mT \in \mathcal{P}_{\Theta'}(^mE'_{\mathrm{b}};F'_{\mathrm{b}}) \to \|T\| \in \mathbb{R}$$

defines a norm $\|.\|_{\Theta'}$ on $\mathcal{P}_{\Theta'}(^mE'_{\mathrm{b}};F'_{\mathrm{b}})$. Thus $(\mathcal{P}_\Theta(^mE;F)',\|\cdot\|)$ is isometrically isomorphic to $(\mathcal{P}_{\Theta'}(^mE'_{\mathrm{b}};F'_{\mathrm{b}}),\|\cdot\|_{\Theta'})$.

(ii) A function $f \in \mathcal{H}(E'_{\mathrm{b}})$ is said to be of *Θ'-exponential type* if $\hat{d}^mf(0) \in \mathcal{P}_{\Theta'}(^mE'_{\mathrm{b}})$ for every $m \in \mathbb{N}_0$, and there are $C, c \geqslant 0$ such that $\|\hat{d}^mf(0)\|_{\Theta'} \leqslant Cc^m$ for every $m \in \mathbb{N}_0$. The vector space of all entire functions of Θ'-exponential type on E'_{b} is denoted by $\mathrm{Exp}_{\Theta'}(E'_{\mathrm{b}})$ (see p. 915 of [22]).

By Theorem 2.1 of [22] the *holomorphic Borel transform*

$$\mathcal{B} : [\mathcal{H}_{\Theta\mathrm{b}}(E)]'_{\mathrm{b}} \to \mathrm{Exp}_{\Theta'}(E'_{\mathrm{b}}),$$

which is defined by

$$(\mathcal{B}T)(\phi) = T(e^\phi) \quad \text{for every } T \in [\mathcal{H}_{\Theta\mathrm{b}}(E)]'_{\mathrm{b}} \text{ and } \phi \in E'_{\mathrm{b}},$$

is a vector space isomorphism.

Let E be a locally convex space such that there exists a fundamental family $D \subset cs(E)$ such that $(\mathcal{P}_\Theta(^mE_p))_{m=0}^{\infty}$ is a π_1-holomorphy type for every $p \in D$. Then it is clear that for each $\phi \in E'_{\mathrm{b}}$ there exist $p \in cs(E)$ and $\phi_p \in (E_p)'_{\mathrm{b}}$ such that $\phi = \phi_p \circ \pi_p$. Thus, for $T \in \mathcal{H}_{\mathrm{u}\Theta\mathrm{b}}(E)'$ we have

$$T(e^\phi) = T(e^{\phi_p \circ \pi_p}) = T \circ \pi_p^*(e^{\phi_p}) = T_p(e^{\phi_p}),$$

with $T_p = T \circ \pi_p^* \in \mathcal{H}_{\Theta\mathrm{b}}(E_p)$. By the preceding definition the function $\phi_p \in (E_p)'_{\mathrm{b}} \to T_p(e^{\phi_p}) \in \mathbb{C}$ belongs to $\mathrm{Exp}_{\Theta'}((E_p)'_{\mathrm{b}})$.

Now it makes sense to give the next definition.

DEFINITION 1.9. Let E be a locally convex space such that there exists a fundamental family $D \subset cs(E)$ such that $(\mathcal{P}_{\Theta}(^m E_p))_{m=0}^{\infty}$ is a π_1-holomorphy type for every $p \in D$. We denote by $\text{Exp}_{\Theta'}(E'_{\text{b}})$ the subspace of all $f \in \mathcal{H}(E'_{\text{b}})$ such that $f \circ \pi_p^* \in \text{Exp}_{\Theta'}((E_p)'_{\text{b}})$ for some $p \in cs(E)$. We define the *holomorphic Borel transform*

$$\mathcal{B} : [\mathcal{H}_{\text{u}\Theta\text{b}}(E)]'_{\text{b}} \to \text{Exp}_{\Theta'}(E'_{\text{b}})$$

by

$$(\mathcal{B}T)(\phi) = T(e^{\phi}) \quad \text{for every } T \in [\mathcal{H}_{\text{u}\Theta\text{b}}(E)]'_{\text{b}}, \phi \in E'_{\text{b}}.$$

It follows that $\mathcal{B}$ is well-defined, that is $\mathcal{B}T \in \mathcal{H}(E'_{\text{b}})$. In fact,

$$(\mathcal{B}T)(\phi) = \sum_{m=0}^{\infty} \frac{T(\phi^m)}{m!}$$

is the Taylor series expansion of $\mathcal{B}T$ around 0 and this implies that $\mathcal{B}T$ is holomorphic (see, for instance Example 5.4 of [38]).

Finally we recall the concept of convolution operator on $\mathcal{H}_{\Theta\text{b}}(E)$ when E is a normed space and we introduce convolution operators on $\mathcal{H}_{\text{u}\Theta\text{b}}(E)$ in the case where E is a locally convex space.

DEFINITION 1.10 ([22], Definition 3.1). Let E be a normed space.

(i) A *convolution operator* on $\mathcal{H}_{\Theta\text{b}}(E)$ is a continuous linear mapping

$$L : \mathcal{H}_{\Theta\text{b}}(E) \to \mathcal{H}_{\Theta\text{b}}(E)$$

such that $L(\tau_a f) = \tau_a(Lf)$ for every $f \in \mathcal{H}_{\Theta\text{b}}(E)$ and $a \in E$. We recall that $(\tau_a f)(x) = f(x - a)$ for every $x \in E$ and we denote by $\mathcal{A}_{\Theta\text{b}}(E)$ the vector space of all convolution operators on $\mathcal{H}_{\Theta\text{b}}(E)$.

(ii) The linear mapping

$$\Gamma : \mathcal{A}_{\Theta\text{b}}(E) \to \mathcal{H}_{\Theta\text{b}}(E)'$$

is defined by

$$(\Gamma L)(f) = (Lf)(0) \quad \text{for every } L \in \mathcal{A}_{\Theta\text{b}}(E) \text{ and } f \in \mathcal{H}_{\Theta\text{b}}(E).$$

DEFINITION 1.11. Let E be a locally convex space. A *convolution operator* on $\mathcal{H}_{\text{u}\Theta\text{b}}(E)$ is a continuous linear mapping

$$L : \mathcal{H}_{\text{u}\Theta\text{b}}(E) \to \mathcal{H}_{\text{u}\Theta\text{b}}(E)$$

such that $L(\tau_a f) = \tau_a(Lf)$ for every $f \in \mathcal{H}_{\text{u}\Theta\text{b}}(E)$ and $a \in E$.

We denote by $\mathcal{A}_{\text{u}\Theta\text{b}}(E)$ the vector space of all convolution operators on $\mathcal{H}_{\text{u}\Theta\text{b}}(E)$. The linear mapping

$$\Gamma : \mathcal{A}_{\text{u}\Theta\text{b}}(E) \to \mathcal{H}_{\text{u}\Theta\text{b}}(E)'$$

is defined by

$$(\Gamma L)(f) = (Lf)(0) \quad \text{for every } L \in \mathcal{A}_{\text{u}\Theta\text{b}}(E) \text{ and } f \in \mathcal{H}_{\text{u}\Theta\text{b}}(E).$$

2. CONVOLUTION OPERATORS ON SPACES OF ENTIRE FUNCTIONS ON (DFN)-SPACES

A (DFN)-space is the strong dual of a Fréchet nuclear space. Nuclear spaces were introduced by Grothendieck [29] and together with normed spaces are the most important classes of locally convex spaces encountered in analysis. A very good reference for the theory of nuclear spaces is the book of Pietsch [45]. Holomorphic functions on nuclear spaces were first investigated by Boland [9], but many other authors have worked in that direction. We mention, among many others, [10], [11], [12].

The main result of this section is the following theorem.

THEOREM 2.1. *Let E be a* (DFN)*-space, and let L be a nontrivial convolution operator on $\mathcal{H}(E)$. Then L is mixing and thus in particular hypercyclic.*

Our proof of Theorem 2.1 rests on the following theorem, which, as mentioned in the Introduction, is due to Costakis and Sambarino [18] and sharpens an earlier result of Kitai [34] and Gethner and Shapiro [24].

THEOREM 2.2. *Let X be a separable Fréchet space. Then a continuous linear mapping $T : X \to X$ is mixing if there are dense subsets Z, Y of X and a mapping $S : Y \to Y$ satisfying the following three conditions:*

(i) $T^n(z) \to 0$ *when* $n \to \infty$ *for every* $z \in Z$.

(ii) $S^n(y) \to 0$ *when* $n \to \infty$ *for every* $y \in Y$.

(iii) $T \circ S(y) = y$ *for every* $y \in Y$.

Before proving Theorem 2.1 we need some auxiliary results.

PROPOSITION 2.3. *Let E be a locally convex space, and assume there is a fundamental family $D \subset cs(E)$ such that the sequence $(\mathcal{P}_\Theta(^m E_p))_{m=0}^\infty$ is a π_1-holomorphy type for every $p \in D$. Then:*

(i) *The set*

$$S_U = \operatorname{span}\{e^\phi : \phi \in U\}$$

is a dense subspace of $\mathcal{H}_{\mathrm{u}\Theta\mathrm{b}}(E)$ for each nonvoid open subset U of E'_{b}.

(ii) *The set*

$$B = \{e^\phi : \phi \in E'_{\mathrm{b}}\}$$

is a linearly independent subset of $\mathcal{H}_{\mathrm{u}\Theta\mathrm{b}}(E)$.

Proof. (i) Let U be a nonvoid open subset of E'_{b}. For each $p \in D$ consider the mapping

$$\pi'_p : \phi_p \in (E_p)'_{\mathrm{b}} \to \phi_p \circ \pi_p \in E'_{\mathrm{b}}$$

and observe that $\pi'_p((E_p)'_{\mathrm{b}}) \subset E'_{\mathrm{b}}$ and π'_p is continuous. Let $U_p = (\pi'_p)^{-1}(U)$. Then U_p is a nonvoid open subset of $(E_p)'_{\mathrm{b}}$. Let

$$S_{U_p} = \operatorname{span}\{e^{\phi_p} : \phi_p \in U_p\}.$$

By Proposition 4.3 of [6] S_{U_p} is a dense subspace of $\mathcal{H}_{\Theta b}(E_p)$. Since

$$\mathcal{H}_{u\Theta b}(E) = \bigcup_{p\in D} \pi_p^*(\mathcal{H}_{\Theta b}(E_p))$$

it follows that S_U is a dense subspace of $\mathcal{H}_{u\Theta b}(E)$.

(ii) If we set

$$B_p = \{e^{\phi_p} : \phi_p \in (E_p)'_b\}$$

for every $p \in D$, then it is clear that $B = \bigcup_{p\in D} \pi_p^*(B_p)$. By Proposition 4.6 of [6], each B_p is a linearly independent subset of $\mathcal{H}_{\Theta b}(E_p)$. Since each π_p^* is injective, it follows that B is a linearly independent subset of $\mathcal{H}_{u\Theta b}(E)$. ∎

LEMMA 2.4. *Let E be a locally convex space, and assume there is a fundamental family $D \subset cs(E)$ such that the sequence $(\mathcal{P}_\Theta(^m E_p))_{m=0}^\infty$ is a π_1-holomorphy type for every $p \in D$. Let L be a convolution operator on $\mathcal{H}_{u\Theta b}(E)$. Then:*

(i) *$L(e^\phi) = \mathcal{B}(\Gamma L)(\phi)e^\phi$ for every $\phi \in E'_b$.*

(ii) *L is a scalar multiple of the identity operator if and only if the entire function $\mathcal{B}(\Gamma L) : E'_b \to \mathbb{C}$ is constant.*

Proof. (i) If $\phi \in E'_b$, then it follows from Definitions 1.9 and 1.11 that

$$\mathcal{B}(\Gamma L)(\phi) = (\Gamma L)(e^\phi) = L(e^\phi)(0).$$

Hence for each $y \in E$ it follows that

$$\begin{aligned}[\mathcal{B}(\Gamma L)(\phi)e^\phi](y) &= \mathcal{B}(\Gamma L)(\phi)e^{\phi(y)} = e^{\phi(y)}(\Gamma L)(e^\phi)\\ &= e^{\phi(y)}(Le^\phi)(0) = [L(e^{\phi(y)}e^\phi)](0) = [L(\tau_{-y}e^\phi)](0)\\ &= [\tau_{-y}(Le^\phi)](0) = L(e^\phi)(y).\end{aligned}$$

(ii) Let $\lambda \in \mathbb{C}$ such that $\mathcal{B}(\Gamma L)(\phi) = \lambda$ for every $\phi \in E'_b$. It follows from (i) that

$$Le^\phi = \mathcal{B}(\Gamma L)(\phi)e^\phi = \lambda e^\phi \quad \text{for every } \phi \in E'_b.$$

Since span $\{e^\phi : \phi \in E'\}$ is dense in $\mathcal{H}_{u\Theta b}(E)$, it follows that $Lf = \lambda f$ for every $f \in \mathcal{H}_{u\Theta b}(E)$.

Conversely let $\lambda \in \mathbb{C}$ such that $Lf = \lambda f$ for every $f \in \mathcal{H}_{u\Theta b}(E)$. It follows from (i) that

$$\lambda e^\phi = Le^\phi = \mathcal{B}(\Gamma L)(\phi)e^\phi,$$

and therefore $\mathcal{B}(\Gamma L)(\phi) = \lambda$ for every $\phi \in E'_b$. ∎

PROPOSITION 2.5. *Let E be a locally convex space and assume there is a fundamental family $D \subset cs(E)$ such that the sequence $(\mathcal{P}_\Theta(^m E_p))_{m=0}^\infty$ is a π_1-holomorphy type for every $p \in D$. Let L be a nontrivial convolution operator on $\mathcal{H}_{u\Theta b}(E)$. Consider the sets*

$$V = \{\phi \in E'_b : |\mathcal{B}(\Gamma L)(\phi)| < 1\} = [\mathcal{B}(\Gamma L)]^{-1}(\Delta)$$

and

$$W = \{\phi \in E'_{\mathrm{b}} : |\mathcal{B}(\Gamma L)(\phi)| > 1\} = [\mathcal{B}(\Gamma L)]^{-1}(\mathbb{C} \setminus \overline{\Delta}).$$

Consider also the sets

$$H_V = \operatorname{span}\{e^{\phi} : \phi \in V\} \quad \text{and} \quad H_W = \operatorname{span}\{e^{\phi} : \phi \in W\}.$$

Then:

(i) *H_V and H_W are dense subspaces of $\mathcal{H}_{\mathrm{u}\Theta\mathrm{b}}(E)$.*

(ii) *$L^n f \to 0$ when $n \to \infty$ for each $f \in H_V$.*

(iii) *If we define*

$$S(e^{\phi}) = \frac{e^{\phi}}{\mathcal{B}(\Gamma L)(\phi)} \quad \text{for every } \phi \in W,$$

then S admits a unique extension to a linear mapping $S : H_W \to H_W$, and $S^n f \to 0$ when $n \to \infty$ for each $f \in H_W$.

(iv) *$L \circ S(f) = f$ for every $f \in H_W$.*

Proof. (i) Since L is not a scalar multiple of the identity, Lemma 2.4(ii) implies that the entire function $\mathcal{B}(\Gamma L) : E'_{\mathrm{b}} \to \mathbb{C}$ is not constant. Hence, it follows from Liouville's theorem that V and W are nonvoid open subsets of E'_{b}. By Proposition 2.3(i) H_V and H_W are dense subspaces of $\mathcal{H}_{\mathrm{u}\Theta\mathrm{b}}(E)$.

(ii) Given $\phi \in V$, Lemma 2.4(i) implies that

$$L(e^{\phi}) = \mathcal{B}(\Gamma L)(\phi)e^{\phi} \in H_V.$$

Since L is linear, it is clear that $L(H_V) \subset H_V$. It is easy to see that

$$L^n(e^{\phi}) = [\mathcal{B}(\Gamma L)(\phi)]^n e^{\phi} \quad \text{for every } \phi \in V, n \in \mathbb{N}.$$

Now let $f \in H_V$, that is $f = \sum_{j=1}^{m} \alpha_j e^{\phi_j}$, with $\alpha_j \in \mathbb{C}$ and $\phi_j \in V$. It follows that

$$L^n(f) = \sum_{j=1}^{m} \alpha_j L^n(e^{\phi_j}) = \sum_{j=1}^{m} \alpha_j [\mathcal{B}(\Gamma L)(\phi_j)]^n e^{\phi_j}.$$

Since $|\mathcal{B}(\Gamma L)(\phi_j)| < 1$ for every $j = 1, \ldots, m$, it follows that $L^n f \to 0$ when $n \to \infty$.

(iii) If $\phi \in W$, then $\mathcal{B}(\Gamma L)(\phi) \neq 0$. Hence we may define

$$S(e^{\phi}) = \frac{e^{\phi}}{\mathcal{B}(\Gamma L)(\phi)} \in H_W.$$

It is easy to see that

$$S^n(e^{\phi}) = \frac{e^{\phi}}{[\mathcal{B}(\Gamma L)(\phi)]^n} \quad \text{for every } \phi \in W, n \in \mathbb{N}.$$

By Proposition 2.3(ii) $\{e^{\phi} : \phi \in W\}$ is a Hamel basis of H_W. Hence S admits a unique extension to a linear mapping $S : H_W \to H_W$. Now let $f \in H_W$, that is

$f = \sum_{j=1}^{m} \alpha_j e^{\phi_j}$, with $\alpha_j \in \mathbb{C}$ and $\phi_j \in W$. It follows that

$$S^n f = \sum_{j=1}^{m} \frac{\alpha_j e^{\phi_j}}{[\mathcal{B}(\Gamma L)(\phi_j)]^n}.$$

Since $|\mathcal{B}(\Gamma L))(\phi_j)| > 1$ for every j, it follows that $S^n f \to 0$ when $n \to \infty$.

(iv) It is clear that $L \circ S(e^{\phi}) = e^{\phi}$ for every $\phi \in W$, and therefore $L \circ S(f) = f$ for every $f \in H_W$. ∎

Proof of Theorem 2.1. By Theorem 6.5 of [17], $\mathcal{H}(E) = \mathcal{H}_{uNb}(E)$ algebraically and topologically. By a result of Boland ([10], Corollary 1.4), $\mathcal{H}(E)$ is a Fréchet nuclear space. In particular $\mathcal{H}_{uNb}(E)$ is a separable Fréchet space. If $D \subset cs(E)$ is any fundamental family, then the sequence $(\mathcal{P}_N(^mE_p))_{m=0}^{\infty}$ is a π_1-holomorphy type for every $p \in D$, by Example 1.7.

By Proposition 2.5 H_V and H_W are dense subspaces of $\mathcal{H}_{uNb}(E)$, and there is a linear mapping $S : H_W \to H_W$ such that:

(a) $L^n f \to 0$ when $n \to \infty$ for every $f \in H_V$;
(b) $S^n f \to 0$ when $n \to \infty$ for every $f \in H_W$;
(c) $L \circ S(f) = f$ for every $f \in H_W$.

By Theorem 2.2 the operator L is mixing. ∎

3. CONVOLUTION OPERATORS ON SPACES OF ENTIRE FUNCTIONS ON (DFC)-SPACES

A (DFC)-space is a locally convex space of the form $E = F_c'$, where F is a Fréchet space. (DFC)-spaces were first studied by Brauner [13] and Höllstein [31], [32]. Holomorphic functions on (DFC)-spaces have been studied by Mujica [37], Valdivia [48], Schottenloher [46], Nachbin [43], Lourenço [35] and Galindo et al. [23].

The main result in this section is the following theorem.

THEOREM 3.1. *Let $E = F_c'$, where F is a separable Fréchet space with the approximation property. Let L be a nontrivial convolution operator on $(\mathcal{H}(E), \tau_0)$. Then L is mixing and thus in particular hypercyclic.*

The proof of Theorem 3.1 rests on Theorem 2.2, but before proving the theorem we need some auxiliary results.

PROPOSITION 3.2. *Let $E = F_c'$, where F is a Fréchet space. Then:*
(i) *E is a semi-Montel, hemicompact k-space.*
(ii) *$(\mathcal{H}(E), \tau_0)$ is a Fréchet space.*

Proof. (i) By Proposition 7.2 of [37] E is a semi-Montel, hemicompact space. By the Banach–Dieudonné theorem (see p. 245, Theorem 1 of [33]) E is a k-space.
(ii) follows at once from (i). ∎

If $E = F'_c$, where F is a Fréchet space, then a result of Schwartz guarantees that F has the approximation property if and only if E has the approximation property (see Exposé n^0 14, Théorème 2 of [47] or Corollary 1.3 of [21]).

PROPOSITION 3.3. *Let $E = F'_c$, where F is a separable Fréchet space with the approximation property. Then $(\mathcal{H}(E), \tau_0)$ is a separable Fréchet space.*

Proof. By considering the Taylor series we see that every $f \in \mathcal{H}(E)$ can be approximated, uniformly on compact sets, by continuous polynomials on E. Since E has the approximation property, it is clear that every continuous polynomial on E can be approximated, uniformly on compact sets, by continuous polynomials of finite type. By the Mackey–Arens theorem (see p. 205, Theorem 1 of [33]) $E'_b = E'_c = F$ is separable, it follows that $(\mathcal{P}_f({}^mE), \tau_0)$ is separable for every $m \in \mathbb{N}_0$. Hence it follows that $(\mathcal{H}(E), \tau_0)$ is separable, as asserted. ∎

DEFINITION 3.4. Let E and F be normed spaces. An operator $T \in \mathcal{L}(E; F)$ is said to be *approximable* if $T \in \overline{E' \otimes F}$.

For the next result recall the definition of the space of entire functions of approximable bounded type in Example 1.7(ii).

LEMMA 3.5. *Let E, F and G be normed spaces, with G complete, and let $T \in \mathcal{L}(F; E)$ be an approximable operator. Then $f \circ T \in \mathcal{H}_{Ab}(F; G)$ for every $f \in \mathcal{H}_b(E; G)$, and the mapping*

$$f \in \mathcal{H}_b(E; G) \to f \circ T \in \mathcal{H}_{Ab}(F; G)$$

is linear and continuous.

Proof. Let $\sum_{m=0}^{\infty} P_m(x)$ denote the Taylor series of f at the origin. Then

$$f \circ T(y) = \sum_{m=0}^{\infty} P_m \circ T(y) \quad \text{for every } y \in F.$$

Since T is approximable, there is a sequence $(T_n)_{n=1}^{\infty} \in F' \otimes E$ such that $\|T - T_n\| \to 0$. Since $P_m \in \mathcal{P}({}^mE)$ for every $m \in \mathbb{N}_0$, it is clear that $P_m \circ T \in \mathcal{P}({}^mF)$ and $\|P_m \circ T\| \leqslant \|P_m\| \|T\|^m$ for every $m \in \mathbb{N}_0$.

Since $f \in \mathcal{H}_b(E; G)$, the Taylor series of f at the origin has an infinite radius of convergence. By the Cauchy–Hadamard formula (see Theorem 4.3 of [38]), $\|P_m\|^{1/m} \to 0$. Hence it follows that

$$\|P_m \circ T\|^{1/m} \leqslant \|P_m\|^{1/m} \|T\| \to 0.$$

Hence the Taylor series of $f \circ T$ at the origin has also an infinite radius of convergence, and therefore $f \circ T \in \mathcal{H}_b(F; G)$.

To show that $f \circ T \in \mathcal{H}_{Ab}(F; G)$ we have to prove that $P_m \circ T \in \mathcal{P}_A(F; G)$ for every $m \in \mathbb{N}_0$. By the Newton binomial formula (see Corollary 1.9 of [38]), for every $y \in F$ with $\|y\| \leqslant 1$ it follows that

$$\begin{aligned}\|P_m \circ T(y) - P_m \circ T_n(y)\| &= \Big\| \sum_{k=1}^{m} \binom{m}{k} \check{P}_m (T_n y)^{m-k} (Ty - T_n y)^k \Big\| \\ &\leqslant \sum_{k=1}^{m} \binom{m}{k} \|\check{P}_m\| \|T_n\|^{m-k} \|T - T_n\|^k \\ &\leqslant e^m \|P_m\| \|T - T_n\| \sum_{k=1}^{m} \binom{m}{k} c^{m-k} d^{k-1}\end{aligned}$$

for suitable positive constants c and d. Therefore $\|P_m \circ T - P_m \circ T_n\| \to 0$. Since $T_n \in F' \otimes E$, it is clear that $P_m \circ T_n \in \mathcal{P}_f({}^m F)$, and therefore $P_m \circ T \in \mathcal{P}_A({}^m F; G)$.

Finally it is clear that the mapping $f \in \mathcal{H}_{\mathrm{b}}(E; G) \to f \circ T \in \mathcal{H}_{A\mathrm{b}}(F; G)$ is continuous, since

$$\|f \circ T\|_\rho = \sum_{m=0}^{\infty} \|P_m \circ T\| \rho^m \leqslant \sum_{m=0}^{\infty} \|P_m\| \|T\|^m \rho^m = \|f\|_{\|T\|\rho}$$

for every $\rho > 0$. ∎

THEOREM 3.6. *Let E be a* (DFC)*-space with the approximation property. Then $\mathcal{H}_{\mathrm{u}A\mathrm{b}}(E) = \mathcal{H}_{\mathrm{ub}}(E) = (\mathcal{H}(E), \tau_0)$ algebraically and topologically.*

Proof. We first establish the continuous inclusions

$$\mathcal{H}_{\mathrm{u}A\mathrm{b}}(E) \hookrightarrow \mathcal{H}_{\mathrm{ub}}(E) \hookrightarrow (\mathcal{H}(E), \tau_0).$$

Since

$$\mathcal{H}_{\mathrm{u}A\mathrm{b}}(E) = \bigcup_{p \in cs(E)} \pi_p^*(\mathcal{H}_{A\mathrm{b}}(E_p))$$

and

$$\mathcal{H}_{\mathrm{ub}}(E) = \bigcup_{p \in cs(E)} \pi_p^*(\mathcal{H}_{\mathrm{b}}(E_p)),$$

it is clear that $\mathcal{H}_{\mathrm{u}A\mathrm{b}}(E) \subset \mathcal{H}_{\mathrm{ub}}(E)$, and the inclusion mapping is continuous. Since the inclusion mapping $\pi_p^*(\mathcal{H}_{\mathrm{b}}(E_p)) \hookrightarrow (\mathcal{H}(E), \tau_0)$ is clearly continuous, we obtain the continuous inclusion $\mathcal{H}_{\mathrm{ub}}(E) \hookrightarrow (\mathcal{H}(E), \tau_0)$.

We next show that $\mathcal{H}_{\mathrm{u}A\mathrm{b}}(E) = (\mathcal{H}(E), \tau_0)$ algebraically and topologically. We know that $\mathcal{H}_{\mathrm{u}A\mathrm{b}}(E)$ is bornological, and $(\mathcal{H}(E), \tau_0)$ is a Fréchet space, in particular bornological. Hence it suffices to show that each bounded subset of $(\mathcal{H}(E), \tau_0)$ is contained and bounded in $\mathcal{H}_{\mathrm{u}A\mathrm{b}}(E)$. Let $\{f_i : i \in I\}$ be a bounded subset of $(\mathcal{H}(E), \tau_0)$. Let $f \in \mathcal{H}(E; \ell_\infty(I))$ be defined by $f(x) = (f_i(x))_{i \in I}$ for every $x \in E$. By a result of Galindo et al. (see Corollary 1 of [23]),

$$(\mathcal{H}(E; \ell_\infty(I)), \tau_0) = \mathcal{H}_{\mathrm{ub}}(E; \ell_\infty(I))$$

algebraically and topologically. In particular $f \in \mathcal{H}_{\mathrm{ub}}(E; \ell_\infty(I))$ and therefore there are $p \in cs(E)$ and $f_p \in \mathcal{H}_{\mathrm{b}}(E; \ell_\infty(I))$ such that $f = f_p \circ \pi_p$. By a result of Lourenço (see Lemma 2.2 of [35]), there are $q \in cs(E)$, $q \geqslant p$ such that the canonical mapping $\pi_{pq} : E_q \to E_p$ is an approximable operator. Let $f_q = f_p \circ \pi_{pq}$. By the preceding lemma $f_q \in \mathcal{H}_{Ab}(E_q; \ell_\infty(I))$ and

$$f = f_p \circ \pi_p = f_p \circ \pi_{pq} \circ \pi_q = f_q \circ \pi_q.$$

Thus $f = (f_i)_{i \in I} \in \mathcal{H}_{uAb}(E; \ell_\infty(I))$, and therefore $\{f_i : i \in I\}$ is a bounded subset of $\mathcal{H}_{uAb}(E)$, as asserted. ∎

Proof of Theorem 3.1. By Proposition 3.3 and Theorem 3.6 we have that $\mathcal{H}_{uAb}(E) = (\mathcal{H}(E), \tau_0)$ is a separable Fréchet space. If $D \subset cs(E)$ is any fundamental family, then it follows from Example 1.7 that $(\mathcal{P}_A(^m E_p))_{m=0}^\infty$ is a π_1-holomorphy type for every $p \in D$. By Proposition 2.5 there are dense subspaces H_V and H_W of $\mathcal{H}_{uAb}(E)$ and there is a linear mapping $S : H_W \to H_V$ such that:

(a) $L^n f \to 0$ when $n \to \infty$ for every $f \in H_V$;
(b) $S^n f \to 0$ when $n \to \infty$ for every $f \in H_W$;
(c) $L \circ S(f) = f$ for every $f \in H_W$.

By Theorem 2.2 the operator L is mixing. ∎

4. CONVOLUTION OPERATORS ON $\mathcal{H}(\mathbb{C}^\mathbb{N})$

We will prove that no convolution operator on $\mathcal{H}(\mathbb{C}^\mathbb{N})$ is hypercyclic. So far we have only considered the compact-open topology τ_0 on the space $\mathcal{H}(E)$. But now we will also consider the compact-ported topology τ_ω introduced by Nachbin [40], and the bornological topology τ_δ introduced by Coeuré [15] and Nachbin [42]. For background information on these topologies we refer the reader to the book of Dineen [20].

THEOREM 4.1. (i) $(\mathcal{H}(\mathbb{C}^\mathbb{N}), \tau_0) = (\mathcal{H}(\mathbb{C}^\mathbb{N}), \tau_\omega) \neq (\mathcal{H}(\mathbb{C}^\mathbb{N}), \tau_\delta) = \mathcal{H}_{ub}(\mathbb{C}^\mathbb{N})$.
(ii) *No convolution operator on* $(\mathcal{H}(\mathbb{C}^\mathbb{N}), \tau)$ *is hypercyclic, for* $\tau = \tau_0$, τ_ω *and* τ_δ.

Proof. (i) By a result of Barroso (see p. 537, Teorema 2.2 of [3]), $\tau_0 = \tau_\omega$ on $\mathcal{H}(\mathbb{C}^\mathbb{N})$. By a result of Dineen (see p. 45, Corollary 3.2 of [19] or Example 3.24(i) of [20]) we have $\tau_\omega \neq \tau_\delta$ on $\mathcal{H}(\mathbb{C}^\mathbb{N})$.

To see the connection with $\mathcal{H}_{ub}(\mathbb{C}^\mathbb{N})$, for each $n \in \mathbb{N}$ and for $\tau = \tau_0$, τ_ω and τ_δ, consider the canonical inclusion $J_n : \mathbb{C}^n \hookrightarrow \mathbb{C}^\mathbb{N}$, the canonical projection $\pi_n : \mathbb{C}^\mathbb{N} \to \mathbb{C}^n$ and the corresponding mappings

$$J_n^* : f \in (\mathcal{H}(\mathbb{C}^\mathbb{N}), \tau) \to f \circ J_n \in (\mathcal{H}(\mathbb{C}^n), \tau)$$

and

$$\pi_n^* : f_n \in (\mathcal{H}(\mathbb{C}^n), \tau) \to f_n \circ \pi_n \in (\mathcal{H}(\mathbb{C}^\mathbb{N}), \tau).$$

Since $J_n^* \circ \pi_n^*$ is the identity on $\mathcal{H}(\mathbb{C}^n)$, it follows that $(\mathcal{H}(\mathbb{C}^n), \tau)$ is topologically isomorphic to a complemented subspace of $(\mathcal{H}(\mathbb{C}^\mathbb{N}), \tau)$. In particular $\pi_n^*(\mathcal{H}(\mathbb{C}^n))$

is a proper closed subspace of $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. By a result of Barroso (see p. 38, Corolário of [3] or Proposition 1.1 of [1])

$$\mathcal{H}(\mathbb{C}^{\mathbb{N}}) = \bigcup_{n=1}^{\infty} \{f_n \circ \pi_n : f_n \in \mathcal{H}(\mathbb{C}^n)\} = \bigcup_{n=1}^{\infty} \pi_n^*(\mathcal{H}(\mathbb{C}^n)).$$

If we define $p_n \in cs(\mathbb{C}^{\mathbb{N}})$ by

$$p_n(x) = \sup_{j \leqslant n} |\xi_j| \quad \text{for every } x = (\xi_j)_{j=1}^{\infty} \in \mathbb{C}^{\mathbb{N}},$$

then we can readily see that the normed space $(\mathbb{C}^{\mathbb{N}})_{p_n}$ is topologically isomorphic to $\mathbb{C}^n$, and therefore

$$\mathcal{H}_{\mathrm{ub}}(\mathbb{C}^{\mathbb{N}}) = \mathrm{ind}(\mathcal{H}((\mathbb{C}^{\mathbb{N}})_{p_n}, \tau_0)) = \mathrm{ind}(\mathcal{H}(\mathbb{C}^n), \tau_0).$$

By a result of Ansemil (see Proposition 1.3 of [1])

$$(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau_\delta) = \mathrm{ind}(\mathcal{H}(\mathbb{C}^n), \tau_0) = \mathcal{H}_{\mathrm{ub}}(\mathbb{C}^{\mathbb{N}}).$$

(ii) Suppose that there exists a hypercyclic convolution operator $L : (\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau) \to (\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. Then there exists $f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ such that the set $\{f, Lf, L^2 f, \ldots\}$ is dense in $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. Let $n \in \mathbb{N}$ be such that $f = f_n \circ \pi_n$, with $f_n \in \mathcal{H}(\mathbb{C}^n)$. We will prove that

$$\{f, Lf, L^2 f, \ldots\} = \pi_n^*(\{f_n, (Lf) \circ J_n, (L^2 f) \circ J_n, \ldots\}).$$

Indeed, since $\tau_a f = f$, for every $a \in \mathbb{C}^{\mathbb{N}}$ such that $\pi_n(a) = 0$, then we have $\tau_a(Lf) = L(\tau_a f) = Lf$, for every $a \in \mathbb{C}^{\mathbb{N}}$ with $\pi_n(a) = 0$. If $x = (x_j) \in \mathbb{C}^{\mathbb{N}}$ is arbitrary and $a = (\underbrace{0, \ldots, 0}_{n}, x_{n+1}, x_{n+2}, \ldots)$, then $\pi_n(a) = 0$ and so

$$\begin{aligned} Lf(x) &= \tau_a(Lf)(x) = Lf(x - a) = Lf(x_1, \ldots, x_n, 0, 0, \ldots) \\ &= Lf(J_n \circ \pi_n(x)) = \pi_n^*((Lf) \circ J_n)(x). \end{aligned}$$

Thus

$$Lf = (Lf) \circ J_n \circ \pi_n = \pi_n^*((Lf) \circ J_n)$$

and the same argument shows that

$$L^2 f = L(Lf) = (L^2 f) \circ J_n \circ \pi_n = \pi_n^*((L^2 f) \circ J_n).$$

Proceeding by induction it follows that

$$L^k f = (L^k f) \circ J_n \circ \pi_n = \pi_n^*((L^k f) \circ J_n),$$

for every $k \in \mathbb{N}$. Therefore

$$\{f, Lf, L^2 f, \ldots\} = \pi_n^*(\{f_n, (Lf) \circ J_n, (L^2 f) \circ J_n, \ldots\}) \subset \pi_n^*(\mathcal{H}(\mathbb{C}^n)),$$

a contradiction, since $\pi_n^*(\mathcal{H}(\mathbb{C}^n))$ is a proper closed subspace of $(\mathcal{H}(\mathbb{C}^{\mathbb{N}}), \tau)$. ∎

COROLLARY 4.2. *No convolution operator on $\mathcal{H}_{\mathrm{ub}}(\mathbb{C}^{\mathbb{N}})$ is hypercyclic.*

Acknowledgements. We are very much indebted to the referee for his/her careful reading and many valuable suggestions. In special for his/her idea which led to a substantial improvement of Theorem 4.1.

The first named author is supported by FAPESP Grant 2014/50536-7; FAPEMIG Grant PPM-00086-14; and CNPq Grants 482515/2013-9, 307517/2014-4.

REFERENCES

[1] J.M. ANSEMIL, Topological duality on the function space $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$, *J. Math. Anal. Appl.* **67**(1979), 188–197.

[2] R. ARON, J. BÈS, Hypercyclic differentiation operators, in *Function Spaces*, Contemp. Math., vol. 232, Amer. Math. Soc., Providence, RI 1999, pp. 39–46.

[3] J.A. BARROSO, Topologias nos espaços de aplicações holomorfas entre espaços localmente convexos, *An. Acad. Brasil. Ciênc.* **43**(1971), 527–546.

[4] F. BAYART, É. MATHERON, *Dynamics of Linear Operators*, Cambridge Tracts in Math., vol. 179, Cambridge Univ. Press, Cambridge 2009.

[5] F. BAYART, É. MATHERON, Mixing operators and small subsets of the circle, *J. Reine Angew. Math.*, to appear.

[6] F. BERTOLOTO, G. BOTELHO, V.V. FÁVARO, A.M. JATOBÁ, Hypercyclicity of convolution operators on spaces of entire functions, *Ann. Inst. Fourier (Grenoble)* **63**(2013), 1263–1283.

[7] J. BÈS, Ö. MARTIN, A. PERIS, S. SHKARIN, Disjoint mixing operators, *J. Funct. Anal.* **263**(2012), 1283–1322.

[8] G.D. BIRKHOFF, Démonstration d'un théorème élémentaire sur les fonctions entières, *C. R. Acad. Sci. Paris* **189**(1929), 473–475.

[9] P.J. BOLAND, Malgrange theorem for entire functions on nuclear spaces, in *Proceedings on Infinite Dimensional Holomorphy*, Lecture Notes in Math., vol. 364, Springer, Berlin 1974, pp. 135–144.

[10] P.J. BOLAND, Holomorphic functions on nuclear spaces, *Trans. Amer. Math. Soc.* **209**(1975), 275–281.

[11] P.J. BOLAND, S. DINEEN, Holomorphic functions on fully nuclear spaces, *Bull. Soc. Math. France* **106**(1978), 311–336.

[12] M. BÖRGENS, R. MEISE, D. VOGT, Entire functions on nuclear sequence spaces, *J. Reine Angew. Math.* **322**(1981), 196–220.

[13] K. BRAUNER, Duals of Fréchet spaces and a generalization of the Banach–Dieudonné theorem, *Duke Math. J.* **40**(1973), 845–855.

[14] D. CARANDO, V. DIMANT, S. MURO, Hypercyclic convolution operators on Fréchet spaces of analytic functions, *J. Math. Anal. Appl.* **336**(2007), 1324–1340.

[15] G. COEURÉ, Fonctions plurisousharmoniques sur les espaces vectoriels topologiques et applications à l'etude des fonctions analytiques, *Ann. Inst. Fourier (Grenoble)* **20**(1970), 361–432.

[16] J.F. COLOMBEAU, M.C. MATOS, Convolution equations in spaces of infinite dimensional entire functions, *Indag. Math. (Proc. A)* **83**(1980), 375–389.

[17] J.F. COLOMBEAU, J. MUJICA, Holomorphic and differentiable mappings of uniform bounded type, in *Functional Analysis, Holomorphy and Approximation Theory*, North-Holland Math. Stud., vol. 71, North-Holland, Amsterdam 1982, pp. 179–200.

[18] G. COSTAKIS, M. SAMBARINO, Topologically mixing operators, *Proc. Amer. Math. Soc.* **132**(2004), 385–389.

[19] S. DINEEN, Holomorphic functions on locally convex topological vector spaces. I. Locally convex topologies on $\mathcal{H}(U)$, *Ann. Inst. Fourier (Grenoble)* **23**(1973), 19–54.

[20] S. DINEEN, *Complex Analysis on Infinite Dimensional Spaces*, Springer, London 1999.

[21] S. DINEEN, J. MUJICA, The approximation property for spaces of holomorphic functions on infinite-dimensional spaces. I, *J. Approx. Theory* **126**(2004), 141–156.

[22] V.V. FÁVARO, A.M. JATOBÁ, Holomorphy types and spaces of entire functions of bounded type on Banach spaces, *Czechoslovak Math. J.* **59**(2009), 909–927.

[23] P. GALINDO, D. GARCIA, M. MAESTRE, Holomorphic mappings of bounded type, *J. Math. Anal. Appl.* **166**(1992), 236–246.

[24] R.M. GETHNER, J.H. SHAPIRO, Universal vector for operators on spaces of holomorphic functions, *Proc. Amer. Math. Soc.* **100**(1987), 281–288.

[25] G. GODEFROY, J.H. SHAPIRO, Operators with dense, invariant, cyclic vector manifolds, *J. Funct. Anal.* **98**(1991), 229–269.

[26] S. GRIVAUX, Hypercyclic operators, mixing operators and the bounded steps problem, *J. Operator Theory* **54**(2005), 147–168.

[27] K.G. GROSSE-ERDMANN, Universal families and hypercyclic operators, *Bull. Amer. Math. Soc.* **36**(1999), 345–381.

[28] K.G. GROSSE-ERDMANN, A. PERIS MANGUILLOT, *Linear Chaos*, Springer, Berlin 2011.

[29] A. GROTHENDIECK, Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.* **16**(1955).

[30] C.P. GUPTA, Convolution operators and holomorphic mappings on a Banach space, Séminaire d'Analyse Moderne, No. 2, Dept. Math., Université de Sherbrooke, Québec 1969.

[31] R. HÖLLSTEIN, (DCF)-Räume und lokalkonvexe tensorprodukte, *Arch. Math. (Basel)* **29**(1977), 524–531.

[32] R. HÖLLSTEIN, Tensorprodukte von steigen linearen Abbildungen in (F)- und (DCF)-Räume, *J. Reine Angew. Math.* **301**(1978), 191–204.

[33] J. HORVÁTH, *Topological Vector Spaces and Distributions*, vol. I, Addison-Wesley, Reading, MA 1966.

[34] C. KITAI, Invariant closed sets for linear operators, Ph.D. Dissertation, Univ. of Toronto, Toronto 1982.

[35] M.L. LOURENÇO, A projective limit representation of (DFC)-spaces with the approximation property, *J. Math. Anal. Appl.* **115**(1986), 422–433.

[36] G.R. MACLANE, Sequences of derivatives and normal families, *J. Anal. Math.* **2**(1952), 72–87.

[37] J. MUJICA, Domains of holomorphy on (DFC)-spaces, in *Functional Analysis, Holomorphy and Approximation Theory*, Lecture Notes in Math., vol. 843, Springer, Berlin 1981, pp. 500–533.

[38] J. MUJICA, *Complex Analysis in Banach spaces*, Dover, Mineola, New York 2010.

[39] S. MURO, D. PINASCO, M. SAVRANSKY, Strongly mixing convolution operators on Fréchet spaces of holomorphic functions, *Integral Equations Operator Theory* **80**(2014), 453–468.

[40] L. NACHBIN, On the topology of the space of all holomorphic functions on a given open subset, *Indag. Math.* **29**(1967), 366–368.

[41] L. NACHBIN, *Topology on Spaces of Holomorphic Mappings*, Springer, New York 1969.

[42] L. NACHBIN, Sur les espaces vectoriels topologiques d'applications continues, *C. R. Acad. Sci. Paris* **271**(1970), 596–598.

[43] L. NACHBIN, On pure uniform holomorphy in spaces of holomorphic germs, *Results Math.* **8**(1985), 117–122.

[44] H. PETERSSON, Hypercyclic subspaces for Fréchet space operators, *J. Math. Anal. Appl.* **319**(2006), 764–782.

[45] A. PIETSCH, *Nuclear Locally Convex Spaces*, Springer, New York 1972.

[46] M. SCHOTTENLOHER, Cartan–Thullen theorem for domains spread over DFM-spaces, *J. Reine Angew. Math.* **345**(1983), 201–220.

[47] L. SCHWARTZ, Produits tensoriels topologiques d'espaces vectoriels topologiques, Espaces vectoriels topologiques nucléaires, Applications, Séminaire Schwartz 1953/1954.

[48] M. VALDIVIA, Interpolation in certain function spaces, *Proc. Roy. Irish Acad. Sec. A* **80**(1980), 173–179.

VINÍCIUS V. FÁVARO, FAMAT, UFU, AV. JOÃO NAVES DE ÁVILA, 2121, 38.400-902, UBERLÂNDIA, MG, BRAZIL
E-mail address: vvfavaro@gmail.com

JORGE MUJICA, IMECC, UNICAMP, RUA SERGIO BUARQUE DE HOLANDA, 651, 13083-859, CAMPINAS, SP, BRAZIL
E-mail address: mujica@ime.unicamp.br

Received October 16, 2015; revised December 19, 2015 and January 8, 2016.