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CONDITIONS IMPLYING COMMUTATIVITY OF UNBOUNDED SELF-ADJOINT OPERATORS AND RELATED TOPICS

KARL GUSTAFSON and MOHAMMED HICHEM MORTAD

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ABSTRACT. Let B be a bounded self-adjoint operator and let A be a nonnegative self-adjoint unbounded operator. It is shown that if BA is normal, it must be self-adjoint and so must be AB. Commutativity is necessary and sufficient for this result. If AB is normal, it must be self-adjoint and BA is essentially self-adjoint. Although the two problems seem to be alike, two different and quite interesting approaches are used to tackle them.

KEYWORDS: *Normal and self-adjoint operators, commutativity, Fuglede theorem.*

MSC (2010): 47B25, 47B15.

INTRODUCTION

In [12], the following result (among others) was proved:

THEOREM 0.1. *Let A be an unbounded self-adjoint operator and let B be a positive (or negative) bounded operator. If AB (respectively BA) is normal, then AB (respectively BA) is self-adjoint.*

The foregoing results have applications for example to the problem of commutativity up to a factor (see [4]). They also provide us with a tool for commutativity of self-adjoint operators (see the proof of Theorem 0.1 in [12]).

The "AB case" was generalized in [12] to the case of two unbounded self-adjoint operators A and B. Later in [13] it was shown that under the same conditions, the normality of BA does not imply anymore its self-adjointness. But, there are still two cases to look at, namely: keeping B bounded, but taking A to be positive (both self-adjoint):

(i) Does AB normal imply AB self-adjoint?
(ii) Does BA normal imply BA self-adjoint?

These are the main questions asked in this paper. Commutativity relations play a vital role in their consideration. Other issues that arise are the domains and closedness of unbounded operators.

To prove the results we first assume basic notions and results on unbounded operator theory. For basic references, see [5], [10], [20], [22] and [24]. Recall that a densely defined unbounded operator A is: normal if it is closed and $AA^* = A^*A$, symmetric if $A \subset A^*$ and self-adjoint if $A = A^*$. Observe that both "symmetric" and "normal" are weaker than "self-adjoint", but one of the easy and nice results is: a normal and symmetric operator must be self-adjoint. Also, remember that the commutativity between a bounded B and an unbounded A is expressed as $BA \subset AB$.

Before finishing the introduction, we recall some other results (needed in the sequel) which cannot be considered as elementary.

The first is the well-known Fuglede–Putnam theorem.

THEOREM 0.2 (for a proof, see [5]). *If T is a bounded operator and and N and M are unbounded and normal, then*

$$TN \subset MT \Longrightarrow TN^* \subset M^*T.$$

COROLLARY 0.3. *If T is a bounded operator and and N and M are unbounded and normal, then*

$$TN = MT \Longrightarrow TN^* = M^*T.$$

The next is a generalization of the Fuglede theorem.

THEOREM 0.4 (Mortad, [12]). *Let T be an unbounded self-adjoint operator with domain $D(T)$. If N is an unbounded normal operator such that $D(N) \subset D(T)$, then*

$$TN \subset N^*T \Longrightarrow TN^* \subset NT.$$

We also note the following.

LEMMA 0.5. *If A and B are densely defined with inverse B^{-1} in $B(H)$, then $(AB)^* = B^*A^*$. In particular, if U is unitary, then*

$$(UAU)^* = U^*(UA)^* = U^*A^*U^*.$$

LEMMA 0.6. *If A and B are densely defined and closed such that either A is invertible or B is bounded, then AB is closed.*

For other related results, see [6], [7], [8], [16] and [21].

1. MAIN RESULTS

THEOREM 1.1. *Let A and B be two self-adjoint operators where only B is bounded. Assume further that A is positive and that BA is normal. Then both BA and AB are self-adjoint. Besides one has $AB = BA$.*

Proof. We may write

$$A(BA) = (AB)A = (BA)^*A.$$

Since BA is normal, $(BA)^*$ is normal too. Since $D(BA) = D(A)$, Theorem 0.4 applies and yields

$$A(BA)^* \subset BAA$$

or

$$A^2B \subset BA^2. \tag{1.1}$$

Let us transform the previous into a commutativity between B and A^2 (i.e. $BA^2 \subset A^2B$).

Since BA and $(BA)^*$ are normal, Corollary 0.3 allows us to write

$$B(BA)^* = B(AB) = (BA)B \Longrightarrow B(BA) = (BA)^*B$$

or

$$B^2A = AB^2. \tag{1.2}$$

This tells us that both B^2A and AB^2 are self-adjoint. Continuing we note that

$$B^2A^2 = AB^2A = A^2B^2$$

and

$$B^2A^4 = B^2A^2A^2 = A^2B^2A^2 = A^4B^2. \tag{1.3}$$

To prove B commutes with A^2, we first show that $\overline{BA}^2$ is normal. We have

$$\begin{aligned}(BA^2)^*BA^2 =& A^2BBA^2\\ \supset& A^2BA^2B \quad \text{(by inclusion (1.1))}\\ \supset& A^2A^2BB \quad \text{(by inclusion (1.1))}\\ =& A^4B^2 = B^2A^4.\end{aligned}$$

Passing to adjoints gives

$$(\overline{BA}^2)^*\overline{BA}^2 = (BA^2)^*\overline{BA}^2 \subset [(BA^2)^*BA^2]^* \subset (B^2A^4)^* = A^4B^2.$$

But A^4B^2 is symmetric by equation (1.3) (it is even self-adjoint). Since $\overline{BA}^2$ is closed, $(\overline{BA}^2)^*\overline{BA}^2$ is self-adjoint, and since self-adjoint operators are maximally symmetric, we immediately obtain

$$(\overline{BA}^2)^*\overline{BA}^2 = A^4B^2. \tag{1.4}$$

Similarly, we may obtain

$$B^2A^4 = A^4B^2 = A^2A^2BB \subset A^2BA^2B \subset BA^2A^2B = BA^2(BA^2)^*$$

and passing to adjoints yields

$$\overline{BA}^2(\overline{BA}^2)^* = \overline{BA}^2(BA^2)^* \subset [BA^2(BA^2)^*]^* \subset (B^2A^4)^* = A^4B^2.$$

Similar arguments as above imply that

$$\overline{BA}^2(\overline{BA}^2)^* = A^4B^2. \tag{1.5}$$

By equations (1.4) and (1.5), we see that $\overline{BA}^2$ is normal and hence we deduce as

$$(\overline{BA}^2)^* = (BA^2)^* = A^2B$$

that A^2B is closed, in fact normal.

Since A^2B is densely defined, we may adjoint relation (1.1) to obtain

$$(A^2B)^* \supset (BA^2)^* = A^2B$$

from which A^2B is symmetric. Since we have just seen that A^2B is normal, we infer that A^2B is self-adjoint. Thus, we have arrived at the basic inclusion and commutativity relation

$$BA^2 \subset \overline{BA}^2 = (A^2B)^* = A^2B = (BA^2)^*.$$

In particular, we then know from Theorem 10 in [3] (or [10]) and the positivity of A that B commutes with A, that is,

$$BA \subset AB \ (= (BA)^*).$$

But both BA and $(BA)^*$ are normal. Since normal operators are maximally normal, we obtain $BA = AB$.

Accordingly,

$$BA = AB = (BA)^* = (AB)^*,$$

and this completes the proof. ∎

Now, we turn to the case of AB normal (keeping all the other assumptions, except BA normal, as those of Theorem 1.1). The proof is very simple if we impose the very strong condition of the closedness of BA. We have the following.

COROLLARY 1.2. *Let A and B be two self-adjoint operators where only B is bounded. Assume further that A is positive, AB is normal and that BA is closed. Then both AB and BA are self-adjoint. Besides one has $AB = BA$.*

Proof. Since AB is normal, and B is bounded, $(BA)^*$ is clearly normal. Hence so is

$$(BA)^{**} = \overline{BA} = BA.$$

By Theorem 1.1, BA is self-adjoint. Therefore,

$$BA = (BA)^* = AB,$$

that is, AB is self-adjoint. ∎

One may wonder that there are so many assumptions that AB is normal would certainly imply that BA is closed. This is not the case as seen just below.

EXAMPLE 1.3. Let A be a self-adjoint, positive and boundedly invertible unbounded operator. Let B be its (bounded) inverse. So B too is self-adjoint. It is then clear

$$AB = I \quad \text{and} \quad BA \subset I.$$

Hence AB is self-adjoint (hence normal!) but BA is not closed.

A natural question is: what if BA is not closed, can we still show that the normality of AB implies its self-adjointness? As in Theorem 1.1, to show that AB is self-adjoint, it suffices to show that $BA \subset AB$. One of the ways of obtaining this is via $BA^2 \subset A^2B$ which may be obtained if for instance we have an intertwining result of the type

$$NA \subset AN^* \Longrightarrow N^*A \subset AN$$

where N is an unbounded normal operator playing the role of AB and A (and also B) is self-adjoint. Such an intertwining relation is, however, not true in general as seen in the next example (we also note that none of the existing unbounded versions of the Fuglede–Putnam theorem, as [12], [14], [17], [18] and [23], allows us to get this desired "inclusion").

EXAMPLE 1.4 (cf. [14]). Define the following operators A and N by

$$Af(x) = (1+|x|)f(x) \quad \text{and} \quad Nf(x) = -\mathrm{i}(1+|x|)f'(x)$$

(with $\mathrm{i}^2 = -1$) respectively on the domains

$$D(A) = \{f \in L^2(\mathbb{R}) : (1+|x|)f \in L^2(\mathbb{R})\}$$

and

$$D(N) = \{f \in L^2(\mathbb{R}) : (1+|x|)f' \in L^2(\mathbb{R})\}.$$

Then A is self-adjoint and positive (admitting even an everywhere defined inverse) and N is normal. We then find that

$$AN^*f(x) = NAf(x) = -\mathrm{i}(1+|x|)\operatorname{sgn}(x)f(x) - \mathrm{i}(1+|x|)^2 f'(x)$$

for any f in the *equal* domains

$$D(AN^*) = D(NA) = \{f \in L^2(\mathbb{R}) : (1+|x|)f \in L^2(\mathbb{R}),\ (1+|x|)^2 f' \in L^2(\mathbb{R})\}$$

and thus

$$AN^* = NA.$$

However

$$AN \not\subset N^*A \quad \text{and} \quad AN \not\supset N^*A \quad \text{for } ANf(x) = -\mathrm{i}(1+|x|)^2 f'(x)$$

whereas

$$N^*Af(x) = -2\mathrm{i}\operatorname{sgn}(x)(1+|x|)f(x) - \mathrm{i}(1+|x|)^2 f'(x).$$

Thus the method of proof of Theorem 1.1 could not be applied to the case AB and the approach then had to be different. After some investigation of possible counterexamples we were able instead to establish the affirmative result as follows.

THEOREM 1.5. *Let A and B be two self-adjoint operators where only B is bounded. Assume further that A is positive and that AB is normal. Then both $\overline{BA}$ and AB are self-adjoint. Besides one has $AB = \overline{BA}$.*

To prove it, we need a few lemmas which are also interesting in their own right.

LEMMA 1.6. *Let A, B be self-adjoint and $B \in B(H)$. If AB is densely defined, then we have:*

$$\overline{BA} = (AB)^*.$$

This easily follows from

$$(BA)^* = AB \Longrightarrow (AB)^* = (BA)^{**} = \overline{BA}.$$

In all the coming lemmas we assume that A and B are two self-adjoint operators such that $B \in B(H)$ and that AB is normal.

LEMMA 1.7. *We have:*

$$|B|A \subset A|B|.$$

Proof. We may write

$$B(AB) = BAB \subset \overline{BA}B.$$

Since both AB and $\overline{BA}$ are normal, Theorem 0.2 yields

$$B(AB)^* \subset (\overline{BA})^*B = (BA)^*B \quad \text{or merely } B^2A \subset AB^2.$$

Finally, by [9] (or [15]), we obtain

$$|B|A \subset A|B|. \quad \blacksquare$$

Before giving the next lemmas, let

$$B = U|B| = |B|U$$

be the polar decomposition of the self-adjoint B, where U is unitary (cf. [20]). Hence

$$B = U^*|B| = |B|U^*.$$

One of the major points is that U is even self-adjoint. To see this, just re-do the proof of Theorem 12.35(b) in [20] in the case of a self-adjoint operator. Then use the (self-adjoint!) functional calculus to get that U is self-adjoint. Another proof may be found in [2]. Therefore, $U = U^*$ and $U^2 = I$.

Let us also agree that any U which appears from now on is the U involved in this polar decomposition of B.

LEMMA 1.8. *We have:*

$$(AB)^* = UABU$$

so that

$$(AB)^*U = UAB \quad and \quad (AB)U = U(AB)^*.$$

Proof. Since $|B|A \subset A|B|$, we have $UBA \subset ABU$. Hence

$$UBAU \subset AB \quad \text{or} \quad (AB)^* \subset (UBAU)^*.$$

Since U is bounded, self-adjoint and invertible, we clearly have (by Lemma 0.5)

$$(UBAU)^* = U(BA)^*U = UABU.$$

Since AB is normal, so are $UABU$ and $(AB)^*$ so that

$$(AB)^* \subset UABU \Longrightarrow (AB)^* = UABU$$

because normal operators are maximally normal. ∎

LEMMA 1.9. *Assume also that $A \geqslant 0$. Then $A|B|$ is positive, self-adjoint and we have:*

$$|AB| = A|B|.$$

Proof. First, remember by Lemma 1.7 that $|B|A \subset A|B|$. Hence $A|B|$ is positive and self-adjoint as both $|B|$ and A are commuting and positive (see e.g. Exercise 23, page 113 of [22]). Now, by Lemma 1.8 we have

$$AB(AB)^* = ABUABU = A|B|A|B| = (A|B|)^2.$$

Since AB is normal, we have

$$|AB|^2 = (AB)^*AB = (A|B|)^2$$

so that (for instance by Theorem 11 of [3])

$$|AB| = A|B|. \quad ∎$$

LEMMA 1.10. *The operator UAB is normal.*

Proof. First, UAB is closed as U is invertible and AB is closed. Now,

$$\begin{aligned} UAB(UAB)^* &= UAB(AB)^*U \\ &= (AB)^*U(AB)^*U \quad \text{(by Lemma 1.8)} \\ &= (AB)^*ABU^2 \quad \text{(by Lemma 1.8)} \\ &= (AB)^*(AB). \end{aligned}$$

On the other hand,

$$(UAB)^*UAB = (AB)^*U^2AB = (AB)^*AB,$$

establishing the normality of UAB. ∎

LEMMA 1.11. *We have:*

$$U|AB| = |AB|U.$$

Proof. Since UAB is normal, we clearly have

$$UAB(AB)^*U = (AB)^*AB$$

which entails

$$UAB(AB)^* = U(AB)^*AB = (AB)^*ABU,$$

i.e.

$$U|AB|^2 = |AB|^2U.$$

Hence (by [3]), we are sure at least that $U|AB| \subset |AB|U$. Since $|AB|$ is self-adjoint, a similar argument to that used in the proof of Lemma 1.8 gives us

$$U|AB| = |AB|U. \quad \blacksquare$$

LEMMA 1.12. *Assume also that $A \geqslant 0$. Then B commutes with A, i.e. $BA \subset AB$.*

Proof. We have by Lemmas 1.9 and 1.11

$$U|AB| = |AB|U \Longleftrightarrow UA|B| = A|B|U \Longleftrightarrow UA|B| = AB.$$

Using Lemma 1.7

$$U|B|A \subset AB \quad \text{or} \quad BA \subset AB. \quad \blacksquare$$

We are now ready to prove Theorem 1.5.

Proof. By Lemma 1.12, $BA \subset AB$ so that

$$(AB)^* \subset AB.$$

Therefore, AB is self-adjoint as we already know that $D(AB) = D[(AB)^*]$. Finally, Lemma 1.6 gives

$$AB = \overline{BA}. \quad \blacksquare$$

The question of the essential self-adjointness of a product of two self-adjoint operators is not easy. In [12], a three page counterexample was constructed to show that if A and B are two unbounded self-adjoint operators such that $B \geqslant 0$, then the normality of $\overline{AB}$ does not entail its self-adjointness. Related to the question of essential self-adjointness of products, the reader may consult [11]. Having said this, now we may rephrase the result of Theorem 1.5 as follows.

COROLLARY 1.13. *Let A and B be two self-adjoint operators where only B is bounded. Assume further that A is positive and that $\overline{BA}$ is normal. Then BA is essentially self-adjoint.*

Proof. Since $\overline{BA}$ is normal, so is $(BA)^*$ or AB. Then by Theorem 1.5, AB is self-adjoint. By Lemma 1.6, $\overline{BA} = (AB)^*$ so that $\overline{BA}$ is self-adjoint. $\blacksquare$

In the end, we give an answer to an open problem from [4] concerning commutativity up to a factor.

PROPOSITION 1.14. *Let A and B be self-adjoint operators where B is bounded. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then $\lambda = 1$ if A is positive.*

Proof. By Proposition 2.2 of [4], we already know that AB is normal. By Theorem 1.5, AB is then self-adjoint. Now,

$$BA \subset \lambda AB \Longrightarrow \frac{1}{\lambda}BA \subset AB.$$

Hence

$$AB = (AB)^* \subset \frac{1}{\lambda}AB.$$

But $D(AB) = D(\alpha AB)$ for any $\alpha \neq 0$. Therefore,

$$AB = \frac{1}{\lambda}AB \text{ or simply } \lambda = 1. \quad \blacksquare$$

CONCLUSION

In this conclusion, we summarize all the related results to the problem considered in this paper. These are gathered from the present paper, [12] and [13]:

THEOREM 1.15. *Let A and B be two self-adjoint operators. Set $N = AB$ and $M = BA$.*

(i) *If $A, B \in B(H)$ (one of them is positive) and N (respectively M) is normal, then N (respectively M) is self-adjoint. In either case, we also have $AB = BA$.*

(ii) *If only $B \in B(H)$, $B \geqslant 0$ and N (respectively M) is normal, then N (respectively M) is self-adjoint. Also $BA \subset AB$ (respectively $BA = AB$).*

(iii) *If $B \in B(H)$, $A \geqslant 0$ and N (respectively M) is normal, then N (respectively M) is self-adjoint. Also $BA \subset AB$ (respectively $BA = AB$).*

(iv) *If $B \in B(H)$ and either A or B is positive, then $\overline{M}$ normal gives the essential self-adjointness of M.*

(v) *If both A and B are unbounded and N is normal, then it is self-adjoint whenever $B \geqslant 0$.*

(vi) *If both A and B are unbounded and $\overline{N}$ is normal, then N need not be essentially self-adjoint even if $B \geqslant 0$.*

(vii) *If both A and B are unbounded and M is normal, then it is not necessarily self-adjoint even when $B \geqslant 0$.*

Acknowledgements. Rehder [19] showed these results for all bounded self-adjoint operators and also provided a counterexample to show the necessity for some positivity. Neither the corresponding author of this paper (back in [12]) nor the authors of [1] were aware of his paper. Therefore Rehder deserves credit for being the first who investigated this topic in the bounded case (in particular his use of the Fuglede–Putnam theorem).

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KARL GUSTAFSON, DEPARTMENT OF MATHEMATICS, UNIV. OF COLORADO AT BOULDER, CAMPUS BOX 395 BOULDER, CO 80309-0395, U.S.A.

E-mail address: Karl.Gustafson@colorado.edu, gustafs@euclid.colorado.edu

MOHAMMED HICHEM MORTAD, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ORAN 1 (AHMED BENBELLA), B.P. 1524, EL MENOUAR, ORAN 31000, ALGERIA *and* B.P. 7085, SEDDIKIA ORAN, 31013 ALGERIA

E-mail address: mhmortad@gmail.com, mortad@univ-oran.dz.

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