ONE MORE PROOF OF THE INDEX FORMULA FOR BLOCK TOEPLITZ OPERATORS

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ABSTRACT. This paper provides a new proof of the index formula for block Toeplitz operators. The idea is to calculate a certain integral formula for the winding number using Fourier series, expressing this in terms of Hankel operators, and producing the expected index.

KEYWORDS: *Index formula, winding number, Toeplitz operator.*

MSC (2010): 47B35, 55M25, 47A53.

INTRODUCTION

Let $S^1 = \mathbb{R}/\mathbb{Z}$ and let $g : S^1 \to M_n(\mathbb{C})$ be a continuous function. We denote by $g_k \in M_n(\mathbb{C})$ the kth Fourier coefficient of g,

$$g_k = \int_0^1 g(t)e^{-2\pi i k t} dt, \quad k \in \mathbb{Z}.$$

The Toeplitz operator $T(g)$ and the Hankel operator $H(g)$ generated by the matrix function g are the bounded linear operators on $\ell^2(\mathbb{N}, \mathbb{C}^n)$ given by the infinite block matrices $(g_{j-k})_{j,k=1}^\infty$ and $(g_{j+k-1})_{j,k=1}^\infty$, respectively. It is well known that $T(g)$ is a Fredholm operator if and only if $\det g(t) \neq 0$ for $t \in S^1$ and that in this case the index Ind $T(g)$ of $T(g)$ equals minus the winding number $W(\det g)$ of $\det g$ about the origin. The earliest proof is [7].

One method of proving the index formula Ind $T(g) = -W(\det g)$, due independently to Atiyah [1] and Douglas [3], is to show these two functions are locally constant, respect composition of loops, and agree on the fundamental loop $g(t) = \{e^{it}, 1, \ldots, 1\}$ in the diagonal matrices. Another proof, which is the one given in [2], is based on Markus and Feldman's theorem [9]. This theorem says that if an operator matrix $T = (T_{jk})_{j,k=1}^n$ is constituted by Hilbert space operators that commute pairwise modulo trace class operators and if T is Fredholm,

then $\det T$ is also Fredholm and $\operatorname{Ind} T = \operatorname{Ind} \det T$. In this way the problem is reduced to the case $n = 1$. Here we give a proof that does not reduce to the case $n = 1$, nor require the topological arguments mentioned. We were motivated to find such a direct algebraic and non-topological proof in our attempt to prove a generalization of this formula involving higher degree differential forms.

The idea behind the proof given here is standard in the pseudodifferential operators community; see, e.g., the classic paper [6] or the lecture notes [8] (where it is "between the lines") or the very recent paper [5]. However, in the case of one-dimensional block Toeplitz operators, this idea can be broken down to an almost elementary and very short reasoning, which encouraged us to publish our proof which works equally well for all cases.

1. MAIN RESULT AND PROOF

It suffices to prove the index formula for continuously differentiable matrix functions g. Therefore, we assume henceforth that g is in C^1. We define $\widetilde{g}$ by $\widetilde{g}(t) = g(-t)$. We have the Fourier series

$$g(t) = \sum_{k \in \mathbb{Z}} g_k e^{2\pi i k t}, \qquad\qquad g^{-1}(t) = \sum_{k \in \mathbb{Z}} h_k e^{2\pi i k t},$$

$$\widetilde{g}(t) = \sum_{k \in \mathbb{Z}} g_{-k} e^{2\pi i k t}, \qquad\qquad \widetilde{g}^{-1}(t) = \sum_{k \in \mathbb{Z}} h_{-k} e^{2\pi i k t}.$$

Since the derivative g' is in L^2, it follows that $\sum_{k \in \mathbb{Z}} |k|^2 |g_k|_2^2 < \infty$. This implies that the Hilbert–Schmidt norm $\|H(g)\|_2$ is finite:

$$\|H(g)\|_2^2 = \sum_{k=1}^{\infty} k |g_k|_2^2 < \infty.$$

Analogously, we see that $H(g^{-1}), H(\widetilde{g}), H(\widetilde{g}^{-1})$, and also the Hankel operators resulting after replacing g by $\det g$, are all Hilbert–Schmidt. Finally, we define $W(g)$ by

$$W(g) = \frac{1}{2\pi i} \int\limits_0^1 \operatorname{tr} g^{-1}(t) g'(t) \, dt.$$

For $n = 1$, this is the usual winding number of g. We will show that if $n > 1$, then $W(g)$ is nothing but the usual winding number of $\det g$.

Here is the main result.

THEOREM. *Let $g : S^1 \to GL_n(\mathbb{C})$ be a C^1 function. Then the following quantities coincide:*
   (i) $-\operatorname{Ind} T(g)$,
   (ii) $-\operatorname{Ind} T(\det g)$,
  (iii) $W(g)$,

(iv) $W(\det g)$,

(v) $\operatorname{tr} H(g)H(\widetilde{g}^{-1}) - \operatorname{tr} H(g^{-1})H(\widetilde{g})$,

(vi) $\operatorname{tr} H(\det g)H(\det \widetilde{g}^{-1}) - \operatorname{tr} H(\det g^{-1})H(\det \widetilde{g})$.

Proof. We have

$$
W(g) = \frac{1}{2\pi i} \int_0^1 \operatorname{tr}\Big(\sum_{j\in\mathbb{Z}} h_j e^{2\pi i j t}\Big)\Big(\sum_{k\in\mathbb{Z}} 2\pi i k g_k e^{2\pi i k t}\Big)dt
$$

$$
= \int_0^1 \operatorname{tr} \sum_{j,k\in\mathbb{Z}} k h_j g_k e^{2\pi i(j+k)t} dt
$$

$$
= \operatorname{tr} \sum_{k\in\mathbb{Z}} k g_k h_{-k} = \operatorname{tr} \sum_{k=1}^{\infty} k g_k h_{-k} - \operatorname{tr} \sum_{k=1}^{\infty} k h_k g_{-k}
$$

$$
= \operatorname{tr} H(g)H(\widetilde{g}^{-1}) - \operatorname{tr} H(g^{-1})H(\widetilde{g}).
$$

A well-known formula due to Cálderon, Hörmander, Fedosov, and probably still others, states that if T is Fredholm and R is an operator such that $I - RT$ and $I - TR$ are trace class, then $\operatorname{Ind} T = \operatorname{tr}(I - RT) - \operatorname{tr}(I - TR)$. Another well-known identity, quoted in Proposition 2.14 of [2], implies that

$$
T(g^{-1})T(g) = I - H(g^{-1})H(\widetilde{g}), \quad T(g)T(g^{-1}) = I - H(g)H(\widetilde{g}^{-1}).
$$

As the Hankel operators are Hilbert–Schmidt, we deduce from this general index formula that

$$
-\operatorname{Ind} T(g) = \operatorname{tr}\left[I - T(g)T(g^{-1})\right] - \operatorname{tr}\left[I - T(g^{-1})T(g)\right]
$$

$$
= \operatorname{tr} H(g)H(\widetilde{g}^{-1}) - \operatorname{tr} H(g^{-1})H(\widetilde{g}).
$$

To complete the proof we are left with verifying the equality $W(g) = W(\det g)$, which is a consequence of the identity

$$
\frac{(\det g)'}{\det g} = \operatorname{tr}(g^{-1}g'),
$$

or, more generally, $d\log\det g = \operatorname{tr}(g^{-1}dg)$. This identity can easily be proved by elementary linear algebra. Another argument is as follows. Both sides are left invariant 1-forms on $GL_n(\mathbb{C})$ which equal the trace map on the Lie algebra $M_n(\mathbb{C})$. And, if $\gamma(t) = B + tA$ for $B \in GL_n(\mathbb{C})$ and $A \in M_n(\mathbb{C})$, then $\gamma(t) \in GL_n(\mathbb{C})$ for t small, and $\operatorname{tr}(\gamma^{-1}(0)\gamma'(0)) = \operatorname{tr}(B^{-1}A)$, while on the other hand,

$$
\frac{d}{dt}\log\det(\gamma(t))\Big|_{t=0} = \frac{1}{\det(\gamma(0))}\frac{d}{dt}\Big(\det(B)\det(I+tB^{-1}A)\Big)\Big|_{t=0} = \operatorname{tr}(B^{-1}A). \quad \blacksquare
$$

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