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CONTRACTIONS WITH POLYNOMIAL CHARACTERISTIC FUNCTIONS. II. ANALYTIC APPROACH

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This paper is dedicated to our lifelong friend Ron Douglas on the occasion of his upcoming 80th birthday

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ABSTRACT. The simplest and most natural examples of completely nonunitary contractions on separable complex Hilbert spaces which have polynomial characteristic functions are the nilpotent operators. The main purpose of this paper is to prove the following theorem: let T be a completely nonunitary contraction on a Hilbert space $\mathcal{H}$. If the characteristic function Θ_T of T is a polynomial of degree m, then there exist a Hilbert space $\mathcal{M}$, a nilpotent operator N of order m, a coisometry $V_1 \in \mathcal{L}(\overline{\operatorname{ran}}(I - NN^*) \oplus \mathcal{M}, \overline{\operatorname{ran}}(I - TT^*))$, and an isometry $V_2 \in \mathcal{L}(\overline{\operatorname{ran}}(I - T^*T), \overline{\operatorname{ran}}(I - N^*N) \oplus \mathcal{M})$, such that

$$\Theta_T = V_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} V_2.$$

KEYWORDS: *Characteristic function, model, nilpotent operators, operator valued polynomials.*

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INTRODUCTION

This is a sequel to our paper [4], where we identified the structure of the completely nonunitary contractions on a Hilbert space that have a polynomial characteristic function. Namely, we proved that the characteristic function Θ_T of a completely nonunitary contraction T on a separable, infinite dimensional, complex Hilbert space $\mathcal{H}$ is a polynomial if and only if there exist three closed subspaces $\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_{-1}$ of $\mathcal{H}$ with $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$, a pure isometry S in $\mathcal{L}(\mathcal{H}_1)$, a nilpotent N in $\mathcal{L}(\mathcal{H}_0)$, and a pure coisometry C in $\mathcal{L}(\mathcal{H}_{-1})$, such that T has the matrix representation

$$T = \begin{bmatrix} S & * & * \\ 0 & N & * \\ 0 & 0 & C \end{bmatrix}.$$

Moreover, the multiplicities of S and C, in other words, $\dim\ker S^*$ and $\dim\ker C$ are unitary invariants of T, and the nilpotent operator is uniquely determined by T up to a quasi-similarity. For earlier results on contractions with constant characteristic functions see [1], [9] and [10].

Recall that a pure isometry is a unilateral shift of some multiplicity and a pure coisometry is the adjoint of a pure isometry. Recall also that a contraction T on a Hilbert space $\mathcal{H}$, (i.e., $\|Th\| \leqslant \|h\|$ for all h in $\mathcal{H}$) is completely nonunitary (c.n.u.) if there is no nontrivial reducing subspace $\mathcal{M}$ of $\mathcal{H}$ for T such that $T|_{\mathcal{M}}$ is unitary.

In this paper we shall adopt a second approach to prove the theorem stated in the abstract, based essentially on new factorizations of characteristic functions of upper triangular 3×3 block contractions (see Theorem 1.3).

Before we continue we recall the notion of the characteristic function of a contraction. Consider a contraction T on a Hilbert space $\mathcal{H}$. The defect operators D_T and D_{T^*} and the defect spaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ of T are defined by

$$\begin{aligned} &D_T = (I_{\mathcal{H}} - T^*T)^{1/2}, \quad D_{T^*} = (I_{\mathcal{H}} - TT^*)^{1/2}, \quad \text{and} \\ &\mathcal{D}_T = \overline{\operatorname{Ran} D_T}, \quad \mathcal{D}_{T^*} = \overline{\operatorname{Ran} D_{T^*}}, \end{aligned}$$

respectively. Then the *characteristic function* of the contraction T is the $\mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*})$-valued contractive analytic function defined by

$$\Theta_T(z) = [-T + zD_{T^*}(I_{\mathcal{H}} - zT^*)^{-1}D_T]|_{\mathcal{D}_T} \quad (z \in \mathbb{D}).$$

In particular, Θ_T is an $\mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*})$-valued bounded analytic function on $\mathbb{D}$ (see [8]). Moreover, the characteristic function Θ_T is purely contractive, that is,

$$\|\Theta_T(0)\eta\| < \|\eta\| \quad (\eta \in \mathcal{D}_T, \eta \neq 0).$$

Let $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{M}, \mathcal{M}_*)$ and $\Psi : \mathbb{D} \to \mathcal{L}(\mathcal{N}, \mathcal{N}_*)$ be two operator valued analytic functions on $\mathbb{D}$. We say that Θ and Ψ coincide and write $\Theta \cong \Psi$ if there exist two unitary operators $\tau : \mathcal{M} \to \mathcal{N}$ and $\tau_* : \mathcal{M}_* \to \mathcal{N}_*$ such that

$$\Theta(z) = \tau_*^{-1}\Psi(z)\tau \quad (z \in \mathbb{D}),$$

or, equivalently, for all $z \in \mathbb{D}$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Theta(z)} & \mathcal{M}_* \\ \tau\Big\downarrow & & \tau_*\Big\downarrow \\ \mathcal{N} & \xrightarrow{\Psi(z)} & \mathcal{N}_* \end{array}$$

The characteristic function is a complete unitary invariant in the following sense (see Theorem 3.4 of [8]): two c.n.u. contractions T on $\mathcal{H}$ and R on $\mathcal{K}$ are unitarily equivalent (that is, there is a unitary operator U from $\mathcal{H}$ to $\mathcal{K}$ such that $T = U^*RU$) if and only if

$$\Theta_T \cong \Theta_R.$$

Moreover, for a given $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued purely contractive analytic function Θ defined on $\mathbb{D}$, there exists a c.n.u. contraction T on some Hilbert space, explicitly determined by Θ, such that Θ_T coincides with Θ.

Contractive operator valued analytic functions play an important role in operator theory and serve as a bridge between operator theory and function theory in terms of systems theory and interpolation theory (cf. [2], [6], [8]).

The class of nilpotent contractions yields a natural set of examples of operators that have polynomial characteristic functions. Indeed, let N be a contraction and a nilpotent operator of order m, $m \geqslant 1$, that is, $\|N\| \leqslant 1$, $N^m = 0$, and $N^{m-1} \neq 0$. The characteristic function Θ_N of N is given by

$$\begin{aligned}\Theta_N(z) &= [-N + zD_{N^*}(I_{\mathcal{H}} - zN^*)^{-1}D_N]|_{\mathcal{D}_N} \\ &= \Big[-N + \sum_{p=0}^{\infty} z^{p+1} D_{N^*} N^{*p} D_N\Big]\Big|_{\mathcal{D}_N} \\ &= \Big[-N + \sum_{p=0}^{m-1} z^{p+1} D_{N^*} N^{*p} D_N\Big]\Big|_{\mathcal{D}_N},\end{aligned}$$

for all $z \in \mathbb{D}$. Therefore Θ_N is a polynomial in z of degree at most m with operator coefficients.

From this viewpoint, it is important to understand, up to unitary equivalence, the analytic structure of polynomial characteristic functions of contractions. The main goal of the present paper is to address this issue. More specifically, in Theorem 2.2 we prove: if the characteristic function Θ_T of a c.n.u. contraction T is a polynomial of degree m, then there exist a Hilbert space $\mathcal{M}$, a nilpotent operator N of order m, a coisometry $V_1 \in \mathcal{L}(\mathcal{D}_{N^*} \oplus \mathcal{M}, \mathcal{D}_{T^*})$, and an isometry $V_2 \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_N \oplus \mathcal{M})$, such that

$$\Theta_T = V_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} V_2.$$

Along the way we prove the following factorization result for characteristic functions (see Theorem 1.3): let $\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_{-1}$ be three Hilbert spaces and set $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$. Let

$$T = \begin{bmatrix} S & * & * \\ 0 & N & * \\ 0 & 0 & C \end{bmatrix}$$

be any contraction on $\mathcal{H}$ with the above matricial form. Then the characteristic function Θ_T of T and

$$\begin{bmatrix} \Theta_C & 0 \\ 0 & I_{\mathcal{E}_1} \end{bmatrix} U_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} U_2 \begin{bmatrix} \Theta_S & 0 \\ 0 & I_{\mathcal{E}_2} \end{bmatrix}$$

coincide, where $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{M}$ are Hilbert spaces, and $U_1 \in \mathcal{L}(\mathcal{D}_{N^*} \oplus \mathcal{M}, \mathcal{D}_C \oplus \mathcal{E}_1)$ and $U_2 \in \mathcal{L}(\mathcal{D}_{S^*} \oplus \mathcal{E}_2, \mathcal{D}_N \oplus \mathcal{M})$ are unitary operators.

Our results rely on the upper triangular representation of operators with polynomial characteristic functions (see Theorem 2.1) and a factorization of characteristic functions of upper triangular 2×2 block contractions due to Sz.-Nagy and the first author (see Theorem 1.2).

The rest of this paper is organized as follows: in Section 2, we give the factorization of the characteristic function of an upper triangular 3×3 block contraction on Hilbert space. Our main result is given in Section 3, and provides a complete analytic characterization of polynomial characteristic functions for c.n.u. contractions on Hilbert space.

1. FACTORIZATIONS OF CHARACTERISTIC FUNCTIONS

We start by recalling some known facts about upper triangular 2×2 block contractions, since they will be frequently used in what follows.

The first is a classification of 2×2 block contractions. This is the content of Theorem 1 in [7] (also see Chapter IV, Lemma 2.1 in [3]).

THEOREM 1.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let* $T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ *be a bounded linear operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then T is a contraction if and only if T_1 and T_2 are contractions and*

$$X = D_{T_1^*} \Gamma D_{T_2},$$

for some contraction Γ from $\mathcal{D}_{T_2}$ to $\mathcal{D}_{T_1^*}$.

The second key tool used in our development is the factorization of characteristic functions of 2×2 block contractions (see Theorem 2 in [7]).

THEOREM 1.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, let* $T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ *be a contraction on $\mathcal{H}_1 \oplus \mathcal{H}_2$, and let $X = D_{T_1^*} \Gamma D_{T_2}$ for some contraction $\Gamma \in \mathcal{L}(\mathcal{D}_{T_1^*}, \mathcal{D}_{T_2})$. Then there exist unitary operators $\tau \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T_1} \oplus \mathcal{D}_\Gamma)$ and $\tau_* \in \mathcal{L}(\mathcal{D}_{T^*}, \mathcal{D}_{T_2^*} \oplus \mathcal{D}_{\Gamma^*})$ such that*

$$\Theta_T(z) = \tau_*^{-1} \begin{bmatrix} \Theta_{T_2}(z) & 0 \\ 0 & I_{\mathcal{D}_{\Gamma^*}} \end{bmatrix} J[\Gamma] \begin{bmatrix} \Theta_{T_1}(z) & 0 \\ 0 & I_{\mathcal{D}_\Gamma} \end{bmatrix} \tau \quad (z \in \mathbb{D}),$$

where

$$J[\Gamma] = \begin{bmatrix} \Gamma^* & D_\Gamma \\ D_{\Gamma^*} & -\Gamma \end{bmatrix} \in \mathcal{L}(\mathcal{D}_{T_1^*} \oplus \mathcal{D}_\Gamma, \mathcal{D}_{T_2} \oplus \mathcal{D}_{\Gamma^*}).$$

Recall that if A is a contraction from $\mathcal{H}$ to $\mathcal{K}$ then

$$J[A] = \begin{bmatrix} A^* & D_A \\ D_{A^*} & -A \end{bmatrix} \tag{1.1}$$

is a unitary operator from $\mathcal{K} \oplus \mathcal{D}_A$ to $\mathcal{H} \oplus \mathcal{D}_{A^*}$ (see [5]).

We are now ready to prove our first factorization result.

THEOREM 1.3. *Let $\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_{-1}$ be Hilbert spaces, and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$. Let*

$$T = \begin{bmatrix} S & * & * \\ 0 & N & * \\ 0 & 0 & C \end{bmatrix}$$

be a contraction on $\mathcal{H}$. Then there exist three Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{M}$ and two unitary operators $U_1 \in \mathcal{L}(\mathcal{D}_{N^*} \oplus \mathcal{M}, \mathcal{D}_C \oplus \mathcal{E}_1)$ and $U_2 \in \mathcal{L}(\mathcal{D}_{S^*} \oplus \mathcal{E}_2, \mathcal{D}_N \oplus \mathcal{M})$ such that

$$\Theta_T \cong \begin{bmatrix} \Theta_C & 0 \\ 0 & I_{\mathcal{E}_1} \end{bmatrix} U_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} U_2 \begin{bmatrix} \Theta_S & 0 \\ 0 & I_{\mathcal{E}_2} \end{bmatrix}.$$

Proof. Set

$$\mathcal{K}_1 = \mathcal{H}_1 \oplus \mathcal{H}_0, \quad T = \begin{bmatrix} T_1 & X_1 \\ 0 & C \end{bmatrix} \in \mathcal{L}(\mathcal{K}_1 \oplus \mathcal{H}_{-1}), \quad \text{and}$$

$$T_1 = \begin{bmatrix} S & X \\ 0 & N \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_0),$$

where $X_1 \in \mathcal{L}(\mathcal{H}_{-1}, \mathcal{K}_1)$ and $X \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$. Theorem 1.1 implies that there exist contractions $\Gamma_1 \in \mathcal{L}(\mathcal{D}_C, \mathcal{D}_{T_1^*})$ and $\Gamma \in \mathcal{L}(\mathcal{D}_N, \mathcal{D}_{S^*})$ such that

$$X_1 = D_{T_1^*} \Gamma_1 D_C, \quad \text{and} \quad X = D_{S^*} \Gamma D_N.$$

By Theorem 1.2 there exist unitary operators

$$\tau_1 : \mathcal{D}_T \to \mathcal{D}_{T_1} \oplus \mathcal{D}_{\Gamma_1}, \quad \tau_{1*} : \mathcal{D}_{T^*} \to \mathcal{D}_{C^*} \oplus \mathcal{D}_{\Gamma_1^*}, \quad \text{and} \tag{1.2}$$

$$\tau : \mathcal{D}_{T_1} \to \mathcal{D}_S \oplus \mathcal{D}_\Gamma, \quad \tau_* : \mathcal{D}_{T_1^*} \to \mathcal{D}_{N^*} \oplus \mathcal{D}_{\Gamma^*}, \tag{1.3}$$

such that

$$\Theta_T(z) = \tau_{1*}^{-1} \begin{bmatrix} \Theta_C(z) & 0 \\ 0 & I_{\mathcal{D}_{\Gamma_1^*}} \end{bmatrix} J[\Gamma_1] \begin{bmatrix} \Theta_{T_1}(z) & 0 \\ 0 & I_{\mathcal{D}_{\Gamma_1}} \end{bmatrix} \tau_1, \quad \text{and}$$

$$\Theta_{T_1}(z) = \tau_*^{-1} \begin{bmatrix} \Theta_N(z) & 0 \\ 0 & I_{\mathcal{D}_{\Gamma^*}} \end{bmatrix} J[\Gamma] \begin{bmatrix} \Theta_S(z) & 0 \\ 0 & I_{\mathcal{D}_\Gamma} \end{bmatrix} \tau,$$

for all $z \in \mathbb{D}$, where

$$J[\Gamma_1] = \begin{bmatrix} \Gamma_1^* & D_{\Gamma_1} \\ D_{\Gamma_1^*} & -\Gamma_1 \end{bmatrix} \in \mathcal{L}(\mathcal{D}_{T_1^*} \oplus \mathcal{D}_{\Gamma_1}, \mathcal{D}_C \oplus \mathcal{D}_{\Gamma_1^*}), \quad \text{and}$$

$$J[\Gamma] = \begin{bmatrix} \Gamma^* & D_\Gamma \\ D_{\Gamma^*} & -\Gamma \end{bmatrix} \in \mathcal{L}(\mathcal{D}_{S^*} \oplus \mathcal{D}_\Gamma, \mathcal{D}_N \oplus \mathcal{D}_{\Gamma^*}),$$

are unitary operators (see (1.1)). Now setting

$$\Phi_S(z) = \begin{bmatrix} \Theta_S(z) & 0 \\ 0 & I_{\mathcal{D}_\Gamma} \end{bmatrix}, \quad \Phi_C(z) = \begin{bmatrix} \Theta_C(z) & 0 \\ 0 & I_{\mathcal{D}_{\Gamma_1^*}} \end{bmatrix}, \quad \text{and}$$

$$\Phi_N(z) = \begin{bmatrix} \Theta_N(z) & 0 \\ 0 & I_{\mathcal{D}_{\Gamma^*}} \end{bmatrix},$$

for all $z \in \mathbb{D}$, we get

$$\begin{aligned}\Theta_T(z) &= \tau_{1*}^{-1}\Phi_C(z)J[\Gamma_1]\begin{bmatrix}\Theta_{T_1}(z) & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\tau_1\\
&= \tau_{1*}^{-1}\Phi_C(z)J[\Gamma_1]\begin{bmatrix}\tau_*^{-1}\Phi_N(z)J[\Gamma]\Phi_S(z)\tau & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\tau_1\\
&= \tau_{1*}^{-1}\Phi_C(z)\Big(J[\Gamma_1]\begin{bmatrix}\tau_*^{-1} & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\Big)\begin{bmatrix}\Phi_N(z)J[\Gamma]\Phi_S(z)\tau & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\tau_1\\
&= \tau_{1*}^{-1}\Phi_C(z)U_1\begin{bmatrix}\Phi_N(z)J[\Gamma]\Phi_S(z)\tau & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\tau_1,\end{aligned}$$

for all $z \in \mathbb{D}$, where $U_1 \in \mathcal{L}((\mathcal{D}_{N^*} \oplus \mathcal{D}_{\Gamma^*}) \oplus \mathcal{D}_{\Gamma_1}, \mathcal{D}_C \oplus \mathcal{D}_{\Gamma_1^*})$ is the unitary operator defined by

$$U_1 = J[\Gamma_1]\begin{bmatrix}\tau_*^{-1} & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}.$$

Hence

$$\begin{aligned}\Theta_T(z) &= \tau_{1*}^{-1}\Phi_C(z)U_1\begin{bmatrix}\Phi_N(z)J[\Gamma]\Phi_S(z)\tau & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\tau_1\\
&= \tau_{1*}^{-1}\Phi_C(z)U_1\begin{bmatrix}\Phi_N(z) & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\begin{bmatrix}J[\Gamma] & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\begin{bmatrix}\Phi_S(z) & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\begin{bmatrix}\tau & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\tau_1\\
&= \tau_{1*}^{-1}\Phi_C(z)U_1\begin{bmatrix}\Theta_N(z) & 0\\ 0 & I_{\mathcal{D}_{\Gamma^*}\oplus\mathcal{D}_{\Gamma_1}}\end{bmatrix}\begin{bmatrix}J[\Gamma] & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\\
&\quad\times\begin{bmatrix}\Theta_S(z) & 0\\ 0 & I_{\mathcal{D}_{\Gamma}\oplus\mathcal{D}_{\Gamma_1}}\end{bmatrix}\begin{bmatrix}\tau & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\tau_1,\end{aligned}$$

for all $z \in \mathbb{D}$. Let $U_2 \in \mathcal{L}((\mathcal{D}_{S^*} \oplus \mathcal{D}_{\Gamma}) \oplus \mathcal{D}_{\Gamma_1}, (\mathcal{D}_N \oplus \mathcal{D}_{\Gamma^*}) \oplus \mathcal{D}_{\Gamma_1})$ and $\widetilde{\tau}_1 \in \mathcal{L}(\mathcal{D}_T, (\mathcal{D}_S \oplus \mathcal{D}_{\Gamma}) \oplus \mathcal{D}_{\Gamma_1})$ be unitary operators defined by

$$U_2 = \begin{bmatrix}J[\Gamma] & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}, \quad \text{and} \quad \widetilde{\tau}_1 = \begin{bmatrix}\tau & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1}}\end{bmatrix}\tau_1,$$

respectively. Hence we obtain

$$\Theta_T(z) = \tau_{1*}^{-1}\Big(\begin{bmatrix}\Theta_C(z) & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1^*}}\end{bmatrix}U_1\begin{bmatrix}\Theta_N(z) & 0\\ 0 & I_{\mathcal{D}_{\Gamma^*}\oplus\mathcal{D}_{\Gamma_1}}\end{bmatrix}U_2\begin{bmatrix}\Theta_S(z) & 0\\ 0 & I_{\mathcal{D}_{\Gamma}\oplus\mathcal{D}_{\Gamma_1}}\end{bmatrix}\Big)\widetilde{\tau}_1, \tag{1.4}$$

for all $z \in \mathbb{D}$, and therefore

$$\Theta_T \cong \begin{bmatrix}\Theta_C & 0\\ 0 & I_{\mathcal{D}_{\Gamma_1^*}}\end{bmatrix}U_1\begin{bmatrix}\Theta_N & 0\\ 0 & I_{\mathcal{D}_{\Gamma^*}\oplus\mathcal{D}_{\Gamma_1}}\end{bmatrix}U_2\begin{bmatrix}\Theta_S & 0\\ 0 & I_{\mathcal{D}_{\Gamma}\oplus\mathcal{D}_{\Gamma_1}}\end{bmatrix},$$

holds. Setting $\mathcal{E}_1 = \mathcal{D}_{\Gamma_1^*}$, $\mathcal{M} = \mathcal{D}_{\Gamma^*} \oplus \mathcal{D}_{\Gamma_1}$ and $\mathcal{E}_2 = \mathcal{D}_{\Gamma} \oplus \mathcal{D}_{\Gamma_1}$ in the above, we conclude the proof of the theorem. ∎

Of particular interest is the case when S and C^* are pure isometries.

COROLLARY 1.4. *With the hypotheses of Theorem* 1.3*, let us also assume that S and C^* are pure isometries. Then there exist a Hilbert space $\mathcal{M}$, a coisometry $V_1 \in \mathcal{L}(\mathcal{D}_{N^*} \oplus \mathcal{M}, \mathcal{D}_{T^*})$, and an isometry $V_2 \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_N \oplus \mathcal{M})$, such that*

$$\Theta_T = V_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} V_2.$$

Proof. Notice that since $\mathcal{D}_{C^*} = \{0_{\mathcal{H}_{-1}}\}$ and $\mathcal{D}_S = \{0_{\mathcal{H}_1}\}$, the characteristic functions $\Theta_C : \mathbb{D} \to \mathcal{L}(\mathcal{D}_C, \mathcal{D}_{C^*})$ of C and $\Theta_S : \mathbb{D} \to \mathcal{L}(\mathcal{D}_S, \mathcal{D}_{S^*})$ of S are identically zero, that is,

$$0_C := \Theta_C \equiv 0 : \mathcal{D}_C \to \{0_{\mathcal{H}_{-1}}\}, \quad \text{and} \quad 0_S := \Theta_S \equiv 0 : \{0_{\mathcal{H}_1}\} \to \mathcal{D}_{S^*}.$$

Furthermore, the unitary operators in (1.2) and (1.3) become

$$\tau_1 : \mathcal{D}_T \to \mathcal{D}_{T_1} \oplus \mathcal{D}_{\Gamma_1}, \quad \tau_{1*} : \mathcal{D}_{T^*} \to \{0_{\mathcal{H}_{-1}}\} \oplus \mathcal{D}_{\Gamma_1^*}, \quad \text{and} \tag{1.5}$$

$$\tau : \mathcal{D}_{T_1} \to \{0_{\mathcal{H}_1}\} \oplus \mathcal{D}_{\Gamma}, \quad \tau_* : \mathcal{D}_{T_1^*} \to \mathcal{D}_{N^*} \oplus \mathcal{D}_{\Gamma^*}. \tag{1.6}$$

This along with (1.4) yields

$$\Theta_T = \tau_{1*}^{-1} \begin{bmatrix} 0_C & 0 \\ 0 & I_{\mathcal{D}_{\Gamma_1^*}} \end{bmatrix} U_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{D}_{\Gamma^*} \oplus \mathcal{D}_{\Gamma_1}} \end{bmatrix} U_2 \begin{bmatrix} 0_S & 0 \\ 0 & I_{\mathcal{D}_{\Gamma} \oplus \mathcal{D}_{\Gamma_1}} \end{bmatrix} \widetilde{\tau}_1 = V_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} V_2,$$

where

$$\mathcal{M} = \mathcal{D}_{\Gamma^*} \oplus \mathcal{D}_{\Gamma_1}, \quad V_1 = \tau_{1*}^{-1} \begin{bmatrix} 0_C & 0 \\ 0 & I_{\mathcal{D}_{\Gamma_1^*}} \end{bmatrix} U_1 \in \mathcal{L}((\mathcal{D}_{N^*} \oplus \mathcal{D}_{\Gamma^*}) \oplus \mathcal{D}_{\Gamma_1}, \mathcal{D}_{T^*}) \quad \text{and}$$

$$V_2 = U_2 \begin{bmatrix} 0_S & 0 \\ 0 & I_{\mathcal{D}_{\Gamma} \oplus \mathcal{D}_{\Gamma_1}} \end{bmatrix} \widetilde{\tau}_1 \in \mathcal{L}(\mathcal{D}_T, (\mathcal{D}_N \oplus \mathcal{D}_{\Gamma^*}) \oplus \mathcal{D}_{\Gamma_1}).$$

Now using $0_C 0_C^* = I_{\mathcal{D}_{C^*}} = I_{\{0_{\mathcal{H}_{-1}}\}}$ and $0_S^* 0_S = I_{\mathcal{D}_S} = I_{\{0_{\mathcal{H}_1}\}}$ along with (1.5) and (1.6) we readily see that $V_1 V_1^* = I_{\mathcal{D}_{T^*}}$ and $V_2^* V_2 = I_{\mathcal{D}_T}$. This completes the proof of the corollary. ∎

2. POLYNOMIAL CHARACTERISTIC FUNCTIONS

For the reader's convenience, we first state the main result of [4].

THEOREM 2.1. *Let T be a* c.n.u. *contraction on a Hilbert space $\mathcal{H}$. Then the characteristic function Θ_T of T is a polynomial of degree m if and only if there exist three closed subspaces $\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_{-1}$ of $\mathcal{H}$ with $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$, a pure isometry S in*

$\mathcal{L}(\mathcal{H}_1)$, a nilpotent N of order m in $\mathcal{L}(\mathcal{H}_0)$, and a pure coisometry C in $\mathcal{L}(\mathcal{H}_{-1})$, such that T has the matrix representation

$$T = \begin{bmatrix} S & * & * \\ 0 & N & * \\ 0 & 0 & C \end{bmatrix}.$$

We are now ready for the main theorem on analytic description of contractions which have polynomial characteristic functions.

THEOREM 2.2. *Let T be a* c.n.u. *contraction on a Hilbert space $\mathcal{H}$. If the characteristic function Θ_T of T is a polynomial of degree m, then there exist a Hilbert space $\mathcal{M}$, a nilpotent operator N of order m, a coisometry $V_1 \in \mathcal{L}(\mathcal{D}_{N^*} \oplus \mathcal{M}, \mathcal{D}_{T^*})$, and an isometry $V_2 \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_N \oplus \mathcal{M})$, such that*

$$\Theta_T = V_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} V_2.$$

Proof. Let T be a c.n.u. contraction such that the characteristic function Θ_T of T is a polynomial of degree m. According to Theorem 2.1 there exist closed subspaces $\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_{-1}$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_{-1}$ and such that with respect to that decomposition, T admits the matrix representation

$$T = \begin{bmatrix} S & * & * \\ 0 & N & * \\ 0 & 0 & C \end{bmatrix},$$

where $S \in \mathcal{L}(\mathcal{H}_1)$ is a pure isometry, $N \in \mathcal{L}(\mathcal{H}_0)$ is a nilpotent operator of order m, and $C \in \mathcal{L}(\mathcal{H}_{-1})$ is a pure coisometry. The result now follows from Corollary 1.4. ∎

REMARK 2.3. The converse of the above theorem is not true in full generality: let $\mathcal{M}$ be an infinite dimensional separable Hilbert space, and let T be a c.n.u. contraction with infinite dimensional defect spaces (for example, one can consider $T = S \oplus S^*$ on $H^2_{\mathcal{E}}(\mathbb{D}) \oplus H^2_{\mathcal{E}}(\mathbb{D})$, where $\mathcal{E}$ is an infinite dimensional Hilbert space, $H^2_{\mathcal{E}}(\mathbb{D})$ is the $\mathcal{E}$-valued Hardy space, and S is the shift operator on $H^2_{\mathcal{E}}(\mathbb{D})$). Let N be a nilpotent operator of order m and let

$$V_2 = \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} : \mathcal{D}_T \to \mathcal{D}_N \oplus \mathcal{M},$$

be an isometry, where

$$\|V_{21}\eta\| = \|V_{22}\eta\| \quad (\eta \in \mathcal{D}_T).$$

Also, let $V_1 : \mathcal{D}_{N^*} \oplus \mathcal{M} \to \mathcal{D}_{T^*}$ be a coisometry with $\ker V_1 = \mathcal{D}_{N^*}$. If

$$\Theta_T = V_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} V_2,$$

then Θ_T is a polynomial of degree 0.

However, it is easy to see that the following weak converse of Theorem 2.2 is true: let T be a c.n.u. contraction on a Hilbert space $\mathcal{H}$. Let

$$\Theta_T = V_1 \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} V_2,$$

for some Hilbert space $\mathcal{M}$, nilpotent operator N of order m, coisometry $V_1 \in \mathcal{L}(\mathcal{D}_{N^*} \oplus \mathcal{M}, \mathcal{D}_{T^*})$, and isometry $V_2 \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_N \oplus \mathcal{M})$. Then the characteristic function Θ_T of T is a polynomial of degree less than or equal to m.

It is important to note that the conclusion of Theorem 2.2 depends explicitly on the decomposition of T as used in the proof of Theorem 1.3. With the same setting as in Theorem 2.2, below we will show that the same conclusion holds for the following decomposition of T:

$$T = \begin{bmatrix} S & X_{-1} \\ 0 & T_{-1} \end{bmatrix} = \begin{bmatrix} S & D_{S^*}\Gamma_{-1}D_{T_{-1}} \\ 0 & T_{-1} \end{bmatrix} \in \mathcal{L}(\mathcal{H}_0 \oplus \mathcal{K}_{-1}),$$

where $\mathcal{K}_{-1} = \mathcal{H}_0 \oplus \mathcal{H}_{-1}$,

$$T_{-1} = \begin{bmatrix} N & X \\ 0 & C \end{bmatrix} = \begin{bmatrix} N & D_{N^*}\Gamma D_C \\ 0 & C \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_0), \quad \text{and}$$
$$X_{-1} = D_{S^*}\Gamma_{-1}D_{T_1}, X = D_{N^*}\Gamma D_C,$$

and Γ_{-1} in $\mathcal{L}(\mathcal{D}_{T_{-1}}, \mathcal{D}_{S^*})$ and Γ in $\mathcal{L}(\mathcal{D}_C, \mathcal{D}_{N^*})$ are a pair of contractions. In this case, again by Theorem 1.2, we have

$$\Theta_T = \tau_{-1*}^{-1} \begin{bmatrix} \Theta_{T_{-1}} & 0 \\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}} \end{bmatrix} J[\Gamma_{-1}] \begin{bmatrix} 0_S & 0 \\ 0 & I_{\mathcal{D}_{\Gamma_{-1}}} \end{bmatrix} \tau_{-1}, \quad \text{and} \tag{2.1}$$

$$\Theta_{T_{-1}} = \tau_*^{-1} \begin{bmatrix} 0_C & 0 \\ 0 & I_{\mathcal{D}_{\Gamma^*}} \end{bmatrix} J[\Gamma] \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{D}_\Gamma} \end{bmatrix} \tau, \tag{2.2}$$

where

$$\tau_{-1} : \mathcal{D}_T \to \{0_{\mathcal{H}_1}\} \oplus \mathcal{D}_{\Gamma_{-1}}, \quad \tau_{-1*} : \mathcal{D}_{T^*} \to \mathcal{D}_{T_{-1}^*} \oplus \mathcal{D}_{\Gamma_{-1}^*}, \quad \text{and} \tag{2.3}$$

$$\tau : \mathcal{D}_{T_{-1}} \to \mathcal{D}_N \oplus \mathcal{D}_\Gamma, \quad \tau_* : \mathcal{D}_{T_{-1}^*} \to \{0_{\mathcal{H}_{-1}}\} \oplus \mathcal{D}_{\Gamma^*}, \tag{2.4}$$

are unitary operators. Moreover

$$J[\Gamma_{-1}] = \begin{bmatrix} \Gamma_{-1}^* & D_{\Gamma_{-1}} \\ D_{\Gamma_{-1}^*} & -\Gamma_{-1} \end{bmatrix} \in \mathcal{L}(\mathcal{D}_{S^*} \oplus \mathcal{D}_{\Gamma_{-1}}, \mathcal{D}_{T_{-1}} \oplus \mathcal{D}_{\Gamma_{-1}^*}), \quad \text{and}$$
$$J[\Gamma] = \begin{bmatrix} \Gamma^* & D_\Gamma \\ D_{\Gamma^*} & -\Gamma \end{bmatrix} \in \mathcal{L}(\mathcal{D}_{N^*} \oplus \mathcal{D}_\Gamma, \mathcal{D}_C \oplus \mathcal{D}_{\Gamma^*}).$$

By setting

$$\Psi_0 = \begin{bmatrix} 0_C & 0 \\ 0 & I_{\mathcal{D}_{\Gamma^*}} \end{bmatrix} \quad \text{and} \quad \Psi_N = \begin{bmatrix} \Theta_N & 0 \\ 0 & I_{\mathcal{D}_\Gamma} \end{bmatrix},$$

and using (2.1) and (2.2) we obtain

$$\begin{aligned}
\Theta_T &= \tau_{-1*}^{-1}\begin{bmatrix}\Theta_{T_{-1}} & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix} J[\Gamma_{-1}] \begin{bmatrix}0_S & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}}}\end{bmatrix}\tau_{-1}\\
&= \tau_{-1*}^{-1}\begin{bmatrix}\tau_*^{-1}\Psi_0 J[\Gamma]\Psi_N\tau & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix} J[\Gamma_{-1}] \begin{bmatrix}0_S & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}}}\end{bmatrix}\tau_{-1}\\
&= \tau_{-1*}^{-1}\begin{bmatrix}\tau_*^{-1} & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\begin{bmatrix}\Psi_0 & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\begin{bmatrix}J[\Gamma] & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\begin{bmatrix}\Psi_N & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\\
&\qquad\times\begin{bmatrix}\tau & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix} J[\Gamma_{-1}] \begin{bmatrix}0_S & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}}}\end{bmatrix}\tau_{-1}\\
&= \widetilde{V}_1\begin{bmatrix}\Psi_N & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\widetilde{V}_2 = \widetilde{V}_1\begin{bmatrix}\Theta_N & 0\\ 0 & I_{\mathcal{D}_{\Gamma}\oplus\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\widetilde{V}_2,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{V}_1 &= \tau_{-1*}^{-1}\begin{bmatrix}\tau_*^{-1} & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\begin{bmatrix}\Psi_0 & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\begin{bmatrix}J[\Gamma] & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\\
&= \tau_{-1*}^{-1}\begin{bmatrix}\tau_*^{-1} & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\begin{bmatrix}0_C & 0\\ 0 & I_{\mathcal{D}_{\Gamma^*}\oplus\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\begin{bmatrix}J[\Gamma] & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}, \quad \text{and}\\
\widetilde{V}_2 &= \begin{bmatrix}\tau & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix} J[\Gamma_{-1}] \begin{bmatrix}0_S & 0\\ 0 & I_{\mathcal{D}_{\Gamma_{-1}}}\end{bmatrix}\tau_{-1}.
\end{aligned}$$

Hence

$$\Theta_T = \widetilde{V}_1\begin{bmatrix}\Theta_N & 0\\ 0 & I_{\mathcal{D}_{\Gamma}\oplus\mathcal{D}_{\Gamma_{-1}^*}}\end{bmatrix}\widetilde{V}_2 = \widetilde{V}_1\begin{bmatrix}\Theta_N & 0\\ 0 & I_{\widetilde{\mathcal{M}}}\end{bmatrix}\widetilde{V}_2,$$

where $\widetilde{\mathcal{M}} = \mathcal{D}_\Gamma \oplus \mathcal{D}_{\Gamma_{-1}^*}$. Finally, by virtue of (2.3) and (2.4), we have that $\widetilde{V}_1^*$ and $\widetilde{V}_2$ are isometric operators, that is, $\widetilde{V}_1\widetilde{V}_1^* = I_{\mathcal{D}_{T^*}}$ and $\widetilde{V}_2^*\widetilde{V}_2 = I_{\mathcal{D}_T}$. Yet, we do not know if

$$\dim \widetilde{\mathcal{M}} = \dim \mathcal{M},$$

where $\mathcal{M}$ is as in Theorem 2.2.

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