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INVERTIBILITY AND INVERSES OF TOEPLITZ PLUS HANKEL OPERATORS

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ABSTRACT. Necessary conditions for the invertibility of Toeplitz plus Hankel operators $T(a)+H(b)$ with generating functions $a,b \in L_\infty(\mathbb{T})$ satisfying the relation $a(t)a(1/t)=b(t)b(1/t)$, $t\in\mathbb{T}$ are obtained. In addition, sufficient conditions for the invertibility of such operators are also provided and efficient representations for the corresponding inverses are derived.

KEYWORDS: *Toeplitz plus Hankel operators, matching functions, invertibility, inverses.*

MSC (2010): Primary 47B35; Secondary 47B48.

1. INTRODUCTION

Toeplitz and Hankel operators appear in various fields of mathematics, physics and statistical mechanics [5], [13], and in spite of their remarkably distinctive features, they are closely related to each other. It is not surprising that Toeplitz plus Hankel operators $T(a)+H(b)$ also attracted considerable attention. Some particular points of interest are Fredholm properties, index, and invertibility of such operators.

At present, there is a well-developed Fredholm theory of Toeplitz plus Hankel operators with piecewise continuous generating functions acting on different Banach and Hilbert spaces (see Sections 4.95–4.102 in [5], Sections 4.5 and 5.7 in [14], and [15], [16]), whereas formulas for the index of Toeplitz plus Hankel operators $T(a)+H(b)$ with various assumptions about the generating functions a and b have been established in [6], [16].

The present paper deals with the invertibility of elements from a class of Toeplitz plus Hankel operators $T(a)+H(b)$, acting on classical Hardy spaces $H^p(\mathbb{T})$, $1<p<\infty$ on the unit circle $\mathbb{T}$. For Toeplitz operators $T(a)$ with scalar generating functions a this problem is well studied. There are efficient criteria of invertibility expressed in terms of winding numbers of the operator generating functions. Moreover, the corresponding inverse operators can be readily constructed. For Toeplitz plus Hankel operators the situation is completely different. For example, Fredholm Toeplitz plus Hankel operators are not necessarily one-sided invertible. The behavior of such operators resembles the behavior of matrix Toeplitz operators. Recall that the kernel and cokernel dimension of a matrix Toeplitz operator can be expressed via partial indices of the Wiener–Hopf factorization of the corresponding generating matrix. At the same time, for an arbitrary matrix-function there is no efficient procedure to obtain its Wiener–Hopf factorization or partial indices. Correspondingly, for arbitrary generating functions a and b there is no efficient tool to investigate the invertibility or one-sided invertibility of the operators $T(a) + H(b)$. Nevertheless, for a number of special cases, for example, if

$$b(t) = \pm a(t), \quad \text{or} \quad b(t) = \pm t^{\pm 1} a(t), \quad \text{or} \quad a(t) \equiv 1 \quad \text{and} \quad b(t)b(1/t) = 1,$$

the invertibility of such operators has been discussed in [1], [3], [4], [8], [9]. However, even if the invertibility of the operator $T(a) + H(b)$ is established, the construction of the corresponding inverse is still a challenging problem with no known efficient solution yet. Thus inverses for the operators $T(a) + H(a)$ and $I + H(a)$ acting on $H^2(\mathbb{T})$ are presented in [2], [12] for special functions a. In the more general situation explicit formulas for inverses of Toeplitz plus Hankel operators have been derived only recently and only for operators of the form $I + H(b)$ with generating functions b satisfying the condition $b(t)b(1/t) = 1$ for all $t \in \mathbb{T}$ [4].

In this paper we also deal with the invertibility problem for a special class of Toeplitz plus Hankel operators. Namely, let $\mathfrak{M}^p(L^\infty)$, $1 < p < \infty$ denote the set of Toeplitz plus Hankel operators

$$T(a) + H(b) : H^p(\mathbb{T}) \to H^p(\mathbb{T}),$$

the generating functions of which $a, b \in L^\infty(\mathbb{T})$ satisfy the relation

$$a(t)a(1/t) = b(t)b(1/t), \quad t \in \mathbb{T}. \tag{1.1}$$

Such operators have been studied in [3], [8], [9], [10]. In particular, the dimensions of the subspaces $\ker(T(a) + H(b))$ and $\operatorname{coker}(T(a) + H(b))$ of the operator $T(a) + H(b) \in \mathfrak{M}^p(L^\infty)$ with piecewise continuous generating functions a and b are computed in [3]. Papers [8], [9] are based on the same condition (1.1) but the functions a and b belong to $L^\infty(\mathbb{T})$ and in addition to defect numbers, they also contain an effective description of the subspaces $\ker(T(a) + H(b))$ and $\operatorname{coker}(T(a) + H(b))$. For some operators $T(a) + H(b) \in \mathfrak{M}^p(L^\infty)$ generalized inverses are constructed in [10].

In the present paper, the results of [9] are used in order to describe sets of invertible operators $T(a) + H(b)$ in $\mathfrak{M}^p(L^\infty)$. Moreover, for invertible Toeplitz plus Hankel operators from $\mathfrak{M}^p(L^\infty)$, effective formulas for their inverses are obtained.

Note that this approach can be employed in order to study the invertibility of Wiener–Hopf plus Hankel operators and to obtain efficient representations for their inverses if one uses results of [7], [11].

2. AUXILIARY RESULTS

Let us recall definitions and known results that are needed in what follows.

By $H^p = H^p(\mathbb{T})$ and $\overline{H}^p = \overline{H^p(\mathbb{T})}$ we denote the Hardy spaces of all functions $f \in L^p(\mathbb{T})$ the Fourier coefficients

$$\widehat{f}_n := \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta$$

of which vanish for all $n < 0$ and $n > 0$, respectively. It is well known that the Riesz projection P,

$$P : \sum_{n=-\infty}^{\infty} \widehat{f}_n \mathrm{e}^{\mathrm{i}n\theta} \mapsto \sum_{n=0}^{\infty} \widehat{f}_n \mathrm{e}^{\mathrm{i}n\theta},$$

is bounded on the space $L^p(\mathbb{T})$, $p \in (1,\infty)$ and its range is the whole space H^p. The operator $Q := I - P$ is also a projection and its range is a subspace of the codimension one in $\overline{H}^p$.

Let $J : L^p \mapsto L^p$ be the flip operator,

$$(Jf)(t) := \frac{1}{t} f(1/t), \quad t \in \mathbb{T}.$$

This operator J satisfies the relations

$$J^2 = I, \quad JPJ = Q, \quad JQJ = P,$$

and for any $a \in L^\infty$,

$$JaJ = \widetilde{a}I,$$

where $\widetilde{a}(t) := a(1/t)$, $t \in \mathbb{T}$.

Further, any element $a \in L^\infty$ generates two operators acting on the space H^p, $1 < p < \infty$. Namely, the Toeplitz operator $T(a)$ and the Hankel operator $H(a)$ are defined by

$$T(a) : f \mapsto Paf, \quad H(a) : f \mapsto PaQJf.$$

It is clear that the operators $T(a)$ and $H(a)$ are bounded. Moreover, for any $a, b \in L^\infty$ the identities

$$T(ab) = T(a)T(b) + H(a)H(\widetilde{b}), \quad H(ab) = T(a)H(b) + H(a)T(\widetilde{b}), \tag{2.1}$$

hold.

Consider now Toeplitz plus Hankel operators $T(a) + H(b) \in \mathfrak{M}^p(L^\infty)$ and recall that any duo of functions (a, b) satisfying the relation (1.1) is called the matching pair. It is well known that for any Fredholm operator $T(a) + H(b)$

the element $a \in L^\infty(\mathbb{T})$ is invertible. Therefore, in what follows we always assume that the generating function a belongs to the group GL^∞ of invertible elements from L^∞. Correspondingly, with each matching pair (a,b) one can associate another matching pair (c,d) defined by the relations $c := ab^{-1}(= \widetilde{b}\widetilde{a}^{-1})$, $d := a\widetilde{b}^{-1}(= b\widetilde{a}^{-1})$ and called the subordinated pair for the pair (a,b). The elements c and d of the subordinated pair satisfy the equation

$$c\widetilde{c} = 1, \quad d\widetilde{d} = 1.$$

Any function g satisfying the previous relation is called matching function.

Let us recall some properties of Toeplitz operators with generating matching functions.

PROPOSITION 2.1 ([9]). *Assume that g is a matching function. Then*

(i) *If the operator $T(g) : H^p \to H^p$ is Fredholm, then the function g admits Wiener–Hopf factorization in H^p,*

$$g(t) = \sigma(g)g_+(t)\, t^{-n}\, \widetilde{g}_+^{-1}(t), \quad \widetilde{g}_+^{-1}(\infty) = 1, \tag{2.2}$$

where $g_+ \in H^q$, $g_+^{-1} \in H^p$, $1/p + 1/q = 1$; n is the index of the operator $T(g)$, and $\sigma(g) = \pm 1$ is called the factorization signature of g.

(ii) *If $n = \operatorname{ind} T(g) > 0$, then $T(g)$ is invertible from the right and the operators*

$$\mathbf{P}_g^\pm := \frac{1}{2}(I \pm JQgP) : \ker T(g) \to \ker T(g),$$

considered on the kernel of the operator $T(g)$ are complementary projections on this space.

(iii) *If (2.2) is the Wiener–Hopf factorization of g in H^p and $n > 0$, then the following systems of functions $\mathcal{B}_\pm(g)$ form bases in the spaces $\operatorname{im} \mathbf{P}_g^\pm$:*

(a) *If $n = 2m$, $m \in \mathbb{N}$, then*

$$\mathcal{B}_\pm(g) := \{g_+^{-1}(t^{m-k-1} \pm \sigma(g)t^{m+k}) : k = 0,1,\dots,m-1\}, \quad \textit{and} \quad \dim \operatorname{im} \mathbf{P}_g^\pm = m.$$

(b) *If $n = 2m+1$, $m \in \mathbb{Z}_+$, then*

$$\mathcal{B}_\pm(g) := \{g_+^{-1}(t^{m+k} \pm \sigma(g)t^{m-k}) : k = 0,1,\dots,m\} \setminus \{0\},$$

and

$$\dim \operatorname{im} \mathbf{P}_g^\pm = m + \frac{1 \pm \sigma(g)}{2}. \tag{2.3}$$

3. INVERTIBILITY AND INVERSES OF TOEPLITZ PLUS HANKEL OPERATORS

In this section the invertibility of Toeplitz plus Hankel operators from the set $\mathfrak{M}^p(L^\infty)$ is studied. For any matching pair (a,b) we consider the Toeplitz operators $T(c)$ and $T(d)$, where (c,d) is the subordinated pair for (a,b). In what follows, we always assume that these two operators are Fredholm. It is well known that any Fredholm Toeplitz operator $T(g)$ is one sided invertible and by $T_{\mathrm{r}}^{-1}(g)$ or $T_{\mathrm{l}}^{-1}(g)$ we respectively denote a right or a left inverse for the operator $T(g)$.

We start this section with necessary conditions for the invertibility of operators from $\mathfrak{M}^p(L^\infty)$.

THEOREM 3.1. *Assume that an operator $T(a)+H(b) \in \mathfrak{M}^p(L^\infty)$ is invertible and let $\kappa_1 := \operatorname{ind} T(c)$, $\kappa_2 := \operatorname{ind} T(d)$.*

(i) *if $\kappa_1 \geqslant \kappa_2$ or $\kappa_1\kappa_2 \geqslant 0$, then $|\kappa_1| \leqslant 1$ and $|\kappa_2| \leqslant 1$.*

(ii) *if $\kappa_1 < 0$ and $\kappa_2 > 0$, then*

(a) *if κ_1 and κ_2 are even numbers, then $\kappa_2 = -\kappa_1$;*

(b) *if κ_1 is an odd number and κ_2 is an even one, then $\kappa_2 = -\kappa_1 + \sigma(c)$;*

(c) *if κ_1 is an even number and κ_2 is an odd one, then $\kappa_2 = -\kappa_1 - \sigma(d)$;*

(d) *if κ_1 and κ_2 are odd numbers, then $\kappa_2 = -\kappa_1 + \sigma(c) - \sigma(d)$.*

Proof. Assume first that $\kappa_1 \geqslant \kappa_2$ or $\kappa_1\kappa_2 \geqslant 0$. If $\kappa_1 > 1$, then the operator $T(c)$ is right invertible and if $\kappa_2 \geqslant 1$, then by Theorem 6.1(i) of [9] the kernel of the operator $T(a)+H(b)$ can be represented in the form

$$\ker(T(a)+H(b)) = \operatorname{im} \mathbf{P}_c^- \dotplus \varphi_+(\operatorname{im} \mathbf{P}_d^+),$$

where $\varphi_+ : \ker T(d) \to \ker(T(a)+H(b))$ is the injective operator defined by

$$\varphi_+(s) := \frac{1}{2}(T_{\mathrm{r}}^{-1}(c)T(\widetilde{a}^{-1})s - JQcPT_{\mathrm{r}}^{-1}(c)T(\widetilde{a}^{-1})s + JQ\widetilde{a}^{-1}s).$$

Since $\kappa_1 > 1$, Proposition 2.1 implies that $\ker \mathbf{P}_c^-$ contains non-zero elements, and the operator $T(a)+H(b)$ is not invertible.

Now let $\kappa_1 > 1$ and $\kappa_2 \leqslant 0$. By Theorem 6.1(iii) of [9], we have $\ker(T(a)+H(b)) = \operatorname{im} \mathbf{P}_c^-$. Further, by Proposition 2.1 $\operatorname{im} \mathbf{P}_c^- \neq \{0\}$ and the operator $T(a)+H(b)$ is not invertible either.

Consider next the case where $\kappa_1 < -1$ and $\kappa_2 \leqslant 0$. By Theorem 6.1(ii) of [9], one has

$$\operatorname{coker}(T(a)+H(b)) = \operatorname{im} \mathbf{P}_{\overline{d}}^- \dotplus \varphi_+(\operatorname{im} \mathbf{P}_{\overline{c}}^+).$$

But $\operatorname{ind} T(\overline{c}) = -\kappa_1 > 1$, so that $\operatorname{im} \mathbf{P}_{\overline{c}}^+ \neq \{0\}$ by Proposition 2.1(iii). Since φ_+ is an injective operator, it follows that $\operatorname{coker}(T(a)+H(b)) \neq \{0\}$ and the operator $T(a)+H(b)$ is not invertible. Combining all the above cases together, we obtain that $|\kappa_1| \leqslant 1$.

In order to find the possible range for the index κ_2, we consider the adjoint for the operator $T(a)+H(b)$. Note that the operator $(T(a)+H(b))^* = T(\overline{a}) + H(\widetilde{\overline{b}})$ belongs to the set $\mathfrak{M}^q(L^\infty)$, $1/p+1/q=1$, and $(\overline{d},\overline{c})$ is the subordinated pair for the pair $(\overline{a},\widetilde{\overline{b}})$. Since $\operatorname{ind} T(\overline{d}) = -\kappa_2$, $\operatorname{ind} T(\overline{c}) = -\kappa_1$, and we are still working under the assumption $\kappa_1 \geqslant \kappa_2$ or $\kappa_1\kappa_2 \geqslant 0$, the above considerations entail the inequality $|\kappa_2| \leqslant 1$.

Let us now explore the situation $\kappa_1 < 0$, $\kappa_2 > 0$. As already mentioned, there are four cases to deal with. For definiteness, we consider the case where $\kappa_1 = \operatorname{ind} T(c) < 0$ and $\kappa_2 = \operatorname{ind} T(d) > 0$ are odd numbers. Let n and m be the integers satisfying the relations

$$\kappa_1 + 2n = 1, \quad \kappa_2 = 2m + 1.$$

Further, let c_+ be the plus factor in the Wiener–Hopf factorization of the function c and $\{c_+^{-1}\}$ be the one-dimensional subspace of H^p generated by the function c_+^{-1}. Moreover, let S be the following subspace of H^p,

$$S := \frac{1-\sigma(c)}{2}\{c_+^{-1}\} \dotplus \varphi_+(\operatorname{im} \mathbf{P}_d^+).$$

Assume that

$$n < \dim S.$$

Then there is a function $\psi \in S$, $\psi \neq 0$ such that the first n Fourier coefficients $\widehat{\psi}_0, \widehat{\psi}_1, \dots, \widehat{\psi}_{n-1}$ of ψ are equal to zero. By Theorem 6.2(i) of [9], $\psi \in \ker(T(a) + H(b))$. Thus the kernel of $T(a) + H(b)$ contains a non-zero element, which contradicts the invertibility of this operator. Therefore, the assumption $n < \dim S$ is wrong. Hence, $n \geqslant \dim S$ and using equation (2.3), we obtain

$$n \geqslant \dim\left(\frac{1-\sigma(c)}{2}\{c_+^{-1}\} \dotplus \varphi_+(\operatorname{im} \mathbf{P}_d^+)\right) = \frac{1-\sigma(c)}{2} + m + \frac{1-\sigma(d)}{2},$$

or

$$\frac{1-\kappa_1}{2} \geqslant \frac{1-\sigma(c)}{2} + \frac{\kappa_2 - 1}{2} + \frac{1-\sigma(d)}{2}. \tag{3.1}$$

Similar considerations for the adjoint operator $T(\overline{a}) + H(\overline{\widetilde{b}})$ and the relations $\sigma(\overline{d}) = \sigma(d)$, $\sigma(\overline{c}) = \sigma(c)$ lead to the inequality

$$\frac{\kappa_2+1}{2} \geqslant \frac{1-\sigma(d)}{2} + \frac{-1-\kappa_1}{2} + \frac{1+\sigma(c)}{2}. \tag{3.2}$$

Comparing (3.1) and (3.2), one observes that

$$\sigma(c) - \sigma(d) \leqslant \kappa_2 + \kappa_1 \leqslant \sigma(c) - \sigma(d),$$

and the assertion (ii)(d) of Theorem 3.1 follows.

The other three cases connected with the condition $\kappa_1 < 0$, $\kappa_2 > 0$ can be considered analogously. ∎

REMARK 3.2. It is worth mentioning that if a and b are continuous at one of the points $t = -1$ or $t = 1$, then $\sigma(c) = \sigma(d)$, and the case (ii)(d) produces the condition $\kappa_2 = -\kappa_1$ only.

Indeed, if a and b are continuous, say at $t = 1$, then $c = ab^{-1}$ and $d = a\widetilde{b}^{-1}$ are matching functions also continuous at the same point $t = 1$. By Proposition 5.6 of [9] we have

$$\sigma(c) = a(1)b^{-1}(1) = a(1)\widetilde{b}^{-1}(1) = \sigma(d),$$

since the functions b and $\widetilde{b}$ take the same value at the point $t = 1$.

Our next goal is to obtain sufficient conditions of the invertibility for the operators from $\mathfrak{M}^p(L^\infty)$ and to provide efficient representations for their inverses. In this work we will restrict ourselves to the situation where $|\kappa_1| \leqslant 1$, $|\kappa_2| \leqslant 1$. This situation is represented by nine cases, namely,

(i) $\operatorname{ind} T(c) = 0$, $\quad \operatorname{ind} T(d) = 0$;
(ii) $\operatorname{ind} T(c) = 1$, $\quad \operatorname{ind} T(d) = 0$;
(iii) $\operatorname{ind} T(c) = 0$, $\quad \operatorname{ind} T(d) = 1$;
(iv) $\operatorname{ind} T(c) = 1$, $\quad \operatorname{ind} T(d) = 1$;
(v) $\operatorname{ind} T(c) = 0$, $\quad \operatorname{ind} T(d) = -1$;
(vi) $\operatorname{ind} T(c) = -1$, $\quad \operatorname{ind} T(d) = 0$;
(vii) $\operatorname{ind} T(c) = -1$, $\quad \operatorname{ind} T(d) = -1$;
(viii) $\operatorname{ind} T(c) = 1$, $\quad \operatorname{ind} T(d) = -1$;
(ix) $\operatorname{ind} T(c) = -1$, $\quad \operatorname{ind} T(d) = 1$.

Our first result is concerned with right invertible operators.

THEOREM 3.3. *Let $T(a) + H(b) \in \mathfrak{M}^p(L^\infty)$ and let (c, d) be the corresponding subordinated pair. If the operators $T(c)$ and $T(d)$ are invertible from the right, then the operator $T(a) + H(b)$ is also invertible from the right and the operator*

$$B := (I - H(\widetilde{c}))T_{\mathrm{r}}^{-1}(c)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d) + H(a^{-1})T_{\mathrm{r}}^{-1}(d), \tag{3.3}$$

is one of the right inverses for the operator $T(a) + H(b)$.

Proof. The proof of this result is straightforward. Consider the product $(T(a) + H(b))B$. It is

$$(T(a)+H(b))B=(T(a)+H(b))[(I-H(\widetilde{c}))T_{\mathrm{r}}^{-1}(c)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d)+H(a^{-1})T_{\mathrm{r}}^{-1}(d)].$$

Let us describe how the operators $T(a)$ and $H(b))$ interact with the operators $H(\widetilde{c})$ and $H(a^{-1})$. Taking into account relations (2.1), one obtains

$$H(b)H(\widetilde{c}) = T(bc) - T(b)T(c) = T(a) - T(b)T(c), \tag{3.4}$$

$$T(a)H(\widetilde{c}) = H(a\widetilde{c}) - H(a)T(c) = H(b) - H(a)T(c), \tag{3.5}$$

$$T(a)H(a^{-1}) = H(aa^{-1}) - H(a)T(\widetilde{a}^{-1}) = -H(a)T(\widetilde{a}^{-1}), \tag{3.6}$$

$$H(b)H(a^{-1}) = T(b\widetilde{a}^{-1}) - T(b)T(\widetilde{a}^{-1}) = T(d) - T(b)T(\widetilde{a}^{-1}). \tag{3.7}$$

The relations (3.4)–(3.5) imply that

$$\begin{aligned}
&(T(a) + H(b))(I - H(\widetilde{c})T_{\mathrm{r}}^{-1}(c)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d))\\
&\quad = (T(b)T(c) + H(a)T(c))T_{\mathrm{r}}^{-1}(c)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d)\\
&\quad = (T(b) + H(a))T(c)T_{\mathrm{r}}^{-1}(c)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d)\\
&\quad = T(b)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d) + H(a)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d).
\end{aligned}$$

On the other hand, the identities (3.6)–(3.7) lead to the expression

$$\begin{aligned}
&(T(a)+H(b))H(a^{-1})T_{\mathrm{r}}^{-1}(d)\\
&\qquad = -H(a)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d) + T(d)T_{\mathrm{r}}^{-1}(d) - T(b)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d)\\
&\qquad = I - H(a)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d) - T(b)T(\widetilde{a}^{-1})T_{\mathrm{r}}^{-1}(d).
\end{aligned}$$

Consequently

$$(T(a)+H(b))B = I,$$

and the operator B is a right inverse for the Toeplitz plus Hankel operator $T(a)+H(b)$. ∎

COROLLARY 3.4. *Let $T(a)+H(b) \in \mathfrak{M}^p(L^\infty)$ and let (c,d) be the corresponding subordinated pair. Assume that the operators $T(c)$ and $T(d)$ satisfy one of the conditions:*

(i) $\operatorname{ind} T(c) = 0$, $\operatorname{ind} T(d) = 0$;
(ii) $\operatorname{ind} T(c) = 1$, $\operatorname{ind} T(d) = 0$ *and* $\sigma(c) = 1$;
(iii) $\operatorname{ind} T(c) = 0$, $\operatorname{ind} T(d) = 1$ *and* $\sigma(d) = -1$;
(iv) $\operatorname{ind} T(c) = 1$, $\operatorname{ind} T(d) = 1$, $\sigma(c) = 1$, $\sigma(d) = -1$.

Then the operator $T(a)+H(b)$ is invertible and the inverse operator $(T(a)+H(b))^{-1}$ is given by the formula (3.3) *where the right inverses of the operators $T(c)$ or/and $T(d)$ shall be replaced by the corresponding inverses if necessary.*

Proof. If $\operatorname{ind} T(c) = 0$ and $\operatorname{ind} T(d) = 0$, then the operators $T(c)$ and $T(d)$ are invertible. Relations (3.1) and (3.7) of [9] show that the operator $T(a)+H(b)$ is also invertible and the result follows from Theorem 3.3.

Assume now that $\operatorname{ind} T(c) = 1$, $\operatorname{ind} T(d) = 0$ and $\sigma(c) = 1$. Then by Theorem 6.1(iii) of [9], the operator $T(a)+H(b)$ is right invertible and

$$\ker(T(a)+H(b)) = \operatorname{im} \mathbf{P}_c^-.$$

By Proposition 2.1(iii), the conditions $\operatorname{ind} T(c) = 1$ and $\sigma(c) = 1$ ensure that $\operatorname{im} \mathbf{P}_c^- = \{0\}$. Therefore, the operator $T(a)+H(b)$ is invertible and the inverse operator $(T(a)+H(b))^{-1}$ can be written in the form (3.3).

The consideration of the two remaining cases is based on assertion (i) of Proposition 6.1 in [9]. Thus the kernel of the operator $T(a)+H(b)$ has the form

$$\ker(T(a)+H(b)) = \operatorname{im} \mathbf{P}_c^- \dotplus \varphi_+(\operatorname{im} \mathbf{P}_d^+).$$

By Proposition 2.1(iii), in either case the space $\ker(T(a)+H(b))$ consists of the element 0 only, and the proof is completed. ∎

REMARK 3.5. If the functions a and b admit bounded Wiener–Hopf factorization, then the cases (ii)–(iii) do not appear, since in such situation, $\operatorname{ind} T(c) + \operatorname{ind} T(d)$ is always an even number. Moreover, if a and b are continuous at the point $t=1$ or $t=-1$, then the situation (iv) is not possible either.

Indeed, if a function $u \in L^\infty$, then the representation

$$u(t) = u_+(t)t^n u_-(t), \quad u_-(\infty) = 1, \tag{3.8}$$

where $u_+^{\pm 1} \in H^\infty$ and $u_-^{\pm 1} \in \overline{H}^\infty$ is called the bounded Wiener–Hopf factorization of u. It is well known [5] that if $u \in L^\infty$ admits a bounded Wiener–Hopf factorization (3.8), then for any $p \in (1, \infty)$ the Toeplitz operator $T(u) : H^p \to H^p$ is Fredholm and $\operatorname{ind} T(u) = -n$. Therefore, for a and b admitting bounded Wiener–Hopf factorization, the functions c and d also admit a bounded Wiener–Hopf factorization and

$$\operatorname{ind} T(c) = \operatorname{ind} T(a) - \operatorname{ind} T(b), \quad \operatorname{ind} T(d) = \operatorname{ind} T(a) + \operatorname{ind} T(b),$$

so that $\operatorname{ind} T(c) + \operatorname{ind} T(d) = 2 \operatorname{ind} T(a)$.

As far as the other assertion of Remark 3.5 is concerned, the parity of the factorization signatures has been established earlier (see Remark 3.2).

Let us emphasize that Corollary 3.4 and the representation (3.3) is, in a sense, a very surprising result. There is a vast literature devoted to the study of the Fredholmness and one-sided invertibility of Wiener–Hopf plus Hankel operators but mainly in the situation where the generating functions satisfy the relation $b = a$ or $b = \widetilde{a}$. Of course, such generating functions constitute a matching pair. However, even in these relatively simple situations there are no efficient representations for the operators $(T(a) + H(b))^{-1}$. On the other hand, for a wide class of functions u the one-sided inverses of the Toeplitz operators $T(u)$ can be effectively constructed. Therefore, the above formula (3.3) represents an efficient tool for finding inverses for operators $T(a) + H(b) \in \mathfrak{M}^p(L^\infty)$.

EXAMPLE 3.6. Let us consider the operator $T(a) + H(b)$ in the case where $a = b$. In this situation $c(t) = 1$ and $d(t) = a(t)\widetilde{a}^{-1}(t)$. Hence, $H(\widetilde{c}) = 0$, $T(c) = I$ and if $\operatorname{ind} T(d) = 0$, then the operator $T(a) + H(a)$ is also invertible and

$$(T(a) + H(a))^{-1} = (T(\widetilde{a}^{-1}) + H(a^{-1}))T^{-1}(a\widetilde{a}^{-1}).$$

EXAMPLE 3.7. Consider now the case where $b = \widetilde{a}$. Then $c(t) = a(t)\widetilde{a}^{-1}(t)$ but $d(t) = 1$. Hence, if the operator $T(a\widetilde{a}^{-1})$ is invertible, then the operator $T(a) + H(\widetilde{a})$ is also invertible and

$$(T(a) + H(\widetilde{a}))^{-1} = (I - H(\widetilde{a}a^{-1}))T^{-1}(a\widetilde{a}^{-1})T(\widetilde{a}^{-1}) + H(a^{-1}).$$

EXAMPLE 3.8. Let $a(t) = 1$ and $b(t)\widetilde{b}(t) = 1$ for all $t \in \mathbb{T}$. In this situation, $c(t) = \widetilde{b}(t)$, $d(t) = b(t)$ and if $T(b)$ is invertible, then the operator $I + H(b)$ is also invertible and

$$B = (I + H(b))^{-1} = (I - H(b))T^{-1}(\widetilde{b})T^{-1}(b). \tag{3.9}$$

REMARK 3.9. The inverses of the operators $I + H(b)$ with piecewise continuous matching functions b generating invertible operators $T(b)$ have been derived in the recent paper [4]. Thus if the operator $T(b)$ is invertible, then the operator $(I+H(b))^{-1}$ from [4] has the form

$$B_1 = T^{-1}(\widetilde{b})(I+H(\widetilde{b}))T^{-1}(b). \tag{3.10}$$

Let us show that (3.9) and (3.10) are one and the same operator. Indeed, computing the difference between the operators B_1 and B, we obtain

$$B_1 - B = T^{-1}(\widetilde{b})(H(\widetilde{b}) + T(\widetilde{b})H(b)T^{-1}(\widetilde{b}))T^{-1}(b). \tag{3.11}$$

However, it follows from (2.1) that

$$T(\widetilde{b})H(b) = H(\widetilde{b}b) - H(\widetilde{b})T(\widetilde{b}) = -H(\widetilde{b})T(\widetilde{b}).$$

Hence, the expression in the brackets on the right-hand side of (3.11) is

$$H(\widetilde{b}) + T(\widetilde{b})H(b)T^{-1}(\widetilde{b}) = H(\widetilde{b}) - H(\widetilde{b})T(\widetilde{b})T^{-1}(\widetilde{b}) = 0,$$

so $B = B_1$ and this is how it has to be.

Consider next the case of non-positive indices of the operators $T(c)$ and $T(d)$.

COROLLARY 3.10. *Let $T(a)+H(b) \in \mathfrak{M}^p(L^\infty)$ and let (c,d) be the corresponding subordinated pair. Assume that the operators $T(c)$ and $T(d)$ satisfy one of the conditions:*

(i) $\operatorname{ind} T(c) = -1$, $\operatorname{ind} T(d) = 0$ *and* $\sigma(c) = -1$;
(ii) $\operatorname{ind} T(c) = 0$, $\operatorname{ind} T(d) = -1$ *and* $\sigma(d) = 1$;
(iii) $\operatorname{ind} T(c) = -1$, $\operatorname{ind} T(d) = -1$, $\sigma(c) = -1$, $\sigma(d) = 1$.

Then the operator $T(a)+H(b)$ is invertible and the inverse operator $(T(a)+H(b))^{-1}$ is given by formula

$$\begin{aligned}(T(a)+H(b))^{-1} = &- H(\widetilde{c})(T_1^{-1}(c)T(\widetilde{a}^{-1})T_1^{-1}(d)(I-H(d)) + T_1^{-1}(c)H(\widetilde{a}^{-1}))\\ &+ H(a^{-1})T_1^{-1}(d)(I-H(d)) + T(a^{-1}),\end{aligned}$$

where the left inverses of the operators $T(c)$ or/and $T(d)$ shall be replaced by the corresponding inverses if necessary.

Proof. The invertibility of $T(a)+H(b)$ follows from Corollary 3.4 if one passes to the adjoint operator,

$$(T(a)+H(b))^* = T(\overline{a}) + H(\widetilde{\overline{b}}),$$

and takes into account the fact that $(\overline{d},\overline{c})$ is the subordinated pair for the matching pair $(\overline{a},\widetilde{\overline{b}})$. As soon as the invertibility of the operator $T(a)+H(b)$ is established, the formula (40) of [10] for the generalized inverses of operators from $\mathfrak{M}^p(L^\infty)$ can be used as the representation of the inverse operator. ∎

REMARK 3.11. Another way to obtain a representation for the inverse operator $(T(a)+H(b))^{-1}$ in this case is to pass to the adjoint operator $(T(a)+H(b))^*$, use formula (3.3), and write the adjoint for the operator obtained.

REMARK 3.12. Again, it is worth mentioning that for functions a and b admitting bounded Wiener–Hopf factorization, the situations (i) and (ii) from Corollary 3.10 do not appear (see Remark 3.5).

It remains to consider the cases (viii) and (ix). We start with the situation (viii).

THEOREM 3.13. *Let $T(a)+H(b) \in \mathfrak{M}^p(L^\infty)$ and let (c,d) be the subordinated pair for the pair (a,b). If*

$$\operatorname{ind} T(c) = 1, \quad \operatorname{ind} T(d) = -1, \quad \sigma(c) = 1, \quad \sigma(d) = 1,$$

then the operator $T(a)+H(b)$ is invertible and its inverse is given by the formula

$$\begin{aligned}(T(a)+H(b))^{-1} = &- H(\widetilde{c})(T_{\mathrm{r}}^{-1}(c)T(\widetilde{a}^{-1})T_{\mathrm{l}}^{-1}(d)(I-H(d)) + T_{\mathrm{r}}^{-1}(c)H(\widetilde{a}^{-1}))\\ &+ H(a^{-1})T_{\mathrm{l}}^{-1}(d)(I-H(d)) + T(a^{-1}).\end{aligned}$$

Proof. By Theorem 6.1(iii) of [9] we obtain that

$$\ker(T(a)+H(b)) = \operatorname{im} \mathbf{P}_c^- = \{0\}, \quad \operatorname{coker}(T(a)+H(b)) = \operatorname{im} \mathbf{P}_{\overline{d}}^- = \{0\},$$

so the operator $T(a)+H(b)$ is invertible. (Note that assertion (iii) of Theorem 6.1 in [9] contains typos. Namely, the operator $\mathbf{P}_{\overline{d}}^+$ has to be replaced by $\mathbf{P}_{\overline{d}}^-$ and vice versa.) Moreover, this operator satisfies all conditions of Theorem 6 in [10], hence the generalized inverse for the operator $T(a)+H(b)$ represented by formula (40) from [10] is the inverse operator $(T(a)+H(b))^{-1}$. ∎

EXAMPLE 3.14. Let b be a matching function. Consider the operator

$$A = I + H(b).$$

In this case $a(t) \equiv 1$, $c(t) = \widetilde{b}(t)$, $d(t) = b(t)$ and if $\operatorname{ind} T(b) = -1$ and $\sigma(b) = 1$, then the operator A is invertible and its inverse is given by the formula

$$(I+H(b))^{-1} = I - H(b)T_{\mathrm{r}}^{-1}(\widetilde{b})T_{\mathrm{l}}^{-1}(b)(I-H(b)).$$

Note that for piecewise continuous matching function b this case has been considered in [4] where the operator $(I+H(b))^{-1}$ is represented in a different form.

Consider now the remaining case: $\operatorname{ind} T(c) = -1$, $\operatorname{ind} T(d) = 1$.

THEOREM 3.15. *Let $T(a)+H(b) \in \mathfrak{M}^p(L^\infty)$ and let (c,d) be the subordinated pair for the pair (a,b). If*

$$\operatorname{ind} T(c) = -1, \quad \operatorname{ind} T(d) = 1, \quad \sigma(c) = -1, \quad \sigma(d) = -1,$$

then the operator $T(a)+H(b)$ is invertible and its inverse is given by the formula

$$\begin{aligned}(T(a)+H(b))^{-1} = {} & T(t^{-1})(I - c_+^{-1}tQt^{-1})\\ & \times [(I-H(t^2\widetilde{c}))T_{\mathrm{r}}^{-1}(t^{-2}c)T(\widetilde{a}^{-1}t^{-1})T_{\mathrm{r}}^{-1}(d) + H(a^{-1}t)T_{\mathrm{r}}^{-1}(d)],\end{aligned} \tag{3.12}$$

where c_+ is the plus factor in factorization (2.2) *for the function c.*

Proof. Note that the invertibility of the operator $T(a) + H(b)$ follows from Theorem 6.2(i) and Theorem 6.3(i) of [9]. Thus according to Theorem 6.2(i) of [9], the kernel of the operator $T(a) + H(b)$ is

$$\ker(T(a)+H(b)) = \{\psi \in \{T(t^{-1})u\} : u \in \{zc_+^{-1} \dot{+} \varphi_+(\operatorname{im} \mathbf{P}_d^+)\},\ z \in \mathbb{C} \text{ and } \widehat{u}_0 = 0\},$$

where $\widehat{u}_0$ is the zeroth Fourier coefficient of the function u. The condition $\sigma(d) = -1$ implies that $\operatorname{im} \mathbf{P}_d^+ = \{0\}$, so the zeroth Fourier coefficient of the function $u = zc_+^{-1}$ has to be equal to zero. But c_+ is a factorization factor in the Wiener–Hopf factorization of the function c. Therefore, the corresponding Fourier coefficient of the function u can be equal to zero if and only if $z = 0$. Hence the kernel of the operator $T(a) + H(b)$ consists of the element 0 only. The relation

$$\operatorname{coker}(T(a) + H(b)) = \{0\}$$

can be verified analogously, but instead of Theorem 6.2(i) of [9] one has to use Theorem 6.3(i) of [9]. Thus

$$\ker(T(a) + H(b)) = \{0\}, \quad \operatorname{coker}(T(a) + H(b)) = \{0\},$$

and the operator $T(a) + H(b)$ is invertible.

It remains to derive a formula for the inverse operator. Note that in the case at hand the operator $T(a) + H(b)$ does not satisfy the conditions of Theorem 6 of [10], so that the corresponding formula (40) of [10] cannot be used. Anyway, let us first consider a general situation where an invertible operator A on a Banach space X is represented as the product of two operators, i.e.

$$A = CD, \tag{3.13}$$

with the operators C and D acting on the same Banach space X. It follows from (3.13) that C and D are, correspondingly, right and left invertible operators. It is remarkable that in this case the inverse operator A^{-1} for the operator A can be constructed from one-sided inverses C_{r}^{-1} and D_{l}^{-1} of C and D. If these operators are Fredholm, then their one-sided invertibility and the equation $0 = \operatorname{ind} A = \operatorname{ind} C + \operatorname{ind} D$ show that $\dim\ker C = \dim\ker D^*$. Taking into account that $(\ker C) \cap (\operatorname{im} D) = 0$, one obtains $X = \ker C \dot{+} \operatorname{im} D$. Now we can employ a special projection on the range $\operatorname{im} D$ of the operator D. Indeed, the operator D considered as acting from the space X into the subspace $\operatorname{im} D$ is invertible and let $D_0^{-1} : \operatorname{im} D \to X$ be the corresponding inverse operator. Let P_0 be the projection from X onto $\operatorname{im} D$ parallel to $\ker C$. Then the operator $D_0^{-1} P_0$ is one of the right inverses for the operator $D : X \to \operatorname{im} D$, hence

$$AD_0^{-1}P_0C_{\mathrm{r}}^{-1} = (CD)(D_0^{-1}P_0)C_{\mathrm{r}}^{-1} = C(DD_0^{-1}P_0)C_{\mathrm{r}}^{-1},$$

and taking into account the relation

$$DD_0^{-1}P_0 = P_0 = I - (I - P_0),$$

one obtains

$$AD_0^{-1}P_0C_{\mathrm{r}}^{-1} = CC_{\mathrm{r}}^{-1} - C(I - P_0)C_{\mathrm{r}}^{-1}. \tag{3.14}$$

Since $\operatorname{im}(I - P_0) = \ker C$, the last operator in the right-hand side of (3.14) is just the zero operator. Therefore,

$$AD_0^{-1}P_0C_{\mathrm{r}}^{-1} = CC_{\mathrm{r}}^{-1} = I.$$

Thus the operator $D_0^{-1}P_0C_{\mathrm{r}}^{-1}$ is a right inverse for the operator A. However, the operator A is invertible, hence

$$A^{-1} = D_0^{-1}P_0C_{\mathrm{r}}^{-1}, \tag{3.15}$$

and it does not matter which right inverse C_{r}^{-1} of the operator C is used.

Consider now the operator $T(a) + H(b)$ and represent it in the form

$$A = T(a) + H(b) = A_1D = (T(a_1) + H(b_1))T(t),$$

where $a_1(t) = a(t)t^{-1}$ and $b_1(t) = b(t)t$, $t \in \mathbb{T}$. Recall that $T(t^{-1})$ is a left inverse for the operator $D = T(t)$. Moreover, a right inverse of the operator A_1 can be derived from our previous results. More precisely, the elements c_1 and d_1 of the subordinated pair (c_1, d_1) for the matching pair (a_1, b_1) are

$$\begin{aligned} c_1(t) &= a_1(t)b_1^{-1}(t) = a(t)t^{-1}b^{-1}(t)t^{-1} = c(t)t^{-2}, \\ d_1(t) &= a_1(t)\widetilde{b}_1^{-1}(t) = a(t)t^{-1}\widetilde{b}^{-1}(t)t = d(t). \end{aligned}$$

It follows that

$$\operatorname{ind} T(c_1) = \operatorname{ind} T(c) + 2 = -1 + 2 = 1, \quad \operatorname{ind} T(d_1) = \operatorname{ind} T(d) = 1.$$

Hence, both operators $T(c_1)$ and $T(d_1)$ are invertible from the right. Therefore, by Theorem 3.3 the operator $A_1 : H^p \to H^p$ is right invertible and one of its right inverses has the form

$$A_{1,\mathrm{r}}^{-1} = (I - H(t^2\widetilde{c}))T_{\mathrm{r}}^{-1}(t^{-2}c)T(\widetilde{a}^{-1}t^{-1})T_{\mathrm{r}}^{-1}(d) + H(a^{-1}t)T_{\mathrm{r}}^{-1}(d).$$

Moreover, according to Theorem 6.1 of [9] the kernel of the operator $T(a_1) + H(b_1)$ is

$$\ker(T(a_1) + H(b_1)) = \operatorname{im} \mathbf{P}_{c_1}^{-} + \varphi_+(\operatorname{im} \mathbf{P}_{d_1}^{+}).$$

Further, taking into account the relations

$$\operatorname{ind} T(d_1) = 1, \quad \sigma(d_1) = \sigma(d) = -1,$$

and assertion (iii) of Proposition 2.1, we obtain that $\mathbf{P}_{d_1}^{+}$ is the zero operator. Hence

$$\ker(T(a_1) + H(b_1)) = \operatorname{im} \mathbf{P}_{c_1}^{-} = \{z\, c_+^{-1} : z \in \mathbb{C}\},$$

where c_+ is the plus factor in the Wiener–Hopf factorizations

$$c(t) = \sigma(c)c_+(t)\, t\, \widetilde{c}_+^{-1}(t), \quad c_1(t) = \sigma(c)c_+(t)\, t^{-1}(t)\, \widetilde{c}_+^{-1},$$

for both the function c and the function c_1.

Now we can derive the operator P_0 projecting the space H^p onto the subspace $\operatorname{im} T(t)$ parallel to $\ker(T(a_1) + H(b_1))$, and obtain a representation for the inverse operator $(T(a) + H(b))^{-1}$. Recall that $\operatorname{im} T(t) = \{h \in H^p : \widehat{h}_0 = 0\}$, where $\widehat{h}_0$ is the zeroth Fourier coefficient of the function h. Let us define the operator $P_0 : H^p \to H^p$ by $P_0 := I - c_+^{-1} t Q t^{-1}$, where I is the identity operator and $Q = I - P$. Note that the zeroth Fourier coefficient $\widehat{c_{+0}}$ of the function c_+ is equal to 1 and for any $\varphi \in H^p$ one has $tQt^{-1}\varphi = \widehat{\varphi}_0$. Therefore, the operator P_0 is the projection on $\operatorname{im} T(t)$ parallel to $\ker(T(a_1) + H(b_1))$. Now one can use the representation (3.15), and formula (3.12) is proved. ∎

Thus in the case $|\kappa_1| \leqslant 1$, $|\kappa_2| \leqslant 1$ the problem of the invertibility of Toeplitz plus Hankel operators $T(a) + H(b)$ has been investigated in all details. The remaining cases of possible invertibility, which are mentioned in Theorem 3.1, can be treated analogously. However, the study of such operators requires more effort and the results expected seem to be not as transparent as in the cases considered.

In conclusion let us mention Toeplitz plus Hankel operators from $\mathfrak{M}^p(L^\infty)$ which are not even one-sided invertible.

COROLLARY 3.16. *Let $T(a) + H(b) \in \mathfrak{M}^p(L^\infty)$ and let (c,d) be the corresponding subordinated pair. If the operators $T(c)$ and $T(d)$ satisfy one of the following conditions:*

(i) $\operatorname{ind} T(c) = 1$, $\operatorname{ind} T(d) = -1$, $\boldsymbol{\sigma}(c) = -1$, $\boldsymbol{\sigma}(d) = -1$,

(ii) $\operatorname{ind} T(c) = -1$, $\operatorname{ind} T(d) = 1$, $\boldsymbol{\sigma}(c) = 1$, $\boldsymbol{\sigma}(d) = 1$.

then the operator $T(a) + H(b)$ is not one-sided invertible.

Proof. Indeed, in each case the corresponding operator $T(a) + H(b)$ possesses a non-zero kernel and cokernel. ∎

REMARK 3.17. Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. In the natural basis $\{t^n\}_{n\in\mathbb{Z}_+}$ of the space H^p, $1 < p < \infty$, Toeplitz and Hankel operators with generating function $a \in L^\infty$ can be, respectively, represented as infinite matrices $(\widehat{a}_{k-j})_{k,j=0}^\infty$ and $(\widehat{a}_{k+j+1})_{k,j=0}^\infty$, where $\widehat{a}_k$ is the k-th Fourier coefficient of the function a. In this form, Toeplitz and Hankel operators appear on the spaces $l^p(\mathbb{Z}_+)$ (see Section 2.3 in [5]). The study of Toeplitz plus Hankel operators on the spaces $l^p(\mathbb{Z}_+)$ is much more difficult since it is associated with two specific problems. Namely, with the multiplier and factorization problems. However, Theorem 3.3 and Corollary 3.4(i) do not involve any factorization arguments, so similar assertions are valid for the spaces $l^p(\mathbb{Z}_+)$ if a and b belongs to the corresponding multiplier sets.

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