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PROJECTIVE SPECTRUM AND KERNEL BUNDLE. II

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ABSTRACT. In this paper, we study the projective joint spectrum $P(A)$ and $P(A_*)$ of the operator tuple $A = (A_1, A_2, \dots, A_n)$. We first compute the joint spectrum for the Cuntz tuple. Then we study tuples of compact operators on an infinite dimensional Banach space. We show that if $P(A_*)$ is smooth, then $\bigvee\limits_{z\in P(A_*)} \ker A_*(z)$ forms a holomorphic line bundle over $P(A_*)$. For linearly independent vectors e_1, e_2, e_3 and $A_i = e_i \otimes e_i$, $i = 1, 2, 3$, the smoothness of $P(A_*)$ depends rather subtly on the relative position of the vectors. As an example, we compute the Chern character of the line bundle in the two vector case and show that it is nontrivial.

KEYWORDS: *Projective spectrum, Cuntz tuple, compact operator tuple, Hermitian bundle.*

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1. INTRODUCTION

Consider a complex algebra $\mathcal{B}$ with unit I. The classical spectrum of an element A is the set

$$\sigma(A) = \{\lambda \in \mathbb{C} : \ A - \lambda I \text{ is not invertible in } \mathcal{B}\}.$$

Traditionally, $\sigma(A)$ is viewed as a property of A, and its study is indeed a center piece of operator theory. However, there is a different point of view: $\sigma(A)$ is a gauge of interplay between A and the unit I. This point of view leads to the study of invertibilities of the linear pencil $A_1 - \lambda A_2$ and in more generality the multiparameter pencil

$$A(z) = z_1 A_1 + z_2 A_2 + \cdots + z_n A_n.$$

Indeed the multiparameter pencil $A(z)$ is an important subject in various fields, for example in algebraic geometry [16], mathematical physics [7], [17], PDE [1], [13], group theory [8], etc., and more recently in the settlement of the Kadison–Singer conjecture [11]. Of these studies, the primary interest is in the case when

A is a tuple of self-adjoint operators. For general tuples, the following notion of joint spectrum is defined in [18].

For a tuple $A = (A_1, A_2, \ldots, A_n)$ of elements in a unital Banach algebra $\mathcal{B}$, let $P(A) = \{z \in \mathbb{C}^n : A(z) \text{ is not invertible}\}$ and $p(A) = (P(A) \setminus \{0\})/\mathbb{C}_\times$. Note that $p(A)$ is a subset of the complex projective space $\mathbb{P}^{n-1}$. The set $P(A)$, as well as $p(A)$, is called the *projective joint spectrum* of A (projective spectrum or joint spectrum for short). The *projective resolvent set* refers to their complements $P^c(A) = \mathbb{C}^n \setminus P(A)$ and $p^c(A) = \mathbb{P}^{n-1} \setminus p(A)$.

Clearly, $0 \in \mathbb{C}^n$ is a trivial point in $P(A)$. It is shown in [18] that for every tuple A of elements in a unital Banach algebra $\mathcal{B}$, the projective spectrum $P(A)$ is nontrivial, i.e., containing points other than 0. However, it can happen that $P(A) = \mathbb{C}^n$. This paper will study two examples in this situation, namely the Cuntz tuple of isometries and tuples of compact operators on an infinite dimensional Banach space. In this situation, we consider the extended tuple $\widehat{A} = (I, A_1, A_2, \ldots, A_n)$. Then $P(\widehat{A})$ is the collection of $\widehat{z} \in \mathbb{C}^{n+1}$ such that

$$\widehat{A}(\widehat{z}) = z_0 I + z_1 A_1 + z_2 A_2 + \cdots + z_n A_n = z_0 I + A(z)$$

is not invertible. To avoid the trivial situation $z_0 = 0$, we will set $z_0 = 1$ and call $A_*(z) = I + A(z)$ a normalization of $A(z)$. $A_*(z)$ is obviously invertible when $\|A(z)\| < 1$. Hence $P^c(A_*) = \mathbb{C}^n \setminus P(A_*)$ is always nonempty. In the case $P(A) = \mathbb{C}^n$, the following two identifications are not hard to check:

$$p(\widehat{A}) \cong P(A_*) \cup \{\widehat{z} \in \mathbb{P}^n : z_0 = 0\} \quad \text{and} \tag{1.1}$$

$$p^c(\widehat{A}) \cong P^c(A_*). \tag{1.2}$$

Compared with other notions of joint spectrum, for instance, the Harte spectrum [9] or the Taylor spectrum [15], a notable distinction of the projective spectrum is that it is "base free" in the sense that, instead of using I as a base point and looking at the invertibility of

$$(A_1 - z_1 I, A_2 - z_2 I, \ldots, A_n - z_n I)$$

in various constructions, it considers the invertibility of the homogeneous multiparameter pencil $A(z)$. This simplicity makes it possible to study many interesting noncommuting examples, for instance a tuple of $k \times k$ matrices (cf. [10]), a tuple of compact operators (cf. [5], [14]), the tuple of generating unitaries for the free group von Neumann algebra (cf. [2]) and the tuple (I, a, t) for the infinite dihedral group

$$G = \langle a, t : a^2 = t^2 = 1 \rangle$$

with respect to the left regular representation (cf. [8]). This paper is a continuation of this effort, and in particular of the paper [10], but with a focus on the projective spectrum for the Cuntz tuple and the kernel bundle associated with tuples of compact operators.

2. PROJECTIVE SPECTRUM FOR THE CUNTZ TUPLE

The Cuntz algebra $\mathcal{O}_n$ (cf. [6]) is the universal C^*-algebra generated by n isometries $S_1, S_2, \ldots, S_n$ satisfying

$$S_i^* S_j = \delta_{ij} I \quad \text{for } 1 \leqslant i, j \leqslant n, \tag{2.1}$$

$$\sum_{i=1}^{n} S_i S_i^* = I, \tag{2.2}$$

where I is the identity. The Cuntz algebra is the first concrete example of a separable infinite simple C^*-algebra. In this section, we compute the projective spectrum for the Cuntz tuple $S = (S_1, S_2, \ldots, S_n)$ and its extension $\widehat{S} = (I, S_1, S_2, \ldots, S_n)$. To this end, we first fix a faithful irreducible representation π of $\mathcal{O}_n$ on a Hilbert space $\mathcal{H}$. Clearly, an element $a \in \mathcal{O}_n$ is invertible if and only if $\pi(a)$ is invertible as an operator on $\mathcal{H}$. In other words, the discussion of the projective spectrum is not affected by the choice of such representations.

LEMMA 2.1. *Let $S = (S_1, S_2, \ldots, S_n)$ be the Cuntz tuple. Then $P(S) = \mathbb{C}^n$.*

Proof. If $z = 0$, it is obvious that $S(z) = z_1 S_1 + z_2 S_2 + \cdots + z_n S_n = 0$. In the following, we assume that $z \neq 0$, and there are two cases.

Case 1. n is even.

For a nonzero $x \in \mathcal{H}$, let $y = (\bar{z}_2 S_1 - \bar{z}_1 S_2 + \cdots + \bar{z}_n S_{n-1} - \bar{z}_{n-1} S_n)x$. By (2.1) and (2.2), we see that the $S_j's$ are isometries on $\mathcal{H}$ which have orthogonal ranges. Therefore,

$$\begin{aligned} \|y\|^2 &= \|(\bar{z}_2 S_1 - \bar{z}_1 S_2 + \cdots + \bar{z}_n S_{n-1} - \bar{z}_{n-1} S_n)x\|^2 \\ &= |z_2|^2 \|S_1 x\|^2 + |z_1|^2 \|S_2 x\|^2 + \cdots + |z_n|^2 \|S_{n-1} x\|^2 + |z_{n-1}|^2 \|S_n x\|^2 \\ &= (|z_1|^2 + \cdots + |z_n|^2)\|x\|^2 \neq 0, \end{aligned}$$

that is $y \neq 0$. But

$$\begin{aligned} S^*(z)y &= (\bar{z}_1 S_1^* + \cdots + \bar{z}_n S_n^*)(\bar{z}_2 S_1 - \bar{z}_1 S_2 + \cdots + \bar{z}_n S_{n-1} - \bar{z}_{n-1} S_n)x \\ &= ((\bar{z}_1 \bar{z}_2 S_1^* S_1 - \bar{z}_1 \bar{z}_2 S_2^* S_2) + \cdots + (\bar{z}_{n-1} \bar{z}_n S_{n-1}^* S_{n-1} - \bar{z}_{n-1} \bar{z}_n S_n^* S_n))x = 0. \end{aligned}$$

This shows that $\ker S^*(z) \neq 0$, and hence $S(z)$ is not invertible. Therefore the lemma holds when n is even.

Case 2. n is odd.

For $x \neq 0$, let $y = (\bar{z}_2 S_1 - \bar{z}_1 S_2 + \cdots + \bar{z}_{n-1} S_{n-2} - \bar{z}_{n-2} S_{n-1})x$. Then

$$\|y\|^2 = (|z_1|^2 + \cdots + |z_{n-1}|^2)\|x\|^2.$$

If one of $z_1, z_2, \ldots, z_{n-1}$ is nonzero, then $y \neq 0$. In this case, similar to the arguement in Case 1, we have

$$S^*(z)y = (\bar{z}_1 S_1^* + \cdots + \bar{z}_n S_n^*)(\bar{z}_2 S_1 - \bar{z}_1 S_2 + \cdots + \bar{z}_{n-1} S_{n-2} - \bar{z}_{n-2} S_{n-1})x$$

$$=(\bar{z}_1\bar{z}_2S_1^*S_1-\bar{z}_1\bar{z}_2S_2^*S_2)x+\cdots+(\bar{z}_{n-2}\bar{z}_{n-1}S_{n-2}^*S_{n-2}-\bar{z}_{n-2}\bar{z}_{n-1}S_{n-1}^*S_{n-1})x=0.$$

Hence $S(z)$ is not invertible.

If $z_1 = z_2 = \cdots = z_{n-1} = 0$, then $z_n \neq 0$. But in this case, $S(z) = z_nS_n$ is obviously not invertible. The lemma holds for odd n as well. ∎

Since $P(S) = \mathbb{C}^n$, the resolvent set $P^c(S)$ is empty. To make the study more interesting, we add the identity operator I to the Cuntz tuple and consider the extended Cuntz tuple $\widehat{S} = (I, S_1, S_2, \dots, S_n)$. One sees that the projective spectrum $P(\widehat{S})$ in this case is very different. By definition, $P(\widehat{S})$ is now a proper subset of $\mathbb{C}^{n+1}$ because $(1, 0, \dots, 0) \in P^c(\widehat{S})$. And as is indicated in (1.2),

$$p^c(\widehat{S}) \cong P^c(S_*).$$

THEOREM 2.2. *For the extended Cuntz tuple $\widehat{S} = (I, S_1, S_2, \dots, S_n)$, the projective resolvent set*

$$P^c(S_*) = \mathbb{B}_n,$$

where $\mathbb{B}_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ is the unit ball of $\mathbb{C}^n$.

Proof. If $z = 0$, then $S_*(z) = I + z_1S_1 + z_2S_2 + \cdots + z_nS_n = I$, which is obviously invertible. In the following, we assume that $z \neq 0$.

By (2.1) and (2.2), we easily see that for

$$S(z) = z_1S_1 + z_2S_2 + \cdots + z_nS_n,$$

we have

$$\begin{aligned} S^*(z)S(z) &= (\bar{z}_1S_1^* + \cdots + \bar{z}_nS_n^*)(z_1S_1 + z_2S_2 + \cdots + z_nS_n) \\ &= |z_1|^2S_1^*S_1 + |z_2|^2S_2^*S_2 + \cdots + |z_n|^2S_n^*S_n \\ &= (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)I = \|z\|^2I. \end{aligned}$$

So if we let

$$V(z) = \frac{S(z)}{\|z\|},$$

then $V(z)$ is an isometry for every $z \neq 0$.

By the von Neumann–Wold decomposition theorem, every isometry V is of the form

$$V = U \oplus W,$$

where U is a unitary, and W is a unilateral shift. So for every fixed $z \neq 0$, we have a corresponding decomposition

$$V(z) = U_z \oplus W_z.$$

By Lemma 2.1, $V(z)$ is not invertible and hence it is not a unitary for every $z \neq 0$. So the unilateral shift component W_z is nonzero for each z. It follows that the classical spectrum $\sigma(V(z)) \supseteq \sigma(W_z) = \overline{\mathbb{D}}$. But $V(z)$ itself is an isometry, hence $\sigma(V(z)) \subseteq \overline{\mathbb{D}}$. Thus we have $\sigma(V(z)) = \overline{\mathbb{D}}$.

Since

$$S_*(z) = I + S(z) = I + \|z\|V(z) = \|z\|\Big(V(z) + \frac{1}{\|z\|}I\Big),$$

$S_*(z)$ is invertible if and only if $1/\|z\| > 1$ or equivalently $z \in \mathbb{B}_n$. ∎

3. KERNEL BUNDLE OVER PROJECTIVE SPECTRUM

In this section, we consider the projective spectrum for a tuple A of compact operators. It was shown in [14] that in this case, $P(A_*)$ is a thin set, i.e., it is locally the zero set of one holomorphic function. By a general fact in [18], if A is a commuting tuple, then $P(A_*)$ is a union of hyperplanes. Quite surprisingly, a converse in some sense was proved recently in [5]. In particular, it was shown that for $A = (K, K^*)$, where K is a compact operator on a Hilbert space, K is normal if and only if $P(A_*)$ is a union of hyperplanes. This result indicates that, at least in the compact operator tuple case, the geometry of $P(A_*)$ tells a great deal about the algebraic properties of the tuple. Indeed, there are many appealing questions in this direction. This section studies the smoothness of $P(A_*)$ and shows that if the projective spectrum is smooth, a holomorphic line bundle can be naturally constructed over $P(A_*)$.

Let X be an infinite dimensional Banach space and $A = (A_1, A_2, \dots, A_n)$ be a tuple of compact operators acting on X. Clearly, $P(A) = \mathbb{C}^n$. Just like in the Cuntz tuple case, the extended tuple $\widehat{A} = (I, A_1, A_2, \dots, A_n)$ is more interesting. Recall that

$$P(A_*) = \Big\{z \in \mathbb{C}^n : \ A_*(z) = I + \sum_{j=1}^{n} z_j A_j \text{ is not invertible}\Big\}.$$

As mentioned above, $P(A_*)$ is locally the zero set of one holomorphic function. To see this, for $\lambda \in P(A_*)$, we let $F = (F_1, F_2, \dots, F_n)$ be a tuple of finite rank operators such that $\sum |z_j|\|A_j - F_j\| < 1$ for every point z in a small neighborhood of λ, say $U \subset \mathbb{C}^n$. Then $I + \sum z_j(A_j - F_j)$ is invertible on U. Write

$$\begin{aligned} A_*(z) &= I + \sum z_j(A_j - F_j) + \sum z_j F_j \\ &= \Big(I + \sum z_j(A_j - F_j)\Big)\Big(I + \Big(I + \sum z_j(A_j - F_j)\Big)^{-1} \sum z_j F_j\Big). \end{aligned}$$

For convenience, we let

$$K_{U,F}(z) = I + \Big(I + \sum z_j(A_j - F_j)\Big)^{-1} \sum z_j F_j \quad z \in U. \tag{3.1}$$

For a trace class operator T, the Fredholm determinant is well-defined for $I + T$ (cf. [12]), and it is well-known that $I + T$ is invertible if and only if $\det(I + T) \neq 0$. Clearly, $A_*(z)$ is not invertible if and only if $K_{U,F}(z)$ is not invertible, and this is the case if and only if $\det K_{U,F}(z) = 0$. Hence $U \cap P(A_*)$ is the zero set of the holomorphic function $\det K_{U,F}(z)$.

Recall that for a holomorphic function $h(z)$, its gradient is

$$\nabla h = \Big(\frac{\partial h}{\partial z_1}, \frac{\partial h}{\partial z_2}, \dots, \frac{\partial h}{\partial z_n}\Big).$$

If $\lambda \in \mathbb{C}^n$ is a zero of h, then the tangent plane to the zero set of h at λ is given by the equation $\langle w - \lambda,\ \overline{\nabla h(\lambda)}\rangle = 0,\ w \in \mathbb{C}^n$. We make the following definition to proceed.

DEFINITION 3.1. A point $\lambda \in P(A_*)$ is said to be *regular* if there exists a tuple of finite rank operators $F = (F_1, F_2, \dots, F_n)$ and a neighborhood U of λ in $\mathbb{C}^n$ such that the gradient

$$\nabla(\det K_{U,F})(z) \neq 0 \quad \forall z \in U.$$

$P(A_*)$ is said to be *smooth* if every point of $P(A_*)$ is regular.

It is clear from the definition that the set of regular points in $P(A_*)$ is relatively open in $P(A_*)$. Further, if $\lambda \in P(A_*)$ is regular then there is a small neighborhood V of λ in $\mathbb{C}^n$ such that $V \cap P(A_*)$ is a complex manifold of dimension $n-1$ ([4]). Therefore if $P(A_*)$ is smooth, then it is a complex submanifold of $\mathbb{C}^n$ of complex dimension $n-1$.

Since $A = (A_1, A_2, \dots, A_n)$ is a tuple of compact operators on an infinite dimensional Banach space, the normalized pencil $A_*(z) = I + \sum_{j=1}^{n} z_j A_j$ is always Fredholm with index zero. So it is not invertible if and only if it has a nontrivial kernel. On the dimension of the kernel, we have the following lemma.

LEMMA 3.2. *If λ is a regular point in $P(A_*)$, then* $\ker A_*(\lambda)$ *has dimension* 1.

Proof. A generalization of Jacobi's formula is indicated in [14], namely if $f(z)$ is a trace class operator-valued holomorphic function, then

$$\operatorname{tr}[(I + f(z))^{-1} \mathrm{d} f(z)] = \mathrm{d} \log \det(I + f(z)), \tag{3.2}$$

where for a holomorphic function h, the differential $\mathrm{d}h = \sum_{j=1}^{n} (\partial h/\partial z_j)\mathrm{d}z_j$. For a regular point λ, let U be a neighborhood of λ in $\mathbb{C}^n$ as in Definition 3.1. Consider the linear function

$$z_\lambda(w) = \lambda + w\overline{\nabla(\det K_{U,F})(\lambda)} \quad w \in \mathbb{C}.$$

We pick a small $r > 0$ so that the analytic disk $D_r(\lambda) := \{z_\lambda(w) : \ |w| < r\}$ lies inside U. Further, since the vector $\overline{\nabla(\det K_{U,F})(\lambda)}$ is nonzero and is normal to the tangent plane of $P(A_*)$ at λ, the small disk $D_r(\lambda)$ intersects $P(A_*)$ transversally at λ, and hence the zeros of $\det K_{U,F}(z_\lambda(w))$ are discrete. So we may assume further that r is small enough such that $D_r(\lambda) \cap P(A_*) = \{\lambda\}$. Therefore $K_{U,F}(z_\lambda(w))$ is invertible for each $0 < |w| < r$ and is Fredholm at $w = 0$.

For convenience, we denote $\det K_{U,F}(z_\lambda(w))$ by $g_\lambda(w)$, and take a note that if λ is a regular point in $P(A_*)$, then $g_\lambda(0) = 0$ and $g'_\lambda(0) \neq 0$, that is, $g_\lambda(w)$ has a zero of order 1 at $w = 0$. Further, since $K_{U,F}(z_\lambda(w))$ is a Fredholm operator-valued analytic function on $D_r(\lambda)$, Proposition 3.1 in [3] gives rise to a factorization

$$K_{U,F}(z_\lambda(w)) = h(w)(P_1^\perp + wP_1)\cdots(P_k^\perp + wP_k) \tag{3.3}$$

on $D_r(\lambda)$, where the $P_j's$ are finite rank projections, and $h(w)$ is an operator-valued analytic function and takes invertible values on all of $D_r(\lambda)$.

We will show that in fact in (3.3), $k = 1$ and P_1 is of rank 1. To this end, we observe that from (3.1) we have the operator-valued differential

$$\begin{aligned} \mathrm{d}K_{U,F}(z) &= \mathrm{d}\Big[\Big(I + \sum z_j(A_j - F_j)\Big)^{-1}\Big]\Big(\sum z_jF_j\Big) \\ &\quad + \Big(I + \sum z_j(A_j - F_j)\Big)^{-1}\sum F_j\mathrm{d}z_j \quad z \in U, \end{aligned} \tag{3.4}$$

which is finite rank, and hence trace-class, for each $z \in U$. Therefore

$$K'_{U,F}(z_\lambda(w)) = \frac{\mathrm{d}K_{U,F}(z_\lambda(w))}{\mathrm{d}w}$$

is trace-class operator-valued. On the other hand, by direct computation on (3.3), we have that

$$\begin{aligned} K'_{U,F}(z_\lambda(w)) &= h'(w)(P_1^\perp + wP_1)\cdots(P_k^\perp + wP_k) \\ &\quad + h(w)P_1(P_2^\perp + wP_2)\cdots(P_k^\perp + wP_k) \\ &\quad + \cdots + h(w)(P_1^\perp + wP_1)\cdots(P_{k-1}^\perp + wP_{k-1})P_k. \end{aligned}$$

Since each P_j is finite rank, and

$$(P_1^\perp + wP_1)\cdots(P_k^\perp + wP_k)$$

is invertible for $w \neq 0$ and is Fredholm at $w = 0$, we infer that $h'(w)$ is trace-class for each $w \in D_r(\lambda)$. Moreover, since

$$(P_j^\perp + wP_j)^{-1} = P_j^\perp + \frac{1}{w}P_j \quad w \neq 0,$$

we see by (3.3) that

$$K_{U,F}^{-1}(z_\lambda(w)) = \Big(P_k^\perp + \frac{1}{w}P_k\Big)\cdots\Big(P_1^\perp + \frac{1}{w}P_1\Big)h^{-1}(w) \quad w \in D_r(\lambda),\ w \neq 0.$$

Using the property that $\mathrm{tr}(S^{-1}TS) = \mathrm{tr}T$ for every trace-class operator T and invertible operator S, we compute that

$$\mathrm{tr}(K_{U,F}^{-1}(z_\lambda(w))\mathrm{d}K_{U,F}(z_\lambda(w))) = \mathrm{tr}(h^{-1}(w)h'(w))\mathrm{d}w + \frac{1}{w}\Big(\sum_{j=1}^{k}\mathrm{tr}P_j\Big)\mathrm{d}w. \tag{3.5}$$

Pick $\varepsilon < r$. Using (3.2) and the fact that $g_\lambda(w) = \det K_{U,F}(z_\lambda(w))$ has a single zero at $w = 0$ of order 1 in $D_r(\lambda)$, we have by residue theorem that

$$1 = \frac{1}{2\pi i} \int\limits_{|w|=\varepsilon} \frac{g'_\lambda(w)}{g_\lambda(w)} dw = \frac{1}{2\pi i} \int\limits_{|w|=\varepsilon} d \log g_\lambda(w) = \frac{1}{2\pi i} \int\limits_{|w|=\varepsilon} d \log \det K_{U,F}(z_\lambda(w))$$

$$= \operatorname{tr}\Big(\frac{1}{2\pi i} \int\limits_{|w|=\varepsilon} K^{-1}_{U,F}(z_\lambda(w)) dK_{U,F}(z_\lambda(w)) \Big)$$

$$= \operatorname{tr}\Big(\frac{1}{2\pi i} \int\limits_{|w|=\varepsilon} h^{-1}(w) h'(w) dw \Big) + \Big(\sum_{j=1}^{k} \operatorname{tr} P_j \Big) \frac{1}{2\pi i} \int\limits_{|w|=\varepsilon} \frac{1}{w} dw.$$

Since $h^{-1}(w)h'(w)$ is analytic on $D_r(\lambda)$,

$$\int\limits_{|w|=\varepsilon} h^{-1}(w) h'(w) dw = 0.$$

Therefore,

$$1 = \Big(\sum_{j=1}^{k} \operatorname{tr} P_j \Big) \frac{1}{2\pi i} \int\limits_{|w|=\varepsilon} \frac{1}{w} dw = \sum_{j=1}^{k} \operatorname{tr} P_j,$$

and it follows that $k = 1$ and $\operatorname{tr} P_1 = 1$, i.e.,

$$K_{U,F}(z_\lambda(w)) = h(w)(P^\perp + wP) \quad w \in D_r(\lambda), \tag{3.6}$$

for some rank 1 projection P and some analytic operator-valued function h that is invertible everywhere on $D_r(\lambda)$. It then follows that

$$K_{U,F}(\lambda) = K_{U,F}(z_\lambda(0)) = h(0)P^\perp$$

has one dimensional kernel (which is the range of P). We have already shown that $\ker A_*(\lambda) = \ker K_{U,F}(\lambda)$, so we conclude that $\dim \ker A_*(\lambda) = 1$. ∎

If $P(A_*)$ is smooth, then for every $z \in P(A_*)$, there is an associated vector space $\ker A_*(z)$, and Lemma 3.2 indicates that $\dim \ker A_*(z) = 1$ for every $z \in P(A_*)$. We now consider the disjoint union

$$E_A := \bigvee_{z \in P(A_*)} \ker A_*(z) = \bigcup_{z \in P(A_*)} \{(z, \ker A_*(z))\},$$

and the map $\pi : E_A \to P(A_*)$ defined by $\pi(z, \ker A_*(z)) = z$.

THEOREM 3.3. *If $P(A_*)$ is smooth, then (E_A, π) defines a holomorphic line bundle over $P(A_*)$.*

Proof. It only remains to show that E_A has a locally holomorphic frame at every point $\lambda \in P(A_*)$. Now for every fixed $\lambda \in P(A_*)$, we let λ, U, F_j be as in the proof of Lemma 3.2. For $\tau \in P(A_*)$, let

$$z_\tau(w) = \tau + w\overline{\nabla(\det K_{U,F})(\lambda)} \quad w \in \mathbb{C}.$$

Pick $r > 0$, let $D_r(\tau) := \{z_\tau(w) : |w| < r\}$. Since $D_r(\lambda)$ intersects $P(A_*)$ transversally at λ, there exists a small neighborhood $V \subset P(A_*)$ of λ such that for every $\tau \in V$, $D_r(\tau)$ intersects $P(A_*)$ transversally at τ. We choose V and r small enough so that $V + r\mathbb{B}_n \subset U$, where $\mathbb{B}_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ is the unit ball of $\mathbb{C}^n$, and $D_r(\tau) \cap P(A_*) = \{\tau\}$ for every $\tau \in V$.

In fact, we can make V even smaller so that for each fixed $\tau \in V$, the function $\det K_{U,F}(z_\tau(w))$ has a zero of order 1 at $w = 0$. Then similar arguments as in Lemma 3.2 show that

$$K_{U,F}(z_\tau(w)) = h(w)(P^\perp + wP),$$

where h is analytic and invertible in a neighborhood of 0, and P is a projection onto $\ker A(\tau)$. Clearly, h and P both depends on τ, but we shall see that this dependence is not of concern. Now we can write

$$\begin{aligned} A_*(z_\tau(w)) &= \Big(I + \sum z_{\tau,j}(w)(A_j - F_j)\Big) K_{U,F}(z_\tau(w)) \\ &= \Big(I + \sum z_{\tau,j}(w)(A_j - F_j)\Big) h(w)(P^\perp + wP). \end{aligned}$$

Denote $\big(I + \sum z_{\tau,j}(w)(A_j - F_j)\big)h(w)$ by $\widehat{h}(w)$, and set

$$\omega_A(z) = A_*^{-1}(z)\mathrm{d}A_*(z) = A_*^{-1}(z)\Big(\sum_{j=1}^n A_j \mathrm{d}z_j\Big) \quad z \in \mathbb{C}^n \setminus P(A_*).$$

Pick a small $\varepsilon > 0$ as in the proof of Lemma 3.2. Then for $|w| < \varepsilon$, we have

$$\begin{aligned} \omega_A(z_\tau(w)) &= \Big(P^\perp + \frac{1}{w}P\Big)\widehat{h}^{-1}(w)\mathrm{d}(\widehat{h}(w)(P^\perp + wP)) \\ &= \Big(P^\perp + \frac{1}{w}P\Big)\widehat{h}^{-1}(w)(\widehat{h}'(w)(P^\perp + wP) + \widehat{h}(w)P)\mathrm{d}w \\ &= (P^\perp \widehat{h}^{-1}(w)\widehat{h}'(w)P^\perp + wP^\perp \widehat{h}^{-1}(w)\widehat{h}'(w)P)\mathrm{d}w \\ &\quad + \Big(P\widehat{h}^{-1}(w)\widehat{h}'(w)P + \frac{1}{w}P\widehat{h}^{-1}(w)\widehat{h}'(w)P^\perp + \frac{1}{w}P\Big)\mathrm{d}w. \end{aligned}$$

Since the first three summands are holomorphic in w, we have

$$\begin{aligned} Q(\tau) &:= \frac{1}{2\pi \mathrm{i}}\int_{|w|=\varepsilon} \omega_A(z_\tau(w)) = \frac{1}{2\pi \mathrm{i}}\int_{|w|=\varepsilon} \Big(\frac{1}{w}P\widehat{h}^{-1}(w)\widehat{h}'(w)P^\perp + \frac{1}{w}P\Big)\mathrm{d}w \\ &= P\Big(\frac{1}{2\pi \mathrm{i}}\int_{|w|=\varepsilon} \frac{1}{w}\widehat{h}^{-1}(w)\widehat{h}'(w)\mathrm{d}w\Big)P^\perp + \Big(\frac{1}{2\pi \mathrm{i}}\int_{|w|=\varepsilon} \frac{1}{w}\mathrm{d}w\Big)P \\ &= P\widehat{h}^{-1}(0)\widehat{h}'(0)P^\perp + P. \end{aligned} \tag{3.7}$$

One verifies that

$$Q^2(\tau) = Q(\tau), \quad Q(\tau)P = P, \quad P^\perp Q(\tau) = 0,$$

i.e., $Q(\tau)$ is an idempotent that maps X onto $\ker A(\tau)$. Moreover, since $z_\tau(w)$ is holomorphic in τ for $\tau \in V$, by the first equality in (3.7), $Q(\tau)$ is holomorphic in

τ as well. For a nonzero vector $e \in \ker A(\lambda)$, $Q(\lambda)e = e \neq 0$, and hence $Q(\tau)e$ is nonzero on a small neighborhood V' such that $\lambda \in V' \subset V$, and thus defines a holomorphic frame of E_A over V'. ∎

In conclusion, this section shows that for an n-tuple A of compact operators on a Banach space, if $P(A_*)$ is smooth then it is a complex submanifold in $\mathbb{C}^n$ of dimension $n-1$, and E_A is a natural holomorphic line bundle over $P(A_*)$. Two tempting but seemingly difficult problems follow.

PROBLEM A. Give a condition on the compact tuple A such that $P(A_*)$ is smooth.

PROBLEM B. In the case $P(A_*)$ is smooth, how to compute the curvature of E_A?

4. AN EXAMPLE OF A COMPACT OPERATOR TUPLE

This section makes an initial attempt to address the Problems A and B. Let $\mathcal{H}$ be a Hilbert space, and $e_1, e_2, \ldots, e_n$ be a set of linearly independent unit vectors in $\mathcal{H}$. Let A_i be the rank 1 projection to $\mathbb{C}e_i$, denoted by $A_i = e_i \otimes e_i$. Then $A = (A_1, A_2, \ldots, A_n)$ is a tuple of compact operators.

In this section, we will characterize when the projective spectrum $P(A_*)$ is smooth for $n = 2$ and $n = 3$. It turns out that this depends rather subtly on the relative position of the vectors. If the projective spectrum is smooth, by Theorem 3.3, there is a holomorphic line bundle over it. We shall compute its Chern character for the case $n = 2$.

We begin with the smoothness issue for the case $n = 2$.

THEOREM 4.1. *Let $A = (e_1 \otimes e_1, e_2 \otimes e_2)$, and $a = \langle e_1, e_2 \rangle$. Then $P(A_*)$ is smooth if and only if $a \neq 0$.*

Proof. Since $A_*(z) = I + z_1 e_1 \otimes e_1 + z_2 e_2 \otimes e_2$, we have

$$A_*(z)e_j = (I + z_1 e_1 \otimes e_1 + z_2 e_2 \otimes e_2)e_j = e_j + \sum_{i=1}^{2} z_i \langle e_j, e_i \rangle e_i.$$

Let $E = \overline{\text{span}}\{e_1, e_2\}$. With respect to the decomposition $\mathcal{H} = E \oplus E^\perp$, $A_*(z)$ is similar to $W(z) \oplus I_{E^\perp}$, where

$$W(z) = \begin{pmatrix} 1+z_1 & \bar{a}z_1 \\ az_2 & 1+z_2 \end{pmatrix}.$$

Hence $P(A_*) = \{z \in \mathbb{C}^2 : \det W(z) = 0\}$. Let

$$U_1(z) = \begin{pmatrix} 1 & 0 \\ az_2 & 1+z_2 \end{pmatrix}, \quad W_1(z) = \begin{pmatrix} 1 & \bar{a} \\ az_2 & 1+z_2 \end{pmatrix};$$

$$U_2(z) = \begin{pmatrix} 1+z_1 & \bar{a}z_1 \\ 0 & 1 \end{pmatrix}, \quad W_2(z) = \begin{pmatrix} 1+z_1 & \bar{a}z_1 \\ a & 1 \end{pmatrix}.$$

Then

$$\det W(z) = \det U_1(z) + z_1 \det W_1(z) = \det U_2(z) + z_2 \det W_2(z),$$

thus

$$\mathrm{d}(\det W(z)) = \frac{\partial \det W(z)}{\partial z_1}\mathrm{d}z_1 + \frac{\partial \det W(z)}{\partial z_2}\mathrm{d}z_2 = \det W_1(z)\mathrm{d}z_1 + \det W_2(z)\mathrm{d}z_2.$$

Now we consider the set of equations

$$\tag{4.1} \begin{cases} \det W(z) = 0 & \cdots\cdots & (*), \\ \det W_1(z) = 0 & \cdots\cdots & (1), \\ \det W_2(z) = 0 & \cdots\cdots & (2). \end{cases}$$

One sees that $P(A_*)$ is smooth if and only if for any z satisfying $(*)$, z does not satisfy (1) and (2) at the same time. That is, $P(A_*)$ is smooth if and only if (4.1) has no solution.

Note that for $i = 1, 2$, the equation set $(*)$ and (i) is equivalent to $\det U_i(z) = 0$ and $\det W_i(z) = 0$. Then one can easily verify the following two cases.

Case 1. If $a = 0$, the equation set (4.1) has a unique solution $z = (-1, -1)$.

Case 2. $a \neq 0$. If $\det U_1(z) = 0$ then clearly $z_2 = -1$, and hence $\det W_1(z) = |a|^2 \neq 0$. It follows that the equation set (4.1) has no solution, because the equation set $(*)$ and (1) (which is equivalent to $\det U_1(z) = 0$ and $\det W_1(z) = 0$!) has no solution.

Hence the equation set (4.1) has no solution if and only if $a \neq 0$. Therefore $P(A_*)$ is smooth if and only if $a \neq 0$. ∎

EXAMPLE 4.2. Let $A = (e_1 \otimes e_1, e_2 \otimes e_2)$, and $a = \langle e_1, e_2 \rangle \neq 0$. Then $\widehat{A} = (I, e_1 \otimes e_1, e_2 \otimes e_2)$. Recall that $\widehat{A}(\zeta) = \zeta_0 I + \zeta_1 e_1 \otimes e_1 + \zeta_2 e_2 \otimes e_2$, and

$$p(\widehat{A}) = \{\zeta = [\zeta_0, \zeta_1, \zeta_2] \in \mathbb{P}^2 \ : \widehat{A}(\zeta) \text{ is not invertible}\}.$$

Let $U_0 = \{\zeta \in \mathbb{P}^2 : \zeta_0 \neq 0\}$, $z_1 = \zeta_1/\zeta_0$, $z_2 = \zeta_2/\zeta_0$. Then

$$\tag{4.2} p(\widehat{A}) \cap U_0 = P(A_*) = \{z \in \mathbb{C}^2 : A_*(z) \text{ is not invertible}\}.$$

Since $a \neq 0$, Theorem 4.1 indicates that $P(A_*)$ is smooth, and hence by Theorem 3.3 there is a holomorphic line bundle E_A over $P(A_*)$. We shall compute the Chern character of E_A. Consider the vector-valued function

$$\gamma(z) = \begin{pmatrix} 1+z_2 \\ -az_2 \end{pmatrix}.$$

For every $z \in P(A_*)$, one checks easily that $\gamma(z) \in \ker W(z) = \ker A_*(z)$. So $\gamma(z)$ is a holomorphic section of E_A. Since $\gamma(z) \neq 0$ for every $z \in P(A_*)$, $\gamma(z)$ is in fact a frame for the bundle E_A. We can now compute the curvature form of E_A as

$$\begin{aligned}\Theta(z) &= \bar{\partial}\partial\ \log|\gamma(z)|^2 = \bar{\partial}\partial\ \log\left(|1+z_2|^2+|a|^2|z_2|^2\right)\\ &= \bar{\partial}\Big(\frac{(1+\bar{z}_2)\mathrm{d}z_2+|a|^2\bar{z}_2\mathrm{d}z_2}{|1+z_2|^2+|a|^2|z_2|^2}\Big) = \frac{-|a|^2\mathrm{d}z_2\wedge\mathrm{d}\bar{z}_2}{(|1+z_2|^2+|a|^2|z_2|^2)^2}, \quad z_2\in\mathbb{C}.\end{aligned}$$

Then the first Chern class $c_1(E_A) = (\mathrm{i}/2\pi)\Theta$.

We observe further that since $P(A_*)$ is smooth, it is a non-compact submanifold in $\mathbb{P}^2$ of complex dimension 1. Observe also that $p(\widehat{A}) = P(A_*)\cup U_0^c$. Since $U_0^c = \{\zeta_0 = 0\}$ can be viewed as a hyperplane of $\mathbb{P}^2$ at ∞, the set $p(\widehat{A})$ can be viewed as the one point compactification of $P(A_*)$ in $\mathbb{P}^2$. In particular, $p(\widehat{A})$ has no boundary. Moreover, one sees that $\Theta(z)$ converges to 0 as z_2 tends to ∞, and this extends $c_1(E_A)$ to $p(\widehat{A})$. By general theory $c_1(E_A)$ is an element in the cohomology group $H^2(p(\widehat{A}),\mathbb{Z})$. Now we check that it is nontrivial, i.e., non-exact. To this end, we consider the integral

$$\int_{p(\widehat{A})} c_1(E_A).$$

If $c_1(E_A)$ were exact, i.e., $c_1(E_A) = \mathrm{d}F(z)$ for some smooth 1-form F on $p(\widehat{A})$, then by Stokes theorem and the fact $\partial p(\widehat{A}) = \varnothing$, we should have

$$\int_{p(\widehat{A})} c_1(E_A) = \int_{\partial p(\widehat{A})} F = 0.$$

In the following we check that this is not the case here.

We compute that

$$\int_{p(\widehat{A})} -c_1(E_A) = \int_{P(A_*)} -c_1(E_A) = \frac{\mathrm{i}}{2\pi}\int_{1+z_1+z_2+(1-|a|^2)z_1z_2=0} \frac{|a|^2\mathrm{d}z_2\wedge\mathrm{d}\bar{z}_2}{(|1+z_2|^2+|a|^2|z_2|^2)^2}.$$

Since $\mathrm{i}\mathrm{d}z_2\wedge\mathrm{d}\bar{z}_2 = 2\mathrm{d}x_2\wedge\mathrm{d}y_2$ (where $z_2 = x_2+\mathrm{i}y_2$) is a positive form and $|1+z_2|^2 \leqslant (1+|z_2|)^2 \leqslant 2+2|z_2|^2$, we have

$$\begin{aligned}\int_{P(A_*)} -c_1(E_A) &\geqslant \frac{\mathrm{i}}{2\pi}\int_{1+z_1+z_2+(1-|a|^2)z_1z_2=0} \frac{|a|^2\mathrm{d}z_2\wedge\mathrm{d}\bar{z}_2}{(2+2|z_2|^2+|a|^2|z_2|^2)^2}\\ &= \frac{\mathrm{i}}{2\pi}\int_{z_2=-\frac{1+z_1}{1+(1-|a|^2)z_1}} \frac{|a|^2\mathrm{d}z_2\wedge\mathrm{d}\bar{z}_2}{4(1+(1+(|a|^2/2))|z_2|^2)^2}\\ &= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}\setminus\{1/(|a|^2-1)\}} \frac{|a|^2\mathrm{d}z_2\wedge\mathrm{d}\bar{z}_2}{4(1+(1+(|a|^2/2))|z_2|^2)^2}.\end{aligned}$$

Let $w = \sqrt{1+(|a|^2/2)}z_2$. Then

$$\int_{P(A_*)} -c_1(E_A) \geqslant \frac{|a|^2}{4+2|a|^2}\frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{(1+|w|^2)^2} = \frac{|a|^2}{4+2|a|^2} > 0.$$

This shows in particular that $c_1(E_A) \in H^2(p(\widehat{A}), \mathbb{Z})$ is nontrivial.

Next, we proceed to consider the case $n = 3$. Consider $A = (e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3)$, and let $a = \langle e_1, e_2 \rangle$, $b = \langle e_2, e_3 \rangle$, $c = \langle e_3, e_1 \rangle$. Further, let G be the Gramian matrix for e_1, e_2, e_3, i.e.,

$$G = \begin{pmatrix} 1 & \langle e_2, e_1 \rangle & \langle e_3, e_1 \rangle \\ \langle e_1, e_2 \rangle & 1 & \langle e_3, e_2 \rangle \\ \langle e_1, e_3 \rangle & \langle e_2, e_3 \rangle & 1 \end{pmatrix} = \begin{pmatrix} 1 & \overline{a} & c \\ a & 1 & \overline{b} \\ \overline{c} & b & 1 \end{pmatrix}.$$

Obviously, G is invertible, since e_1, e_2, e_3 are linearly independent.

Since $A_*(z) = I + z_1 e_1 \otimes e_1 + z_2 e_2 \otimes e_2 + z_3 e_3 \otimes e_3$, we have

$$A_*(z)e_j = (I + z_1 e_1 \otimes e_1 + z_2 e_2 \otimes e_2 + z_3 e_3 \otimes e_3)e_j = (1+z_j)e_j + \sum_{i=1, i\neq j}^{3} z_i \langle e_j, e_i \rangle e_i.$$

Let $E = \overline{\operatorname{span}}\{e_1, e_2, e_3\}$. With respect to the decomposition $\mathcal{H} = E \oplus E^\perp$, $A_*(z)$ is similar to $W(z) \oplus I_{E^\perp}$, where

$$W(z) = \begin{pmatrix} 1+z_1 & \overline{a}z_1 & cz_1 \\ az_2 & 1+z_2 & \overline{b}z_2 \\ \overline{c}z_3 & bz_3 & 1+z_3 \end{pmatrix}. \tag{4.3}$$

Hence $P(A_*) = \{z \in \mathbb{C}^3 : \det W(z) = 0\}$. Now we characterize when $P(A_*)$ is smooth.

THEOREM 4.3. *For $A = (e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3)$, the following four conditions are equivalent:*

(i) *$P(A_*)$ is smooth;*

(ii) *$a = \overline{bc} \neq 0$ or $b = \overline{ac} \neq 0$ or $c = \overline{ab} \neq 0$ or abc is not real;*

(iii) *$\operatorname{rank}(W(z)) = 2$, $\forall\, z \in P(A_*)$;*

(iv) *$\operatorname{rank}(G^{-1} + \operatorname{diag}(z_1, z_2, z_3)) \geqslant 2$, $\forall\, z \in \mathbb{C}^3$.*

Proof. We first prove (i) $\Leftrightarrow$ (ii). Let

$$U_1(z) = \begin{pmatrix} 1 & 0 & 0 \\ az_2 & 1+z_2 & \overline{b}z_2 \\ \overline{c}z_3 & bz_3 & 1+z_3 \end{pmatrix}, \quad W_1(z) = \begin{pmatrix} 1 & \overline{a} & c \\ az_2 & 1+z_2 & \overline{b}z_2 \\ \overline{c}z_3 & bz_3 & 1+z_3 \end{pmatrix};$$

$$U_2(z) = \begin{pmatrix} 1+z_1 & \overline{a}z_1 & cz_1 \\ 0 & 1 & 0 \\ \overline{c}z_3 & bz_3 & 1+z_3 \end{pmatrix}, \quad W_2(z) = \begin{pmatrix} 1+z_1 & \overline{a}z_1 & cz_1 \\ a & 1 & \overline{b} \\ \overline{c}z_3 & bz_3 & 1+z_3 \end{pmatrix};$$

$$U_3(z)=\begin{pmatrix}1+z_1 & \bar{a}z_1 & cz_1\\ az_2 & 1+z_2 & \bar{b}z_2\\ 0 & 0 & 1\end{pmatrix},\quad W_3(z)=\begin{pmatrix}1+z_1 & \bar{a}z_1 & cz_1\\ az_2 & 1+z_2 & \bar{b}z_2\\ \bar{c} & b & 1\end{pmatrix}.$$

Then

$$\begin{aligned}\det W(z) &= \det U_1(z)+z_1\det W_1(z)=\det U_2(z)+z_2\det W_2(z)\\ &= \det U_3(z)+z_3\det W_3(z),\end{aligned}$$

thus

$$d(\det W(z))=\sum_{i=1}^{3}\frac{\partial\det W(z)}{\partial z_i}dz_i=\sum_{i=1}^{3}\det W_i(z)dz_i. \tag{4.4}$$

Now we consider the set of equations

$$\begin{cases}\det W(z)=0 & \cdots\cdots & (*),\\ \det W_1(z)=0 & \cdots\cdots & (1),\\ \det W_2(z)=0 & \cdots\cdots & (2),\\ \det W_3(z)=0 & \cdots\cdots & (3).\end{cases} \tag{4.5}$$

One sees that $P(A_*)$ is smooth if and only if for any z satisfying $(*)$, z does not satisfy (1), (2) and (3) simultaneously. That is, $P(A_*)$ is smooth if and only if (4.5) has no solution.

Note that for $i=1,2,3$, the equation set $(*)$ and (i) is equivalent to the equation set $\det U_i(z)=0$ and $\det W_i(z)=0$. Now we solve (4.5) in the following three cases.

Case 1. If $abc=0$, the equation set (4.5) has solutions. For example, if $a=0$, $z=(-1,-1,0)$ is a solution.

Case 2. If $abc\neq 0$ and $a=\overline{bc}$, the equation set (4.5) has no solution. To see this, we consider the equation set $(*)$ and (1), which is equivalent to the equation set $\det U_1(z)=0$ and $\det W_1(z)=0$. Put $a=\overline{bc}$ into the equation set, and note that $c\neq 0$, we get the following

$$1+z_2+z_3+(1-|b|^2)z_2z_3=0,\quad |b|^2z_2+z_3+(1-|b|^2)z_2z_3=0.$$

One easily sees that $z_2=1/(|b|^2-1)$. But if we put $z_2=1/(|b|^2-1)$ into the above equation set, we get $b=0$, a contradiction. Hence the equation set $(*)$ and (1) has no solution, which leads to the conclusion that the equation set (4.5) has no solution.

Similarly, if $abc\neq 0$ and $b=\overline{ac}$ or $c=\overline{ab}$, the equation set (4.5) has no solution either.

Case 3. If $abc\neq 0$ and $a\neq\overline{bc}$, $b\neq\overline{ac}$, $c\neq\overline{ab}$, let

$$\lambda_1=\frac{\bar{b}}{ac-\bar{b}},\quad \lambda_2=\frac{\bar{c}}{ab-\bar{c}},\quad \lambda_3=\frac{\bar{a}}{bc-\bar{a}}. \tag{4.6}$$

One can verify that the equation set $(*)$ and (1) has solutions $(z_1,\lambda_2,\overline{\lambda}_3)$ and $(z_1,\overline{\lambda}_2,\lambda_3)$, where $z_1\in\mathbb{C}$, the equation set $(*)$ and (2) has solutions $(\lambda_1,z_2,\overline{\lambda}_3)$

and $(\overline{\lambda}_1, z_2, \lambda_3)$, where $z_2 \in \mathbb{C}$, and the equation set $(*)$ and (3) has solutions $(\lambda_1, \overline{\lambda}_2, z_3)$ and $(\overline{\lambda}_1, \lambda_2, z_3)$, where $z_3 \in \mathbb{C}$. Hence the equation set (4.5) has solutions if and only if λ_1, λ_2, λ_3 are all real, or equivalently abc is real. In this case the equation set (4.5) has a unique solution $(\lambda_1, \lambda_2, \lambda_3)$.

We conclude that $P(A_*)$ is smooth if and only if (4.5) has no solution, and this holds if and only if $a = \overline{bc} \neq 0$ or $b = \overline{ac} \neq 0$ or $c = \overline{ab} \neq 0$ or abc is not real, completing the proof of (i) $\Leftrightarrow$ (ii).

Now we prove (i) $\Rightarrow$ (iii). If $P(A_*)$ is smooth, by definition, every $z \in P(A_*)$ is regular, i.e., for each $z \in P(A_*)$, $\mathrm{d}(\det W(z)) \neq 0$. By (4.4), we have $\det W_i(z) \neq 0$ for some i. Since two rows of $W_i(z)$ coincides with that of $W(z)$, at least two rows of $W(z)$ are linearly independent, that is $\operatorname{rank}(W(z)) \geqslant 2$. But $z \in P(A_*)$ means $\det W(z) = 0$. Hence $\operatorname{rank}(W(z)) = 2$.

We then prove (iii) $\Rightarrow$ (i). If $P(A_*)$ is not smooth, there exist non-regular points in $P(A_*)$, and these points are solutions for the equation set (4.5). We will show that at these non-regular points, $\operatorname{rank}(W(z)) < 2$.

Case 1. $abc \neq 0$. In this case, we have shown in the proof of (i) $\Leftrightarrow$ (ii) that the equation set (4.5) has solutions if and only if abc is real and $a \neq \overline{bc}$, $b \neq \overline{ac}$, $c \neq \overline{ab}$, and the unique solution for (4.5) is $z = (\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_1, \lambda_2, \lambda_3$ are as in (4.6). One verifies by direct computation that at this point z, all the 2×2 submatrices in $W(z)$ have determinant 0, hence $\operatorname{rank}(W(z)) < 2$.

Case 2. One of a, b, c is zero, the other two are nonzero, say $a = 0$, $bc \neq 0$. In this case, the solution for (4.5) is $z = (-1, -1, 0)$. At this point z, $\operatorname{rank}(W(z)) = 1 < 2$.

Case 3. Two of a, b, c are zeros, the other is nonzero, say $a = b = 0$, $c \neq 0$. In this case, the solutions for (4.5) are $z = (z_1, -1, z_3)$, where $z_1, z_3 \in \mathbb{C}$. One verifies that at these points, $\operatorname{rank}(W(z)) = 1 < 2$.

Case 4. $a = 0$, $b = 0$, $c = 0$. Let $z = (z_1, z_2, z_3)$ be a solution for (4.5). If $z_1 = -1$, then $z_2 = -1$ or $z_3 = -1$, and then $\operatorname{rank}(W(z)) < 2$. If $z_1 \neq -1$, then $z_2 = -1$ and $z_3 = -1$, and we still have $\operatorname{rank}(W(z)) = 1 < 2$.

Combining the above four cases, we obtain (iii) $\Rightarrow$ (i).

Finally, we prove (iii) $\Leftrightarrow$ (iv). Let $\mathbf{v}_j$ be the j-th row of G, $\mathbf{w}_j$ be the j-th row of $W(z)$, and ε_j be the j-th row of the 3×3 identity matrix I_3.

Condition (iii) says that for any $z \in P(A_*)$, $\operatorname{rank}(W(z)) = 2$. But for $z \notin P(A_*)$, it is obvious that $\operatorname{rank}(W(z)) = 3$, because $\det W(z) \neq 0$. Therefore, (iii) is equivalent to $\operatorname{rank}(W(z)) \geqslant 2$ for any $z \in \mathbb{C}^3$. Note that

$$\begin{aligned} W(z) &= \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 + z_1 \mathbf{v}_1 \\ \varepsilon_2 + z_2 \mathbf{v}_2 \\ \varepsilon_3 + z_3 \mathbf{v}_3 \end{pmatrix} = I_3 + \operatorname{diag}(z_1, z_2, z_3) G \\ &= (G^{-1} + \operatorname{diag}(z_1, z_2, z_3)) G. \end{aligned}$$

Since G is invertible, we have that

$$\operatorname{rank}(G^{-1} + \operatorname{diag}(z_1, z_2, z_3)) = \operatorname{rank}(W(z)) \geqslant 2$$

for any $z \in \mathbb{C}^3$, and this is just (iv). Therefore, (iii) and (iv) are equivalent.

The proof is complete. ∎

We remark that condition (ii) in Theorem 4.3 is most interesting because it only depends on the relative position of the three vectors. Also observe that the four conditions in (ii) of Theorem 4.3 are mutually exclusive. For instance, if $a = \overline{bc}$ and $b = \overline{ac}$ both hold, then $a = a|c|^2$. Since $a \neq 0$, we have $|c| = 1$ which contradicts with the fact that e_1, e_2, e_3 are linearly independent unit vectors. Condition (ii) is somewhat mysterious to us and appears hard to generalize. But the following corollary is immediate.

COROLLARY 4.4. *If abc is real and non-positive then $P(A_*)$ is not smooth.*

Condition (iv) in Theorem 4.3 is suitable for generalization, so we conclude the paper with the following conjecture.

CONJECTURE 4.5. Let $A = (e_1 \otimes e_1, e_2 \otimes e_2, \ldots, e_n \otimes e_n)$. Then $P(A_*)$ is smooth if and only if $\operatorname{rank}(G^{-1} + \operatorname{diag}(z_1, z_2, \ldots, z_n)) \geqslant n - 1$ for every $z \in \mathbb{C}^n$, where G is the Gramian matrix of the vectors.

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