J. OPERATOR THEORY
78:2(2017), 435–472
doi: 10.7900/jot.2016oct03.2155

ON THE KK-THEORY OF ELLIOTT–THOMSEN ALGEBRAS

QINGNAN AN and GEORGE A. ELLIOTT

Communicated by Marius Dădârlat

ABSTRACT. This paper concerns the KK-theory of the class $\mathcal{C}$ of Elliott –Thomsen algebras, with special emphasis on the problem of when a KK-element can be represented by a homomorphism between two such C^*-algebras (allowing the tensor product with a matrix algebra for the codomain algebra), and gives an existence theorem for a certain subclass of $\mathcal{C}$ which we denote by $\mathcal{C}_{\mathcal{O}}$.

KEYWORDS: *Elliott–Thomsen algebra,* KK-*theory,* KK-*lifting,* mod-p K-*theory.*

MSC (2010): Primary 46L80; Secondary 19K14, 19K33, 19K35.

INTRODUCTION

It has been shown that a large number of simple C^*-algebras can be classified by the standard Elliott invariant. Gong [12] first presented an example to show that the ordered graded K-group is not sufficient any more for non-simple AH algebras of real rank zero (and no dimension growth). Elliott, Gong, and Su [9] constructed such examples for AD algebras by using AH algebras (indirectly). Then Dădârlat and Loring [4] also gave such an example for AD algebras directly. In 1997, Dădârlat and Gong [1] classified approximately homogeneous C^*-algebras of real rank zero (and no dimension growth) by means of ordered total K-theory together with a certain order structure (see also [4] and also [5]).

It is natural to try to relate the order on the total K-theory to the possibility of lifting a KK-element to a homomorphism (let us call this the KK-lifting problem), in particular, in the setting of dimension drop interval algebras (see [3]). Jiang and Su studied a larger class in [14], which we shall call generalized dimension drop interval algebras, and gave a criterion for KK-lifting. In [10], Elliott and Li reported that a KK-element preserving the Dădârlat–Loring order may not always have a lifting to a homomorphism.

In this paper, we consider the algebras introduced by Elliott and Thomsen in [11] (see also [8] and [18]). Such an algebra is now sometimes called a one dimensional non-commutative finite CW complex (see [6]). In this paper, we shall give a description of the KK-group of two Elliott–Thomsen algebras. This description will help us to understand the structure of the KK-group and enable us to give a criterion for KK-lifting.

The paper is organized as follows. In Section 2, we list some preliminaries concerning the class $\mathcal{C}$ of Elliott–Thomsen algebras and mod-p K-theory with the Dădârlat–Loring order. In Section 3, we formulate a description (as a quotient group) of the KK-group of two Elliott–Thomsen algebras. In Section 4, we give a useful sufficient condition for KK-lifting for Elliott–Thomsen algebras which gives a complete criterion for a certain subclass, denoted by $\mathcal{C}_{\mathcal{O}}$. (The sufficiency will be useful, when we check the Dădârlat–Loring order.) In Section 5, we prove that for the subclass $\mathcal{C}_{\mathcal{O}}$ of Elliott–Thomsen algebras, which in fact includes the generalized dimension drop interval algebras, a KK-element preserving the Dădârlat–Loring order can be lifted to a homomorphism — contrary to what was stated in Theorem 1.1 of [10]. We show that, as was suggested by the work [10], there is a genuine difficulty present, and the lifting theorem does not hold for all of $\mathcal{C}$. (Possibly, lifting holds for a KK-class suitably compatible, in an approximate sense, with traces and algebraic K_1.)

1. PREMIMINARIES

1.1. ([4], [7]) Consider the algebra

$$I_p = \{f \in M_p(C_0(0,1]) : f(1) = \lambda \cdot 1_p, 1_p \text{ is the identity of } M_p\},$$

and the algebra $\widetilde{I}_p$ obtained by adjoining a unit to I_p.

1.2. ([1]) For a C^*-algebra A, the total K-theory of A is defined by

$$\underline{\mathrm{K}}(A) = \bigoplus_{p=0}^{\infty} \mathrm{K}_*(A;\mathbb{Z}_p),$$

with $\mathrm{K}_*(A;\mathbb{Z}_p) = \mathrm{K}_*(A)$ for $p = 0$, $\mathrm{K}_*(A;\mathbb{Z}_p) = 0$ for $p = 1$, and $\mathrm{K}_*(A;\mathbb{Z}_p) = \mathrm{KK}(I_p, A \otimes C(S^1))$ for $p \geqslant 2$.

1.3. ([1]) We will consider the group

$$\mathrm{K}_*(A;\mathbb{Z} \oplus \mathbb{Z}_p) = \mathrm{K}_*(A) \oplus \mathrm{K}_*(A;\mathbb{Z}_p).$$

By Section 4 of [1],

$$\mathrm{K}_*(A;\mathbb{Z} \oplus \mathbb{Z}_p) \cong \mathrm{KK}(\widetilde{I}_p, A \otimes C(S^1)).$$

1.4. (Dădârlat–Loring order [4]) Define $\mathrm{K}_*(A;\mathbb{Z} \oplus \mathbb{Z}_p)^+$ as the image of the abelian semigroup $[\widetilde{I}_p, A \otimes C(S^1) \otimes \mathscr{K}]$ in $\mathrm{KK}(\widetilde{I}_p, A \otimes C(S^1)) \cong \mathrm{K}_*(A;\mathbb{Z} \oplus \mathbb{Z}_p)$.

1.5. Let F_1 and F_2 be two finite dimensional C^*-algebras, and let $\varphi_0, \varphi_1 : F_1 \to F_2$ be two unital homomorphisms. Set

$$\begin{aligned} A &= A(F_1, F_2, \varphi_0, \varphi_1) \\ &= \{(f,a) \in C([0,1], F_2) \oplus F_1 : f(0) = \varphi_0(a) \text{ and } f(1) = \varphi_1(a)\}. \end{aligned}$$

The C^*-algebras constructed in this way have been studied by Elliott and Thomsen (they are sometimes called Elliott–Thomsen algebras). Let us use $\mathcal{C}$ to denote the class of all unital such C^*-algebras $A = A(F_1, F_2, \varphi_0, \varphi_1)$ (up to isomorphism). Following [13], let us say that a unital C^*-algebra $A \in \mathcal{C}$ is minimal, or a minimal block, if it is indecomposable, i.e., not the direct sum of two or more C^*-algebras in $\mathcal{C}$.

Throughout this paper, when talking about $\mathrm{KK}(A, B)$ with $A, B \in \mathcal{C}$, we shall assume the notational convention that

$$A = A(F_1, F_2, \varphi_0, \varphi_1), \quad B = B(F_1', F_2', \varphi_0', \varphi_1'),$$

with

$$F_1 = \bigoplus_{i=1}^{p} M_{k_i}(\mathbb{C}), \quad F_2 = \bigoplus_{j=1}^{l} M_{h_j}(\mathbb{C}), \quad \text{and} \quad F_1' = \bigoplus_{i'=1}^{p'} M_{k'_{i'}}(\mathbb{C}), \quad F_2' = \bigoplus_{j'=1}^{l'} M_{h'_{j'}}(\mathbb{C}).$$

1.6. ([7], [14]) A dimension drop interval algebra, denoted by $I[m_0, m, m_1]$, is the C^*-algebra (in the class $\mathcal{C}$)

$$I[m_0, m, m_1] = \{f \in M_m(C([0,1])) : f(0) = a_0 \otimes 1_{m/m_0}, f(1) = a_1 \otimes 1_{m/m_1}\},$$

where m_0, m_1 divide m, a_0, a_1 (for a given f) belong to M_{m_0}, M_{m_1}, respectively, and $1_{m/m_0}$, $1_{m/m_1}$ are the identity elements of M_{m/m_0}, M_{m/m_1}, respectively. (The algebras $\widetilde{I}_p$ are already dimension drop interval algebras; the more general algebras just constructed are sometimes called generalized dimension drop interval algebras.)

1.7. As pointed out in [13], for a minimal block $A = A(F_1, F_2, \varphi_0, \varphi_1)$, we have $\ker \varphi_0 \cap \ker \varphi_1 = \{0\}$. Let us use $\mathcal{C}_{\mathcal{O}}$ to denote the class of all unital minimal block C^*-algebras $A = A(F_1, F_2, \varphi_0, \varphi_1)$, where $F_2 = M_r(\mathbb{C})$, for some integer r, and $\ker \varphi_0 \oplus \ker \varphi_1 = F_1$ (there is no block of F_1 mapping into both 0 and 1). This subclass was studied by Li in [15]. Note that $I[m_0, m, m_1] \in \mathcal{C}_{\mathcal{O}}$.

The following notions come from [13].

1.8. For $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$, consider the short exact sequence

$$0 \to SF_2 \xrightarrow{\iota} A \xrightarrow{\pi} F_1 \to 0,$$

where $SF_2 = C_0(0,1) \otimes F_2$ is the suspension of F_2, ι is the embedding map, and $\pi(f,a) = a, (f,a) \in A$. Then one has the six-term exact sequence

$$0 \to \mathrm{K}_0(A) \xrightarrow{\pi_*} \mathrm{K}_0(F_1) \xrightarrow{\partial} \mathrm{K}_0(F_2) \xrightarrow{\iota_*} \mathrm{K}_1(A) \to 0,$$

where $\partial = \alpha - \beta$, with α, β the maps $\mathrm{K}_0(\varphi_0), \mathrm{K}_0(\varphi_1)$, respectively. Hence,

$\mathrm{K}_0(A) = \ker(\alpha - \beta) \subset \mathbb{Z}^p$, $\quad \mathrm{K}_1(A) = \mathbb{Z}^l / \mathrm{Im}(\alpha - \beta)$, $\quad \mathrm{K}_0^+(A) = \ker(\alpha - \beta) \cap \mathrm{K}_0^+(F_1)$.

1.9. Denote by $\theta_1, \theta_2, \ldots, \theta_p$ the spectrum of F_1. Numbering the blocks of F_2 from 1 to j, we have

$$\mathrm{Sp}(C([0,1], F_2)) = \coprod_{j=1}^{l} \{(t,j), 0 \leqslant t \leqslant 1\}.$$

With the identifications $\varphi_0(a) = f(0)$ and $\varphi_1(a) = f(1)$ for $(f,a) \in A$, as in Section 13 of [13], $(0,j) \in \mathrm{Sp}(C([0,1], F_2))$ is identified with the set with multiplicities

$$(\theta_1{}^{\sim \alpha_{j1}}, \theta_2{}^{\sim \alpha_{j2}}, \ldots, \theta_p{}^{\sim \alpha_{jp}}) \subset \mathrm{Sp}(F_1),$$

and $(1,j) \in \mathrm{Sp}(C([0,1], F_2))$ is identified with the set with multiplicities

$$(\theta_1{}^{\sim \beta_{j1}}, \theta_2{}^{\sim \beta_{j2}}, \ldots, \theta_p{}^{\sim \beta_{jp}}) \subset \mathrm{Sp}(F_1).$$

Also, $\mathrm{Sp}(A) = \mathrm{Sp}(F_1) \cup \coprod_{j=1}^{l} (0,1)_j$.

1.10. Let $A \in \mathcal{C}$ be a minimal block, and let $\phi : A \to M_n(\mathbb{C})$ be a homomorphism. Then there exists a unitary u such that

$$\phi(f,a) = u \cdot \mathrm{diag}(\underbrace{a(\theta_1), \ldots, a(\theta_1)}_{t_1}, \ldots, \underbrace{a(\theta_p), \ldots, a(\theta_p)}_{t_p}, f(y_1), \ldots, f(y_r)) \cdot u^*$$

with $y_1, y_2, \ldots, y_r \in \coprod_{j=1}^{l} (0,1)_j$, for some integer r. We write

$$\mathrm{Sp}\phi = \{\theta_1{}^{\sim t_1}, \theta_2{}^{\sim t_2}, \ldots, \theta_p{}^{\sim t_p}, y_1, \ldots, y_r\} \tag{1.1}$$

with $y_k \in \coprod_{j=1}^{l} (0,1)_j$; in other words (as usual), $\mathrm{Sp}\phi$ is the set of irreducible components of ϕ, with multiplicity.

1.11. Consider the suspension of A, $SA = C_0(0,1) \otimes A$. Since we have $C_0(0,1) \cong C_0(S^1 \backslash \{1\})$, the spectrum of SA is given by

$$\mathrm{Sp}(SA) = \mathrm{Sp}(A) \times \{\mathrm{e}^{2\pi \mathrm{i}\omega}, 0 < \omega < 1\} = \Big\{\mathrm{Sp}(F_1) \cup \coprod_{j=1}^{l} (0,1)_j\Big\} \times \{\mathrm{e}^{2\pi \mathrm{i}\omega}, 0 < \omega < 1\}.$$

Let us write the elements of $\mathrm{Sp}(F_1) \times \{\mathrm{e}^{2\pi \mathrm{i}\omega}, 0 < \omega < 1\}$ as $(\theta_i, \mathrm{e}^{2\pi \mathrm{i}\omega})$ with $i = 1, 2, \ldots, p, 0 < \omega < 1$, and the elements of $[0,1]_j \times \{\mathrm{e}^{2\pi \mathrm{i}\omega}, 0 < \omega < 1\}$ as $(t, j, \mathrm{e}^{2\pi \mathrm{i}\omega})$ with $t \in [0,1], j = 1, 2, \ldots, l, 0 < \omega < 1$. Then the element $(0, j, \mathrm{e}^{2\pi \mathrm{i}\omega}) \in [0,1]_j \times \{\mathrm{e}^{2\pi \mathrm{i}\omega}, 0 < \omega < 1\}$ is identified with

$$\{(\theta_1, \mathrm{e}^{2\pi \mathrm{i}\omega})^{\sim \alpha_{j1}}, (\theta_2, \mathrm{e}^{2\pi \mathrm{i}\omega})^{\sim \alpha_{j2}}, \ldots, (\theta_p, \mathrm{e}^{2\pi \mathrm{i}\omega})^{\sim \alpha_{jp}}\},$$

and $(1, j, \mathrm{e}^{2\pi \mathrm{i}\omega})$ is identified with

$$\{(\theta_1, \mathrm{e}^{2\pi \mathrm{i}\omega})^{\sim\beta_{j1}}, (\theta_2, \mathrm{e}^{2\pi \mathrm{i}\omega})^{\sim\beta_{j2}}, \ldots, (\theta_p, \mathrm{e}^{2\pi \mathrm{i}\omega})^{\sim\beta_{jp}}\},$$

with (α_{ji}) and (β_{ji}) the matrices representing $(\varphi_0)_*$ and $(\varphi_1)_* : \mathrm{K}_0(F_1) = \mathbb{Z}^p \to \mathrm{K}_0(F_2) = \mathbb{Z}^l$. Also, let us write the elements of SA as (f, a) with $f : [0,1] \times S^1 \to F_2$ and $a : S^1 \to F_1$ such that

$$f(t, 1) = 0, \quad t \in [0,1], \quad \text{and} \quad a(1) = 0,$$
$$f(0, \mathrm{e}^{2\pi \mathrm{i}\omega}) = \varphi_0(a(\mathrm{e}^{2\pi \mathrm{i}\omega})) \quad \text{and} \quad f(1, \mathrm{e}^{2\pi \mathrm{i}\omega}) = \varphi_1(a(\mathrm{e}^{2\pi \mathrm{i}\omega})).$$

Since in the pair (f, a), a is completely determined by f (as A is a minimal block), we may simplify (f, a) as f if there is no confusion.

1.12. A homomorphism $\phi : SA \to M_n(\mathbb{C})$ is given by

$$\phi(f, a) = u \cdot \mathrm{diag}(a(x_1), a(x_2), \ldots, a(x_r), f(y_1), f(y_2), \ldots, f(y_m), 0, 0, \ldots, 0) \cdot u^*$$

for some unitary $u \in M_n(\mathbb{C})$, where $x_k \in \mathrm{Sp}(F_1) \times \{\mathrm{e}^{2\pi \mathrm{i}\omega}, 0 < \omega < 1\}$ are of the form $(\theta_i, \mathrm{e}^{2\pi \mathrm{i}\omega})$ with $i = 1, 2, \ldots, p$, $0 < \omega < 1$, and $y_k \in \coprod_{j=1}^{l} (0,1)_j \times \{\mathrm{e}^{2\pi \mathrm{i}\omega}, 0 < \omega < 1\}$ are of the form $(t, j, \mathrm{e}^{2\pi \mathrm{i}\omega})$ with $t \in (0,1)$, $j = 1, 2, \ldots, l$, $0 < \omega < 1$, and r, m are both some integers.

2. KK-THEORY FOR ELLIOTT–THOMSEN ALGEBRAS

REMARK 2.1. Let $\phi : A \to M_n(\mathbb{C})$ be as described in Paragraph 1.10, with $\mathrm{Sp}(\phi)$ as in (1.1). Even though in general the point $y_i \in [0,1]_j$ (in $\mathrm{Sp}(\phi)$ as in (1.1) of Paragraph 1.10) may not be the endpoint 0_j or 1_j, the homomorphism defined by evaluating at this point is homotopic to the homomorphism defined by evaluating at 0_j or 1_j. Consequently we can find a new homomorphism $\widetilde{\phi}$ with

$$\mathrm{KK}(\phi) = \mathrm{KK}(\widetilde{\phi}), \quad \mathrm{Sp}(\widetilde{\phi}) \subset \mathrm{Sp}(F_1).$$

Now, let us extend this procedure to a homomorphism between two Elliott–Thomsen algebras, as a prelude to describing concretely the KK-group of these two C^*-algebras.

REMARK 2.2. Let $A(F_1, F_2, \varphi_0, \varphi_1), B(F_1', F_2', \varphi_0', \varphi_1')$ be in $\mathcal{C}$, let $\varphi : A \to B$ be a homomorphism, and consider the maps $\pi_0', \pi_1' : B \to F_2'$, where $\pi_t'(f, a) = f(t) = \varphi_t'(a)$, $t = 0$ or 1. Then we can always choose a new homomorphism $\psi : A \to B$ such that

$$\psi \sim_{\mathrm{h}} \phi, \quad \mathrm{KK}(\psi) = \mathrm{KK}(\phi), \quad \text{and} \quad \mathrm{Sp}(\pi_0' \circ \psi), \quad \mathrm{Sp}(\pi_1' \circ \psi) \subset \mathrm{Sp}(F_1).$$

The above condition on $\mathrm{Sp}(\pi_0' \circ \psi), \mathrm{Sp}(\pi_1' \circ \psi)$ is equivalent to $\psi(SF_2) \subset SF_2'$. Hence, we have a commutative diagram as follows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{0*}} & & \downarrow{\scriptstyle \psi_{0**}} & & \downarrow{\scriptstyle \psi_{1**}} & & \downarrow{\scriptstyle \psi_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi_*']{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota_*']{} & \mathrm{K}_1(B) & \longrightarrow & 0.
\end{array}$$

REMARK 2.3. If $\phi, \psi : A \to B$ satisfy

$$\mathrm{Sp}(\pi_0' \circ \phi), \quad \mathrm{Sp}(\pi_1' \circ \phi) \subset \mathrm{Sp}(F_1), \quad \mathrm{Sp}(\pi_0' \circ \psi), \quad \mathrm{Sp}(\pi_1' \circ \psi) \subset \mathrm{Sp}(F_1),$$

respectively, let us define the sum of the diagrams

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_{0*}} & & \downarrow{\scriptstyle \phi_{0**}} & & \downarrow{\scriptstyle \phi_{1**}} & & \downarrow{\scriptstyle \phi_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi_*']{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota_*']{} & \mathrm{K}_1(B) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{0*}} & & \downarrow{\scriptstyle \psi_{0**}} & & \downarrow{\scriptstyle \psi_{1**}} & & \downarrow{\scriptstyle \psi_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi_*']{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota_*']{} & \mathrm{K}_1(B) & \longrightarrow & 0
\end{array}$$

as the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_{0*}+\psi_{0*}} & & \downarrow{\scriptstyle \phi_{0**}+\psi_{0**}} & & \downarrow{\scriptstyle \phi_{1**}+\psi_{1**}} & & \downarrow{\scriptstyle \phi_{1*}+\psi_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi_*']{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota_*']{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

which is just the commutative diagram induced by $\phi \oplus \psi$.

REMARK 2.4. Let $\phi \in \mathrm{Hom}(A, B)$. By Remark 2.2, we can find a homomorphism $\widetilde{\phi}$ homotopic to ϕ satisfying $\widetilde{\phi}(SF_2) \subset SF_2'$. Let us associate the diagram of such a map $\widetilde{\phi}$ to the homomorphism ϕ. Note that the commutative diagram we get depends on the choice of the map $\widetilde{\phi}$. Here we would like to describe the difference between such commutative diagrams corresponding to two different choices of homomorphisms for $\widetilde{\phi}$. Since $\widetilde{\phi}$ is homotopic to ϕ, two different choices of $\widetilde{\phi}$ are also homotopic. For convenience, we shall use τ, ψ to denote the two choices of $\widetilde{\phi}$. Suppose that $\tau, \psi : A \to B$ are two homomorphisms with a homotopy path Φ_t with $\Phi_0 = \tau$ and $\Phi_1 = \psi$. We have both

$$\mathrm{Sp}(\pi_0' \circ \tau), \quad \mathrm{Sp}(\pi_1' \circ \tau) \subset \mathrm{Sp}(F_1) \quad \text{and} \quad \mathrm{Sp}(\pi_0' \circ \psi), \quad \mathrm{Sp}(\pi_1' \circ \psi) \subset \mathrm{Sp}(F_1).$$

Then, the difference between the diagrams

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau_{0*}} & & \downarrow{\scriptstyle \tau_{0**}} & & \downarrow{\scriptstyle \tau_{1**}} & & \downarrow{\scriptstyle \tau_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{0*}} & & \downarrow{\scriptstyle \psi_{0**}} & & \downarrow{\scriptstyle \psi_{1**}} & & \downarrow{\scriptstyle \psi_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0
\end{array}$$

is equal to

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

where there exists $\mu \in \mathrm{Hom}(\mathrm{K}_1(SF_2), \mathrm{K}_0(F_1'))$ with $\mu_0 = \mu \circ (\alpha - \beta)$, $\mu_1 = (\alpha' - \beta') \circ \mu$. Namely, we can choose μ to be induced by $\pi' \circ \Phi_t$, where $\pi'(f, a) = a$, for all $(f, a) \in B$.

If a map between $\mathrm{K}_0(F_1)$ and $\mathrm{K}_0(F_1')$ is induced by a homomorphism from A to $M_r(B)$, for some integer r, then the map is positive — all entries of the matrix of the map are positive (or zero). We need a lemma in order to construct such a homomorphism.

LEMMA 2.5. *Let $A, B \in \mathcal{C}$ be minimal. Let λ be a commutative diagram*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_{0*}} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

where the map λ_0 is positive. If the map

$$\tau = \bigoplus_{j'=1}^{l'} \tau_{j'} : A \to C([0,1], F_2')$$

is such that $\mathrm{Sp}(\pi_0' \circ \tau), \mathrm{Sp}(\pi_1' \circ \tau) \subset \mathrm{Sp}(F_1)$ *and*

$$\mathrm{K}_0(\pi_0' \circ \tau) = \alpha' \circ \lambda_0 \quad \textit{and} \quad \mathrm{K}_0(\pi_1' \circ \tau) = \beta' \circ \lambda_0,$$

then there exists a unitary $u \in C([0,1], F_2)$ such that $\mathrm{Ad}u \circ \tau$ *gives a homomorphism from A to B.*

Proof. For any $j' \in \{1, 2, \ldots, l'\}$, with notation as in Paragraph 1.5, write

$$\varphi_{0j'}'(a') = f'(0, j') \quad \text{and} \quad \varphi_{1j'}'(a') = f'(1, j'),$$

and denote by $\tau_{j'}^0, \tau_{j'}^1$ the evaluation maps of $\tau_{j'}$ at 0 and 1. Let $\gamma : F_1 \to F_1'$ be a homomorphism with $\mathrm{K}_0(\gamma) = \lambda_0$. Since

$$\mathrm{K}_0(\tau_{j'}^0) = \mathrm{K}_0(\varphi_{0j'}' \circ \gamma) \quad \text{and} \quad \mathrm{K}_0(\tau_{j'}^1) = \mathrm{K}_0(\varphi_{1j'}' \circ \gamma),$$

we can find unitaries $U_{j'}, V_{j'} \in M_{h'_{j'}}(\mathbb{C})$ such that

$$\mathrm{Ad}U_{j'} \circ \tau_{j'}^0 = \varphi_{0j'}' \circ \gamma \quad \text{and} \quad \mathrm{Ad}V_{j'} \circ \tau_{j'}^1 = \varphi_{1j'}' \circ \gamma.$$

Connect $U_{j'}$ and $V_{j'}$ by a unitary path $W_{j'}(t) \in M_{h'_{j'}}(C[0,1])$. With

$$u = \bigoplus_{j'=1}^{l'} W_{j'}(t),$$

at the endpoints 0 and 1, we have

$$\mathrm{Ad}u(0) \circ \bigoplus_{j'=1}^{l'} \tau_{j'}^0 = \varphi_0' \circ \gamma \quad \text{and} \quad \mathrm{Ad}u(1) \circ \bigoplus_{j'=1}^{l'} \tau_{j'}^1 = \varphi_1' \circ \gamma.$$

Then $\mathrm{Ad}u \circ \tau$ gives a homomorphism from A to B. ∎

REMARK 2.6. Let λ be a commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_{0*}} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi_*']{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota_*']{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

with the map λ_0 not necessarily positive. Transform it into the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & K_1(SA) & \xrightarrow{\pi_*} & K_1(SF_1) & \xrightarrow{\alpha-\beta} & K_0(S^2F_2) & \xrightarrow{\iota_*} & K_0(SA) & \longrightarrow 0 \\ & \downarrow{\scriptstyle \lambda_{0*}} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_{1*}} & \\ 0 \longrightarrow & K_1(SB) & \xrightarrow[\pi'_*]{} & K_1(SF'_1) & \xrightarrow[\alpha'-\beta']{} & K_0(S^2F'_2) & \xrightarrow[\iota'_*]{} & K_0(SB) & \longrightarrow 0. \end{array}$$

Let us construct a homomorphism from SA to $M_r(SB)$ to realise the above diagram. For the above commutative diagram λ, we define

$$\Gamma^0_{j'i} = \sum_{\lambda^1_{j'j}>0} \alpha_{ji}\lambda^1_{j'j} - \sum_{\lambda^1_{j'j}<0} \beta_{ji}\lambda^1_{j'j}, \quad \Gamma^1_{j'i} = -\sum_{\lambda^1_{j'j}<0} \alpha_{ji}\lambda^1_{j'j} + \sum_{\lambda^1_{j'j}>0} \beta_{ji}\lambda^1_{j'j}, \quad \text{and}$$
$$\Gamma^0 = \max_{j',i} \Gamma^0_{j'i}, \quad \Gamma^1 = \max_{j',i} \Gamma^1_{j'i}, \quad \Gamma = \max\{\Gamma^0, \Gamma^1\},$$

with $\lambda^1_{j'j}$ the $(j',j)^{\text{th}}$ entry of λ_1. Then we have

$$\lambda^0_{i'i} = \Gamma + (\lambda^0_{i'i} - \Gamma),$$

where $\lambda^0_{i'i}$ is the $(i',i)^{\text{th}}$ entry of λ_0. Define homomorphisms

$$D_{j'j} : SA \to M_r(SM_{h'_{j'}}(C[0,1]))$$

(for some integer r) as follows: if $\lambda^1_{j'j} \geqslant 0$, then

$$SA \ni (f,a) \stackrel{D_{j'j}}{\longmapsto} g_{j'j} \in M_r(SM_{h'_{j'}}(C[0,1])),$$

where

$$g_{j'j}(t, e^{2\pi i\omega}) = \operatorname{diag}\{\underbrace{f(t,j,e^{2\pi i\omega}) \oplus \cdots \oplus f(t,j,e^{2\pi i\omega})}_{\lambda^1_{j'j}}\};$$

if $\lambda^1_{j'j} < 0$, then

$$SA \ni (f,a) \stackrel{D_{j'j}}{\longmapsto} h_{j'j} \in M_r(SM_{h'_{j'}}(C[0,1])),$$

where

$$h_{j'j}(t, e^{2\pi i\omega}) = \operatorname{diag}\{\underbrace{f(1-t,j,e^{2\pi i\omega}) \oplus \cdots \oplus f(1-t,j,e^{2\pi i\omega})}_{-\lambda^1_{j'j}}\}.$$

Define homomorphisms $R_{j'i} : SA \to M_r(SM_{h'_{j'}}(C[0,1]))$ (for some integer r) as below:

$$SA \ni (f,a) \stackrel{R_{j'i}}{\longmapsto} r_{j'i} \in M_r(SM_{h'_{j'}}(C[0,1])),$$

where

$$\begin{aligned} r_{j'i}(t,\mathrm{e}^{2\pi\mathrm{i}\omega}) = \mathrm{diag}\Big\{ & \underbrace{a(\theta_i,\mathrm{e}^{2\pi\mathrm{i}\omega})\oplus\cdots\oplus a(\theta_i,\mathrm{e}^{2\pi\mathrm{i}\omega})}_{\sum_{i'=1}^{p'}\alpha_{j'i'}\Gamma-\Gamma^0_{j'i}} \\ & \oplus \bigoplus_{\lambda^0_{i'i}-\Gamma\geqslant 0} \underbrace{a(\theta_i,\mathrm{e}^{2\pi\mathrm{i}\omega})\oplus\cdots\oplus a(\theta_i,\mathrm{e}^{2\pi\mathrm{i}\omega})}_{\alpha_{j'i'}(\lambda^0_{i'i}-\Gamma)} \\ & \oplus \bigoplus_{\lambda^0_{i'i}-\Gamma< 0} \underbrace{a(\theta_i,\mathrm{e}^{-2\pi\mathrm{i}\omega})\oplus\cdots\oplus a(\theta_i,\mathrm{e}^{-2\pi\mathrm{i}\omega})}_{-\alpha_{j'i'}(\lambda^0_{i'i}-\Gamma)} \Big\}, \end{aligned}$$

$(f,a)\in SA$. Define

$$\zeta_\lambda = \bigoplus_{j'=1}^{l'}\Big(\bigoplus_{j=1}^{l} D_{j'j}\oplus\bigoplus_{i=1}^{p} R_{j'i}\Big).$$

As in Lemma 2.5, there is a unitary $u\in M_r(F_2\otimes C[0,1])$ such that $\mathrm{Ad}u\circ\zeta_\lambda$ is a homomorphism from SA to $M_r(SB)$, which induces a commutative diagram λ:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_1(SA) & \xrightarrow{\pi_*} & \mathrm{K}_1(SF_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_0(S^2F_2) & \xrightarrow{\iota_*} & \mathrm{K}_0(SA) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \lambda_{0*}} & & \Big\downarrow{\scriptstyle \lambda_0} & & \Big\downarrow{\scriptstyle \lambda_1} & & \Big\downarrow{\scriptstyle \lambda_{1*}} & & \\ 0 & \longrightarrow & \mathrm{K}_1(SB) & \xrightarrow[\pi'_*]{} & \mathrm{K}_1(SF'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_0(S^2F'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_0(SB) & \longrightarrow & 0. \end{array}$$

For convenience, we shall still use ζ_λ to denote this homomorphism of exact sequences.

2.1. Denote by $C(A,B)$ the set of all the commutative diagrams

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \lambda_{0*}} & & \Big\downarrow{\scriptstyle \lambda_0} & & \Big\downarrow{\scriptstyle \lambda_1} & & \Big\downarrow{\scriptstyle \lambda_{1*}} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0, \end{array}$$

and by $M(A,B)$ the subset of $C(A,B)$ of all the commutative diagrams

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \mu_0} & & \Big\downarrow{\scriptstyle \mu_1} & & \Big\downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0 \end{array}$$

such that there exists $\mu \in \mathrm{Hom}(\mathrm{K}_1(SF_2), \mathrm{K}_0(F_1'))$ satisfying $\mu_0 = \mu \circ (\alpha - \beta)$, $\mu_1 = (\alpha' - \beta') \circ \mu$. Since such a diagram is completely determined by μ, we may denote it by λ_μ.

2.2. For two commutative diagrams $\lambda_{\mathrm{I}}, \lambda_{\mathrm{II}} \in C(A, B)$,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{I}0*}} & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{I}0}} & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{I}1}} & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{I}1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{II}0*}} & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{II}0}} & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{II}1}} & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{II}1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

define the sum of λ_{I} and λ_{II} as

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{I}0*}+\lambda_{\mathrm{II}0*}} & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{I}0}+\lambda_{\mathrm{II}0}} & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{I}1}+\lambda_{\mathrm{II}1}} & & \Big\downarrow{\scriptstyle \lambda_{\mathrm{I}1*}+\lambda_{\mathrm{II}1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0.
\end{array}$$

Note that $\lambda_{\mathrm{I}} + \lambda_{\mathrm{II}} \in C(A, B)$. The diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

to be denoted by 0, is the (unique) zero element of $C(A, B)$. (Clearly, $\lambda + 0 = \lambda$ for $\lambda \in C(A, B)$.)

Given a commutative diagram $\lambda \in C(A, B)$,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \lambda_{0*}} & & \Big\downarrow{\scriptstyle \lambda_{0}} & & \Big\downarrow{\scriptstyle \lambda_{1}} & & \Big\downarrow{\scriptstyle \lambda_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

the inverse of λ, to be denoted by $-\lambda$, is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(A) & \xrightarrow{\pi_*} & K_0(F_1) & \xrightarrow{\alpha-\beta} & K_1(SF_2) & \xrightarrow{\iota_*} & K_1(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle -\lambda_{0*}} & & \downarrow{\scriptstyle -\lambda_0} & & \downarrow{\scriptstyle -\lambda_1} & & \downarrow{\scriptstyle -\lambda_{1*}} & & \\ 0 & \longrightarrow & K_0(B) & \xrightarrow[\pi'_*]{} & K_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & K_1(SF'_2) & \xrightarrow[\iota'_*]{} & K_1(B) & \longrightarrow & 0. \end{array}$$

Note that $-\lambda \in C(A,B)$, and $\lambda + (-\lambda) = 0$.

Now we get the following proposition.

PROPOSITION 2.7. *Assume that $A, B \in \mathcal{C}$. Then $C(A,B)$ is an Abelian group, and $M(A,B)$ is a subgroup of $C(A,B)$.*

2.3. Define a map $\mathfrak{R} : C(A,B) \to [SA, SB \otimes \mathscr{K}]$ by

$$\lambda \overset{\mathfrak{R}}{\longmapsto} [\zeta_\lambda].$$

Note that, for $\lambda_{\mathrm{I}}, \lambda_{\mathrm{II}} \in C(A,B)$, we may not have $\zeta_{\lambda_{\mathrm{I}}} \oplus \zeta_{\lambda_{\mathrm{II}}} = \zeta_{\lambda_{\mathrm{I}}+\lambda_{\mathrm{II}}}$. Nevertheless, the homomorphism $SA \ni (f,a) \mapsto g \in SM_m(C[0,1])$ defined by

$$g(t, \mathrm{e}^{2\pi \mathrm{i}\omega}) = f(t, j, \mathrm{e}^{2\pi \mathrm{i}\omega}) \oplus f(1-t, j, \mathrm{e}^{2\pi \mathrm{i}\omega})$$

and the homomorphism $SA \ni (f,a) \mapsto h \in SM_m(C[0,1])$ defined by

$$h(t, \mathrm{e}^{2\pi \mathrm{i}\omega}) = \bigoplus_{i=1}^{p} \underbrace{a(\theta_i, \mathrm{e}^{2\pi \mathrm{i}\omega}) \oplus \cdots \oplus a(\theta_i, \mathrm{e}^{2\pi \mathrm{i}\omega})}_{\alpha_{ji}+\beta_{ji}}$$

are homotopic to each other as homomorphisms from A to $SM_m(C[0,1])$. Also, the homomorphism $SA \ni (f,a) \mapsto l \in SM_m(\mathbb{C})$ defined by

$$l(t, \mathrm{e}^{2\pi \mathrm{i}\omega}) = a(\theta_i, \mathrm{e}^{2\pi \mathrm{i}\omega}) \oplus a(\theta_i, \mathrm{e}^{-2\pi \mathrm{i}\omega})$$

is homotopic to 0 as a homomorphism from A to $SM_m(\mathbb{C})$. With the aid of these two facts, it is easy to check that

$$[\zeta_{\lambda_{\mathrm{I}}}] \oplus [\zeta_{\lambda_{\mathrm{II}}}] = [\zeta_{\lambda_{\mathrm{I}}+\lambda_{\mathrm{II}}}],$$

which means that $\mathfrak{R}$ is a homomorphism.

Denote by $\mathfrak{N}$ the natural map from $[SA, SB \otimes \mathscr{K}]$ to $\mathrm{KK}(A,B)$; the composed map $\mathfrak{N} \circ \mathfrak{R}$ is then a homomorphism from $C(A,B)$ to $\mathrm{KK}(A,B)$.

2.4. Let $A, B \in \mathcal{C}$, $\alpha \in \mathrm{KK}(A,B)$. Then, in view of the short exact sequences

$$0 \to SF_2 \xrightarrow{\iota} A \xrightarrow{\pi} F_1 \to 0 \quad \text{and} \quad 0 \to SF'_2 \xrightarrow{\iota'} B \xrightarrow{\pi'} F'_1 \to 0,$$

we have KK-theory six-term exact sequences as follows:

$$0 \to \mathrm{KK}^1(SF_2, B) \to \mathrm{KK}^1(SF_2, F'_1) \to \mathrm{KK}(SF_2, SF'_2) \to \mathrm{KK}(SF_2, B) \to 0 \quad \text{and}$$
$$0 \leftarrow \mathrm{KK}(A, F'_1) \leftarrow \mathrm{KK}(F_1, F'_1) \leftarrow \mathrm{KK}^1(SF_2, F'_1) \leftarrow \mathrm{KK}^1(A, F'_1) \leftarrow 0.$$

Then there exist

$$\alpha_1 \in \mathrm{KK}(SF_2, SF_2') = \mathrm{Hom}(\mathrm{K}_0(F_2), \mathrm{K}_0(F_2')), \quad \text{and}$$
$$\alpha_0 \in \mathrm{KK}(F_1, F_1') = \mathrm{Hom}(\mathrm{K}_0(F_1), \mathrm{K}_0(F_1'))$$

such that, with respect to the Kasparov product,

$$\alpha_1 \times \mathrm{KK}(\iota') = \mathrm{KK}(\iota) \times \alpha \in \mathrm{KK}(SF_2, B), \quad \alpha \times \mathrm{KK}(\pi') = \mathrm{KK}(\pi) \times \alpha_0 \in \mathrm{KK}(A, F_1'),$$

and $\alpha, \alpha_0, \alpha_1$ induce a commutative diagram $\lambda_{(\alpha,\alpha_0,\alpha_1)} \in C(A, B)$:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\iota_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\pi_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{*0}} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_{*1}} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\iota'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\pi'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0. \end{array}$$

Next, we will show a useful lemma of KK-theory.

LEMMA 2.8. *Let $A, B \in \mathcal{C}$, $\alpha \in \mathrm{KK}(A, B)$. If there are $\alpha_1 \in \mathrm{KK}(SF_2, SF_2')$ and $\alpha_0 \in \mathrm{KK}(F_1, F_1')$ such that*

$$\alpha_1 \times \mathrm{KK}(\iota') = \mathrm{KK}(\iota) \times \alpha \quad \text{and} \quad \alpha \times \mathrm{KK}(\pi') = \mathrm{KK}(\pi) \times \alpha_0,$$

then

$$\mathfrak{N} \circ \mathfrak{R}(\lambda_{(\alpha,\alpha_0,\alpha_1)}) = \alpha,$$

where the homomorphism $\mathfrak{N} \circ \mathfrak{R}$ is as defined in Paragraph 2.3.

Proof. Set

$$\mathfrak{N} \circ \mathfrak{R}(\lambda_{(\alpha,\alpha_0,\alpha_1)}) = \widetilde{\alpha}.$$

By Proposition 2.9 of [3], we only need to show that $\alpha - \widetilde{\alpha}$ induces the zero map from $\underline{\mathrm{K}}(A)$ to $\underline{\mathrm{K}}(B)$. Note that

$$(\alpha - \widetilde{\alpha}) \times \mathrm{KK}(\pi') = \mathrm{KK}(\pi) \times 0 \quad \text{and} \quad 0 \times \mathrm{KK}(\iota') = \mathrm{KK}(\iota) \times (\alpha - \widetilde{\alpha}).$$

Then we have a commutative diagram, or a homomorphism, of exact sequences,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\iota_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\pi_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{*0}-\widetilde{\alpha}_{*0}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \alpha_{*1}-\widetilde{\alpha}_{*1}} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\iota'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\pi'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0, \end{array}$$

where $\alpha_{*0} - \widetilde{\alpha}_{*0}, \alpha_{*1} - \widetilde{\alpha}_{*1}$ are induced by $\alpha - \widetilde{\alpha}$. As $\alpha_{*0} - \widetilde{\alpha}_{*0}$ is the restriction of 0, and $\alpha_{*1} - \widetilde{\alpha}_{*1}$ is the quotient map of 0, we have

$$\alpha_{*0} - \widetilde{\alpha}_{*0} = 0 \quad \text{and} \quad \alpha_{*1} - \widetilde{\alpha}_{*1} = 0.$$

Since

$$\mathrm{KK}(I_p, SF_2) = 0, \quad \text{and} \quad \mathrm{KK}^1(I_p, F_1) = 0,$$

we also have the homomorphism of exact sequences

$$\begin{array}{ccccccccc} 0 \to & \mathrm{KK}(I_p, A) & \to & \mathrm{KK}(I_p, F_1) & \to & \mathrm{KK}^1(I_p, SF_2) & \to & \mathrm{KK}^1(I_p, A) & \to 0 \\ & \downarrow{\scriptstyle \alpha^p_{*0}-\widetilde{\alpha}^p_{*0}} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \alpha^p_{*1}-\widetilde{\alpha}^p_{*1}} & \\ 0 \to & \mathrm{KK}(I_p, B) & \to & \mathrm{KK}(I_p, F_1') & \to & \mathrm{KK}^1(I_p, SF_2') & \to & \mathrm{KK}^1(I_p, B) & \to 0, \end{array}$$

where $\alpha^p_{*0} - \widetilde{\alpha}^p_{*0}, \alpha_{*1} - \widetilde{\alpha}^p_{*1}$ are induced by $\alpha - \widetilde{\alpha}$. Since $\alpha^p_{*0} - \widetilde{\alpha}^p_{*0}$ is the restriction of 0 and $\alpha^p_{*1} - \widetilde{\alpha}^p_{*1}$ is the quotient map of 0, we have

$$\alpha^p_{*0} - \widetilde{\alpha}^p_{*0} = 0 \quad \text{and} \quad \alpha^p_{*1} - \widetilde{\alpha}^p_{*1} = 0.$$

In summary, we have $\alpha = \widetilde{\alpha}$. ∎

THEOREM 2.9. *Let $A, B \in \mathcal{C}$. Then we have a natural isomorphism of groups*

$$\mathrm{KK}(A, B) \cong C(A, B)/M(A, B).$$

Proof. Recall that in Paragraph 2.3, we obtained a homomorphism $\mathfrak{N} \circ \mathfrak{R}$ from $C(A, B)$ to $\mathrm{KK}(A, B)$. From Paragraph 2.4 and Lemma 2.8, we know that $\mathfrak{N} \circ \mathfrak{R}$ is surjective. We only need to show that

$$\ker \mathfrak{N} \circ \mathfrak{R} = M(A, B).$$

Recall that if $\mu \in \mathrm{Hom}(\mathrm{K}_1(SF_2), \mathrm{K}_0(F_1'))$, then any element $\lambda_\mu \in M(A, B)$ is defined as the diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{K}_1(SA) & \xrightarrow{\pi_*} & \mathrm{K}_1(SF_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_0(S^2F_2) & \xrightarrow{\iota_*} & \mathrm{K}_0(SA) & \to 0 \\ & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle 0} & \\ 0 \to & \mathrm{K}_1(SB) & \xrightarrow[\pi'_*]{} & \mathrm{K}_1(SF_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_0(S^2F_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_0(SB) & \to 0, \end{array}$$

where $\mu_0 = \mu \circ (\alpha - \beta)$, $\mu_1 = (\alpha' - \beta') \circ \mu$. From the well-known (six-term) exact sequences of KK-theory,

$$0 \to \mathrm{KK}^1(SF_2, B) \xrightarrow{\mathrm{KK}(\pi')} \mathrm{KK}^1(SF_2, F_1') \xrightarrow{\mathrm{KK}(\partial')} \mathrm{KK}(F_2, F_2') \xrightarrow{\mathrm{KK}(\iota')} \mathrm{KK}(SF_2, B) \to 0$$

and

$$0 \leftarrow \mathrm{KK}(A, F_1') \xleftarrow{\mathrm{KK}(\pi)} \mathrm{KK}(F_1, F_1') \xleftarrow{\mathrm{KK}(\partial)} \mathrm{KK}^1(SF_2, F_1') \xleftarrow{\mathrm{KK}(\iota)} \mathrm{KK}^1(A, F_1') \leftarrow 0,$$

we have $\mathrm{KK}(\pi) \times \mu_0 = \mathrm{KK}(\pi) \times \mathrm{KK}(\partial) \times \mu = 0$ and $\mu_1 \times \mathrm{KK}(\iota') = \mu \times \mathrm{KK}(\partial') \times \mathrm{KK}(\iota') = 0$.

Then we have $\mathfrak{N} \circ \mathfrak{R}(\lambda_\mu) = 0$ by Lemma 2.8.

The proof of the converse inclusion,

$$\ker \mathfrak{N} \circ \mathfrak{R} \subset M(A, B),$$

divides naturally into two cases.

Case 1. $\lambda \in C(A,B)$ is the following commutative diagram such that $\mathfrak{N} \circ \mathfrak{R}(\lambda) = 0$:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0.
\end{array}
$$

Note that here the map $\lambda_0 : \mathrm{K}_0(F_1) \to \mathrm{K}_0(F'_1)$ is assumed to be the zero map. As λ is commutative, we have

$$\mathrm{KK}(\partial) \times \lambda_1 = 0 \times \mathrm{KK}(\partial') = 0 \in \mathrm{KK}^1(F_1, SF'_2).$$

Consider the diagram

$$
\tag{2.1}
\begin{array}{ccccccc}
0 & & 0 & & \mathrm{KK}(F_1,B) & \xrightarrow{\mathrm{KK}(\pi')} & \mathrm{KK}(F_1,F'_1) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{KK}(\pi)} & & \downarrow{\scriptstyle \mathrm{KK}(\pi)} \\
\mathrm{KK}^1(A,F'_1) & \xrightarrow{\mathrm{KK}(\partial')} & \mathrm{KK}(A,SF'_2) & \xrightarrow{\mathrm{KK}(\iota')} & \mathrm{KK}(A,B) & \xrightarrow{\mathrm{KK}(\pi')} & \mathrm{KK}(A,F'_1) \\
\downarrow{\scriptstyle \mathrm{KK}(\iota)} & & \downarrow{\scriptstyle \mathrm{KK}(\iota)} & & \downarrow{\scriptstyle \mathrm{KK}(\iota)} & & \downarrow \\
\mathrm{KK}^1(SF_2,F'_1) & \xrightarrow{\mathrm{KK}(\partial')} & \mathrm{KK}(SF_2,SF'_2) & \xrightarrow{\mathrm{KK}(\iota')} & \mathrm{KK}(SF_2,B) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{KK}(\partial)} & & \downarrow{\scriptstyle \mathrm{KK}(\partial)} & & & & \\
\mathrm{KK}(F_1,F'_1) & \xrightarrow{\mathrm{KK}(\partial')} & \mathrm{KK}^1(F_1,SF'_2) & & & & \\
\downarrow{\scriptstyle \mathrm{KK}(\pi)} & & & & & & \\
\mathrm{KK}(A,F'_1) & & & & & & \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}
$$

where $\mathrm{KK}(\partial) \in \mathrm{KK}^1(F_1, SF_2)$ and $\mathrm{KK}(\partial') \in \mathrm{KK}^1(F'_1, SF'_2)$. By the exactness of the second column of (2.1), there exists a unique $\delta \in \mathrm{KK}(A, SF'_2)$ such that

$$\mathrm{KK}(\iota) \times \delta = \lambda_1 \in \mathrm{KK}(SF_2, SF'_2).$$

Note that $\delta \times \mathrm{KK}(\iota') \in \mathrm{KK}(A,B)$ satisfies

$$\delta \times \mathrm{KK}(\iota') \times \mathrm{KK}(\pi') = 0 = \mathrm{KK}(\pi) \times 0 \quad \text{and} \quad \mathrm{KK}(\iota) \times \delta \times \mathrm{KK}(\iota') = \lambda_1 \times \mathrm{KK}(\iota').$$

From $\mathfrak{N} \circ \mathfrak{R}(\lambda) = 0$, by Lemma 2.8, it follows that

$$\delta \times \mathrm{KK}(\iota') = 0.$$

By the exactness of the second line, there exists $\gamma \in \mathrm{KK}^1(A, F_1')$ such that

$$\gamma \times \mathrm{KK}(\partial') = \delta \in \mathrm{KK}(A, SF_2').$$

Then the exactness of the first column of (2.1) implies

$$\mathrm{KK}(\partial) \times \mathrm{KK}(\iota) \times \gamma = 0 \in \mathrm{KK}(A, B),$$

and we also have

$$\mathrm{KK}(\iota) \times \gamma \times \mathrm{KK}(\partial') = \lambda_1 \in \mathrm{KK}(SF_2, SF_2'),$$

which leads to the conclusion that the KK-element $\mathrm{KK}(\iota) \times \gamma \in \mathrm{KK}^1(SF_2, F_1')$ induces $\lambda \in M(A, B)$.

Case 2. Let us consider the general case. Let $\lambda \in C(A, B)$ be given,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_{0*}} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

such that $\mathfrak{N} \circ \mathfrak{R}(\lambda) = 0$. Then

$$\mathrm{KK}(\pi) \times \lambda_0 = 0 \times \mathrm{KK}(\pi') = 0 \in \mathrm{KK}(SF_2, F_1').$$

By the exactness of the first column of (2.1), there exists $\mu \in \mathrm{KK}^1(SF_2, F_1')$ such that

$$\lambda_0 = \mathrm{KK}(\partial) \times \mu \in \mathrm{KK}(F_1, F_1').$$

Note that $\mu \in \mathrm{KK}^1(SF_2, F_1')$ induces the following diagram $\lambda_\mu \in M(A, B)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle (\alpha'-\beta')\circ\mu} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

where $(\alpha' - \beta') \circ \lambda_0 = (\alpha' - \beta') \circ \mu \circ (\alpha - \beta)$. Then

$$\mathfrak{N} \circ \mathfrak{R}(\lambda_\mu) = 0.$$

As $\mathfrak{N} \circ \mathfrak{R}$ is a homomorphism, we have $\mathfrak{N} \circ \mathfrak{R}(\lambda - \lambda_\mu) = 0$. Then from what we have shown in Case 1, it follows that

$$\lambda - \lambda_\mu \in M(A, B).$$

In particular, $\lambda \in M(A, B)$.

In summary, we have $C(A, B)/M(A, B) \cong \mathrm{KK}(A, B)$. ∎

From now on, let us use χ to denote the isomorphism (the inverse of $\mathfrak{N}\circ\mathfrak{R}$ induced by the map obtained in Paragraph 2.4) from $\mathrm{KK}(A,B)$ to $C(A,B)/M(A,B)$; that is,

$$\chi(\alpha)=\lambda_{(\alpha,\alpha_0,\alpha_1)}+M(A,B).$$

In fact, for an element of $C(A,B)$,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\lambda_{0*}} & & \downarrow{\scriptstyle\lambda_0} & & \downarrow{\scriptstyle\lambda_1} & & \downarrow{\scriptstyle\lambda_{1*}} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

all the information is contained in the smaller commutative diagram

$$\begin{array}{ccc}
\mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2)\\
\downarrow{\scriptstyle\lambda_0} & & \downarrow{\scriptstyle\lambda_1}\\
\mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2).
\end{array}$$

So if it makes no confusion, we will use the smaller one for short.

REMARK 2.10. In our case, every KK-element can be realized by a homomorphism between the suspensions of the algebras (by Paragraph 2.3 and Lemma 2.8). One can also realize the KK-element by a difference of two homomorphisms from A to $M_r(B)$. But we should point out that if two homomorphisms from A to $M_r(B)$ determine the same KK-element, sometimes they are not homotopic to each other, but are such that on adding the same homomorphism to each, they become homotopic to each other. We present an example here.

With $F_1=\mathbb{C}\oplus\mathbb{C}$, $F_2=M_2(\mathbb{C})$,

$$\varphi_0(a\oplus b)=\begin{pmatrix} a & \\ & a\end{pmatrix},\quad \varphi_1(a\oplus b)=\begin{pmatrix} a & \\ & b\end{pmatrix},$$

$B=\mathbb{C}$, and $A=A(F_1,F_2,\varphi_0,\varphi_1)$, define two homomorphisms δ_1, $\delta_2: A\to B$:

$$\delta_1(f,a\oplus b)=a,\quad (f,a\oplus b)\in A,\quad\text{and}\quad \delta_2(f,a\oplus b)=b,\quad (f,a\oplus b)\in A.$$

Then δ_1, δ_2 induce the two diagrams

$$\begin{array}{ccc}
\mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(1,-1)} & \mathbb{Z}\\
\downarrow{\scriptstyle(1,0)} & & \downarrow{\scriptstyle 0}\\
\mathbb{Z} & \xrightarrow[0]{} & 0
\end{array}\quad\text{and}\quad
\begin{array}{ccc}
\mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(1,-1)} & \mathbb{Z}\\
\downarrow{\scriptstyle(0,1)} & & \downarrow{\scriptstyle 0}\\
\mathbb{Z} & \xrightarrow[0]{} & 0.
\end{array}$$

At the same time, we have

$$\delta_1\not\sim_{\mathrm{h}}\delta_2\quad\text{but}\quad\delta_1\oplus\delta_1\sim_{\mathrm{h}}\delta_2\oplus\delta_1.$$

Denoting $S\delta_1, S\delta_2$ by the homomorphisms from SA to SB induced by δ_1, δ_2, we also have $S\delta_1 \sim_{\mathrm{h}} S\delta_2$.

EXAMPLE 2.11. Let us consider the group $C(A, \mathbb{C}), A \in \mathcal{C}$. The elements of $C(A, \mathbb{C})$ are the diagrams

$$\begin{array}{ccc} \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) \\ \lambda_0 \big\downarrow & & \big\downarrow 0 \\ \mathbb{Z} & \xrightarrow[0]{} & 0. \end{array}$$

The subgroup $M(A, \mathbb{C})$ consists of the diagrams

$$\begin{array}{ccc} \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) \\ \mu\circ(\alpha-\beta) \big\downarrow & & \big\downarrow 0 \\ \mathbb{Z} & \xrightarrow[0]{} & 0. \end{array}$$

Then we have

$$\mathrm{K}^0(A) \cong \mathrm{KK}(A, \mathbb{C}) \cong C(A, \mathbb{C})/M(A, \mathbb{C}) \cong \mathrm{coker}(\alpha-\beta)^{\mathrm{T}}.$$

EXAMPLE 2.12. Let us consider the group $C(C(S^1), A), A \in \mathcal{C}$. The elements of $C(C(S^1), A)$ are the diagrams

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\ \lambda_0 \big\downarrow & & \big\downarrow \lambda_1 \\ \mathrm{K}_0(F_1) & \xrightarrow[\alpha-\beta]{} & \mathrm{K}_1(SF_2) \end{array}$$

with $(\alpha-\beta)\circ\lambda_0 = 0$. The subgroup $M(C(S^1), A)$ consists of the diagrams

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\ 0 \big\downarrow & & \big\downarrow (\alpha-\beta)\circ\mu \\ \mathrm{K}_0(F_1) & \xrightarrow[\alpha-\beta]{} & \mathrm{K}_1(SF_2). \end{array}$$

Then we have

$$C(C(S^1), A)/M(C(S^1), A) \cong \ker(\alpha-\beta) \oplus \mathrm{coker}(\alpha-\beta).$$

EXAMPLE 2.13. Let us consider the quotient group

$$C(\widetilde{I}_p, \widetilde{I}_p)/M(\widetilde{I}_p, \widetilde{I}_p).$$

$C(\widetilde{I}_p, \widetilde{I}_p)$ consists of the diagrams

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(p,-p)} & \mathbb{Z} \\ {\scriptstyle \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle \rho} \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[(p,-p)]{} & \mathbb{Z} \end{array}$$

where $\rho = a - c = d - b$, and the subgroup $M(A, B)$ consists of the diagrams

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(p,-p)} & \mathbb{Z} \\ {\scriptstyle \begin{pmatrix} \mu_1 p & -\mu_1 p \\ \mu_2 p & -\mu_2 p \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle \mu_1 p - \mu_2 p} \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[(p,-p)]{} & \mathbb{Z} \end{array}$$

where $\mu_1, \mu_2 \in \mathbb{Z}$.

Considering the homomorphism γ from $C(\widetilde{I}_p, \widetilde{I}_p)$ to $\mathbb{Z} \bigoplus \mathbb{Z}_p \bigoplus \mathbb{Z}_p$ such that

$$\gamma(\lambda) = (a + b, \overline{b}, \overline{d}),$$

where the diagram of λ is

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(p,-p)} & \mathbb{Z} \\ {\scriptstyle \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle \rho} \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[(p,-p)]{} & \mathbb{Z}, \end{array}$$

we get $C(\widetilde{I}_p, \widetilde{I}_p) / M(\widetilde{I}_p, \widetilde{I}_p) \cong \mathbb{Z} \bigoplus \mathbb{Z}_p \bigoplus \mathbb{Z}_p$.

EXAMPLE 2.14. Let us consider the quotient group

$$C(C(S^1), C(S^1)) / M(C(S^1), C(S^1)).$$

The group $C(C(S^1), C(S^1))$ consists of the diagrams

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(C(S^1)) & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{1-1=0} & \mathbb{Z} & \xrightarrow{1} & \mathrm{K}_1(C(S^1)) & \longrightarrow & 0 \\ & & {\scriptstyle m} \downarrow & & {\scriptstyle m} \downarrow & & {\scriptstyle n} \downarrow & & {\scriptstyle n} \downarrow & & \\ 0 & \longrightarrow & \mathrm{K}_0(C(S^1)) & \xrightarrow[1]{} & \mathbb{Z} & \xrightarrow[1-1=0]{} & \mathbb{Z} & \xrightarrow[1]{} & \mathrm{K}_1(C(S^1)) & \longrightarrow & 0, \end{array}$$

and $M(C(S^1), C(S^1)) = \{0\}$.

Considering the homomorphism γ from $C(C(S^1), C(S^1))$ to $\mathbb{Z} \bigoplus \mathbb{Z}$ such that

$$\gamma(\lambda_{(m,n)}) = (m, n),$$

where the diagram of $\lambda_{(m,n)}$ is

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K_0(C(S^1)) & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{1-1=0} & \mathbb{Z} & \xrightarrow{1} & K_1(C(S^1)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \\
0 & \longrightarrow & K_0(C(S^1)) & \xrightarrow[1]{} & \mathbb{Z} & \xrightarrow[1-1=0]{} & \mathbb{Z} & \xrightarrow[1]{} & K_1(C(S^1)) & \longrightarrow & 0,
\end{array}$$

we get $KK(A,B) \cong \mathbb{Z} \bigoplus \mathbb{Z}$.

3. THE KK-LIFTING PROBLEM FOR ELLIOTT–THOMSEN ALGEBRAS

DEFINITION 3.1. Let $A, B \in \mathcal{C}$. Define $KK^+(A,B)$ as the image of the abelian semigroup (of homotopy classes of homomorphisms) $[A, B \otimes \mathscr{K}]$ in the group of $KK(A,B)$. We shall say that $\alpha \in KK(A,B)$ is *positive*, if $\alpha \in KK^+(A,B)$.

In [14], Jiang and Su investigated the generalized dimension drop interval algebras, and obtained a characterization of positive KK-elements in terms of ordered K-homology. Recall that they defined an order structure on the K-homology group of their building blocks:

$$K^0_+(A) := \{[\phi] \in K^0(A) : \varphi \text{ is a finite dimensional representation of } A\}.$$

Then they proved the following criterion for KK-lifting.

THEOREM 3.2 ([14], Theorem 3.7). *Let A, B be generalized dimension drop interval algebras, and $\alpha \in KK(A,B)$. The KK-element α can be lifted to a homomorphism if and only if α^* is positive from $K^0(B)$ to $K^0(A)$, where α^* is the operation of Kasparov product of α with K-homology elements.*

Recall that in the previous section, we gave a description of the KK-group for two Elliott–Thomsen algebras as the quotient group $C(A,B)/M(A,B)$, which (as we shall see) makes the calculation easier. Before we give a different criterion with this new description (in which we will require that $A \in \mathcal{C}_{\mathcal{O}}$, $B \in \mathcal{C}$), we shall define an order on this quotient group (as the image of of the natural order on $C(A,B)$).

DEFINITION 3.3. Let $A, B \in \mathcal{C}$ be minimal and $\lambda \in C(A,B)$:

$$\begin{array}{ccc}
K_0(F_1) & \xrightarrow{\alpha-\beta} & K_1(SF_2) \\
{\scriptstyle \lambda_0}\downarrow & & \downarrow{\scriptstyle \lambda_1} \\
K_0(F_1') & \xrightarrow[\alpha'-\beta']{} & K_1(SF_2').
\end{array}$$

Let us say that λ is *positive* if λ_0 has no negative entry and $\lambda_0 \neq 0$ or λ is the zero element. And let us say that λ is *positive modulo $M(A,B)$*, and that $\lambda + M(A,B)$

is *positive*, if there exists $\lambda_\mu \in M(A,B)$,

$$\begin{array}{ccc} \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) \\ \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu_1} \\ \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2'), \end{array}$$

such that $\lambda + \lambda_\mu$ is a positive.

REMARK 3.4. By Definition 3.3, in the case of Example 2.14, the diagram $\lambda_{(0,n)}$, $n \neq 0$, is not positive modulo $M(C(S^1), C(S^1)) = \{\lambda_{(0,0)}\}$.

If $A = \bigoplus A^i, B = \bigoplus B^j$ with each A^i and B^j a minimal Elliott–Thomsen algebra, then we shall say that $\lambda + M(A,B)$ is positive, where $\lambda \in C(A,B)$ is determined by $\lambda_{ij} \in C(A^i, B^j)$, if $\lambda_{ij} + M(A^i, B^j)$ is positive for each i, j. Let us write

$$(C(A,B)/M(A,B))^+ = \{\lambda + M(A,B) : \lambda + M(A,B) \text{ is positive}\}$$

for the positive cone of $C(A,B)/M(A,B)$.

For any A, B in $\mathcal{C}$, recall that by Theorem 2.9, as groups, $\mathrm{KK}(A,B)$ and $C(A,B)/M(A,B)$ are isomorphic. By the universal multi-coefficient theorem of [1] (see Theorem 4.4 below), $\mathrm{KK}(A,B)$ is also naturally isomorphic as a group to $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$. This is without considering the order structures of these groups. It is important for our purpose to study the relations between these order structures.

By Remark 2.1 and Definition 3.3, we get directly the following proposition.

PROPOSITION 3.5. *Let $A \in \mathcal{C}$. The commutative diagram*

$$\begin{array}{ccc} \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) \\ \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle 0} \\ \mathbb{Z} & \xrightarrow[0]{} & 0. \end{array}$$

can be lifted to a representation if and only if λ is positive module $M(A,\mathbb{C})$.

The following lemma gives a sufficient condition for KK-lifting.

LEMMA 3.6. *Let $A, B \in \mathcal{C}$ be minimal. Let the diagram $\lambda \in C(A,B)$,*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda_{0*}} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_{1*}} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0, \end{array}$$

be given, such that λ is positive, then λ_0 is positive. If for any $i \in \{1,2,\ldots,p\}, j' \in \{1,2,\ldots,l'\}$,

$$(\alpha' \circ \lambda_0)_{j'i} \geqslant \sum_{\alpha_{ji}\cdot\lambda^1_{j'j}\geqslant 0} \alpha_{ji}\cdot\lambda^1_{j'j} - \sum_{\beta_{ji}\cdot\lambda^1_{j'j}\leqslant 0} \beta_{ji}\cdot\lambda^1_{j'j} \quad \textit{and}$$
$$(\beta' \circ \lambda_0)_{j'i} \geqslant - \sum_{\alpha_{ji}\cdot\lambda^1_{j'j}\leqslant 0} \alpha_{ji}\cdot\lambda^1_{j'j} + \sum_{\beta_{ji}\cdot\lambda^1_{j'j}\geqslant 0} \beta_{ji}\cdot\lambda^1_{j'j},$$

then there is a homomorphism from A to $M_r(B)$ for some integer r inducing the diagram λ.

Proof. For any $j' \in \{1,2,\ldots,l'\}$, define a homomorphism from A to the algebra $M_r(M_{h'_{j'}}(C[0,1]))$ (for some integer r):

$$A \ni (f,a) \overset{\phi_{j'}}{\longmapsto} g_{j'} \in M_r(M_{h'_{j'}}(C[0,1]))$$

with

$$\begin{aligned} g_{j'}(t) = &\bigoplus_{\lambda_{j'j}\geqslant 0} \mathrm{diag}\{\underbrace{f(t,j), f(t,j), \ldots, f(t,j)}_{\lambda_{j'j}}\} \\ &\oplus \bigoplus_{\lambda_{j'j}\leqslant 0} \mathrm{diag}\{\underbrace{f(1-t,j), f(1-t,j), \ldots, f(1-t,j)}_{-\lambda_{j'j}}\} \\ &\oplus \bigoplus_{i=1}^{p} \mathrm{diag}\{\underbrace{a(\theta_i), a(\theta_i), \ldots, a(\theta_i)}_{\eta_{j'i}}\}, \end{aligned}$$

where

$$\eta_{j'i} = (\alpha' \circ \lambda_0)_{j'i} - \Big(\sum_{\alpha_{ji}\cdot\lambda^1_{j'j}\geqslant 0} \alpha_{ji}\cdot\lambda^1_{j'j} - \sum_{\beta_{ji}\cdot\lambda^1_{j'j}\leqslant 0} \beta_{ji}\cdot\lambda^1_{j'j} \Big).$$

Then, with

$$\phi = \bigoplus_{j'=1}^{l'} \phi_{j'},$$

by Lemma 2.5 we have a unitary u such that $\mathrm{Ad}u \circ \phi$ is a homomorphism from A to $M_r(B)$ inducing the commutative diagram λ. ∎

We have shown that the condition in Lemma 3.6 is sufficient, but it is also necessary in the following special case. (This will be used in Remark 3.11 and Example 4.7.)

COROLLARY 3.7. *Let $A, B \in \mathcal{C}$ be minimal, with $F_2 = M_n(\mathbb{C})$. Let λ be a positive element of $C(A,B)$, that is, a commutative diagram*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda_{0*}} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_{1*}} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0 \end{array}$$

with λ_0 positive, and assume that $\lambda_1 = (1,1,\dots,1)^{\mathrm{T}}$. Then, there is a homomorphism from A to $M_r(B)$, for some integer r, inducing the diagram λ, if and only if for any $i \in \{1,2,\dots,p\}, j' \in \{1,2,\dots,l'\}$,

$$(\alpha' \circ \lambda_0)_{j'i} \geqslant \alpha_{1i} \quad \textit{and} \quad (\beta' \circ \lambda_0)_{j'i} \geqslant \beta_{1i}.$$

Proof. If there is a homomorphism ϕ from A to $M_r(B)$, for some integer r, inducing the diagram λ, then ϕ must be homotopic to a homomorphism $\widetilde{\phi}$, where $\widetilde{\phi}(f,a) = g \in M_r(B)$, with

$$g(t,j') = u_{j'} \cdot \operatorname{diag}(f(t), \underbrace{a(\theta_1),\dots,a(\theta_1)}_{\eta_{j'1}}, \dots, \underbrace{a(\theta_p),\cdots,a(\theta_p)}_{\eta_{j'p}}) \cdot u_{j'}.$$

Immediately, we have

$$(\alpha' \circ \lambda_0)_{j'i} \geqslant \alpha_{1i} \quad \text{and} \quad (\beta' \circ \lambda_0)_{j'i} \geqslant \beta_{1i}. \quad \blacksquare$$

Using Lemma 3.6, we shall now give a criterion for KK-lifting for $A \in \mathcal{C}_{\mathcal{O}}$.

THEOREM 3.8. *Let $A, B \in \mathcal{C}$ be minimal, with $A \in \mathcal{C}_{\mathcal{O}}$. A* KK*-element $\alpha \in$* KK(A,B) *can be lifted to a homomorphism if and only if $\chi(\alpha) \in C(A,B)/M(A,B)$ is positive, where χ is the isomorphism defined following Theorem* 2.9.

Proof. The necessity comes from Remark 2.2. Here we only need to show that if $\chi(\alpha)$ is positive, then there exists a homomorphism realising it. By Definition 3.3, there is a positive commutative diagram $\lambda \in \chi(\alpha)$,

$$\begin{array}{ccc} \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) \\ \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_1} \\ \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2), \end{array}$$

with λ_0 a positive map. We have $\mathrm{K}_1(SF_2) = \mathbb{Z}$ ($A \in \mathcal{C}_{\mathcal{O}}$ — see Paragraph 1.7),

$$\lambda_1 = (\lambda^1_{11}, \cdots, \lambda^1_{l'1})^{\mathrm{T}}, \quad \text{and} \quad \alpha - \beta = (\alpha_{11} - \beta_{11}, \alpha_{12} - \beta_{12}, \dots, \alpha_{1p} - \beta_{1p}).$$

(We shall show later that it is necessary to have $\mathbb{Z}$ itself here, not a sum of copies of $\mathbb{Z}$.) From $\lambda_1 \circ (\alpha - \beta) = (\alpha' - \beta') \circ \lambda_0$, we have

$$(\alpha' \circ \lambda_0)_{j'i} - (\beta' \circ \lambda_0)_{j'i} = \lambda^1_{j'1} \cdot \alpha_{1i} - \lambda^1_{j'1} \cdot \beta_{1i}.$$

Note that for any $i \in \{1, 2, \dots, p\}$, from the definition of $\mathcal{C}_{\mathcal{O}}$, at least one of α_{1i} and β_{1i} is 0. For this case we automatically have

$$(\alpha' \circ \lambda_0)_{j'i} \geqslant \sum_{\alpha_{ji} \cdot \lambda^1_{j'j} \geqslant 0} \alpha_{ji} \cdot \lambda^1_{j'j} - \sum_{\beta_{ji} \cdot \lambda^1_{j'j} \leqslant 0} \beta_{ji} \cdot \lambda^1_{j'j} \quad \text{and}$$

$$(\beta' \circ \lambda_0)_{j'i} \geqslant - \sum_{\alpha_{ji} \cdot \lambda^1_{j'j} \leqslant 0} \alpha_{ji} \cdot \lambda^1_{j'j} + \sum_{\beta_{ji} \cdot \lambda^1_{j'j} \geqslant 0} \beta_{ji} \cdot \lambda^1_{j'j}.$$

(At most one of the four numbers

$$\sum_{\alpha_{ji} \cdot \lambda^1_{j'j} \geqslant 0} \alpha_{ji} \cdot \lambda^1_{j'j}, \quad \sum_{\beta_{ji} \cdot \lambda^1_{j'j} \leqslant 0} \beta_{ji} \cdot \lambda^1_{j'j}, \quad \sum_{\alpha_{ji} \cdot \lambda^1_{j'j} \leqslant 0} \alpha_{ji} \cdot \lambda^1_{j'j} \quad \text{and} \quad \sum_{\beta_{ji} \cdot \lambda^1_{j'j} \geqslant 0} \beta_{ji} \cdot \lambda^1_{j'j}$$

is non-zero.) By Lemma 3.6, there exists a homomorphism φ from A to $M_r(B)$ for some integer r inducing the diagram λ, which also realizes the KK-element α. ∎

We should note that the condition of Lemma 3.6 is just a sufficient condition. (But it is interesting as it shows that a given KK-element, if it induces a large enough map between the K_0-groups compared with the K_1 map, in a suitable sense, can be lifted.) Even if we required that A should be just $C(S^1)$, some liftable KK-elements do not satisfy this sufficient condition.

THEOREM 3.9. *Consider the case $A = C(S^1)$ (not in $\mathcal{C}_{\mathcal{O}}$), $B \in \mathcal{C}$ is minimal, $F_2' = M_n(\mathbb{C})$, for some integer n. Then $\alpha \in \mathrm{KK}(C(S^1), B)$ can be lifted to a homomorphism if and only if $\chi(\alpha) \in C(A, B)/M(A, B)$ is positive.*

Proof. As we did in Theorem 3.8, choose a positive commutative diagram $\lambda \in \chi(\alpha)$,

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1-1=0} & \mathbb{Z} \\ {\scriptstyle \lambda_0}\downarrow & & \downarrow{\scriptstyle \lambda_1} \\ \mathrm{K}_0(F_1) & \xrightarrow[\alpha' - \beta']{} & \mathbb{Z}, \end{array}$$

with λ_0 a positive map. Note that $\lambda_1 \in \mathbb{Z}$, $\alpha' \circ \lambda_0 = \beta' \circ \lambda_0 \in \mathbb{Z}$.

If ϕ is a homomorphism from $C(S^1)$ to $M_r(M_n(C[0,1]))$, for some integer r,

$$z \overset{\phi}{\longmapsto} Z,$$

where

$$z(\mathrm{e}^{2\pi \mathrm{i}\theta}) = \mathrm{e}^{2\pi \mathrm{i}\theta} \quad \text{and} \quad Z(t) = \mathrm{e}^{2\lambda_1 \pi \mathrm{i} t} \oplus \mathrm{diag}\{\underbrace{1, 1, \dots, 1}_{\alpha' \circ \lambda_0 - 1}\},$$

then by Lemma 2.5, $\mathrm{Ad}u \circ \phi$ is a homomorphism from $C(S^1)$ to $M_n(B)$, for some unitary u. Evidently, $\mathrm{Ad}u \circ \phi$ induces the commutative diagram λ. ∎

Let us look at some examples.

REMARK 3.10. Consider the case of Example 2.14. From Theorem 3.9, we get

$$\mathrm{KK}^+(C(S^1), C(S^1)) \cong \{(m,n) \in \mathbb{Z} \oplus \mathbb{Z} : (m,n) = (0,0), \text{ or } m > 0\}.$$

REMARK 3.11. Note that in Theorem 3.9, we required that $F_2' = M_n(\mathbb{C})$, for some integer n, besides that B is minimal. Let us consider the following example: $A = C(S^1)$, $F_1' = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, $F_2' = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ (at this time, F_2' has two blocks rather than one, which we required in Theorem 3.9),

$$\varphi_0'(a \oplus b \oplus c) = \begin{pmatrix} a & \\ & b \end{pmatrix} \oplus \begin{pmatrix} a & \\ & c \end{pmatrix}, \quad \text{and} \quad \varphi_1'(a \oplus b \oplus c) = \begin{pmatrix} a & \\ & b \end{pmatrix} \oplus \begin{pmatrix} a & \\ & c \end{pmatrix},$$

and $B = B(F_1', F_2', \varphi_0', \varphi_1') \in \mathcal{C}$. Then

$$\alpha' = \beta' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Consider the commutative diagram $\lambda \in C(A, B)$,

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1-1=0} & \mathbb{Z} \\ {\scriptstyle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \big\downarrow & & \big\downarrow {\scriptstyle \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[\alpha' - \beta' = 0]{} & \mathbb{Z} \oplus \mathbb{Z}. \end{array}$$

Note that $M(A, B) = \{0\}$, B is minimal, the diagram λ is positive. But the KK-element $\mathrm{KK}(\lambda)$ cannot be lifted. (From Remark 2.2, if the KK-element $\mathrm{KK}(\lambda)$ can be lifted, there will be a homomorphism inducing λ, which is contradictory to Corollary 3.7.)

We should also mention that $\mathrm{KK}(\lambda_{(1,1)}) \in \mathrm{KK}^+(C(S^1), C(S^1))$ (see Remark 3.10), but $\mathrm{KK}(\lambda_{(1,1)}) \times \mathrm{KK}(\lambda) \notin \mathrm{KK}^+(C(S^1), B)$. ($\mathrm{KK}(\lambda_{(1,1)}) \times \mathrm{KK}(\lambda)$ still corresponds the class of the diagram $\lambda \in C(C(S^1), B)$

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1-1=0} & \mathbb{Z} \\ {\scriptstyle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \big\downarrow & & \big\downarrow {\scriptstyle \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[\alpha' - \beta' = 0]{} & \mathbb{Z} \oplus \mathbb{Z}, \end{array}$$

which cannot be lifted as we have just shown.) Then we have

$$\mathrm{KK}(\lambda)(\mathrm{K}_*^+(A)) \nsubseteq \mathrm{K}_*^+(B),$$

which says that $\mathrm{KK}(\lambda)$ does not respect the Dădârlat–Loring order.

It is not hard to see that Theorem 2.9 and Theorem 3.8 also hold in the non-unital case. In particular, the following result of Dădârlat and Loring (see [2]) can be proved (in a new way).

THEOREM 3.12. *Let $A \in \mathcal{C}$. There is a natural isomorphism of groups*

$$\mathrm{KK}(I_p, A) \cong [I_p, A \otimes \mathscr{K}]$$

Proof. For a commutative diagram $\lambda \in C(I_p, A)$,

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\;p\;} & \mathbb{Z} \\ {\scriptstyle \lambda_0}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_1} \\ \mathrm{K}_0(F_1) & \xrightarrow[\alpha-\beta]{} & \mathrm{K}_1(SF_2), \end{array}$$

denote by $\lambda_\mu \in M(I_p, A)$ the diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\;p\;} & \mathbb{Z} \\ {\scriptstyle p(|\lambda_0|+I)}\big\downarrow & & \big\downarrow{\scriptstyle (\alpha-\beta)(|\lambda_0|+I)} \\ \mathrm{K}_0(F_1) & \xrightarrow[\alpha-\beta]{} & \mathrm{K}_1(SF_2), \end{array}$$

where $|\lambda_0| = (|\lambda^0_{11}|, |\lambda^0_{12}|, \ldots, |\lambda^0_{1p}|)^{\mathrm{T}}$, $I = (1, 1, \ldots, 1)^{\mathrm{T}}$. Then $\lambda_0 + p(|\lambda_0| + I)$ is a positive map, which means that λ is positive modulo $M(I_p, A)$. From Theorem 3.8, we have a homomorphism inducing λ. ∎

We shall now give a (new) proof of the criterion for KK-lifting of Jiang and Su. (In fact, a generalization of Theorem 3.2 above.)

THEOREM 3.13. *Let $A, B \in \mathcal{C}$, and suppose that both $\mathrm{K}_1(A)$ and $\mathrm{K}_1(B)$ are torsion groups. Then there are natural group isomorphisms*

$$\mathrm{KK}(A, B) \cong \mathrm{Hom}(\mathrm{K}^0(B), \mathrm{K}^0(A)) \cong C(A, B)/M(A, B).$$

Furthermore, if $A \in \mathcal{C}_{\mathcal{O}}$, then the isomorphisms respect the order structures.

Proof. For $A, B \in \mathcal{C}$, a commutative diagram $\lambda \in C(A, B)$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & {\scriptstyle \lambda_{0*}}\big\downarrow & & {\scriptstyle \lambda_0}\big\downarrow & & {\scriptstyle \lambda_1}\big\downarrow & & {\scriptstyle \lambda_{1*}}\big\downarrow & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0, \end{array}$$

is dual to

$$\begin{array}{ccccccc} 0 \longleftarrow & \mathrm{K}^0(A) & \xleftarrow{\pi^*} & \mathrm{K}^0(F_1) & \xleftarrow{\alpha^{\mathrm{T}}-\beta^{\mathrm{T}}} & \mathrm{K}^1(SF_2) & \xleftarrow{\iota^*} & \mathrm{K}^1(A)(=0) \\ & \big\uparrow{\scriptstyle \lambda^{0*}} & & \big\uparrow{\scriptstyle \lambda_0^{\mathrm{T}}} & & \big\uparrow{\scriptstyle \lambda_1^{\mathrm{T}}} & & \big\uparrow{\scriptstyle \lambda^{1*}} \\ 0 \longleftarrow & \mathrm{K}^0(B) & \xleftarrow[\pi'^*]{} & \mathrm{K}^0(F'_1) & \xleftarrow[\alpha'^{\mathrm{T}}-\beta'^{\mathrm{T}}]{} & \mathrm{K}^1(SF'_2) & \xleftarrow[\iota'^*]{} & \mathrm{K}^1(B)(=0). \end{array}$$

If $\mathrm{K}_1(A)$ and $\mathrm{K}_1(B)$ are both torsion groups, we have $\mathrm{rank}(\alpha - \beta) = p$ and $\mathrm{rank}(\alpha' - \beta') = p'$. Then $\alpha^{\mathrm{T}} - \beta^{\mathrm{T}}$, ${\alpha'}^{\mathrm{T}} - \beta' T$ are injections. So we may regard $\mathrm{K}^1(SF_2), \mathrm{K}^1(SF_2')$ as subgroups of $\mathrm{K}^0(F_1), \mathrm{K}^0(F_1')$, respectively, and λ_1^{T} as the restriction of λ_0^{T}.

Let $\mu \in \mathrm{Hom}(\mathrm{K}_1(SF_2), \mathrm{K}_0(F_1'))$; then the element $\lambda_\mu \in M(A, B)$,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F_1') & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF_2') & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0,
\end{array}$$

is dual to

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathrm{K}^0(A) & \xleftarrow{\pi^*} & \mathrm{K}^0(F_1) & \xleftarrow{\alpha^{\mathrm{T}}-\beta^{\mathrm{T}}} & \mathrm{K}^1(SF_2) & \xleftarrow{\iota^*} & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \mu_0^{\mathrm{T}}} & & \uparrow{\scriptstyle \mu_1^{\mathrm{T}}} & & \uparrow{\scriptstyle 0} \\
0 & \longleftarrow & \mathrm{K}^0(B) & \xleftarrow[{\pi'}^*]{} & \mathrm{K}^0(F_1') & \xleftarrow[{\alpha'}^{\mathrm{T}}-{\beta'}^{\mathrm{T}}]{} & \mathrm{K}^1(SF_2') & \xleftarrow[{\iota'}^*]{} & 0,
\end{array}$$

where $\mu_1^{\mathrm{T}} = \mu^{\mathrm{T}} \circ ({\alpha'}^{\mathrm{T}} - {\beta'}^{\mathrm{T}})$ and $\mu_0^{\mathrm{T}} = (\alpha^{\mathrm{T}} - \beta^{\mathrm{T}}) \circ \mu^{\mathrm{T}}$.

Define a map $\mathfrak{H} : C(A, B) \to \mathrm{Hom}(\mathrm{K}^0(B), \mathrm{K}^0(A))$ by

$$\lambda \overset{\mathfrak{H}}{\longmapsto} \lambda^{0*}.$$

Obviously, $\mathfrak{H}$ is a homomorphism. Firstly note that $\mathrm{K}^0(F_1), \mathrm{K}^1(SF_2), \mathrm{K}^0(F_1')$ and $\mathrm{K}^1(SF_2')$ are free abelian groups, which have the projective property, so that for any $\theta : \mathrm{K}^0(B) \to \mathrm{K}^0(A)$, we can construct a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathrm{K}^0(A) & \xleftarrow{\pi^*} & \mathrm{K}^0(F_1) & \xleftarrow{\alpha^{\mathrm{T}}-\beta^{\mathrm{T}}} & \mathrm{K}^1(SF_2) & \xleftarrow{\iota^*} & 0 \\
 & & \uparrow{\scriptstyle \theta} & & \uparrow{\scriptstyle \theta^{0*}} & & \uparrow{\scriptstyle \theta^{1*}} & & \uparrow{\scriptstyle 0} \\
0 & \longleftarrow & \mathrm{K}^0(B) & \xleftarrow[{\pi'}^*]{} & \mathrm{K}^0(F_1') & \xleftarrow[{\alpha'}^{\mathrm{T}}-{\beta'}^{\mathrm{T}}]{} & \mathrm{K}^1(SF_2') & \xleftarrow[{\iota'}^*]{} & 0.
\end{array}$$

This shows that $\mathfrak{H}$ is surjective. Secondly, we see that, for any commutative diagram

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathrm{K}^0(A) & \xleftarrow{\pi^*} & \mathrm{K}^0(F_1) & \xleftarrow{\alpha^{\mathrm{T}}-\beta^{\mathrm{T}}} & \mathrm{K}^1(SF_2) & \xleftarrow{\iota^*} & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \mu_0^{\mathrm{T}}} & & \uparrow{\scriptstyle \mu_1^{\mathrm{T}}} & & \uparrow{\scriptstyle 0} \\
0 & \longleftarrow & \mathrm{K}^0(B) & \xleftarrow[{\pi'}^*]{} & \mathrm{K}^0(F_1') & \xleftarrow[{\alpha'}^{\mathrm{T}}-{\beta'}^{\mathrm{T}}]{} & \mathrm{K}^1(SF_2') & \xleftarrow[{\iota'}^*]{} & 0,
\end{array}$$

there exists a map μ from $\mathrm{K}_1(SF_2)$ to $\mathrm{K}_0(F_1')$ such that

$$\mu_1^{\mathrm{T}} = \mu^{\mathrm{T}} \circ (\alpha'^{\mathrm{T}} - \beta'^{\mathrm{T}}), \quad \mu_0^{\mathrm{T}} = (\alpha^{\mathrm{T}} - \beta^{\mathrm{T}}) \circ \mu^{\mathrm{T}},$$

which shows that $\ker \mathfrak{H} = M(A,B)$.

Thus, $\mathrm{Hom}(\mathrm{K}^0(B), \mathrm{K}^0(A))$ and the quotient $C(A,B)/M(A,B)$ are isomorphic as groups — in fact, inspection of the proof shows as ordered groups.

It follows by Theorem 2.9 that, as groups, with respect to the natural map,

$$\mathrm{KK}(A,B) \cong \mathrm{Hom}(\mathrm{K}^0(B), \mathrm{K}^0(A)).$$

If, in addition, $A \in \mathcal{C}_{\mathcal{O}}$, then by Theorem 3.8, the isomorphism respects the order structures. ∎

4. THE VARIOUS ORDER STRUCTURES ON THE KK-GROUP FOR ELLIOTT–THOMSEN ALGEBRAS

Using the criterion for KK-lifting given in the previous section (Theorem 3.8 and Theorem 3.9), we shall now consider in addition the Dădârlat–Loring order structure on the KK-group for $A, B \in \mathcal{C}$. We shall show that there is an existence theorem for $A \in \mathcal{C}_{\mathcal{O}}$, $B \in \mathcal{C}$. But this theorem does not hold for certain special cases $A, B \in \mathcal{C}$, even if we assume that $F_2 = M_r(\mathbb{C})$ (recall that in Paragraph 1.7, for A to be in $\mathcal{C}_{\mathcal{O}}$ we required that both $F_2 = M_r(\mathbb{C})$ and $\ker \varphi_0 \oplus \ker \varphi_1 = F_1$). One will see that the tool we gave (in Theorem 3.8) makes the checking of KK-lifting easier, especially when we need to compose KK-elements. First, let us consider the class of Paragraph 1.6.

THEOREM 4.1. *Let $A = I[m_0, m, m_1]$, $B = I[n_0, n, n_1]$. If α is a* KK*-element in* $\mathrm{KK}(A,B)$ *satisfying*

$$\alpha(\underline{\mathrm{K}}^+(A)) \subset \underline{\mathrm{K}}^+(B),$$

then α can be lifted to a homomorphism.

Proof. By Theorem 2.9, there exists a commutative diagram $\lambda \in C(A,B)$,

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(m/m_0, -m/m_1)} & \mathbb{Z} \\ {\scriptstyle \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle s} \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[(n/n_0, -n/n_1)]{} & \mathbb{Z}, \end{array}$$

such that $\lambda + M(A,B) = \chi(\alpha)$. From

$$\alpha(\underline{\mathrm{K}}^+(A)) \subset \underline{\mathrm{K}}^+(B),$$

it follows from Section 3 in [3] that

$$\alpha(\mathrm{KK}^+(\widetilde{I}_m, A)) \subset \mathrm{KK}^+(\widetilde{I}_m, B).$$

Let

$$\iota : \widetilde{I}_m \to A$$

denote the natural embedding. Then the homomorphism ι induces the following diagram $\lambda_\iota \in C(\widetilde{I}_m, A)$:

$$\begin{CD} \mathbb{Z}\oplus\mathbb{Z} @>{(m,-m)}>> \mathbb{Z} \\ @V{\begin{pmatrix} m_0 & \\ & m_1 \end{pmatrix}}VV @VV{1}V \\ \mathbb{Z}\oplus\mathbb{Z} @>>{(m/m_0,-m/m_1)}> \mathbb{Z}. \end{CD}$$

Also, we have

$$\alpha([\iota]) \in \mathrm{KK}^+(\widetilde{I}_m, B).$$

Note that $\alpha([\iota])$ coincides with the class of the diagram $\lambda_{\alpha([\iota])} \in C(\widetilde{I}_m, B)$:

$$\begin{CD} \mathbb{Z}\oplus\mathbb{Z} @>{(m,-m)}>> \mathbb{Z} \\ @V{\begin{pmatrix} am_0 & bm_1 \\ cm_0 & dm_1 \end{pmatrix}}VV @VV{s}V \\ \mathbb{Z}\oplus\mathbb{Z} @>>{(n/n_0,-n/n_1)}> \mathbb{Z}. \end{CD}$$

From Theorem 3.8, the rest of the proof is divided into two cases.

Case 1. Suppose that $\alpha([\iota])$ is lifted as the zero map from $\widetilde{I}_m$ to B. In this case the diagram $\lambda_{\alpha([\iota])}$,

$$\begin{CD} \mathbb{Z}\oplus\mathbb{Z} @>{(m,-m)}>> \mathbb{Z} \\ @V{\begin{pmatrix} am_0 & bm_1 \\ cm_0 & dm_1 \end{pmatrix}}VV @VV{s}V \\ \mathbb{Z}\oplus\mathbb{Z} @>>{(n/n_0,-n/n_1)}> \mathbb{Z}, \end{CD}$$

belongs to $M(\widetilde{I}_m, B)$. That is, there exist $\mu_1, \mu_2 \in \mathbb{Z}$ such that

$$am_0 = \mu_1 m, \quad bm_1 = -\mu_1 m, \quad cm_0 = \mu_2 m, \quad dm_1 = -\mu_2 m, \quad s = \frac{\mu_1 n}{n_0} - \frac{\mu_2 n}{n_1}.$$

Then, we have

$$a = \frac{\mu_1 m}{m_0}, \quad b = -\frac{\mu_1 m}{m_1}, \quad c = \frac{\mu_2 m}{m_0}, \quad d = -\frac{\mu_2 m}{m_1}, \quad s = \frac{\mu_1 n}{n_0} - \frac{\mu_2 n}{n_1}.$$

Consequently, the diagram λ, given by

$$\begin{CD} \mathbb{Z}\oplus\mathbb{Z} @>{(m/m_0,-m/m_1)}>> \mathbb{Z} \\ @V{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}VV @VV{s}V \\ \mathbb{Z}\oplus\mathbb{Z} @>>{(n/n_0,-n/n_1)}> \mathbb{Z}, \end{CD}$$

belongs to $M(A,B)$. So we have lifted α as the zero map from A to B.

Case 2. Suppose that $\alpha([\iota])$ is lifted as the non-zero homomorphism from $\widetilde{I}_m$ to B. In this case, the diagram $\lambda_{\alpha([\iota])}$,

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(m,-m)} & \mathbb{Z} \\ {\scriptstyle \begin{pmatrix} am_0 & bm_1 \\ cm_0 & dm_1 \end{pmatrix}}\downarrow & & \downarrow{\scriptstyle s} \\ \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[(n/n_0,-n/n_1)]{} & \mathbb{Z}, \end{array}$$

is positive modulo $M(\widetilde{I}_m, I[n_0,n,n_1])$; that is, there exist $\mu_1,\mu_2\in\mathbb{Z}$ such that

$$am_0+\mu_1 m\geqslant 0,\quad bm_1-\mu_1 m\geqslant 0,\quad cm_0+\mu_2 m\geqslant 0,\quad dm_1-\mu_2 m\geqslant 0.$$

Then, we have

$$a+\frac{\mu_1 m}{m_0}\geqslant 0,\quad b-\frac{\mu_1 m}{m_1}\geqslant 0,\quad c+\frac{\mu_2 m}{m_0}\geqslant 0,\quad d-\frac{\mu_2 m}{m_1}\geqslant 0.$$

Note that

$$\begin{pmatrix} a+\mu_1 m/m_0 & b-\mu_1 m/m_1 \\ c+\mu_1 m/m_0 & d-\mu_2 m/m_1 \end{pmatrix}\neq 0.$$

That is, λ is positive modulo $M(I[m_0,m,m_1], I[n_0,n,n_1])$. So from Theorem 3.8, we have α lifted as a homomorphism from A to $M_r(B)$ for some integer r. ∎

Now we can generalize Theorem 4.1 as follows.

THEOREM 4.2. *Let $A\in\mathcal{C}_{\mathcal{O}}$ and $B\in\mathcal{C}$. Then a* KK*-element $\gamma\in\mathrm{KK}(A,B)$ can be lifted to a homomorphism if and only if*

$$\gamma(\underline{\mathrm{K}}^+(A))\subset\underline{\mathrm{K}}^+(B).$$

Proof. Here we only need to prove the sufficiency. Suppose that

$$F_1 = M_{0_1}\oplus M_{0_2}\oplus\cdots\oplus M_{0_s}\oplus M_{1_1}\oplus M_{1_2}\oplus\cdots\oplus M_{1_t},$$

where $0_i, 1_j$ are positive integers and

$$\ker\varphi_1 = M_{0_1}\oplus M_{0_2}\oplus\cdots\oplus M_{0_s},\quad \ker\varphi_0 = M_{1_1}\oplus M_{1_2}\oplus\cdots\oplus M_{1_t}\quad s+t=p.$$

Write

$$\alpha=(\alpha_{11},\ldots,\alpha_{1s},0,\ldots,0)\quad\text{and}\quad \beta=(0,\ldots,0,\beta_{1(s+1)},\ldots,\beta_{1(s+t)}).$$

Then for any $x\in\{1,2,\ldots,s\}$ and $y\in\{1,2,\ldots,t\}$, let $m_{xy}=\alpha_{1x}\cdot\beta_{1(s+y)}\cdot 0_x\cdot 1_y\in\mathbb{N}$. By Theorem 3.8, there exists a homomorphism η_{xy} from $\widetilde{I}_{m_{xy}}$ to $M_r(A)$

inducing the following commutative diagram $\lambda_{xy} \in C(\widetilde{I}_{m_{xy}}, A)$:

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(m_{xy}, -m_{xy})} & \mathbb{Z} \\ \lambda_0^{xy} \downarrow & & \downarrow 1 \\ \mathbb{Z}^p & \xrightarrow[\alpha-\beta]{} & \mathbb{Z} \end{array}$$

where

$$\lambda_0^{xy} = \begin{pmatrix} 0 & \vdots \\ m_{xy}/\alpha_{1x} & 0 \\ 0 & m_{xy}/\beta_{1(s+y)} \\ \vdots & 0 \end{pmatrix} \tag{4.1}$$

(the $p \times 2$ matrix with all entry 0 except for $(x, 1)^{\text{th}}$ entry m_{xy}/α_{1x} and $(s + y, 2)^{\text{th}}$ entry $m_{xy}/\beta_{1(s+y)}$). By Theorem 2.9 there is a commutative diagram $\lambda \in C(A, B)$:

$$\begin{array}{ccc} \mathbb{Z}^p & \xrightarrow{(m_{xy}, -m_{xy})} & \mathbb{Z} \\ (a_{i'i})_{p' \times p} \downarrow & & \downarrow S \\ \mathbb{Z}^{p'} & \xrightarrow[\alpha'-\beta']{} & \mathbb{Z}^{l'} \end{array}$$

such that $\lambda + M(A, B) = \chi(\gamma)$. Since

$$\gamma(\underline{\mathrm{K}}^+(A)) \subset \underline{\mathrm{K}}^+(B),$$

we have

$$\gamma(\mathrm{KK}^+(\widetilde{I}_{m_{xy}}, A)) \subset \mathrm{KK}^+(\widetilde{I}_{m_{xy}}, B).$$

It follows that

$$\gamma([\eta_{xy}]) \in \mathrm{KK}^+(\widetilde{I}_{m_{xy}}, B).$$

$\gamma([\eta_{xy}])$ coincides with the class of the following diagram $\lambda_{\gamma([\iota_{xy}])} \in C(\widetilde{I}_{m_{xy}}, B)$:

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(m_{xy}, -m_{xy})} & \mathbb{Z} \\ (a_{i'i})_{p' \times p}\lambda_0^{xy} \downarrow & & \downarrow S \\ \mathbb{Z}^{p'} & \xrightarrow[\alpha'-\beta']{} & \mathbb{Z}^{l'}. \end{array}$$

By Theorem 3.8, there exists $\mu_{xy} = (\mu_{xy}^1, \mu_{xy}^2, \ldots, \mu_{xy}^{p'})^{\mathrm{T}} \in \mathrm{Hom}(\mathrm{K}_1(SF_2), \mathrm{K}_0(F_1'))$ such that the map

$$(a_{i'i})_{p' \times p}\lambda_0^{xy} + \mu_{xy}(m_{xy}, -m_{xy})$$

is positive, i.e., for any $i' \in \{1, 2, \ldots, p'\}$,

$$\frac{a_{i'x} m_{xy}}{\alpha_{1x}} + \mu^{i'}_{xy} m_{xy} \geqslant 0, \quad \frac{a_{i'(s+y)} m_{xy}}{\beta_{1(s+y)}} - \mu^{i'}_{xy} m_{xy} \geqslant 0.$$

Then for any $i' \in \{1, 2, \ldots, p'\}$, we have

$$a_{i'x} + \mu^{i'}_{xy} \alpha_{1x} \geqslant 0, \quad a_{i's+y} - \mu^{i'}_{xy} \beta_{1(s+y)} \geqslant 0.$$

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_{p'})^{\mathrm{T}}$, where

$$\mu_{i'} = \min_y \max_x \mu^{i'}_{xy}, \quad i' \in 1, 2, \ldots, p'.$$

Then for any $i' \in \{1, 2, \ldots, p'\}$, we have

$$a_{i'x} + \mu_{i'} \alpha_{1x} \geqslant 0, \quad a_{i'(s+y)} - \mu_{i'} \beta_{1(s+y)} \geqslant 0,$$

in other words, the map

$$(a_{i'i})_{p' \times p} + \mu(\alpha - \beta)$$

is positive.

Let $\widetilde{\lambda} \in C(A, B)$ denote the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{\lambda}_{0*}} & & \downarrow{\scriptstyle \widetilde{\lambda}_0} & & \downarrow{\scriptstyle \widetilde{\lambda}_1} & & \downarrow{\scriptstyle \widetilde{\lambda}_{1*}} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0, \end{array}$$

where $\widetilde{\lambda}_0 = (0, 0, \ldots, 0)^{\mathrm{T}}$. Note that

$$\mathrm{K}_1(SF_2) = \mathbb{Z} \quad \text{and} \quad \alpha - \beta = (\alpha_{11}, \ldots, \alpha_{1s}, -\beta_{1(s+1)}, \ldots, -\beta_{1(s+t)}).$$

Immediately we will have $\widetilde{\lambda}_1 = (0, 0, \ldots, 0)^{\mathrm{T}}$.

In summary, $\chi(\gamma)$ is positive. It follows from Theorem 3.8 that $\chi(\gamma)$ can be lifted as a homomorphism. ∎

COROLLARY 4.3. *Assume that $A, B \in \mathcal{C}$ are minimal and $F_2 = M_n(\mathbb{C})$. If γ is a* KK-*element in* $\mathrm{KK}(A, B)$ *satisfying*

$$\gamma(\underline{\mathrm{K}}^+(A)) \subset \underline{\mathrm{K}}^+(B),$$

then $\chi(\gamma)$ is positive.

Proof. As $F_2 = M_n(\mathbb{C})$, $\alpha - \beta$ is a $1 \times p$ matrix. Transform $\alpha - \beta$ into

$$(p_1, p_2, \ldots, p_\triangle, n_1, n_2, \ldots, n_\triangledown, 0, 0, \ldots, 0)$$

where $\triangle$ and $\triangledown$ are integers, and

$$p_k > 0, \quad k \in \{1, 2, \ldots, \triangle\} \quad \text{and} \quad n_m < 0, \quad m \in \{1, 2, \ldots, \triangledown\}.$$

For p_k, n_m, we do the same thing as what we did in the proof of Theorem 4.2 and for the 0 in $(p_1, p_2, \ldots, p_\triangle, n_1, n_2, \ldots, n_\triangledown, 0, 0, \ldots, 0)$, we construct a map from $C(S^1)$ to A. Then we have that $\chi(\gamma)$ is positive. ∎

The following theorem is a corollary of the universal coefficient theorem proved in [1].

THEOREM 4.4. *Let A, B be C^*-algebras. Suppose that $A \in \mathcal{N}$ (where $\mathcal{N}$ is the "bootstrap" category defined in* [17]*), $\mathrm{K}_*(A)$ is finitely generated, and B is σ-unital. Then the natural map*

$$\Gamma : \mathrm{KK}(A, B) \to \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$$

is a group isomorphism.

As $\mathcal{C} \subset \mathcal{N}$, Theorem 4.2 can be expressed as the following theorem.

THEOREM 4.5. *Assume that $A \in \mathcal{C}_\mathcal{O}$ and $B \in \mathcal{C}$. Then the natural map*

$$\Gamma : \mathrm{KK}(A, B) \to \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$$

is an order isomorphism. Here $\mathrm{Hom}^+_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$ is the cone consisting of the classes which preserves the Dădârlat–Loring order.

REMARK 4.6. Here we should point out that Theorem 3.13 and Theorem 4.5 still hold if we allow A to be the direct sum of blocks in $\mathcal{C}_\mathcal{O}$; Corollary 4.3 still holds if A is the direct sum of some minimal blocks in the class of blocks $A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ with $F_2 = M_n(\mathbb{C})$.

Note that for any $A \in \mathcal{C}, \mathrm{K}_0(A)$ is torsion free. It follows that for any C^*-algebras $A, B \in \mathcal{C}$, the requirement

$$\alpha(\underline{\mathrm{K}}^+(A)) \subset \underline{\mathrm{K}}^+(B)$$

is equivalent to

$$\alpha(\mathrm{K}_0^+(A; \mathbb{Z} \oplus \mathbb{Z}_p)) \subset \mathrm{K}_0^+(B; \mathbb{Z} \oplus \mathbb{Z}_p), \quad p \geqslant 2, \quad \text{and} \quad \alpha(\mathrm{K}_*^+(A)) \subset \mathrm{K}_*^+(B).$$

If we further assume that

$$\mathrm{K}_1(A) = \mathrm{Torsion}(\mathrm{K}_1(A)), \quad \text{and} \quad \mathrm{K}_1(B) = \mathrm{Torsion}(\mathrm{K}_1(B)),$$

then the inclusion

$$\alpha(\underline{\mathrm{K}}^+(A)) \subset \underline{\mathrm{K}}^+(B)$$

is equivalent to the inclusion

$$\alpha(\mathrm{K}_0^+(A; \mathbb{Z} \oplus \mathbb{Z}_p)) \subset \mathrm{K}_0^+(B; \mathbb{Z} \oplus \mathbb{Z}_p).$$

Note that the C^*-algebras of the example in [10] are in the class of Theorem 4.1 and therefore cannot serve the purpose intended in Theorem 1.1 of [10]. The intuition of the authors is, still, quite correct. We present a new example here, in the setting of the general class of $\mathcal{C}$, in which the lifting is indeed not possible.

EXAMPLE 4.7. Let $F_1 = \mathbb{C} \oplus \mathbb{C}$, $F_2 = M_4(\mathbb{C})$,

$$\varphi_0(a \oplus b) = \mathrm{diag}\{a, a, a, a\}, \quad \varphi_1(a \oplus b) = \mathrm{diag}\{a, a, b, b\},$$

$B = \widetilde{I}_2$, and $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$.
Consider the following commutative diagram $\lambda \in C(A, \widetilde{I}_2)$:

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(2,-2)} & \mathbb{Z} \\ {\scriptstyle \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle 1} \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[(2,-2)]{} & \mathbb{Z}. \end{array}$$

With Theorem 3.8, it can be easily checked that the related KK-element preserves the Dădârlat–Loring order of K-theory with coefficients. Note that λ is the unique element in $\lambda + M(A, \widetilde{I}_2)$ with a positive map λ_0 from $K_0(F_1)$ to $K_0(F_1')$. Then with Remark 2.2 and Corollary 3.7, we get that the KK-element in question cannot be lifted. (This means that the existence theorem does not hold directly even for some special cases of $A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ with $F_2 = M_n(\mathbb{C})$.)

In general, Theorem 4.5 and Theorem 3.8 tell us, when $A \in \mathcal{C}_{\mathcal{O}}$, that the three ordered groups $\mathrm{KK}(A, B)$, $C(A, B)/M(A, B)$ and $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$ are isomorphic. And if we require the weaker condition that A have only one generic block, i.e., that $F_2 = M_n(\mathbb{C})$, then by Theorem 2.9 and Theorem 4.4, we still have (as groups, with the natural maps)

$$\mathrm{KK}(A, B) \cong C(A, B)/M(A, B) \cong \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)),$$

and by Corollary 4.3 we have

$$\mathrm{KK}^+(A, B) \subset \mathrm{Hom}_\Lambda^+(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)) \subset (C(A, B)/M(A, B))^+.$$

In the case of Example 4.7, $\mathrm{KK}^+(A, B) \subsetneqq \mathrm{Hom}_\Lambda^+(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$. And in the case of Remark 3.11, $\mathrm{Hom}_\Lambda^+(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)) \subsetneqq (C(A, B)/M(A, B))^+$.

REMARK 4.8. What needs to be pointed out that both of the order structures coinciding with $\mathrm{Hom}_\Lambda^+(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$ and $(C(A, B)/M(A, B))^+$ are much stronger than the order structure on $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$ used in [16], where an element is positive if and only if its restriction to the K_0 part is positive. The order structure in [16] in general is not sufficient for KK-lifting, even for dimension drop algebras. We show the following example.

Let $A = \widetilde{I}_3$ and $B = \mathbb{C}$. Consider the diagram $\lambda \in C(\widetilde{I}_3, \mathbb{C})$

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(3,-3)} & \mathbb{Z} \\ {\scriptstyle (2,-1)} \downarrow & & \downarrow \\ \mathbb{Z} & \xrightarrow{\qquad} & 0. \end{array}$$

One can easily check that $\mathrm{KK}(\lambda)(\mathrm{K}_0^+(\widetilde{I}_3)\backslash\{0\}) \subset \mathrm{K}_0^+(\mathbb{C})\backslash\{0\}$, while it can not be lifted.

At last, we should point out that the condition

$$F_2 = M_n(\mathbb{C})$$

on $A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ in Corollary 4.3 is necessary. Consider the following example with A minimal and $F_2 = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$.

EXAMPLE 4.9. Let $F_1 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, $F_2 = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$,

$\varphi_0(a \oplus b \oplus c) = \mathrm{diag}\{a, a\} \oplus \mathrm{diag}\{b, b\}$, $\quad \varphi_1(a \oplus b \oplus c) = \mathrm{diag}\{b, b\} \oplus \mathrm{diag}\{c, c\}$,

$B = \mathbb{C}$ and $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$. Then we have

$$\alpha = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Consider the following commutative diagram in $C(A, B)$:

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\alpha-\beta} & \mathbb{Z} \oplus \mathbb{Z} \\ {\scriptstyle (1,-1,1)}\downarrow & & \downarrow{\scriptstyle 0} \\ \mathbb{Z} & \xrightarrow[0]{} & 0. \end{array}$$

Denote the related KK-element by γ. Now let us check whether γ preserves the Dădârlat–Loring order.

Case 1. Let $\lambda \in C(C(S^1), A)$ be the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1-1=0} & \mathbb{Z} \\ {\scriptstyle \begin{pmatrix} x \\ y \\ z \end{pmatrix}}\downarrow & & \downarrow{\scriptstyle \begin{pmatrix} s \\ t \end{pmatrix}} \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[\alpha-\beta]{} & \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

where $x, y, z \geqslant 0$. Then from the commutativity of λ, we immediately get $x = y = z$. Further more, the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1-1=0} & \mathbb{Z} \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle 0} \\ \mathbb{Z} & \xrightarrow[0]{} & 0 \end{array}$$

corresponds to the class of $\mathrm{KK}(\lambda) \times \gamma$, which can be lifted.

This implies

$$\gamma(\mathrm{K}_*^+(A)) \subset \mathrm{K}_*^+(B).$$

Case 2. Let $\lambda \in C(\widetilde{I}_p, A)$ be the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(p,-p)} & \mathbb{Z} \\ {\scriptsize\begin{pmatrix} e & x\\ f & y\\ g & z\end{pmatrix}}\Big\downarrow & & \Big\downarrow{\scriptsize\begin{pmatrix} s\\ t\end{pmatrix}} \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[\alpha-\beta]{} & \mathbb{Z}\oplus\mathbb{Z}\end{array}$$

where $e, f, g, x, y, z \geqslant 0$. Then from the commutativity of λ, we have

$$e - f = y - x = \frac{sp}{2} =: \Delta \quad \text{and} \quad f - g = z - y = \frac{tp}{2} =: \Theta.$$

Note that the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(p,-p)} & \mathbb{Z} \\ {\scriptstyle (e-f+g,x-y+z)}\Big\downarrow & & \Big\downarrow{\scriptstyle 0} \\ \mathbb{Z} & \xrightarrow[0]{} & 0\end{array}$$

corresponds to the class of $\mathrm{KK}(\lambda) \times \gamma$.

(i) If t is even, the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(p,-p)} & \mathbb{Z} \\ {\scriptstyle (\Theta,-\Theta)}\Big\downarrow & & \Big\downarrow{\scriptstyle 0} \\ \mathbb{Z} & \xrightarrow[0]{} & 0\end{array}$$

is contained in $M(\widetilde{I}_p, B)$. As

$$e - f + g + \Theta = e \geqslant 0 \quad \text{and} \quad x - y + z - \Theta = x \geqslant 0,$$

we get that λ is positive modulo $M(\widetilde{I}_p, B)$. By Proposition 3.5, $\mathrm{KK}(\lambda) \times \gamma$ can be lifted to a representation.

(ii) If s is even, the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(p,-p)} & \mathbb{Z} \\ {\scriptstyle (-\Delta,\Delta)}\Big\downarrow & & \Big\downarrow{\scriptstyle 0} \\ \mathbb{Z} & \xrightarrow[0]{} & 0\end{array}$$

is contained in $M(\widetilde{I}_p, B)$. As

$$e - f + g - \Delta = g \geqslant 0 \quad \text{and} \quad x - y + z + \Delta = z \geqslant 0,$$

we get that λ is positive modulo $M(\widetilde{I}_p, B)$. By Proposition 3.5, $\mathrm{KK}(\lambda) \times \gamma$ can be lifted.

(iii) If both s and t are odd, the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(p,-p)} & \mathbb{Z} \\ {\scriptstyle(\Theta-\Delta,\Delta-\Theta)}\downarrow & & \downarrow{\scriptstyle 0} \\ \mathbb{Z} & \xrightarrow[0]{} & 0 \end{array}$$

is contained in $M(\widetilde{I}_p, B)$. As

$$e-f+g+\Theta-\Delta=f\geqslant 0 \quad \text{and} \quad x-y+z+\Delta-\Theta=y\geqslant 0,$$

we get that λ is positive modulo $M(\widetilde{I}_p, B)$. By Proposition 3.5, $\mathrm{KK}(\lambda)\times\gamma$ can be lifted.

From (i), (ii) and (iii), we have

$$\alpha(\mathrm{K}_0^+(A;\mathbb{Z}\oplus\mathbb{Z}_p))\subset \mathrm{K}_0^+(B;\mathbb{Z}\oplus\mathbb{Z}_p).$$

In summary, γ preserves the Dădârlat–Loring order, but $\chi(\gamma)$ is not positive (by definition) and so cannot be lifted (by Proposition 3.5). (γ does not preserve the order structure of K-homology, either. Note that this is different from the case in Corollary 4.3. This is the reason we consider the subclass $A\in\mathcal{C}$ with $F_2=M_n(\mathbb{C})$, and Example 4.7 and Theorem 4.5 have shown that, in this subclass, $\mathcal{C}_{\mathcal{O}}$ is even more special.)

Acknowledgements. The authors would like to thank Guihua Gong for helpful discussion.

The research of the second author was supported by the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

[1] M. DĂDÂRLAT, G. GONG, A classification result for approximately homogeneous C^*-algebras of real rank zero, *Geom. Funct. Anal.* **7**(1997), 646–711.

[2] M. DĂDÂRLAT, T.A. LORING, K-homology, asymptotic representations, and unsuspended E-theory, *J. Funct. Anal.* **126**(1994), 367–383.

[3] M. DĂDÂRLAT, T.A. LORING, A universal multicoefficient theorem for the Kasparov groups, *Duke Math. J.* **84**(1996), 355–377.

[4] M. DĂDÂRLAT, T.A. LORING, Classifying C^*-algebras via ordered mod-p K-theory, *Math. Ann.* **305**(1999), 601–616.

[5] S. EILERS, A complete invariant for AD algebras with bounded torsion in K_1, *J. Funct. Anal.* **139**(1991), 325–348.

[6] S. EILERS, T.A. LORING, G.K. PEDERSEN, Fragility of subhomogeneous C^*-algebras with one-dimensional spectrum, *Bull. London Math. Soc.* **31**(1999), 337–344.

[7] G.A. ELLIOTT, On the classification of C^*-algebras of real rank zero, *J. Reine Angew. Math.* **443**(1993), 179–219.

[8] G.A. ELLIOTT, Invariant for simple C^*-algebras, Canadian Math. Soc., vol. 3, Canadian Math. Soc., Ottawa, ON 1945–1995, pp. 61–90.

[9] G.A. ELLIOTT, G. GONG, H. SU, On the classification of C^*-algebras of real rank zero. IV. Reduction to local spectrum of dimension two, in *Operator Algebras and their Applications, (Waterloo, ON, 1994/1995).* II, Fields Inst. Commun., vol. 20, Amer. Math. Soc., Providence, RI 1998, pp. 73–95.

[10] G.A. ELLIOTT, Z. LI, The KK-lifting problem for dimension drop interval algebras, *J. Funct. Anal* **268**(2015), 847–868.

[11] G.A. ELLIOTT, K. THOMSEN, The state space of the K_0-group of a simple separable C^*-algebra, *Geom. Funct. Anal.* **4**(1994), 522–538.

[12] G. GONG, Classification of C^*-algebras of real rank zero and unsuspended E-equivalence types, *J. Funct. Anal.* **152**(1998), 281–329.

[13] G. GONG, H. LIN, Z. NIU, Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras, arXiv:1501.00135v6.

[14] X. JIANG, H. SU, On a simple unital projectionless C^*-algebra, *Amer. J. Math.* **121**(1999), 359–413.

[15] Z. LI, On the simple inductive limits of splitting interval algebras with dimension drops, *Canad. J. Math.* **64**(2012), 544–572.

[16] H. LIN, Z. NIU, Lifting KK-elements, asymptotic unitary equivalence and classification of simple C^*-algebras, *Adv. Math.* **219**(2008), 1729–1769.

[17] J. ROSENBERG, C. SCHOCHET, The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor, *Duke Math. J.* **55**(1987), 431–474.

[18] K. THOMSEN, From trace states to states on the K_0 group of a simple C^*-algebra, *Bull. London Math. Soc.* **28**(1996), 66–72.

QINGNAN AN, DEPARTMENT OF MATHEMATICS, JILIN UNIVERSITY, CHANGCHUN, 130012, P. R. CHINA
E-mail address: 369824037@qq.com

GEORGE A. ELLIOTT, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, CANADA M5S 2E4
E-mail address: elliott@math.toronto.edu

Received October 3, 2016; revised June 5, 2017.