LOWER BOUNDS FOR UNBOUNDED OPERATORS AND SEMIGROUPS

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Communicated by Florian-Horia Vasilescu

ABSTRACT. Let A be an unbounded operator on a Banach space X. It is sometimes useful to improve the operator A by extending it to an operator B on a larger Banach space Y with smaller spectrum. It would be preferable to do this with some estimates for the resolvent of B, and also to extend bounded operators related to A, for example a semigroup generated by A. When X is a Hilbert space, one may also want Y to be Hilbert space. Results of this type for bounded operators have been given by Arens, Read, Müller and Badea, and we give some extensions of their results to unbounded operators and we raise some open questions. A related problem is to improve properties of a C_0-semigroup satisfying lower bounds by extending it to a C_0-group on a larger space or by finding left-inverses. Results of this type for Hilbert spaces have been obtained by Louis and Wexler, and by Zwart, and we give some additional results.

KEYWORDS: *Semigroup, operator, approximate point spectrum, lower bound, expansive, extension, quasi-hyperbolic.*

MSC (2010): Primary 47A20; Secondary 34D20 47B44 47D03.

1. INTRODUCTION

Let U be a bounded operator on a Banach space X, and suppose that there is a constant $c > 0$ such that

$$(1.1) \qquad \|Ux\| \geqslant c\|x\| \quad (x \in X).$$

Then U can be extended to a bounded invertible operator on a Banach space Y which contains X as a closed subspace; see Proposition 3.1, for example. One may wish to preserve various properties of U. For this to be most effective it is desirable that bounded operators which commute with U can also be extended to Y. This property is closely related to a result of Arens [2] for commutative Banach algebras. The following theorem is a formulation of Arens's result for a bounded operator U, and $\{U\}'$ denotes the commutant of $\{U\}$ in $\mathcal{B}(X)$. It can be proved in a similar way to Arens's result (see Theorem 3.2). A slightly weaker version with

$\{U\}'$ replaced by any commutative subalgebra can be deduced from the result for Banach algebras, as in [22].

THEOREM 1.1. *Let* $U \in \mathcal{B}(X)$ *and assume that* (1.1) *holds for some* $c > 0$. *Then there exist a Banach space* $Y \supseteq X$ *and a unital isometric algebra homomorphism* $\varphi : \{U\}' \rightarrow \mathcal{B}(Y)$ *such that:*

(i) $\varphi(U)$ *is invertible and* $\|\varphi(U)^{-1}\| \leqslant c^{-1}$;

(ii) *for each* $V \in \{U\}'$, $\varphi(V)$ *is an extension of* V.

In general, one cannot simultaneously extend two commuting operators each of which satisfy a lower bound so that both operators have inverses with the optimal norm (see Theorem 2.1 of [7] and Example 2.3 of [5]). Nevertheless Read extended Arens's result firstly in the context of commutative Banach algebras [24] and then for operators [26] in the form of Theorem 1.2 below. Again a slightly weaker version for operators can be quickly deduced from the version for Banach algebras (see Chapter 2, Theorem 22 of [23]). The full statement about the homomorphism φ is not included in these references, but it can be seen from the proof in [26].

THEOREM 1.2. *Let* $U \in \mathcal{B}(X)$. *There is a Banach space* $Y \supseteq X$ *and a unital isometric algebra homomorphism* $\varphi : \{U\}' \rightarrow \mathcal{B}(Y)$, *such that* $\varphi(W)$ *is an extension of* W *for all* $W \in \{U\}'$ *and* $\sigma(\varphi(U)) = \sigma_{\mathrm{ap}}(U)$.

Arens's theorem provides the optimal estimate for the norm of $\varphi(U)^{-1}$ whereas Read's theorem and its proof do not provide estimates for the resolvent of $\varphi(U)$. The following result of Badea and Müller ([3], Theorem 3.1) concerns the norms of the powers of the inverse of the extension of a bounded operator.

THEOREM 1.3. *Let* $c : \mathbb{N} \rightarrow (0, \infty)$ *be a submultiplicative sequence. A bounded operator* $U \in \mathcal{B}(X)$ *has an invertible extension* V *on a Banach space* $Y \supseteq X$ *with* $\|V^{-j}\| \leqslant c_j$ *for all* $j \geqslant 1$ *if and only if*

$$\|x\| \leqslant c_n \|x_0\| + c_{n-1} \|x_1\| + \cdots + c_1 \|x_{n-1}\|$$

whenever $n \in \mathbb{N}$, $x, x_j \in X$ *and* $U^n x = x_0 + U x_1 + \cdots + U^{n-1} x_{n-1}$.

When these conditions are satisfied, one may choose Y *and* V *in such a way that the following hold:*

(i) $\|V^j\| = \|U^j\|$ *and* $\|V^{-j}\| \leqslant c_j$ *for all* $j \geqslant 1$; *and*

(ii) *there is a unital isometric algebra homomorphism* $\varphi : \{U\}' \rightarrow \mathcal{B}(Y)$ *with* $\varphi(U) = V$ *and* $\varphi(W)$ *is an extension of* W *for all* $W \in \{U\}'$.

In this paper we consider questions of this type in two further contexts, firstly replacing the single bounded operator U by an unbounded operator A, and secondly replacing U by a C_0-semigroup. The common theme is that the operators should satisfy lower bounds, and they should be extended to operators on a larger Banach space with corresponding inverses, and preserving other properties as far as possible. There is an elementary construction in [21] which extends

many bounded operators which are bounded below and it can be extended to unbounded operators (see Proposition 3.1). However we seek to extend operators in the commutant of A and/or to remove large parts of the spectrum, and to obtain estimates for the norms of associated bounded operators on the extended space, similarly to the theorems stated above. The homomorphisms on $\{A\}'$ will be used to show that the constructions of Arens and Read can be carried through for generators of C_0-semigroups (Sections 3 and 4, respectively).

For a C_0-semigroup T on X we consider lower bounds on the semigroup, of the form

$$\|T(t)x\| \geqslant c(t)\|x\| \quad (x \in X, t \geqslant 0).$$

In this context we seek an extension to a C_0-group S on a larger space Y, as a natural analogue of the discrete case. Alternatively one might look for a C_0-semigroup L of left inverses on Y, or even on X itself, so that $L(t)T(t)x = x$ for all $x \in X$ and $t \geqslant 0$. There are existing results in the literature for the case when $c(t) = 1$ (see Proposition 5.1) and for a more general case which is a continuous analogue of Theorem 1.3 (see Theorem 5.2). When $c(t)$ is a constant in $(0, 1)$ we show that one can obtain S with the exponential growth bound of $\|S(-t)\|$ arbitrarily small (see Proposition 5.3), but in general not with $\|S(-t)\|$ bounded (see Example 5.6). We also characterise when $-A$ is dissipative in an equivalent norm on X (see Theorem 5.4).

In Sections 6 and 7 we consider questions of the same type in the context of operators on Hilbert spaces. Most of the theorems above have versions in which both X and Y are Hilbert spaces, although some changes of detail are needed. In addition the Hilbert space structure allows different approaches. For example in Proposition 6.1 we present a construction, via polar decomposition, of an extension of any closed, densely defined operator on Hilbert space satisfying a lower bound. In Theorem 7.1 we show that many dissipative operators A on Hilbert space X have extensions to generators B of contraction semigroups on larger Hilbert spaces with $D(B) \cap X = D(A)$. Lower bounds for an operator semigroup on Hilbert space, and the possibility of finding a left-inverse semigroup on the same space, have already been considered in the literature relating to the Weiss conjecture on admissibility of observation operators in control theory [20], [28], [29], [31]. In Theorem 7.3 we show how our results on extensions of semigroups with lower bounds relate to some of those results.

2. PRELIMINARIES

In this paper, X, Y and Z will denote complex Banach spaces, and $\mathcal{B}(X)$ will denote the space of bounded linear operators on X. We shall write $X \subseteq Y$, or $Y \supseteq X$, to mean that X is a closed subspace of Y with the same norm. We shall also consider embeddings $\pi : X \to Y$. When π is isometric we regard X as being a subspace of Y by identifying X with $\pi(X)$. Occasionally we will allow

embeddings which are isomorphisms, or continuous injections, and we will say this explicitly whenever it arises.

Given Banach spaces X_n ($n \in \mathbb{N}$), we may consider their ℓ^p-direct sum for $1 \leqslant p < \infty$, or their c_0-direct sum, in the usual way. In the special case when $X_n = X$ for all n, we will denote this space by $\ell_p(X)$ or $c_0(X)$.

An operator A on X should be taken to be unbounded unless specified otherwise. Thus the domain of A is a subspace of X and A is a linear mapping into X. We denote the domain, kernel, range, spectrum and resolvent set of A by $D(A)$, Ker A, Ran A, $\sigma(A)$ and $\rho(A)$ respectively, and we put $R(\lambda, A) = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. Recall that a *pseudo-resolvent* on X is a function $R : \Omega \subseteq \mathbb{C} \to \mathcal{B}(X)$ which satisfies the resolvent identity

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \quad (\lambda, \mu \in \Omega).$$

Then Ker $R(\lambda)$ and Ran $R(\lambda)$ are both independent of λ. Moreover, R is the resolvent of some operator A on X if and only if Ker $R(\lambda) = \{0\}$, and then $D(A) = $ Ran $R(\lambda)$. If $R(\lambda) = R(\lambda, A)$ for some $\lambda \in \Omega$, then $\Omega \subseteq \rho(A)$ and $R_\lambda = R(\lambda, A)$ for all $\lambda \in \Omega$ ([30], Section VIII.4).

We will say that A is *bounded below* if there exists $c > 0$ such that

(2.1) $$\|Ax\| \geqslant c\|x\| \quad (x \in D(A)).$$

We denote by $\{A\}'$ the commutator algebra of A in $\mathcal{B}(X)$ defined in the following way:

$$\{A\}' = \{U \in \mathcal{B}(X) : \text{for all } x \in D(A), Ux \in D(A) \text{ and } AUx = UAx\}.$$

If $\lambda \in \rho(A)$, then $\{A\}' = \{R(\lambda, A)\}'$ ([1], Proposition B.7).

Let A be an operator on X and Y be a Banach space with $X \subseteq Y$. An operator B on Y is an *extension* of A if $D(A) \subseteq D(B)$ and $Bx = Ax$ for all $x \in D(A)$. We shall say that B is an *outer extension* of A if $D(A) = D(B) \cap X$ and $Bx = Ax$ for all $x \in D(A)$.

Let T be a C_0-semigroup on X, so that $T : [0, \infty) \to \mathcal{B}(X)$ is continuous in the strong operator topology, with $T(0) = I$ and $T(s)T(t) = T(s + t)$ $(s, t \geqslant 0)$, and let A be the generator of T. We refer the reader to Section 3.1 of [1] or Chapter 2 of [13] for standard properties of C_0-semigroups and their generators. It is easily seen that

$$\{A\}' = \{U \in \mathcal{B}(X) : T(t)U = UT(t) \text{ for all } t \geqslant 0\}.$$

If S is a C_0-semigroup on $Y \supseteq X$ and T is a C_0-semigroup on X, we shall say that S is an *extension* of T if $S(t)x = T(t)x$ for all $x \in X$ and $t \geqslant 0$.

We note the following elementary facts.

PROPOSITION 2.1. *Let A be the generator of a C_0-semigroup T on a Banach space X and B be the generator of a C_0-semigroup S on a Banach space $Y \supseteq X$.*

(i) *The following are equivalent:*

    (a) *B is an extension of A;*

    (b) *$S(t)x = T(t)x$ for all $x \in X$, $t \geqslant 0$;*

(c) *B is an outer extension of A.*

(ii) *The following are equivalent:*

    (a) *B is an extension of* $-A$;

    (b) $S(t)T(t)x = x$ *for all* $x \in X$, $t \geqslant 0$.

Proof. (i) Suppose that B is an extension of A, and let $x \in D(A)$, $t \geqslant 0$. Then, for $0 \leqslant s \leqslant t$,

$$\frac{\mathrm{d}}{\mathrm{d}s} S(t-s)T(s)x = -S(t-s)BT(s)x + S(t-s)AT(s)x = 0.$$

Hence $S(t)x = T(t)x$ for all $x \in D(A)$ and then for all $x \in X$ by density of $D(A)$ in X. Thus (a) $\Rightarrow$ (b). The proofs of (b) $\Rightarrow$ (c) $\Rightarrow$ (a) are very simple.

(ii) The proof that (a) $\Rightarrow$ (b) is similar to (i), showing that $S(t)T(t)x$ has derivative 0 when $x \in D(A)$, and then using the density of $D(A)$. Now assume (b), and let $x \in D(A)$ and $\tau > 0$. Then for $0 < t < \tau$,

$$t^{-1}(S(t)T(\tau)x - T(\tau)x) = t^{-1}(T(\tau-t)x - T(\tau)x) \to -T(\tau)Ax$$

as $t \to 0+$. Hence $T(\tau)x \in D(B)$ and $BT(\tau)x = -T(\tau)Ax$. Letting $\tau \to 0+$, and using that B is closed, it follows that $x \in D(B)$ and $Bx = -Ax$. ∎

PROPOSITION 2.2. *Let A be the generator of a C_0-semigroup T on a Banach space X, let Y be a Banach space with $X \subseteq Y$, and let $\varphi : \{A\}' \to \mathcal{B}(Y)$ be an isometric unital algebra homomorphism such that $\varphi(U)$ extends U for all $U \in \{A\}'$. Then there exists a Banach space Y_0 with $X \subseteq Y_0 \subseteq Y$ and a C_0-semigroup S on Y_0, with generator B, such that:*

  (i) *S extends T;*

  (ii) *Y_0 is invariant under $\varphi(U)$ for all $U \in \{A\}'$, and hence the map $\varphi_0 : U \mapsto \varphi(U)|_{Y_0}$ is an isometric unital homomorphism of $\{A\}'$ into $\mathcal{B}(Y_0)$;*

  (iii) *$\sigma(\varphi_0(U)) \subseteq \sigma(\varphi(U))$ for all $U \in \{A\}'$;*

  (iv) *$\sigma(B) \subseteq \sigma(A)$ and $R(\lambda, B) = \varphi_0(R(\lambda, A))$ for all $\lambda \in \rho(A)$.*

Proof. For $t \geqslant 0$, $T(t) \in \{A\}'$. Let $\widetilde{S}(t) = \varphi(T(t))$. Then $\widetilde{S}$ satisfies the semigroup property and is locally bounded. Let

$$Y_0 = \Big\{ y \in Y : \lim_{t \to 0+} \|\widetilde{S}(t)y - y\| = 0 \Big\}.$$

Then Y_0 is a closed $\widetilde{S}$-invariant subspace of Y, containing X. For $U \in \{A\}'$, $\varphi(U)$ commutes with $\widetilde{S}(t)$, so Y_0 is invariant under $\varphi(U)$ and under $R(\lambda, \varphi(U))$ for all $\lambda \in \rho(\varphi(U))$. Define $\varphi_0(U) = \varphi(U)|_{Y_0}$, and $S(t) = \varphi_0(T(t))$. Then φ_0 is a homomorphism and S is a C_0-semigroup on Y_0. Moreover, $\lambda \in \rho(\varphi_0(U))$ and $R(\lambda, \varphi_0(U)) = R(\lambda, \varphi(U))|_{Y_0}$.

Now, take $\lambda \in \rho(A)$ and let $R_\lambda = \varphi_0(R(\lambda, A))$. Now

$$\|t^{-1}(\mathrm{e}^{-\lambda t}S(t) - I)R_\lambda^2 + R_\lambda\| = \|t^{-1}(\mathrm{e}^{-\lambda t}T(t) - I)R(\lambda, A)^2 + R(\lambda, A)\|$$

$$= \left\| \frac{1}{t} \int_0^t (I - e^{-\lambda s} T(s)) R(\lambda, A) ds \right\| \to 0 \quad \text{as } t \to 0+.$$

This shows that $R_\lambda = (\lambda - B) R_\lambda^2$ on Y_0, and hence $y = (\lambda - B) R_\lambda y$ for y in the range of R_λ and then for $y \in Y_0$, since B is closed. Hence $R(\lambda, B) = \varphi_0(R(\lambda, A))$ for all $\lambda \in \rho(A)$. ∎

In the notation of the proof above, for sufficiently large real λ,

$$\| \widetilde{S}(t) \varphi(R(\lambda, A)) - \varphi(R(\lambda, A)) \| = \| T(t) R(\lambda, A) - R(\lambda, A) \| \leqslant \int_0^t \| e^{-\lambda s} T(s) \| ds \to 0$$

as $t \to 0+$. It follows that Y_0 contains the range of $\varphi(R(\lambda, A))$. On the other hand, Y_0 is the closure of the range of $R(\lambda, B)$ which is contained in the range of $\varphi(R(\lambda, A))$. So Y_0 is the closure of the range of $\varphi(R(\lambda, A))$ for any $\lambda \in \rho(A)$.

3. LOWER BOUNDS FOR UNBOUNDED OPERATORS

There is an elementary construction which gives an invertible extension of a closed operator A which is bounded below and has complemented range, but it lacks the homomorphism of the commutant. For bounded operators this construction has been given in Theorem 3 of [21] (see also [6] and [10]). Our context is different and we shall need some additional properties of the construction which are not explicit in those references, so we give the details in Proposition 3.1.

When A is closed and bounded below, Ran A is a closed subspace of X. We assume also that Ran A is complemented in X, so that there is a closed subspace Y of X such that $X = \text{Ran } A \oplus Y$. Equivalently there is a left-inverse operator $L \in \mathcal{B}(X)$ such that L maps X into $D(A)$ and $LAx = x$ for all $x \in X$. Then let Z be the ℓ_p-direct sum, or the c_0-direct sum, of X and countably many copies of Y. Thus Z consists of appropriate sequences

$$z = (x, y_1, y_2, \dots)$$

where $x \in X$ and $y_n \in Y$, and

$$\|z\| = \|(\|x\|, \|y_1\|, \|y_2\|, \dots)\|_{\ell_p}.$$

Then $X \subseteq Z$, via the isometric embedding $\pi : x \mapsto (x, 0, 0, \dots)$.

PROPOSITION 3.1. *Let A be a closed operator on a Banach space X, satisfying (2.1), and assume that* Ran A *is complemented in X. Let Y and Z be as above, and define an operator B on Z by*

$$D(B) = \{(x, y_1, y_2, \dots) : x \in D(A), y_j \in Y\},$$
$$B(x, y_1, y_2, \dots) = (Ax + cy_1, cy_2, cy_3, \dots).$$

Then B has the following properties:

(i) *B is an outer extension of A.*
(ii) *B is invertible.*
(iii) $\sigma(B) \subseteq \sigma(A)$.
(iv) *If A generates a C_0-semigroup on X, then B generates a C_0-semigroup on Z.*
(v) *If A is bounded, then B is bounded.*
(vi) *If $|\lambda| = c$, the following hold:*
  (a) $\mathrm{Ker}(\lambda - A) = \mathrm{Ker}(\lambda - B)$;
  (b) $\mathrm{Ran}(\lambda - A) = \mathrm{Ran}(\lambda - B) \cap X$;
  (c) $\mathrm{Ran}(\lambda - B)$ *is dense in Z if* $\mathrm{Ran}(\lambda - A)$ *is dense in X.*

Moreover, B is a minimal invertible extension of A in the sense that there is no proper closed subspace of Z which contains X and is invariant under B^{-1}.

Proof. By replacing A by $c^{-1}A$, we may assume that $c = 1$. If A is invertible then $Y = \{0\}$ and all properties are trivial.

Properties (i) and (v) are immediate. For (ii), B^{-1} is given by

$$B^{-1}(Ax + y_0, y_1, y_2, \dots) = (x, y_0, y_1, y_2, \dots).$$

The assumption (2.1) implies that any point λ with $|\lambda| < 1$ is not in $\sigma_{\mathrm{ap}}(A)$ and hence is not in the boundary of $\sigma(A)$. If A is not invertible, it follows that any point $\lambda \in \rho(A)$ satisfies $|\lambda| > 1$. Then $R(\lambda, B)$ is given by

$$R(\lambda, B)(x, y_1, y_2, \dots)$$
$$= \left(R(\lambda, A)\left(x + \sum_{n=0}^{\infty} \lambda^{-(n+1)}y_n \right), \sum_{n=0}^{\infty} \lambda^{-(n+1)}y_{n+1}, \sum_{n=0}^{\infty} \lambda^{-(n+1)}y_{n+2}, \dots \right).$$

This establishes (iii).

For (vi), consider λ with $|\lambda| = 1$. If $z = (x, y_1, y_2, \dots) \in D(B)$, and $(\lambda - B)z \in \pi(X)$ then $|y_n| = |y_{n+1}|$ for all $n \geqslant 1$. This implies that $y_n = 0$, $z = \pi(x)$ and $(\lambda - B)z = \pi((\lambda - A)x)$. In particular, if $(\lambda - B)z = 0$ then $(\lambda - A)x = 0$. For any vector in Z of the form

$$(x, y_1, y_2, \dots, y_k, 0, 0, \dots),$$

let $y'_n = 0$ for $n > k$, and define $y'_n \in Y$ recursively for $n = k, k-1, \dots, 1$ by $\lambda y'_n = y'_{n+1} + y_n$. If $\mathrm{Ran}(\lambda - A)$ is dense we may then choose $x' \in D(A)$ such that $\|\lambda x' - Ax' - y'_1 - x\|$ is arbitrarily small. Hence $\mathrm{Ran}(\lambda - B)$ is dense in Z.

If A generates a C_0-semigroup on X, then the operator $z \mapsto (Ax, 0, 0, \dots)$ generates a C_0-semigroup on Z. Since B is a bounded perturbation of this operator, B also generates a C_0-semigroup.

Finally, the span of the union of $\{B^{-k}(X) : k \geqslant 0\}$ contains all vectors $(x, y_1, y_2, \dots)$ where $y_n = 0$ for all except finitely many n. These vectors are dense in Z, and the minimality follows. ∎

The following is a version of Theorem 1.1 for semigroup generators. The proof here is an adaptation of Arens's proof in the context of Banach algebras.

THEOREM 3.2. *Let A be the generator of a C_0-semigroup on X, and assume that A satisfies (2.1) for some $c > 0$. Then there exist a Banach space $Y \supseteq X$ and an operator B on Y with the following properties:*

(i) *B is the generator of a C_0-semigroup on Y;*

(ii) *B is an outer extension of A;*

(iii) *B is invertible with $\|B^{-1}\| \leqslant c^{-1}$, and $\sigma(B) \subseteq \sigma(A)$;*

(iv) *there is a unital isometric algebra homomorphism $\varphi : \{A\}' \to \mathcal{B}(Y)$ such that $\varphi(U)$ is an extension of U for all $U \in \{A\}'$ and $\varphi(R(\lambda, A)) = R(\lambda, B)$ for all $\lambda \in \rho(A)$.*

Proof. Replacing A by $c^{-1}A$ we may assume that $c = 1$. Let T be the C_0-semigroup generated by A. We first lift T to the space $\ell_1(X)$.

Define
$$\widetilde{T}(t)f = (T(t)x_n) \quad (f = (x_n) \in \ell_1(X), t \geqslant 0).$$
Then $\widetilde{T}$ is a C_0-semigroup on $\ell_1(X)$ and its generator C is given by
$$D(C) = \{f = (x_n) \in \ell_1(X) : x_n \in D(A) \ (n \in \mathbb{N}), (Ax_n) \in \ell_1(X)\},$$
$$Cf = (Ax_n).$$
Moreover, $\sigma(C) = \sigma(A)$ and the resolvent of C is given by
$$R(\lambda, C)f = (R(\lambda, A)x_n) \quad (\lambda \in \rho(A), f = (x_n) \in \ell_1(X)).$$
Let $\overline{R}$ be the right shift on $\ell_1(X)$, and note that $\overline{R} \in \{C\}'$. Let J be the closure of $\{f - C\overline{R}f : f \in D(C)\}$ in $\ell_1(X)$. Then J is invariant under $\widetilde{T}, \overline{R}$ and $R(\lambda, C)$.

Let $Y = \ell_1(X)/J$ and define $\pi : X \to Y$ by $\pi(x) = x\mathbf{e}_0 + J$. Here $x\mathbf{e}_0$ is the sequence f with $f(0) = x$ and $f(n) = 0$ for $n \geqslant 1$. It is clear that $\|\pi(x)\| \leqslant \|x\|$. Using the triangle inequality and the lower bound (2.1) for A repeatedly, for any $f = (x_n) \in D(C)$, we have
$$\|x\| \leqslant \|x - x_0\| + \|Ax_0\| \leqslant \|x - x_0\| + \|x_1 - Ax_0\| + \|Ax_1\| \leqslant \cdots$$
$$\leqslant \|x - x_0\| + \sum_{n=1}^{\infty} \|x_n - Ax_{n-1}\| = \|x\mathbf{e}_0 - (I - C\overline{R})f\|_{\ell_1(X)}.$$
Thus $\|x\| \leqslant \|\pi(x)\|$ and π is isometric.

Let $q : \ell_1(X) \to Y$ be the quotient map $q(f) = f + J$. Since J is invariant under $\widetilde{T}(t)$, there are operators on Y defined by
$$S(t)q(f) = q(\widetilde{T}(t)f) \quad (f \in \ell_1(X), t \geqslant 0).$$
Then S is a C_0-semigroup and its generator B is given by $D(B) = q(D(C))$, $Bq(f) = q(Cf)$ for $f \in D(C)$. Moreover, $\sigma(B) \subseteq \sigma(C) = \sigma(A)$ and $R(\lambda, B)q(f) = q(R(\lambda, C)f)$ for $f \in \ell_1(X)$ (see I.5.13, II.2.4 and IV.2.15 of [13]). For $x \in X$, we have
$$S(t)\pi(x) = \pi(T(t)x).$$
When we identify X with its image under π, this shows that B is an outer extension of A (Proposition 2.1(i)).

Next we show that B has a bounded inverse. Define V on Y by

$$Vq(f) = q(\overline{R}f) \quad (f \in \ell_1(X)).$$

Since J is invariant under $\overline{R}$, the operator V is well-defined and $\|V\| \leqslant \|\overline{R}\| = 1$. Let $y \in D(B)$ and choose $f \in D(C)$ such that $y = q(f)$. Since V commutes with $S(t)$, $Vy \in D(B)$ and

$$BVy = VBy = q(C\overline{R}f) = q(f) - q(f - C\overline{R}f) = y.$$

Since B is closed, $D(B)$ is dense in Y and V is bounded, we deduce that $Vy \in D(B)$ and $BVy = y$ for all $y \in Y$, and V is the inverse of B.

For $U \in \{A\}'$ and $f = (x_n) \in \ell_1(X)$, define

$$\varphi(U)q(f) := q((Ux_n)).$$

Since J is invariant under the map $f \mapsto (Ux_n)$, $\varphi(U)$ is a well-defined, bounded operator on Y with $\|\varphi(U)\| \leqslant \|U\|$ and $\varphi(U)\pi(x) = \pi(Ux)$ for $x \in X$, so $\varphi(U)$ is an extension of U when we identify X with $\pi(X) \subseteq Y$. Moreover, φ is a unital algebra homomorphism. For $\lambda \in \rho(A)$ and $f \in \ell_1(X)$, we have

$$\varphi(R(\lambda, A))q(f) = q((R(\lambda, A)x_n)) = q(R(\lambda, C)f) = R(\lambda, B)q(f).$$

So $\varphi(R(\lambda, A)) = R(\lambda, B)$. ∎

It is plausible that Theorem 3.2 can be extended to larger classes of unbounded operators than generators of C_0-semigroups, but this is not straightforward. In trying to extend the proof of Theorem 3.2, one can define an operator C on $\ell_1(X)$ and the corresponding space J. Then the resolvent of A induces operators R_λ on $\ell_1(X)/J$ which form a pseudo-resolvent. The technical problem which arises is to show that R_λ is injective so that the pseudo-resolvent is the resolvent of an operator. This can be achieved in the proof of Theorem 3.2, because C generates a C_0-semigroup which leaves J invariant, so that if $f \in D(C) \cap J$ then $Cf \in J$ (Proposition 2.1). A similar problem arises if one assumes that $\rho(A)$ is non-empty and applies Theorem 1.1 to $U := \mu^{-1} - R(\mu, A)$ where $\mu \in \rho(A)$.

In seeking extensions it is natural to consider the class of generators of integrated semigroups, but we are able to obtain only a weak result (Proposition 3.5) for once integrated semigroups, with a weaker conclusion giving a continuous embedding of X in Y.

Let $k \geqslant 1$. A function $T : [0, \infty) \to \mathcal{B}(X)$ which is continuous in the strong operator topology, is said to be a *k-times integrated semigroup* if $T(0) = 0$ and there exist $\omega \geqslant 0$ and $M \geqslant 1$ such that

$$\left\| \int_0^t T(s)x \, ds \right\| \leqslant Me^{\omega t}\|x\| \quad (x \in X, t \geqslant 0),$$

and the (improper) integral

$$R(\lambda)x := \lambda^k \int_0^\infty e^{-\lambda t} T(t)x \, dt \quad (x \in X, \lambda > \omega)$$

defines a pseudo-resolvent. These integrals may be improper, but the integrals over $[0, \tau]$ converge in operator-norm (i.e., uniformly for $\|x\| \leqslant 1$), as $\tau \to \infty$ ([1], Remark 1.4.6). Then T is *non-degenerate* if any one or all of the following equivalent conditions is satisfied:

(i) $R(\lambda)$ is injective;
(ii) there is an operator A such that $R(\lambda, A) = R(\lambda)$;
(iii) if $x \in X$ and $T(t)x = 0$ for all $t \geqslant 0$ then $x = 0$.

See Proposition 3.2.9 of [1]. Then A is the *generator* of the integrated semigroup.

When A generates a non-degenerate (once) integrated semigroup T on X and Y is a closed T-invariant subspace of X, there may exist $y \in Y \cap D(A)$ with $Ay \notin Y$, and the induced integrated semigroup on X/Y is then degenerate. This causes the proof of Theorem 3.2 to break down for integrated semigroups. In the special case of once integrated semigroups we can partially avoid this obstruction as shown in Proposition 3.5.

Let S be a (degenerate) k-times integrated semigroup on X, and N be its *degeneration space*

$$N := \{y \in X : S(t)y = 0 \text{ for all } t \geqslant 0\}.$$

This is a closed subspace of X, invariant under $S(t)$. Let $\widetilde{S}(t)$ be the operator on X/N induced by $S(t)$. It is readily verified that $\widetilde{S}$ is a k-times integrated semigroup.

LEMMA 3.3. *If S is a once integrated semigroup on X, then $\widetilde{S}$ is a non-degenerate once integrated semigroup on X/N.*

Proof. We have to show that $\widetilde{S}$ is non-degenerate, that is, if $x \in X$ and $S(t)x \in N$ for all $t \geqslant 0$, then $x \in N$. Since $S(t)x \in N$, $S(s)S(t)x = 0$ for all $s, t \geqslant 0$. By Proposition 3.2.4 of [1],

$$\int_s^{s+t} S(r)x \, dr = \int_0^t S(r)x \, dr.$$

Differentiation at $t = 0$ gives $S(s)x = S(0)x = 0$ for all s, so $x \in N$. ∎

Lemma 3.3 does not extend to k-times integrated semigroups, where $k \geqslant 2$.

EXAMPLE 3.4. Let $k \geqslant 2$, $X = \mathbb{C}^2$ and

$$S(t) = \begin{pmatrix} 0 & t^{k-1} \\ 0 & 0 \end{pmatrix} \quad (t \geqslant 0).$$

Then $S(0) = 0$ and

$$R(\lambda) = \lambda^k \int_0^\infty e^{-\lambda t} S(t)\, dt = \begin{pmatrix} 0 & (k-1)! \\ 0 & 0 \end{pmatrix} \quad (\lambda > 0).$$

This is a pseudo-resolvent, so S is a k-times integrated semigroup. The degeneration space N is spanned by $(1,0)$. The induced integrated semigroup on X/N is 0, so it is degenerate.

PROPOSITION 3.5. *Let A be the generator of a non-degenerate once integrated semigroup on X, and assume that* (2.1) *holds for some $c > 0$. Then there exist a Banach space Z, a continuous embedding $\pi : X \to Z$ and an operator B on Z with the following properties:*

  (i) *B is the generator of a non-degenerate once integrated semigroup on Z;*
  (ii) *B is invertible with $\|B^{-1}\| \leqslant c^{-1}$, and $\sigma(B) \subseteq \sigma(A)$;*
  (iii) *if $x \in D(A)$ then $\pi(x) \in D(B)$ and $B\pi(x) = \pi(Ax)$.*

  Proof. The proof initially follows the lines of Theorem 3.2. Let T be the once integrated semigroup generated by A. Define Y, C, $\overline{R}$, J, $\widetilde{T}(t)$ and $S(t)$ as in the proof of Theorem 3.2. They have the same properties as in Theorem 3.2, except that $\widetilde{T}$ and S are now once integrated semigroups, $\widetilde{T}$ is non-degenerate with generator C, and S may be degenerate. Nevertheless there exist bounded operators R_λ ($\lambda \in \rho(A)$) on Y such that $R_\lambda(f + J) = (R(\lambda, A)x_n) + J$ for $f = (x_n) \in \ell_1(X)$. They form a pseudo-resolvent, but they are not necessarily injective.

  Let N be the degeneration space of S, and let $Z = Y/N$. Since R_λ commutes with $S(t)$, N is invariant under R_λ, and therefore R_λ induces a bounded operator $\widetilde{R}_\lambda$ on Z, and $\{\widetilde{R}_\lambda : \lambda \in \rho(A)\}$ is a pseudo-resolvent. By Lemma 3.3, S induces a non-degenerate once integrated semigroup $\widetilde{S}$ on Z. Let $q(f) = (f + J) + N$ for $f \in \ell_1(X)$, and $\pi(x) = q(x\mathbf{e}_0)$ for $x \in X$. Let $x \in \operatorname{Ker} \pi$, so $S(t)(x\mathbf{e}_0 + J) = 0$ for all $t \geqslant 0$. This means that $T(t)x\mathbf{e}_0 \in J$, and then $T(t)x = 0$ for all $t \geqslant 0$ since $y \to y\mathbf{e}_0 + J$ is injective on X. Since T is non-degenerate, this implies that $x = 0$. Thus π is a continuous embedding of X into Z.

  Define $V : Z \to Z$ by

$$Vq(f) = q(\overline{R}f) \quad (f \in \ell_1(X)).$$

This is well-defined and contractive, since $f + J \in N$ implies that $\widetilde{T}(t)f \in J$ for all $t \geqslant 0$, and then $\widetilde{T}(t)\overline{R}f = \overline{R}\widetilde{T}(t)f \in J$, so $\overline{R}f + J \in N$. Let B be the generator of $\widetilde{S}$, take $\lambda > \omega$, and consider $z \in D(B)$. Then

$$z = R(\lambda, B)y = \lambda \int_0^\infty e^{-\lambda t} \widetilde{S}(t)y\, dt$$

for some $y \in Z$. Take $g \in \ell_1(X)$ such that $y = q(g)$, and let $f = R(\lambda, C)g$. Then $z = q(f)$ and

$$VBz = \lambda Vz - Vy = \lambda Vz - q\overline{R}g = \lambda Vz - q\overline{R}(\lambda - C)f = qC\overline{R}f = q(f).$$

Moreover, for any $z = q(f)$ and $\lambda > \omega$,

$$R(\lambda, B)(\lambda Vz - z) = qR(\lambda, C)(\lambda \overline{R}f - f) = \lambda q\overline{R}R(\lambda, C)f - q\overline{R}CR(\lambda, C)f = q\overline{R}f = Vz.$$

Hence $Vz \in D(B)$ and $BVz = z$. So B is invertible with $B^{-1} = V$. Moreover, for $f \in \ell_1(X)$ and $\lambda > \omega$,

$$qR(\lambda, C)f = q\left(\lim_{\tau \to \infty} \left(\lambda \int_0^\tau e^{-\lambda t} \widetilde{T}(t)f \, dt \right) \right) = \lambda \lim_{\tau \to \infty} \int_0^\tau \widetilde{S}(t)q(f) \, dt = R(\lambda, B)q(f).$$

Since $\{\widetilde{R}_\lambda : \lambda \in \rho(A)\}$ is a pseudo-resolvent, it follows that $\sigma(B) \subseteq \sigma(A)$.

For $x \in D(A)$, we have $x = R(\lambda, A)(\lambda x - Ax)$, so

$$\pi(x) = \widetilde{R}_\lambda \pi(\lambda x - Ax) = R(\lambda, B)\pi(\lambda x - Ax).$$

It follows that $\pi(x) \in D(B)$ and $B\pi(x) = \pi(Ax)$. ∎

4. REDUCING THE SPECTRUM

Here we adapt Read's theorem (Theorem 1.2) itself to the case of the generator of a C_0-semigroup.

THEOREM 4.1. *Let A be the generator of a C_0-semigroup on X. Then there exist a Banach space $Y \supseteq X$ and an operator B on Y with the following properties:*
   (i) *B is the generator of a C_0-semigroup on Y;*
   (ii) *B is an outer extension of A;*
   (iii) *$\sigma(B) = \sigma_{ap}(A)$.*
Furthermore there is a unital isometric algebra homomorphism

$$\varphi : \{A\}' \to \mathcal{B}(Y)$$

such that $\varphi(U)$ is an extension of U for all $U \in \{A\}'$ and $\varphi(R(\lambda, A)) = R(\lambda, B)$ for all $\lambda \in \rho(A)$.

Proof. Let T be the C_0-semigroup generated by A, and take $\mu > \omega_0(T)$, the growth bound of T, so $\mu \in \rho(A)$. By applying Theorem 1.2 to $R(\mu, A)$, and then applying Proposition 2.2, there exist a Banach space $Y \supseteq X$ and a C_0-semigroup S on Y, extending T, such that the generator B of S satisfies

$$\sigma(R(\mu, B)) \subseteq \sigma_{ap}(R(\mu, A)).$$

Since $R(\mu, B)$ extends $R(\mu, A)$, we have

$$\sigma(R(\mu, B)) = \sigma_{ap}(R(\mu, A)).$$

By a standard spectral property ([1], Proposition B.2),

$$\sigma_{\mathrm{ap}}(A) = \{\lambda \in \mathbb{C} : (\mu - \lambda)^{-1} \in \sigma_{\mathrm{ap}}(R(\mu, A))\}$$
$$= \{\lambda \in \mathbb{C} : (\mu - \lambda)^{-1} \in \sigma(R(\mu, B))\} = \sigma(B). \quad \blacksquare$$

A class of "uniformly expansive" operators was introduced in [12] and [19], and the following slightly larger class was considered in [4]. An operator $U \in \mathcal{B}(X)$ is said to be *quasi-hyperbolic* if the following equivalent conditions are satisfied. Here, $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$:

(i) There exists $n \in \mathbb{N}$ such that $\max(\|U^{2n}x\|, \|x\|) \geqslant 2\|U^n x\|$ for all $x \in X$.

(ii) $\sigma_{\mathrm{ap}}(U) \cap \mathbb{T} = \varnothing$.

(iii) There exists $c > 0$ such that

(4.1) $$\|(U - \lambda)x\| \geqslant c\|x\| \quad (x \in X, \lambda \in \mathbb{T}).$$

(iv) U is the restriction of an operator $V \in \mathcal{B}(Y)$ on a Banach space $Y \supseteq X$ with $\mathbb{T} \subseteq \rho(V)$ (i.e., V is hyperbolic).

The implication (ii) $\Rightarrow$ (i) was essentially shown in [19], and (ii) $\Rightarrow$ (iv) is a special case of Theorem 1.2. The other implications are very elementary.

Although the proof of Read's theorem is substantially simpler in the restricted form needed here than in its full generality, it does not appear to give any estimate for $\|R(\lambda, V)\|$. Nevertheless it is possible to show that there exists K depending only on $\|U\|$ and c, such that V can be constructed as in (iv) and $\|R(\lambda, V)\| \leqslant K$ for all $\lambda \in \mathbb{T}$. To see this, we argue by contradiction.

Let $M > 1$ and $c > 0$. Assume that for each $n \in \mathbb{N}$, there is an operator U_n on a Banach space X_n such that U_n satisfies (4.1), $\|U_n\| \leqslant M$ and for every extension V of U_n with $\mathbb{T} \subseteq \rho(V)$, there exists $\lambda \in \mathbb{C}$ such that $\|R(\lambda, V)\| > n$. Let X be the c_0-direct sum $X = \bigoplus_{n \geqslant 1} X_n$ and $U = \bigoplus_{n \geqslant 1} U_n$. Then $\|U\| \leqslant M$ and

$$\|(U - \lambda)(x_n)\| \geqslant c\|(x_n)\| \quad ((x_n) \in X, \lambda \in \mathbb{T}).$$

By Theorem 1.2 there is an extension V of U with $\mathbb{T} \subseteq \rho(V)$. Since $\mathbb{T}$ is compact there exists K such that $\|R(\lambda, V)\| \leqslant K$ for all $n \geqslant 1$. Now X_n is isometrically embedded in X and V is an extension of U_n. For each $n \geqslant 1$, $n < \|R(\lambda_n, V)\| \leqslant K$ for some $\lambda_n \in \mathbb{T}$. This is a contradiction.

This raises the following problem.

OPEN QUESTION 4.2. Find $K : (0,1) \times (1,\infty) \to (1,\infty)$ such that the following holds. If U is a bounded operator satisfying (4.1), there is an extension V of U with $\|V\| = \|U\|$, $\mathbb{T} \subseteq \rho(V)$ and

$$\sup_{\lambda \in \mathbb{T}} \|R(\lambda, V)\| \leqslant K(c, \|U\|).$$

Weighted shift operators provide examples of quasi-hyperbolic operators which are not hyperbolic. The following gives an explicit description of extensions of weighted shifts on $\ell_1(\mathbb{Z})$ to hyperbolic operators.

EXAMPLE 4.3. Let $X = \ell_1(\mathbb{Z})$ and $T_w \in \mathcal{B}(X)$ be the weighted shift given by

$$T_w(\mathbf{e}_n) = w_n \mathbf{e}_{n+1} \quad (n \in \mathbb{Z}),$$

where $\mathbf{e}_n$ are the standard basis vectors and $0 < w_n \leqslant M$ for all n. We assume that

$$w_n \geqslant 1 \quad (n \geqslant 0), \quad w_n < 1 \quad (n < 0), \quad 0 \leqslant r^-(T_w) < 1 < i^+(T_w).$$

Here

$$i^+(T_w) = \lim_{n \to \infty} \inf_{k > 0} (w_k \cdots w_{k+n-1})^{1/n}, \quad r_-(T_w) = \lim_{n \to \infty} \sup_{k < 0} (w_{k-n} \cdots w_{k-1})^{1/n}.$$

For the corresponding operator on $\ell_2(\mathbb{Z})$, Ridge [27] (see also Proposition 2.5 of [4]) showed that $\sigma_{\mathrm{ap}}(T_w)$ consists of two (possibly degenerate) annuli centred at 0, one with outer radius $r^-(T_w)$, and the other with inner radius $i^+(T_w)$, while $\sigma(T_w)$ is the smallest annulus containing $\sigma_{\mathrm{ap}}(T_w)$. In particular, T_w is quasi-hyperbolic, but $\mathbb{T} \subseteq \sigma(T_w)$.

In Section 3.6.1 of [14] an isomorphic embedding $\pi : X \to Y := \ell_1(\mathbb{Z}) \oplus \ell_1(\mathbb{Z})$, and weights α and β on $\mathbb{Z}$, are constructed such that

$$(4.2) \qquad\qquad\qquad \pi \circ T_w = S \circ \pi$$

where $S = T_\alpha \oplus T_\beta$. Moreover, the weights α and β are chosen so that $\sigma(T_\alpha)$ is inside the open unit disc and $\sigma(T_\beta)$ is outside the closed unit disc. In particular, $\sigma(S) \cap \mathbb{T}$ is empty. Furthermore it is shown that if $c > 0$ is chosen so that T_w satisfies (4.1) then

$$(4.3) \qquad\qquad \|R(\lambda, S)\| \leqslant \frac{e\widetilde{M}^{2/c+1} \log \widetilde{M}}{c} \quad (\lambda \in \mathbb{T}),$$

where $\widetilde{M} = \max(2, \|T_w\|)$.

The embedding π of X in Y here is not isometric but $\|x\|_* := \|\pi(x)\|_Y$ is a weighted ℓ_1-norm satisfying $\|x\|_X \leqslant \|x\|_* \leqslant 2\|x\|_X$. One may vary the definition of π so that it becomes isometric. Then one can vary the definition of α and β in order to satisfy (4.2).

Other constructions or calculations may provide sharper estimates than (4.3). On the other hand, it is plausible that weighted shifts will have the worst behaviour for this type of problem. Thus we conjecture that there exist positive constants a, b such that

$$K(c, \tau) := a\tau^{b/c}$$

has the properties required in Open Question 4.2.

There is a related question for generators of quasi-hyperbolic C_0-semigroups as defined in [4]. A C_0-semigroup T is quasi-hyperbolic if and only if $T(t)$ is quasi-hyperbolic for some, or equivalently all, $t > 0$.

Let A be the generator of a C_0-semigroup T. When T is quasi-hyperbolic, it was shown in Proposition 3.2 of [4] that

$$(4.4) \qquad\qquad \|(A - is)x\| \geqslant c\|x\| \quad (x \in D(A), s \in \mathbb{R})$$

for some $c > 0$. On the other hand, the full converse is not true, as (4.4) does not imply that T is quasi-hyperbolic in general. A possible partial converse was raised as a question in Section 4 of [4]; it remains an open question whether (4.4) implies that there is a continuous injection of X into a Banach space Y such that $\pi \circ T(t) = S(t) \circ \pi$ for some hyperbolic semigroup S on Y.

The following question asks about another possible partial converse property of a different type.

OPEN QUESTION 4.4. Let A be the generator of a C_0-semigroup on a Banach space X, and assume that (4.4) holds for some $c > 0$. Does A have an extension B which is the generator of a C_0-semigroup on a Banach space $Y \supseteq X$, with $i\mathbb{R} \subseteq \rho(B)$ and $\sup_{s\in\mathbb{R}} \|R(is, B)\| < \infty$?

This question has a positive answer on Hilbert spaces. For a C_0-semigroup T on a Hilbert space, the following properties are equivalent:

(i) (4.4) holds;

(ii) T is quasi-hyperbolic;

(iii) T is the restriction of a hyperbolic C_0-semigroup on a Hilbert space to a closed invariant subspace;

(iv) A has an extension B which is the generator of a C_0-semigroup on a Hilbert space $Y \supseteq X$, with $i\mathbb{R} \subseteq \rho(B)$ and $\sup_{s\in\mathbb{R}} \|R(is, B)\| < \infty$.

The implication (i) $\Rightarrow$ (ii) was shown in Corollary 3.1 of [4]. Read's theorem may be applied as in Theorem 2.2 of [4] to show that (ii) $\Rightarrow$ (iii). Elementary theory of hyperbolic semigroups as in Section V.1c of [13] shows that (iii) $\Rightarrow$ (iv), and (iv) $\Rightarrow$ (i) is elementary.

5. LOWER BOUNDS FOR SEMIGROUPS

Let T be a C_0-semigroup on a Banach space X, and assume that, for some $\tau > 0$, there exists $c(\tau) > 0$ such that

$$\|T(\tau)x\| \geqslant c(\tau)\|x\| \quad (x \in X).$$

Using the semigroup property and local boundedness of $\|T(t)\|$, it is easy to see that

(5.1) $$\|T(t)x\| \geqslant c(t)\|x\| \quad (x \in X, t \geqslant 0),$$

where $c : (0, \infty) \to (0, \infty)$. The function

$$\gamma_T(t) := (\inf\{\|T(t)x\| : x \in X, \|x\| = 1\})^{-1},$$

is submultiplicative, i.e., $\gamma_T(s + t) \leqslant \gamma_T(s)\gamma_T(t)$, and locally bounded, and it satisfies $\gamma_T(t) \leqslant Me^{\alpha t}$ for some $M > 0$ and $\alpha \in \mathbb{R}$. Now (5.1) holds if and only if $c(t) \geqslant 1/\gamma_T(t)$.

In some parts of the literature, a C_0-semigroup is said to be "left-invertible" if (5.1) holds with $c(t) > 0$, but such terminology for semigroups might be interpreted in several different ways for semigroups. For greater precision, we shall say that T *satisfies lower bounds* when (5.1) holds, with $c(t) > 0$. Now we describe various notions of left-inverse for T.

If each operator $T(t)$ has a bounded left-inverse on X, then T satisfies lower bounds. When T satisfies lower bounds, each $T(t)$ is injective with closed range, so there is a bounded left-inverse operator $L_0(t) : \operatorname{Ran} T(t) \to X$ such that $L_0(t)T(t) = I$. In many cases, $L_0(t)$ can be extended to a bounded operator $L(t)$ on X, and then $L(t)T(t) = I$. However this extension process may be quite arbitrary, and L may not satisfy the semigroup property on X.

We shall say that a C_0-semigroup L on X is a *left-inverse semigroup* on X for T if $L(t)T(t)x = x$ for all $x \in X$, $t \geqslant 0$. More generally, a C_0-semigroup L on a Banach space $Y \supseteq X$ is a *left-inverse semigroup* on Y for T if $L(t)T(t)x = x$ for all $x \in X$, $t \geqslant 0$. Proposition 2.1 shows that L is a left-inverse semigroup for T if and only if the generator B of L is an extension of $-A$, where A is the generator of T. It is known that any C_0-semigroup satisfying lower bounds on a Hilbert space X has a left-inverse semigroup on X (see Theorem 7.3).

A particular case of a left-inverse semigroup occurs when T extends to a C_0-group S on $Y \supseteq X$ in the sense that X is invariant under $S(t)$ and $T(t) = S(t)|_X$, for all $t \geqslant 0$. In this case, we shall say that S is a C_0-*group extension* of T on Y. Then $L(t) := S(-t)$ is a left-inverse semigroup for T on Y, and the generator of S is an extension of A. However X will not be invariant under $S(-t)$ for $t > 0$ unless T is itself a C_0-group. We shall see in Proposition 5.3 that every C_0-semigroup satisfying lower bounds has a C_0-group extension on a larger Banach space.

When constructing left-inverse C_0-semigroups L or C_0-group extensions S for T, it is desirable also to keep control on their growth. Ideally the growth of $\|L(t)\|$ would be comparable with the growth of $\gamma_T(t)$, but this cannot be achieved precisely in general (see Proposition 5.3 and Example 5.6). In the case of group extensions, it would also be desirable that $\|S(t)\|$ is comparable to $\|T(t)\|$ for large $t > 0$.

We now recall some known results, beginning with the case when $c(t) = 1$ for all t. In the following proposition the equivalence of (i) and (ii) is a counterpart of the Lumer–Phillips theorem, from pp. 419, 420 of [15], and the equivalence of (i) and (iii) is from Proposition 2.2, Theorem 3.3 of [5].

PROPOSITION 5.1. *Let A be the generator of a C_0-semigroup T on a Banach space X. The following are equivalent:*

(i) *T is expansive, i.e., $\|T(t)x\| \geqslant \|x\|$ for all $x \in X$, $t \geqslant 0$.*

(ii) *$-A$ is dissipative.*

(iii) *There is a C_0-group extension S of T on a Banach space Y containing X such that $\|S(-t)\| \leqslant 1$ for all $t \geqslant 0$.*

The following theorem is a compilation of known results giving a version of Proposition 5.1 allowing arbitrary (submultiplicative) growth of the left-inverses. The property (i) of Theorem 5.2 implies (5.1) for $c(t) = \gamma(t)^{-1}$, by putting $n = 1$ and $t_1 = 0$. The converse (that (5.1) implies (i)) holds when $c(t) = 1$, or $c(t) = e^{-\alpha t}$ (see Proposition 5.1), but not in general (see Example 5.6). The implication (i) $\Rightarrow$ (ii) was noted in Remarks 3.2(iv) of [3], where details of the discrete counterpart were given. For (iii) $\Rightarrow$ (i), applying $L(t)$ to (5.2) gives

$$x = L(t - t_1)x_1 + \cdots + L(t - t_n)x_n,$$

and (i) follows.

THEOREM 5.2. *Let $\gamma : \mathbb{R}_+ \to (0, \infty)$ be a submultiplicative function. Let T be a C_0-semigroup on X. The following properties are equivalent:*
   (i) *Whenever $x, x_1, \ldots, x_n \in X$, $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t$, and*

(5.2) $$T(t)x = T(t_1)x_1 + \cdots + T(t_n)x_n,$$

we have

(5.3) $$\|x\| \leqslant \gamma(t - t_1)\|x_1\| + \cdots + \gamma(t - t_n)\|x_n\|.$$

   (ii) *There is a C_0-group extension S of T on a Banach space Y containing X such that $\|S(-t)\| \leqslant \gamma(t)$ for all $t \geqslant 0$.*
   (iii) *There is a left-inverse semigroup L for T on a Banach space Y containing X such that $\|L(t)\| \leqslant \gamma(t)$ for all $t \geqslant 0$.*

Zwart ([31], Section 3) has asked whether every C_0-semigroup satisfying lower bounds on a Banach space X has a left-inverse semigroup on X. The following new result shows that any semigroup satisfying lower bounds has a C_0-group extension S on a larger Banach space such that the exponential growth bound of $\{S(-t) : t \geqslant 0\}$ is arbitrarily close to the exponential growth bound of γ_T. In Example 5.6 we show that it may not be possible to arrange that $\|S(-t)\| = O(c_T(t)^{-1})$ as $t \to \infty$.

PROPOSITION 5.3. *Let T be a C_0-semigroup on a Banach space X satisfying lower bounds, and let $c > 0$ and $\alpha \in \mathbb{R}$ be such that*

$$\|T(t)x\| \geqslant ce^{-\alpha t}\|x\| \quad (x \in X, t > 0).$$

For each $\omega > \alpha$, there exist a C_0-group extension S on a Banach space Y containing X and $M \geqslant 1$ such that $\|S(t)\| = \|T(t)\|$ and $\|S(-t)\| \leqslant Me^{\omega t}$ for all $t > 0$.

Proof. By replacing $T(t)$ by $e^{\alpha t}T(t)$ and ω by $\omega - \alpha$, we may assume that $\alpha = 0$. Let $\omega > 0$ and $\tau = \omega^{-1}\log(c^{-1})$. By Theorem 1.1, there exist $Y \supseteq X$ and an isometric homomorphism $\varphi : \{T(\tau)\}' \to \mathcal{B}(Y)$ such that $\varphi(T(t))$ is an extension of $T(t)$ for all t, $\varphi(T(\tau))$ is invertible and $\|(\varphi(T(\tau)))^{-1}\| \leqslant c^{-1}$. By applying Proposition 2.2 and changing Y if necessary, we may arrange that there is a C_0-semigroup S on Y given by

$$S(t) = \varphi(T(t)) \quad (t \geqslant 0).$$

Since $S(\tau)$ is invertible, S is a C_0-group on Y extending T. Moreover

$$\|S(-\tau n)\| \leqslant c^{-n} = e^{\omega \tau n} \quad (n \geqslant 1).$$

This implies that

$$\|S(-t)\| \leqslant M e^{\omega t} \quad (t \geqslant 0),$$

where $M \geqslant 1$ is chosen suitably. ∎

Xu and Shang ([29], Theorem 2.4) have stated the following theorem.

THEOREM 5.4. *Let T be a C_0-semigroup on a Banach space X, with generator A. The following properties are equivalent:*

(i) *T satisfies lower bounds;*

(ii) *there exists an equivalent norm $\| \cdot \|_*$ on X such that, for some $\omega \in \mathbb{R}$, $-(A + \omega)$ is dissipative on $(X, \| \cdot \|_*)$.*

The proof of (i) $\Rightarrow$ (ii) in [29] appears to be incomplete, as the equivalent norm is taken to be

(5.4) $$\|x\|_* = \inf_{t \geqslant 0} \|e^{\alpha t} T(t) x\| \quad (x \in X),$$

where

(5.5) $$\|T(t)x\| \geqslant c e^{-\alpha t} \|x\| \quad (x \in X).$$

The authors then take $\omega = \alpha$ in (ii). However the definition of $\| \cdot \|_*$ in (5.4) may not satisfy the triangle inequality. So we give here a proof that (i) implies (ii), based on Proposition 5.3. This proof shows that when (5.5) holds for some c and α, then ω in (ii) can be any number with $\omega > \alpha$. We shall see in Example 5.6 that it may not be possible to take $\omega = \alpha$.

Proof. (i) $\Rightarrow$ (ii) Let c and α be as in (5.5), and $\omega > \alpha$. Let S be a C_0-group extension of T on $Y \supseteq X$ as in Proposition 5.3. By a standard renorming in semigroup theory ([13], Lemma II.3.10), there is an equivalent norm on Y given by

$$\|y\|_* = \sup\{e^{-\omega t} \|S(-t)y\| : t \geqslant 0\} \quad (y \in Y),$$

and $e^{-\omega t} S(-t)$ is a C_0-semigroup of contractions on $(Y, \| \cdot \|_*)$, so its generator is dissipative in that space. The generator is $-(B + \omega)$, where B is the generator of the C_0-semigroup S. By Proposition 2.1, B is an extension of A. Hence $-(A + \omega)$ is dissipative on $(X, \| \cdot \|_*)$. ∎

When A generates a C_0-semigroup T, the following result gives extension properties of T which characterise when there is an equivalent norm for which $-A$ is dissipative.

THEOREM 5.5. *Let T be a C_0-semigroup on a Banach space X, with generator A. The following are equivalent:*

(i) *There is an equivalent norm $\| \cdot \|_*$ on X such that $-A$ is dissipative with respect to $(X, \| \cdot \|_*)$.*

(ii) *There is a Banach space Y in which X is isomorphically embedded, and a C_0-group S on Y such that $\|S(-t)\|_{\mathcal{B}(Y)} \leqslant 1$ and $S(t)x = T(t)x$ for all $x \in X$, $t > 0$.*

(iii) *There is a C_0-group extension S of T on a Banach space $Z \supseteq X$ and a constant κ such that $\|S(-t)\|_{\mathcal{B}(Z)} \leqslant \kappa$ for all $t \geqslant 0$.*

(iv) *There exists κ such that*

$$\|x\| \leqslant \kappa(\|x_1\| + \cdots + \|x_n\|)$$

whenever $x_1, \ldots, x_n \in X$, $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t$ and

$$T(t)x = T(t_1)x_1 + \cdots + T(t_n)x_n.$$

Proof. (i) $\Rightarrow$ (ii) Apply Proposition 5.1 in the space $(X, \|\cdot\|_*)$.

(ii) $\Rightarrow$ (i) Let $\|\cdot\|_*$ be the norm of Y restricted to X. The generator of $\{S(-t) : t \geqslant 0\}$ is an extension of $-A$, and it is dissipative with respect to $\|\cdot\|_Y$.

(ii) $\Rightarrow$ (iii) Take $\kappa_1, \kappa_2 > 0$ such that $\kappa_1\|x\|_Y \leqslant \|x\|_X \leqslant \kappa_2\|x\|_Y$ for all $x \in X$. For $y \in Y$, let

$$\|y\|_Z = \inf\{\|x\|_X + \kappa_2\|y - x\|_Y : x \in X\}.$$

It is readily verified that $\|\cdot\|_Z$ is a norm on Y, equivalent to $\|\cdot\|_Y$, and $\|x\|_Z = \|x\|_X$ for all $x \in X$. Then we can take $Z = (Y, \|\cdot\|_Z)$.

(iii) $\Rightarrow$ (ii) Let $Y = (Z, \|\cdot\|_Y)$ where

$$\|y\|_Y = \sup\{\|S(-t)y\|_Z : t \geqslant 0\}.$$

This norm is equivalent to $\|\cdot\|_Z$ on Z, and hence equivalent to $\|\cdot\|_X$ on X, and $S(-t)$ is contractive with respect to $\|\cdot\|_Y$.

(iii) $\Leftrightarrow$ (iv) This follows from Theorem 5.2, taking $\gamma(t) = \kappa \geqslant 1$ for all $t \geqslant 0$. ∎

The following example shows that (5.1) for $c(t) = c$ does not imply the equivalent conditions of Theorem 5.5. Consequently, (5.1) for $c(t) = c$ does not imply that T is expansive in an equivalent norm on X, and one cannot take $\omega = \alpha$ in Proposition 5.3.

EXAMPLE 5.6. Let $c \in (0, 1)$ and $M > 1$ be given. There exists a C_0-semigroup T on a Hilbert space X with the following properties:

(i) $c\|x\| \leqslant \|T(t)x\| \leqslant \sqrt{2}M^t\|x\|$ for all $t \geqslant 0$, $x \in X$;

(ii) if L is any left-inverse semigroup for T on a Banach space $Y \supseteq X$ then $\{\|L(t)\| : t \geqslant 0\}$ is unbounded.

We begin by presenting examples satisfying a weaker version of (ii).

Given $M > 1$ and $c \in (0, 1)$, take $\varepsilon \in (0, c)$. Take $k_\varepsilon > 0$ and $\varepsilon' > 0$ such that

$$c^k < \varepsilon, \quad \frac{M^k(\varepsilon - c^k)}{2} > \frac{Mc}{1 - c}, \quad \frac{\varepsilon - c^k}{2} < \frac{M}{1 - c}\varepsilon' < \varepsilon - c^k.$$

Then let

$$\Omega = \{(x,y) \in \mathbb{R}_+^2 : x = 0 \text{ or } y \in k_\varepsilon \mathbb{N}\},$$

$$w(x,y) = \begin{cases} c^{\lceil y/k_\varepsilon \rceil - 1} & \text{if } 0 \leqslant x \leqslant t < x + y, \\ c^{y/k_\varepsilon} M^x \varepsilon' & \text{if } x > 0. \end{cases}$$

Let μ be one-dimensional Hausdorff measure on Ω, and let X_ε be the Hilbert space $L^2(\Omega, w, \mu)$ of all f for which

$$\|f\|^2 := \int_\Omega |f|^2 w^2 \, d\mu < \infty.$$

Let

$$(T_\varepsilon(t)f)(x,y) = \begin{cases} f(x - t, y) & \text{if } 0 \leqslant t < x, \\ f(0, y - t + x) & \text{if } 0 \leqslant x \leqslant t < x + y, \\ 0 & \text{otherwise.} \end{cases}$$

Then T_ε is a C_0-semigroup with $\|T_\varepsilon(t)\| \leqslant \sqrt{2} M^t$ and $\|T_\varepsilon(t)f\| \geqslant c\|f\|$ for all $f \in X$ and $t \geqslant 0$. Take $f \in X$ which is supported on $\{0\} \times (0, k_\varepsilon)$, with $\|f\| = 1$. Then there exist $g, f_i \in X$ with

$$T_\varepsilon(k_\varepsilon^2)f = g + \sum_{i=1}^{k_\varepsilon} T_\varepsilon(k_\varepsilon(k_\varepsilon - i))f_i, \quad \|g\| + \sum_{i=1}^{k_\varepsilon} \|f_i\| \leqslant \varepsilon.$$

We refer the reader to Section 3.2 of [14] for the detailed justification of these properties. It follows that for any left-inverse C_0-semigroup L on a Banach space $Y \supseteq X_\varepsilon$,

$$\sup_{0 \leqslant t \leqslant k_\varepsilon^2} \|L(t)\| \geqslant \varepsilon^{-1}.$$

To complete the construction, take the ℓ_2-direct sum X of the spaces X_{n-1} [3] ($n > 1/c$) as constructed above with $\varepsilon = n^{-1}$, and $k_\varepsilon = \lceil \log(2n)/\log(1/c) \rceil$ and let T be the direct sum of the C_0-semigroups T_{n-1}. Any left-inverse semigroup for T on $Y \supseteq X$ is then a left-inverse semigroup for T_{n-1} on X_{n-1} (regarded as a subspace of X in the natural way), so there exist t_n such that

$$0 \leqslant t_n \leqslant \left(1 + \frac{\log(2n)}{\log(1/c)}\right)^2, \quad \|L(t_n)\| \geqslant n \quad \left(n \geqslant \frac{1}{c}\right).$$

This implies that

$$\limsup_{t \to \infty} e^{-\sqrt{t}} \|L(t)\| > 0.$$

In particular $\|L(t)\|$ is not polynomially bounded in t.

6. OPERATORS ON HILBERT SPACES

In this section we give results for operators on Hilbert spaces, where the extensions should also be operators on Hilbert spaces. We start by reformulating Proposition 3.1 in a form which is specific to Hilbert spaces.

When X is a Hilbert space, we may take $Y = X \ominus \operatorname{Ran} A$ and $p = 2$ in the construction of Proposition 3.1. Then Z is also a Hilbert space, and the extension B constructed there satisfies $\|B^{-1}\| \leqslant c^{-1}$ and $\|B\| = \|A\|$ if A is bounded (note that $\|A\| \geqslant c$). If in addition A is densely defined we can construct the same (up to unitary equivalence) invertible extension of A via the polar decomposition of A, $A = PU$ where $P = (A^*A)^{1/2}$ is self-adjoint and U is a partial isometry. When A is bounded below as in (2.1), P is invertible and U is a unitary operator of X onto the closed subspace $\operatorname{Ran} A$.

PROPOSITION 6.1. *Let A be a closed, densely defined operator on a Hilbert space X, and assume that (2.1) holds for some $c > 0$. Let $A = UP$ be the polar decomposition of A. Let $\widetilde{Z}$ be the Hilbert space $X \times X$, and $\widetilde{B}$ be the operator on $\widetilde{Z}$ given by*

$$\widetilde{B} = \begin{pmatrix} A & c(I - UU^*) \\ 0 & cU^* \end{pmatrix},$$

with domain $D(\widetilde{B}) = D(A) \times X$. Then $\widetilde{B}$ has the following properties:
   (i) *$\widetilde{B}$ is an outer extension of A (when X is identified with $X \times \{0\}$);*
  (ii) *$\widetilde{B}$ is invertible and $\|\widetilde{B}^{-1}\| \leqslant c^{-1}$;*
 (iii) *if A is not invertible, then $\sigma(\widetilde{B}) \subseteq \sigma(A)$;*
 (iv) *if A generates a C_0-semigroup on X, then $\widetilde{B}$ generates a C_0-semigroup on Z;*
  (v) *if A is bounded, then $\widetilde{B}$ is bounded and $\|\widetilde{B}\| = \|A\|$.*

Proof. Replacing A by $c^{-1}A$ we will assume that $c = 1$, which simplifies the presentation.

It is clear that $(x, 0) \in D(\widetilde{B})$ if and only if $x \in D(A)$ and then $\widetilde{B}(x, 0) = (Ax, 0)$. Moreover, $\widetilde{B}$ is invertible with

$$\widetilde{B}^{-1} = \begin{pmatrix} P^{-1}U^* & 0 \\ I - UU^* & U \end{pmatrix},$$

and $\|\widetilde{B}^{-1}\| \leqslant 1$ because

$$\begin{aligned}
\|\widetilde{B}(x_1, x_2)\|^2 &= \|Ax_1 + (I - UU^*)x_2\|^2 + \|U^*x_2\|^2 \\
&= \|Ax_1\|^2 + \|(I - UU^*)x_2\|^2 + \|UU^*x_2\|^2 \\
&= \|Ax_1\|^2 + \|x_2\|^2 \geqslant \|x_1\|^2 + \|x_2\|^2.
\end{aligned}$$

Assume that A is not invertible. The assumption (2.1) on A implies that the approximate point spectrum of A is contained in $\{\lambda \in \mathbb{C} : |\lambda| \geqslant 1\}$. Since the boundary of $\sigma(A)$ consists of approximate eigenvalues and $0 \in \sigma(A)$ by assumption, it follows that the closed unit disc is contained in $\sigma(A)$. Thus if $\lambda \in \rho(A)$

then $|\lambda| > 1$, so $\lambda \in \rho(U^*)$, $R(\lambda, U^*) = \sum\limits_{j=0}^{\infty} \lambda^{-(j+1)}(U^*)^j$ and the operator

$$(6.1) \qquad R(\lambda, \widetilde{B}) = \begin{pmatrix} R(\lambda, A) & R(\lambda, A)(I - UU^*)R(\lambda, U^*) \\ 0 & R(\lambda, U^*) \end{pmatrix}$$

is a two-sided inverse of $\lambda - \widetilde{B}$. Thus $\lambda \in \rho(\widetilde{B})$. So $\sigma(\widetilde{B}) \subseteq \sigma(A)$.

The proof of (iv) is essentially the same as in Proposition 3.1, as the operator $\widetilde{B}$ is a bounded perturbation of

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

If A is bounded, then

$$\|\widetilde{B}(x_1, x_2)\|^2 = \|Ax_1\|^2 + \|x_2\|^2 \leqslant \|A\|^2 \, \|(x_1, x_2)\|^2,$$

since (2.1) for $c = 1$ implies that $\|A\| \geqslant 1$. ∎

The extension of A to $\widetilde{B}$ in Proposition 6.1 is not necessarily minimal. In order to achieve the minimal extension B, we should replace $\widetilde{Z}$ by the closure of $\bigcup\limits_{k \geqslant 0} \widetilde{B}^{-k}(X \times \{0\})$. This space is identified in the following way.

PROPOSITION 6.2. *Let X, $\widetilde{Z}$, A, c and $\widetilde{B}$ be as in Proposition 6.1, and identify X with the subspace $X \times \{0\}$ of $\widetilde{Z}$. Let Z be the closure of $\bigcup\limits_{k \geqslant 0} \widetilde{B}^{-k}(X)$ in $\widetilde{Z}$.*

For $k \geqslant 0$, let

$$\widetilde{Z}_k = \mathrm{Ran}(U^k(I - UU^*)), \quad Z_k = \{0\} \times \widetilde{Z}_k.$$

The subspaces $\widetilde{Z}_k$ of X are closed and pairwise orthogonal. Moreover

$$(6.2) \qquad Z = X \oplus \bigoplus_{k=0}^{\infty} Z_k,$$

as an orthogonal direct sum.

Let B be the restriction of $\widetilde{B}$ to $D(B) := D(A) \oplus \bigoplus\limits_{k=0}^{\infty} Z_k$, considered as an operator on Z. The following properties hold:

 (i) *B satisfies the properties (i)–(vi) of Proposition 3.1, when X is identified with $X \times \{0\}$, and $\|B^{-1}\| \leqslant c^{-1}$.*

 (ii) *$c^{-1}B$ is a unitary operator of Z_k onto Z_{k-1} for each $k \geqslant 0$, where $Z_{-1} = (X \ominus \mathrm{Ran}\, A) \subseteq X$.*

 (iii) *Let $W \supseteq X$ be a Hilbert space with closed subspaces W_k such that*

$$W = X \oplus \bigoplus_{k=0}^{\infty} W_k,$$

as an orthogonal direct sum, and let C be an invertible operator on W such that $\|C^{-1}\| \leqslant c^{-1}$, C extends A, $D(C) = D(A) \oplus (W \ominus X)$, and C maps W_k onto W_{k-1} for $k \geqslant 0$, where $W_{-1} = X \ominus \mathrm{Ran}\, A$. Then there is a contraction $\pi : Z \to W$ which acts as

the identity on X, maps $D(B)$ into $D(C)$, and satisfies $C \circ \pi = \pi \circ B$ on $D(B)$. If $c^{-1}C : W_k \to W_{k-1}$ is unitary for each $k \geqslant 0$, then $\pi : Z \to W$ is unitary.

Proof. We again assume that $c = 1$.

It is elementary that the spaces $\widetilde{Z}_k$ are closed and orthogonal in X and so the spaces Z_k are closed and orthogonal in $\widetilde{Z}$. It is easily seen that $\widetilde{B}$ maps Z_k isometrically onto Z_{k-1} for $k \geqslant 0$, and $\widetilde{B}^{-k}(X \times \{0\}) = X \times \widetilde{Z}_{k-1}$, for all $k \geqslant 1$. This establishes that (6.2) and property (ii) hold. Moreover, Z is invariant under $\widetilde{B}$ and $\widetilde{B}^{-1}$, and under $R(\lambda, \widetilde{B})$ for $|\lambda| > 1$. So properties (i)–(v) established in Proposition 6.1 for $\widetilde{B}$ transfer to B. To see (vi), one may use (iii), proved in the next paragraph, to deduce that the extension (Z, B) is unitarily equivalent to the extension (Z, B) obtained from Proposition 3.1 with $p = 2$ and $Y = Z_{-1}$, so (vi) transfers from the corresponding statement in Proposition 3.1. Alternatively one may carry out the calculations directly in the space Z (see [14]).

Let W, W_k and C be as in (iii). For each $k \geqslant 0$ the map $C^{-(k+1)}B^{k+1}$ is a contraction of Z_k onto W_k. Together with the identity map on X these maps define a contraction π of Z into W. The property that $C \circ \pi = \pi \circ B$ is easily seen to hold on X and each Z_k. If $C : W_k \to W_{k-1}$ is isometric and surjective for each k, then so is π. So (iii) is established. ∎

As in the case of Banach spaces, it would be most useful to have a Hilbert space construction in which a given operator A is extended to an invertible operator with preservation of its spectrum and of norms of associated bounded operators, and also including the existence of a suitable homomorphism φ from $\{A\}'$. This would be similar to Theorem 1.1 but with X and Y both Hilbert spaces. By adapting Arens's method, Badea and Müller ([3], Corollary 4.8) obtained the following result for bounded operators. It has most of the desired features but the estimates for the norms of the powers of the inverse are not as sharp as one might expect.

THEOREM 6.3. *Let U be a bounded operator on a Hilbert space X and assume that*

$$\|Ux\| \geqslant c\|x\| \quad (x \in X)$$

for some $c > 0$. Then there exists a Hilbert space $Y \supseteq X$ and a unital homomorphism $\varphi : \{U\}' \to \mathcal{B}(Y)$ such that:
   (i) *$\varphi(U)$ is invertible and $\|\varphi(U)^{-k}\| \leqslant c^{-k}2^{(k+1)/2}$ for all $k \geqslant 1$;*
   (ii) *for each $V \in \{U\}'$, $\varphi(V)$ is an extension of V and $\|\varphi(V)\| \leqslant \sqrt{2}\|V\|$.*

We note also that Read [25] gave a version of Theorem 1.2 for bounded operators on Hilbert space, in which the norm of the extensions was almost preserved.

OPEN QUESTION 6.4. For which bounded operators U on a Hilbert space satisfying (2.1) do there exist an invertible extension V of U on a Hilbert space $Y \supseteq X$ and an isometric unital homomorphism $\varphi : \{U\}' \to \mathcal{B}(Y)$ with $\varphi(U) = V$?

7. SEMIGROUPS ON HILBERT SPACES

Recall that a densely defined operator A on a Hilbert space X generates a C_0-semigroup of contractions if and only if A is a maximal dissipative operator, i.e., A is dissipative and it has no proper extension which is a dissipative operator on X ([9], Theorem 10.4.2). Moreover, any dissipative operator on X has a maximal dissipative extension on X (see Theorem 3.1.2 of [16] for a simple proof).

The following result gives a different type of extension, by showing that any dissipative operator on a Hilbert space has an *outer* extension which generates a contraction semigroup on a larger Hilbert space.

THEOREM 7.1. *Let A be a closed, densely defined, dissipative operator on a Hilbert space X, and assume that there exists $\mu \in \mathbb{C}$ such that $\operatorname{Re}\mu < 0$ and $\mu \notin \sigma_{\mathrm{p}}(A)$. Then there exist a Hilbert space $Y \supseteq X$ and an outer extension B of A such that B generates a C_0-semigroup of contractions on Y.*

Proof. Substituting A by $aA + ib$ for some $a > 0$ and $b \in \mathbb{R}$ if necessary, we may assume that $-1 \notin \sigma_{\mathrm{p}}(A)$. Consider the Cayley transform of $-A$:

$$C := (I - A)(I + A)^{-1}, \quad D(C) = \operatorname{Ran}(I + A).$$

Then C is closed, $I + C = 2(I + A)^{-1}$ is injective with dense range $D(A)$, and $A = (I - C)(I + C)^{-1}$ on $D(A)$. Moreover, $\|Cx\| \geqslant \|x\|$ for all $x \in X$, by putting $y = (I + A)^{-1}x$ in

$$\|(I - A)y\|^2 - \|(I + A)y\|^2 = -4\operatorname{Re}\langle Ay, y \rangle \geqslant 0.$$

We apply Proposition 6.2 to the operator C, putting $c = 1$ and $\lambda = -1$ in the statement of Proposition 3.1(vi). Then C has an extension to a bounded invertible operator G with $\|G^{-1}\| \leqslant 1$ on some Hilbert space $Y \supseteq X$, $\operatorname{Ker}(I + G) = \{0\}$ and $\operatorname{Ran}(I + G)$ is dense in Y, and $\operatorname{Ran}(I + G) \cap X = \operatorname{Ran}(I + C)$. Thus the operator

$$B := (I - G)(I + G)^{-1}, \quad D(B) = \operatorname{Ran}(I + G),$$

is well-defined with dense domain. Moreover

$$I + B = 2(I + G)^{-1}, \quad I - B = 2G(I + G)^{-1},$$

so $1 \in \rho(B)$, $(I - B)^{-1} = (1/2)(G^{-1} + I)$, and the Cayley transform of B is

$$(I + B)(I - B)^{-1} = G^{-1}.$$

Since the Cayley transform of B is contractive, B is dissipative. Thus B generates a C_0-semigroup of contractions on Y, by the Lumer–Phillips theorem.

To show that B extends A, we have

$$D(B) \cap X = \operatorname{Ran}(I + G) \cap X = \operatorname{Ran}(I + C) = D(A).$$

For $x \in D(A)$, $x = (I + C)x'$ for some $x' \in X$ and

$$(I + G)(x + Bx) = (I + G)(2x') = 2x = (I + C)(2x')$$
$$= (I + C)(x + Ax) = (I + G)(x + Ax).$$

Since $I + G$ is injective, $Bx = Ax$. ∎

Now we turn to lower bounds for semigroups on Hilbert space. The following is a Hilbert space version of Proposition 5.1, in which the left inverses act on the same space as the original operator.

PROPOSITION 7.2. *Let A be the generator of a C_0-semigroup T on a Hilbert space X. The following are equivalent:*
(i) *T is expansive, i.e., $\|T(t)x\| \geqslant \|x\|$ for all $x \in X, t \geqslant 0$;*
(ii) *$-A$ is dissipative;*
(iii) *there is a left-inverse semigroup for T consisting of contractions on X.*

Proof. The equivalence of (i) and (ii) was established in Proposition 5.1, and it is elementary that (iii) implies (i).

Assume that $-A$ is dissipative. Then $-A$ has a maximal dissipative extension B which generates a C_0-semigroup L of contractions on X. By Proposition 2.1(ii), L is a left-inverse semigroup for T. ∎

We do not know whether the properties in Proposition 7.2 are also equivalent to the existence of a C_0-group extension S of T on a Hilbert space $Y \supseteq X$ such that $\|S(-t)\| \leqslant 1$ for all $t \geqslant 0$.

The following includes results of Louis and Wexler ([20], Corollary, p. 260) and Zwart [31].

THEOREM 7.3. *Let T be a C_0-semigroup on a Hilbert space X, with generator A. The following are equivalent:*
(i) *T satisfies lower bounds;*
(ii) *there is a left-inverse semigroup for T on X;*
(iii) *there is a C_0-group extension S of T on a Hilbert space $Y \supseteq X$;*
(iv) *there exists $Q \in \mathcal{B}(X)$ and an equivalent inner product on X such that the C_0-semigroup generated by $A + Q$ is isometric in the equivalent norm.*
If $c > 0$ and $\alpha \in \mathbb{R}$ are such that $\|T(t)x\| \geqslant ce^{-\alpha t}\|x\|$ for all $x \in X$ and $t > 0$, and $\omega > \alpha$, then Y and S in (iii) may be chosen so that $\|S(t)\| \leqslant \sqrt{2}\|T(t)\|$ and $\|S(-t)\| \leqslant Me^{\omega t}$ for all $t > 0$, for some constant M.

Proof. (i) $\Rightarrow$ (ii) This was proved in Corollary, p. 260 of [20], and an alternative proof was given in Theorem 1 of [31].

(i) $\Rightarrow$ (iii) This can be proved in a very similar way to Proposition 5.3, using Theorem 6.3 instead of Theorem 1.1.

(i) $\Rightarrow$ (iv) This was proved in Theorem 3 of [31].

(iv) ⇒ (iii) Let $\widetilde{X}$ be the space X with the equivalent inner product and $\widetilde{T}$ be the isometric semigroup generated by $A + Q$ on $\widetilde{X}$. By a result originally due to Cooper [8] (see also [11]), $\widetilde{T}$ has an extension to a unitary group $\widetilde{S}$ on a Hilbert space $\widetilde{Y} \supseteq \widetilde{X}$. Let $\| \cdot \|_{\widetilde{Y}}$ be the norm on $\widetilde{Y}$ and P be the orthogonal projection of Y onto $\widetilde{X}$. Define

$$\|y\|_Y^2 = \|Py\|_{\widetilde{X}}^2 + \|(I-P)y\|_{\widetilde{Y}}^2 \quad (y \in \widetilde{Y}).$$

This is an equivalent norm on $\widetilde{Y}$ and it coincides with $\| \cdot \|_X$ on X. The space $Y :=$ $(\widetilde{Y}, \| \cdot \|_Y)$ is a Hilbert space in the norm $\| \cdot \|_Y$. Let $\widetilde{B}$ be the generator of $\widetilde{S}$ and let S be the C_0-group on Y generated by the bounded perturbation $B := \widetilde{B} - QP$. Then B extends A and S extends T.

As in Section 5 either of (ii) or (iii) implies (i). ∎

The proofs in [20] and [31] use ideas from control theory. In contrast to Proposition 5.3, these arguments give little information about the norm of $\|S(t)\|$ for either $t > 0$ or $t < 0$.

We conclude with some remarks concerning the relations between Zwart's result ([31], Theorem 3) (the implication (i) ⇒ (iv) in Theorem 7.3) and a similar result of Haase ([17], Theorem 3.1; [18], Theorem 7.2.8). Haase's result says that if B is the generator of a C_0-group on a Hilbert space Y then B is a bounded perturbation of the generator of a group of unitaries with respect to an equivalent inner product on Y. In an earlier version of this paper, we raised the question whether the two results are logically related, and we are grateful to Abraham Ng for providing the key to the following arguments.

First, we assume Zwart's result, and we will show that Haase's result follows. Let B generate a C_0-group on a Hilbert space X. Then the semigroup generated by B satisfies lower bounds, so Zwart's result implies that there exist $Q \in \mathcal{B}(X)$ and an equivalent inner product on X such that $B - Q$ generates a semigroup of isometries in the new norm. In addition, $B - Q$ generates a C_0-group, so it generates a C_0-group of unitaries in the new norm. This gives Haase's result.

Now, we assume Haase's result, and we will use it to show the implication (iii) ⇒ (iv) in Theorem 7.3. Let A be the generator of a C_0-semigroup T on a Hilbert space X, with a C_0-group extension S on a Hilbert space $Y \supseteq X$, with generator B. By Haase's result, there exists $Q \in \mathcal{B}(Y)$ such that $B - Q$ generates a C_0-group of unitaries for an equivalent inner product on Y. Let $\widetilde{Y}$ and $\widetilde{X}$ be the spaces Y and X with the equivalent inner product, and let P be the orthogonal projection of $\widetilde{Y}$ onto $\widetilde{X}$. For $x \in D(A) = D(B) \cap X$,

$$Ax = PAx = P(B-Q)x + PQx.$$

Let $Q_1 \in \mathcal{B}(X)$ be the restriction of PQ to X, and let A_1 be the restriction of $P(B - Q)$ to the domain $D(A)$. Then $A_1 = A - Q_1$ is the generator of a C_0-semigroup on X. Since $B - Q$ is skew-symmetric on $\widetilde{Y}$, A_1 is skew-symmetric on

$\widetilde{X}$. Hence the semigroup generated by A_1 is a semigroup of isometries on $\widetilde{X}$. This establishes (iv).

Finally, suppose that $\omega > \alpha$ as in Theorem 7.3 and in addition ω is greater than the exponential growth bound of T. Then the equivalent inner product and the bounded operator Q in (iv) can be chosen so that Q is self-adjoint on $\widetilde{X}$ and $\|Q\|_{\mathcal{B}(\widetilde{X})} \leqslant \omega$. This follows from the corresponding statements in Haase's result combined with the proof in the paragraph above.

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