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C^*-ALGEBRAS ASSOCIATED WITH ENDOMORPHISMS OF GROUPS

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ABSTRACT. In this work we construct a C^*-algebra from an injective endomorphism of a discrete group G allowing the endomorphism to have infinite cokernel. We generalize some results obtained by I. Hirshberg and by J. Cuntz together with A. Vershik. For certain cases of the above construction, we show that Kirchberg's classification theorem can be applied.

KEYWORDS: *Group, endomorphism, semigroup C^*-algebra, K-theory, crossed product.*

MSC (2010): 47L65, 46L80.

INTRODUCTION

In [13] Hirshberg defined a C^*-algebra associated with endomorphisms of groups with finite cokernel. A natural continuation of that paper is to construct the same C^*-algebra for endomorphisms with infinite cokernel. So in this paper we define and study a universal C^*-algebra constructed from an injective endomorphism φ with infinite cokernel of a discrete countable group. In other words, the main difference between our work and Hirshberg's ([13]) is that we allow

$$\Big|\frac{G}{\varphi(G)}\Big| = \infty.$$

In order to generalize the constructions, we also associate the C^*-algebra to a family B of subgroups of G and call it $\mathbb{U}[\varphi, B]$. Their rôle is to naturally implement the multiplication rule inside $\mathbb{U}[\varphi, B]$, because here we do not have finitely many projections summing up to one. The relations defining $\mathbb{U}[\varphi, B]$ are dictated by the natural representations of φ, B and G on the Hilbert space $l^2(G)$. The group elements are represented by unitaries, the elements of B are associated with projections and an isometry represents φ. All these operators generate a concrete C^*-subalgebra of $\mathcal{L}(l^2(G))$. We prove that, in some cases, this concrete C^*-subalgebra is isomorphic to the C^*-algebra $\mathbb{U}[\varphi, B]$.

Constructions like the one presented in Hirshberg's paper [13] have been studied before by various authors [3], [8], [10], [11], [17] and [18]. In particular, somewhat similar C^*-algebras have been associated with endomorphisms of abelian groups and also with semigroups. Also the ring C^*-algebras studied in [8], [10] arise in a similar way. Constructions along the same lines are considered in [23].

The group elements give rise to unitary operators $\{U_g\}_{g\in G}$ acting on $l^2(G)$ by left multiplication, and the endomorphism induces an isometry S acting on $l^2(G)$ through φ. Denoting by $\{\xi_h : h \in G\}$ the canonical orthonormal basis of $l^2(G)$, S is defined by $S(\xi_h) := \xi_{\varphi(h)}$. For every element H of B, consider the projection $E_{[H]}$ so that

$$E_{[H]}(\xi_g) = \begin{cases} \xi_g & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

The C^*-subalgebra of $\mathcal{L}(l^2(G))$ generated by the above operators is denoted by $C^*_{\mathrm{r}}[\varphi, B]$ (Definition 1.1). We note that $E_{[H]}$ satisfy some natural relations, and we use these relations to define the universal C^*-algebra $\mathbb{U}[\varphi, B]$, associated with φ. Particularly, the sum condition which appears in [11] does not hold in our setting (we would have an inequality of the form $< \infty$). However, the projections associated with the subgroups of G in $\mathbb{U}[\varphi, B]$ play a key role to prove our main results.

Our most important contribution is given in Theorem 5.9. We prove that $\mathbb{U}[\varphi, B]$ is a Kirchberg algebra under the assumptions that G is amenable, that the intersection of the elements of B contains some image of G through φ and that φ is pure, the latter meaning that $\bigcap_{n\in\mathbb{N}} \varphi^n(G) = \{e\}$, the unit of G. In particular, this implies that in this case $\mathbb{U}[\varphi, B]$ and $C^*_{\mathrm{r}}[\varphi, B]$ are isomorphic. This result also extends the ones obtained by Hirshberg in [13] and by Cuntz and Vershik in [11].

In order to prove the statement above, it is crucial to use a semigroup crossed product description of $\mathbb{U}[\varphi, B]$. Here we consider the definition of a semigroup crossed product presented by Li in Appendix A of [17] using covariant representations. The semigroup implementing the crossed product can be the semidirect product $S := G \rtimes_\varphi \mathbb{N}$ or the semigroup of natural numbers $\mathbb{N}$. Such a description allows us to use the six term exact sequence presented by Khoshkam and Skandalis [14] to calculate the K-theory of our C^*-algebra, in a similar way as in [11].

The above semigroup crossed product description implies the existence of a (full corner) group crossed product description of $\mathbb{U}[\varphi, B]$ ([6], [7] and [16]), using the group of integers $\mathbb{Z}$. It allows one to use the classical Pimsner–Voiculescu exact sequence [20] to calculate their K-groups.

We will see that if we start with B containing any subsets (instead of subgroups) of the type $g\varphi^n(G)$, $g \in G$, it leads to the same C^*-algebra. In particular, we can choose $B = \{G\}$. It gives interesting examples, and we then denote the corresponding C^*-algebra by $\mathbb{U}[\varphi]$. In this case, the isomorphism above is not the only way to represent it as a crossed product: analogously to the work of G. Boava and R. Exel in [1] one can show that $\mathbb{U}[\varphi]$ has a partial group crossed product description, which can also be related to an inverse semigroup crossed product by [12]. Apart from giving $\mathbb{U}[\varphi]$ another description by an established structure, this result also provides another way to prove the simplicity of $\mathbb{U}[\varphi]$ in some cases.

It can be noted that a particular semigroup is very important in our constructions, namely the semigroup $S = G \rtimes_\varphi \mathbb{N}$. We prove that when the group G is amenable and φ is pure, the three semigroup C^*-algebras defined by Li in [18] — namely $C^*(S)$, $C^*_{\mathrm{s}}(S)$ and $C^*_{\mathrm{r}}(S)$ — associated with the semigroup S are isomorphic to $\mathbb{U}[\varphi]$. We also prove that they are nuclear, simple and purely infinite (Theorem 6.4), answering partially to one open question in [18].

To finish, using the semigroup crossed product description of $\mathbb{U}[\varphi]$, we study its K-theory. Using a natural split exact sequence and the six term exact sequence provided by Khoshkam and Skandalis [14] we easily conclude that the K-groups of $\mathbb{U}[\varphi]$ are the same as the ones of $C^*(G)$. This implies that, imposing some extra conditions, $\mathbb{U}[\varphi]$ is a Kirchberg algebra which satisfies UCT.

A weaker version of the result concerning the K-theory of $\mathbb{U}[\varphi]$ can be obtained independently using some recent results by Cuntz, Echterhoff and Li in [9].

1. DEFINITIONS AND BASIC RESULTS

Throughout the paper, G will be a discrete countable group with unit e and φ an injective endomorphism (monomorphism) of G with infinite cokernel. When necessary, we require the amenability of G or φ to be pure. We want to construct a C^*-algebra associated with φ. To generalize Hirshberg's constructions even more, we also want to associate the C^*-algebra with some set B of subgroups of G which contains G. We consider it to have a natural behavior of the multiplication rule inside the C^*-algebra.

We now define $C(B)$ as the smallest set of subsets of G containing B and closed under finite unions, finite intersections, complements and under images of φ. These conditions will be called *regularity conditions*. Thus we have the following concrete C^*-algebra.

DEFINITION 1.1. Consider B a family of subgroups of G (containing G) defined as above. Let $C^*_{\mathrm{r}}[\varphi, B]$ denote the *reduced C^*-algebra* generated by the projections $\{E_{[X]} : X \in C(B)\}$, the unitaries $\{U_g : g \in G\}$ and the isometry S.

By studying the properties of the operators above, it is natural to define the universal version of that C^*-algebra.

DEFINITION 1.2. As above choose a set B of subgroups of G (containing G) and construct the family $C(B)$. Then $\mathbb{U}[\varphi, B]$ is the *universal C^*-algebra* generated by the projections $\{e_{[X]} : X \in C(B)\}$, the unitaries $\{u_g : g \in G\}$ and one isometry s satisfying:

(i) $u_g s^n u_h s^m = u_{g\varphi^n(h)} s^{n+m}$;
(ii) $u_g s^n e_{[X]} s^{*n} u_{g^{-1}} = e_{[g\varphi^n(X)]}$;
(iii) $e_{[G]} = 1$;
(iv) $e_{[X]} e_{[Y]} = e_{[X\cap Y]}$ and
(v) $e_{[X]} + e_{[Y]} = e_{[X\cup Y]} + e_{[X\cap Y]}$.

Since $u_g s^n s^{*n} u_{g^{-1}} = e_{[g\varphi^n(G)]}$, the projections $u_g s^n s^{*n} u_{g^{-1}}$ commute and considering $n \geqslant m$, we have:

$$\begin{aligned} u_g s^n s^{*n} u_{g^{-1}} u_h s^m s^{*m} u_{h^{-1}} &= e_{[g\varphi^n(G)]} e_{[h\varphi^m(G)]} = e_{[g\varphi^n(G)\cap h\varphi^m(G)]} \\ &= \begin{cases} e_{[g\varphi^n(G)]} & \text{if } h \in g\varphi^m(G), \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} u_g s^n s^{*n} u_{g^{-1}} & \text{if } h \in g\varphi^m(G), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

REMARK 1.3. If we consider any set C of subsets of G closed under the regularity conditions, the construction above also can be done.

First of all we will see that only the initial set B is important to generate the C^*-algebras above, and the fact that in $C(B)$ some elements are not subgroups of G is not a problem. Note that some elements of $C(B)$ are given by

$$g \bigcap_{i=1}^{m} \varphi^{n_i}(H_i)$$

with $g \in G$, $n_i \in \mathbb{N}$ and $H_i \in B$. In fact, we can use these elements to describe the $*$-algebra $\operatorname{span}(\{e_{[X]} : X \in C(B)\})$.

LEMMA 1.4. *Define*

$$B' := \Big\{ \bigcap_{i=1}^{m} \varphi^{n_i}(H_i) : H_i \in B, n_i \in \mathbb{N} \Big\}.$$

Then $\operatorname{span}(\{e_{[X]} : X \in C(B)\}) \cong \operatorname{span}(\{e_{[gH']} : g \in G, H' \in B'\}) =: D'$.

Proof. $\supseteq$ Obvious.

$\subseteq$ Let us call $K' := \{X \subseteq G : e_{[X]} \in D'\}$. It is obvious that $B \subseteq K'$. Moreover K' is closed under:

(•) $\bigcap_{i=1}^{n}$: By definition, for $X_1, X_2 \in K'$ it holds that

$$e_{[X_1\cap X_2]} = e_{[X_1]} e_{[X_2]}.$$

(•) Complements: For $X \in K'$:

$$e_{[X^c]} = 1 - e_{[X]} = e_{[G]} - e_{[X]} \in D'.$$

(•) $\bigcup_{i=1}^{n}$: Note that $X \cup Y = [X^c \cap Y^c]^c \in K', \forall X, Y \in K'$.

(•) And if $X \in K'$, $g \in G$ and $n \in \mathbb{N}$, the injectivity of φ implies $g\varphi^n(X) \in K'$. Therefore K' satisfies the regularity conditions, and $C(B) \subseteq K'$, because $C(B)$ is the smallest set containing B satisfying it. Then $\mathrm{span}(\{e_{[X]} : X \in C(B)\}) \subseteq D'$. ∎

This result leads to an important and simpler way to describe $\mathbb{U}[\varphi, B]$.

PROPOSITION 1.5. *The universal C^*-algebra $\mathbb{U}[\varphi, B]$ is generated by*

$$\{e_{[H]}, u_g, s : H \in B, g \in G\}.$$

Proof. Due to last lemma we only have to prove that

$$\mathrm{span}(\{e_{[gH']} : g \in G, H' \in B'\}) \subseteq \mathrm{span}\{e_{[H]}, u_g, s : H \in B, g \in G\}.$$

But $e_{[gH']} = u_g e_{[H']} u_{g^{-1}}$ and for $H' = \bigcap_{i=1}^{n} \varphi^{n_i}(H_i) \in B'$ with $n_i \in \mathbb{N}$ and $H_i \in B \cup \{G\}$ we have

$$e_{[H']} = \prod_{i=1}^{n} e_{[\varphi^{n_i}(H_i)]} = \prod_{i=1}^{n} s^{n_i} e_{[H_i]} s^{*n_i}.$$

Therefore, $e_{[gH']} \in C^*(\{e_{[H]}, u_g, s : H \in B, g \in G\})$. ∎

REMARK 1.6. Note that the lemma and the proposition above hold for any choice of B (i.e, even if it does not consist of subgroups).

Another interesting basic result is the following.

PROPOSITION 1.7. *Consider $\overline{B}$ containing only sets of the form $g_i\varphi^n(H_i)$, with $g_i \in G$ and H_i subsets of G. Then*

$$\mathbb{U}[\varphi, \overline{B}] \cong \mathbb{U}[\varphi, B]$$

where B contains only the subsets H_i.

Proof. By Proposition 1.5 (and the remark above),

$$\mathbb{U}[\varphi, \overline{B}] = C^*(\{e_{[g_iH_i]}, u_g, s : g_iH_i \in \overline{B}, g \in G\}),$$
$$\mathbb{U}[\varphi, B] = C^*(\{e_{[H_i]}, u_g, s : H_i \in B, g \in G\}).$$

But since

$$e_{[g_i\varphi^n(H_i)]} = u_{g_i} s^n e_{[H_i]} s^{*n} u_{g_i^{-1}},$$

both C^*-algebras are isomorphic. ∎

REMARK 1.8. If we choose $B = \{G\}$ then $\mathbb{U}[\varphi, B]$ is generated only by the unitary elements $\{u_g : g \in G\}$ and the isometry s. Furthermore, it can be viewed as a natural generalization of the constructions in [13] and [11]. This case will be studied in Section 5.

2. CROSSED PRODUCT DESCRIPTIONS

Define

$$D[\varphi, B] := C^*(\{u_g s^n e_{[H]} s^{*n} u_{g^{-1}} : g \in G, n \in \mathbb{N}, H \in B\})$$

and note that it is a commutative C^*-subalgebra of $\mathbb{U}[\varphi, B]$ because we have $u_g s^n e_{[H]} s^{*n} u_{g^{-1}} = e_{[g\varphi^n(H)]}$. We can define an action of the semigroup $S = G \rtimes_\varphi \mathbb{N}$ on $D[\varphi, B]$ via

$$\begin{aligned} \alpha : S &\to \operatorname{End}(D[\varphi, B]) \\ (g, n) &\mapsto u_g s^n (\cdot) s^{*n} u_{g^{-1}}. \end{aligned}$$

PROPOSITION 2.1. *The C^*-algebra $\mathbb{U}[\varphi, B]$ is isomorphic to $D[\varphi, B] \rtimes_\alpha S$.*

Proof. In this proof we use the universality of both C^*-algebras to find the desired isomorphism. Recall from [17], that $D[\varphi, B] \rtimes_\alpha S$ together with

$$\begin{aligned} \iota_D : D[\varphi, B] &\to D[\varphi, B] \rtimes_\alpha S \\ x &\mapsto \iota_D(x) \end{aligned}$$

and

$$\begin{aligned} \iota_S : S &\to \operatorname{Isom}(D[\varphi, B] \rtimes_\alpha S) \\ (g, n) &\mapsto \iota_S(g, n) \end{aligned}$$

satisfying

$$\iota_D(u_g s^n x s^{*n} u_{g^{-1}}) = \iota_S(g, n)\iota_D(x)\iota_S(g, n)^*$$

is the semigroup crossed product of the dynamic system $(D[\varphi, B], S, \alpha)$. But note that $\mathbb{U}[\varphi, B]$ together with

$$\begin{aligned} \pi : D[\varphi, B] &\to \mathbb{U}[\varphi, B] \\ x &\mapsto x \end{aligned}$$

and

$$\begin{aligned} \rho : S &\to \operatorname{Isom}(\mathbb{U}[\varphi, B]) \\ (g, n) &\mapsto u_g s^n \end{aligned}$$

is a covariant representation of $(D[\varphi, B], S, \alpha)$, since

$$\rho(g, n)\pi(x)\rho(g, n)^* = u_g s^n x s^{*n} u_{g^{-1}} = \pi(\alpha_{(g,n)}(x)).$$

So we conclude that there exists a $*$-homomorphism

$$\Phi : D[\varphi, B] \rtimes_\alpha S \to \mathbb{U}[\varphi, B] \tag{2.1}$$

such that $\Phi \circ \iota_D = \pi$ and $\Phi \circ \iota_S = \rho$.

On the other hand, it is well known that the crossed product $D[\varphi, B] \rtimes_\alpha S$ is generated as a C^*-algebra by elements of the form $\iota_S(g, n)$ and $\iota_D(e_{[H]})$ with $H \in B$. Identifying $\iota_S(g, n)$ with $u_g s^n$ and $\iota_D(e_{[H]})$ with $e_{[H]}$, it is easy to check that they satisfy conditions (i)–(v) of Definition 1.2 which generate $\mathbb{U}[\varphi, B]$.

Therefore, we have a $*$-homomorphism

$$\begin{aligned}\Delta : \mathbb{U}[\varphi, B] &\to D[\varphi, B] \rtimes_\alpha S \\ u_g s^n &\mapsto \iota_S(g, n) \\ e_{[H]} &\mapsto \iota_D(e_{[H]}).\end{aligned} \tag{2.2}$$

We now show that (2.1) and (2.2) are inverses of each other:

$$\begin{aligned}\Phi \circ \Delta(u_g) &= \Phi(\iota_S(g, 0)) = \rho(g, 0) = u_g, \\ \Phi \circ \Delta(s) &= \Phi(\iota_S(0, 1)) = \rho(0, 1) = s, \\ \Phi \circ \Delta(e_{[H]}) &= \Phi(\iota_D(e_{[H]})) = \pi(e_{[H]}) = e_{[H]},\end{aligned}$$

and on the other side

$$\begin{aligned}\Delta \circ \Phi(\iota_S(g, n)) &= \Delta(\rho(g, n)) = \Delta(u_g s^n) = \iota_S(g, n), \\ \Delta \circ \Phi(\iota_D(e_{[H]})) &= \Delta(\pi(e_{[H]})) = \Delta(\iota_D(e_{[H]})) = \iota_D(e_{[H]}).\end{aligned}$$

Thus, $\mathbb{U}[\varphi, B]$ and $D[\varphi, B] \rtimes_\alpha S$ are isomorphic. ∎

REMARK 2.2. Note that $\mathbb{U}[\varphi, B]$ is also isomorphic to $(D[\varphi, B] \rtimes_\omega G) \rtimes_\tau \mathbb{N}$:

$$\begin{aligned}\omega : G &\to \mathrm{Aut}(D[\varphi, B]) \\ g &\mapsto u_g(\cdot)u_{g^{-1}}, \\ \tau : \mathbb{N} &\to \mathrm{End}(D[\varphi, B] \rtimes_\omega G) \\ n &\mapsto s^n(\cdot)s^{*n}\end{aligned}$$

where for $a_g\delta_g$ of $D[\varphi, B] \rtimes_\omega G$, $\tau_n(a_g\delta_g) = s^n a_g s^{*n} \delta_{\varphi^n(g)}$.

Using the minimal automorphic dilation presented by Laca in [16] it is possible to see the C^*-algebra $\mathbb{U}[\varphi, B]$ as a corner of a group crossed product. For this, we need to prove the following proposition.

PROPOSITION 2.3. *The semidirect product $S = G \rtimes_\varphi \mathbb{N}$ is an Ore semigroup, i.e, it is cancellative and right-reversible.*

Proof. Consider $(g_i, n_i) \in S$ for $i \in \{1, 2, 3\}$. S is cancellative:

$$\begin{aligned}(g_1, n_1)(g_3, n_3) = (g_2, n_2)(g_3, n_3) &\Rightarrow (g_1\varphi^{n_1}(g_3), n_1 + n_3) = (g_2\varphi^{n_2}(g_3), n_2 + n_3) \\ &\Rightarrow n_1 = n_2 \quad \text{and} \\ g_1\varphi^{n_1}(g_3) = g_2\varphi^{n_1}(g_3) &\Rightarrow g_1 = g_2; \\ (g_1, n_1)(g_2, n_2) = (g_1, n_1)(g_3, n_3) &\Rightarrow (g_1\varphi^{n_1}(g_2), n_1 + n_2) = (g_1\varphi^{n_1}(g_3), n_1 + n_3) \\ &\Rightarrow n_2 = n_3 \quad \text{and} \\ \varphi^{n_1}(g_2) = \varphi^{n_1}(g_3) &\Rightarrow g_2 = g_3 \quad \text{as } \varphi \text{ is injective.}\end{aligned}$$

Also any two principal left ideals of S intersect:

$$\begin{aligned}(\varphi^{n_2}(g_1^{-1}), n_2)(g_1, n_1) &= (e, n_2 + n_1) \\ &= (\varphi^{n_1}(g_2^{-1}), n_1)(g_2, n_2) \in S(g_1, n_1) \cap S(g_2, n_2).\end{aligned}$$ ∎

It follows that the semigroup S can be embedded in a group, called the enveloping group of S. We will denote it as $\mathrm{env}(S)$, such that $S^{-1}S = \mathrm{env}(S)$ ([16], Theorem 1.1.2). It also implies that S is a directed set by the relation defined by $(g,n) < (h,m)$ if $(h,m) \in S(g,n)$. Let us define a candidate for $\mathrm{env}(S)$. Consider

$$\mathbb{G} := \varinjlim\{G_n : \varphi^n\}$$

(with $G_n = G$ for all $n \in \mathbb{N}$) and with the extended automorphism $\overline{\varphi}$ of $\mathbb{G}$ construct the group

$$\overline{S} := \mathbb{G} \rtimes_{\overline{\varphi}} \mathbb{Z}.$$

PROPOSITION 2.4. $\overline{S} \cong \mathrm{env}(S)$.

Proof. For this we need to show that S is a subsemigroup of $\overline{S}$ and $\overline{S} \subset S^{-1}S$ [5].

First, it is obvious that S is a subsemigroup of the group $\overline{S}$ considering the inclusion $(g,n) \mapsto (g_0,n)$, where $g_0 = g \in G = G_0 \hookrightarrow \mathbb{G}$.

Without loss of generality take $(g_i,j) \in \overline{S}$ with $i > |j|$. Then

$$(g_i,j) = (g_i,-i)(e,j+i) = (g_0,i)^{-1}(e,j+i) \in S^{-1}S. \qquad \blacksquare$$

Now consider the inductive system given by

$$\overline{D}[\varphi,B] := \varinjlim\{D[\varphi,B]_{(h,m)} : \alpha_{(h,m)}^{(g\varphi^n(h),n+m)}\}$$

where

$$D[\varphi,B]_{(h,m)} := D[\varphi,B]$$

and

$$\alpha_{(h,m)}^{(g\varphi^n(h),n+m)} : D[\varphi,B]_{(h,m)} \to D[\varphi,B]_{(g,n)(h,m)} = D[\varphi,B]_{(g\varphi^n(h),n+m)},$$

with $\alpha_{(h,m)}^{(g\varphi^n(h),n+m)} := \alpha_{(g,n)}\ \forall\,(h,m),(g,n) \in S$, where the latter was defined before Proposition 2.1. Then the C^*-dynamical system $(\overline{D}[\varphi,B],\overline{S},\overline{\alpha})$ is called the minimal automorphic dilation of $(D[\varphi,B],S,\alpha)$ where

$$\overline{\alpha}_{(g,n)} \circ \iota = \iota \circ \alpha_{(g,n)}, \quad \forall (g,n) \in G \rtimes \mathbb{N}$$

with $\iota : D[\varphi,B] \hookrightarrow D[\varphi,B]_{(e,0)} \to \overline{D}[\varphi,B]$, and

$$\overline{\bigcup_{(g,n)\in S} \overline{\alpha}_{(g,n)}^{-1}(\iota(D[\varphi,B]))} = \overline{D}[\varphi,B].$$

Then, by Theorem 2.2.1 in [16], we have the following lemma.

LEMMA 2.5. *There exists an isomorphism*

$$\Phi : \mathbb{U}[\varphi,B] \cong D[\varphi,B] \rtimes_\alpha S \cong \iota(1)(\overline{D}[\varphi,B] \rtimes_{\overline{\alpha}} \overline{S})\iota(1).$$

Thus, $D[\varphi,B] \rtimes_\alpha S$ is Morita equivalent to $\overline{D}[\varphi,B] \rtimes_{\overline{\alpha}} \overline{S}$, $\Phi|_{D[\varphi,B]} = \iota$ and also $\Phi(u_g s^n) = \iota(1)\overline{U}_{(g,n)}\iota(1)$, where $\overline{U} : \overline{S} \to \mathcal{U}M(\overline{D}[\varphi,B] \rtimes_{\overline{\alpha}} \overline{S})$ (unitary multipliers).

3. SEPARABILITY, NUCLEARITY AND UCT

From Proposition 1.5 we obtain the following proposition.

PROPOSITION 3.1. *If B contains countably many subsets of G, then $\mathbb{U}[\varphi, B]$ is separable.*

Proof. With the condition satisfied we have countably many projections in $\mathbb{U}[\varphi, B]$ and therefore it is generated by countably many elements. ∎

And the group crossed product description obtained in last section implies two properties.

PROPOSITION 3.2. *If G is amenable, $\mathbb{U}[\varphi, B]$ is nuclear.*

Proof. G being amenable implies that $\overline{S}$ is amenable as well (amenability is closed under direct limits by [25] and also closed under semidirect products). But we know that $D[\varphi, B]$ is nuclear because it is commutative, therefore $\overline{D}[\varphi, B] \rtimes_{\overline{\alpha}} \overline{S}$ is nuclear by Proposition 2.1.2 in [21]. Since hereditary C^*-subalgebras of nuclear C^*-algebras are nuclear by Corollary 3.3 (4) in [4], we conclude that

$$\mathbb{U}[\varphi, B] \cong D[\varphi, B] \rtimes_{\alpha} S \cong \iota(1)(\overline{D}[\varphi, B] \rtimes_{\overline{\alpha}} \overline{S})\iota(1)$$

is nuclear. ∎

PROPOSITION 3.3. *If G is amenable, $\mathbb{U}[\varphi, B]$ satisfies the* UCT *property.*

Proof. Since $D[\varphi, B]$ is commutative, $\overline{D}[\varphi, B] \rtimes_{\overline{\alpha}} \overline{S}$ is isomorphic to a groupoid C^*-algebra. When the group G is amenable then $\overline{S}$ also is, and the respective groupoid is also amenable. Therefore, using a result by Tu ([24], Proposition 10.7), the crossed product satisfies UCT. By Morita equivalence, $\mathbb{U}[\varphi, B]$ also satisfies it. ∎

4. PURELY INFINITE AND SIMPLE

To prove that under certain conditions our algebra is purely infinite and simple, we use Proposition 4.1 below, which is proven in Proposition 7 of [17]. The definition of a conditional expectation can be found in Definition 1.5.9 of [2].

PROPOSITION 4.1. *Let $\widetilde{A}$ be a dense $*$-subalgebra of a unital C^*-algebra A. Assume that ϵ is a faithful conditional expectation on A such that, for every $0 \neq x \in \widetilde{A}_+$, there exist finitely many projections $f_i \in A$ with:*

(i) $f_i \perp f_j$, $\forall i \neq j$;

(ii) $\exists s_i$ *isometries such that* $s_i s_i^* = f_i$, $\forall i$;

(iii) $\left\| \sum_i f_i \epsilon(x) f_i \right\| = \|\epsilon(x)\|$;

(iv) $f_i x f_i = f_i \epsilon(x) f_i \in \mathbb{C} f_i$, $\forall i$.

Then A is purely infinite and simple.

So we need to present a dense subalgebra and a conditional expectation of $\mathbb{U}[\varphi, B]$. To find the conditional expectation, it is necessary to suppose in this section that the group G is amenable. And to prove the main theorem, we suppose that φ is pure, i.e:

$$\bigcap_{n\in\mathbb{N}} \varphi^n(G) = \{e\}.$$

The next lemma states that

$$S[\varphi, B] := \operatorname{span}(\{s^{*n}u_{g^{-1}}e_{[H]}u_{g'}s^m : H \in B', g, g' \in G, n, m \in \mathbb{N}\})$$

is dense in $\mathbb{U}[\varphi, B]$.

LEMMA 4.2. *The $*$-subalgebra of $\mathbb{U}[\varphi, B]$ generated by*

$$\{e_{[H]}, u_g, s : H \in B, g \in G\}$$

coincides with $S[\varphi, B]$.

Proof. Note that

$$\{e_{[H]}, u_g, s : H \in B, g \in G\} \subseteq S[\varphi, B] \subseteq \operatorname{span}\{e_{[H]}, u_g, s : H \in B, g \in G\}$$

and $S[\varphi, B]$ is closed under multiplication:

$$\begin{aligned}
&s^{*n}u_{g^{-1}}e_{[H]}u_{g'}s^{n'}s^{*m}u_{h^{-1}}e_{[K]}u_{h'}s^{m'}\\
&= s^{*n}u_{g^{-1}g'}(u_{g'^{-1}}e_{[H]}u_{g'})s^{n'}s^{*n'}s^{*m}s^{n'}(u_{h^{-1}}e_{[K]}u_h)u_{h^{-1}h'}s^{m'}\\
&= s^{*n}u_{g^{-1}g'}s^{*m}(s^m e_{[g'^{-1}H]}s^{*m})s^m s^{n'}s^{*n'}s^{*m}(s^{n'}e_{[h^{-1}K]}s^{*n'})s^{n'}u_{h^{-1}h'}s^{m'}\\
&= s^{*n+m}u_{\varphi^m(g^{-1}g')}e_{[\varphi^m(g'^{-1}H)]}e_{[\varphi^{m+n'}(G)]}e_{[\varphi^{n'}(h^{-1}K)]}u_{\varphi^{n'}(h^{-1}h')}s^{n'+m'}\\
&= s^{*n+m}u_{\varphi^m(g^{-1}g')}e_{[\varphi^m(g'^{-1}H)\cap\varphi^{m+n'}(G)\cap\varphi^{n'}(h^{-1}K)]}u_{\varphi^{n'}(h^{-1}h')}s^{n'+m'}\\
&= s^{*n+m}u_{\varphi^m(g^{-1}g')}e_{[\varphi^m(g'^{-1})\varphi^m(H)\cap\varphi^{m+n'}(G)\cap\varphi^{n'}(h^{-1}K)]}u_{\varphi^{n'}(h^{-1}h')}s^{n'+m'}\\
&= 0 \in S[\varphi, B] \quad \text{or}\\
&= s^{*n+m}u_{\varphi^m(g^{-1}g')}e_{[\tilde{g}(\varphi^m(H)\cap\varphi^{m+n'}(G)\cap\varphi^{n'}(K))]}u_{\varphi^{n'}(h^{-1}h')}s^{n'+m'}\\
&= s^{*n+m}u_{\varphi^m(g^{-1}g')\tilde{g}}\,e_{[\varphi^m(H)\cap\varphi^{m+n'}(G)\cap\varphi^{n'}(K)]}u_{\tilde{g}^{-1}\varphi^{n'}(h^{-1}h')}s^{n'+m'} \in S[\varphi, B].
\end{aligned}$$

The result follows. ∎

Now we just have to define a conditional expectation to use in Proposition 4.1 with the subalgebra defined above. For this, we use the amenability of G. Therefore, $\overline{S}$ is amenable, which implies that both the reduced and the full crossed products by $\overline{S}$ are isomorphic.

Using the isomorphism

$$\Phi : \mathbb{U}[\varphi, B] \cong D[\varphi, B] \rtimes_\alpha S \to \iota(1)(\overline{D}[\varphi, B] \rtimes_{\overline{\alpha}} \overline{S})\iota(1)$$

obtained in Lemma 2.5 we have the easy-to-prove result below.

LEMMA 4.3. *There exists a faithful conditional expectation*

$$\theta : \mathbb{U}[\varphi, B] \to \Phi^{-1}(i(1)\overline{D}[\varphi, B]i(1))$$

$$s^{*n}u_{g^{-1}}e_{[H]}u_{g'}s^{n'} \mapsto \begin{cases} s^{*n}u_{g^{-1}}e_{[H]}u_g s^n & \text{if } n = n' \text{ and } g = g', \\ 0 & \text{otherwise,} \end{cases}$$

for all $H \in B'$, $g, g' \in G$ *and* $n, n' \in \mathbb{N}$.

Now we can prepare to prove that $\mathbb{U}[\varphi, B]$ is simple and purely infinite. For this aim we follow and adapt the proof of Li ([17], Section 5.2) and use the next lemmas.

LEMMA 4.4. *Let* H *and* G_i *be distinct subgroups of* G *with* $\#\left[\frac{H}{H\cap G_i}\right] = \infty$ *for all* $1 \leqslant i \leqslant n$. *Then, for all* $h, g_i \in G$, *we have* $hH \not\subseteq \bigcup_{i=1}^{n} g_i(H \cap G_i)$.

Proof. By induction, for $n = 1$ we have:

$$hH \subseteq g_1(H \cap G_1) \Rightarrow H \subseteq h^{-1}g_1(H \cap G_1) \Rightarrow \#\left[\frac{H}{H \cap G_1}\right] \neq \infty.$$

Assume that the result holds for $n - 1$. Let us prove that it holds for n. Suppose that

$$hH \subseteq \bigcup_{i=1}^{n} g_i(H \cap G_i),$$

for some $h, g_i \in G$, with $1 \leqslant i \leqslant n$. We can consider two possible cases.

Case 1. There exists $1 < j \leqslant n$ with

$$\#\left[\frac{H \cap G_1}{(H \cap G_1) \cap (H \cap G_j)}\right] < \infty.$$

As

$$\frac{(H \cap G_1)(H \cap G_j)}{H \cap G_j} \cong \frac{H \cap G_1}{(H \cap G_1) \cap (H \cap G_j)},$$

it follows that the first one also has cardinality $< \infty$. But the exact sequence

$$\frac{(H \cap G_1)(H \cap G_j)}{H \cap G_j} \hookrightarrow \frac{H}{H \cap G_j} \twoheadrightarrow \frac{H}{(H \cap G_1)(H \cap G_j)}$$

with $\#\left[\frac{(H\cap G_1)(H\cap G_j)}{H\cap G_j}\right] < \infty$ and $\#\left[\frac{H}{H\cap G_j}\right] = \infty$ implies that

$$\#\left[\frac{H}{(H \cap G_1)(H \cap G_j)}\right] = \infty.$$

Define

$$\widetilde{G}_i := \begin{cases} H \cap G_i & \text{if } G_i \neq G_1 \text{ and } G_i \neq G_j, \\ (H \cap G_1)(H \cap G_j) & \text{if } G_i \in \{G_1, G_j\}. \end{cases}$$

Note that

$$\#\Big[\frac{H}{H \cap \widetilde{G}_i}\Big] = \infty \quad \text{and} \quad hH \subseteq \bigcup_{i=1}^{n} g_i(H \cap G_i) \subseteq \bigcup_{i=1}^{n} g_i(H \cap \widetilde{G}_i),$$

but the latter one contradicts our hypothesis, as $\#\{\widetilde{G}_i\} \leqslant n-1$.

Case 2. Now suppose that $\forall 1 < j \leqslant n$,

$$\#\Big[\frac{H \cap G_1}{(H \cap G_1)(H \cap G_j)}\Big] = \infty.$$

As $\#[\frac{H}{H \cap G_1}] = \infty$, we have that $\exists g \in H$ such that $g(H \cap G_1) \neq g_i(H \cap G_i)$ $\forall 1 \leqslant i \leqslant n$. Then we have

$$\begin{aligned} g(H \cap G_1) = g(H \cap G_1) \cap H \subseteq g(H \cap G_1) \cap \bigcup_{i=1}^{n} g_i(H \cap G_i) \\ = \bigcup_{g(H \cap G_1) \cap g_i(H \cap G_i) \neq \varnothing} g(H \cap G_1) \cap g_i(H \cap G_i) \\ = \bigcup_{g(H \cap G_1) \cap g_i(H \cap G_i) \neq \varnothing} \widetilde{g}_i((H \cap G_1) \cap (H \cap G_i)) \end{aligned}$$

and we can conclude that

$$H \cap G_1 \subseteq \bigcup_{g(H \cap G_1) \cap g_i(H \cap G_i) \neq \varnothing} g^{-1}\widetilde{g}_i((H \cap G_1) \cap (H \cap G_i)).$$

But note that, by construction, $g(H \cap G_1) \cap g_i(H \cap G_1) = \varnothing$. So the union has been taken over less than n elements and we have a contradiction. ∎

Let us show that $\mathbb{U}[\varphi, B]$ together with the dense $*$-subalgebra $S[\varphi, B]$ (Lemma 4.2) and the faithful conditional expectation θ defined in Lemma 4.3 satisfy the criteria of Proposition 4.1.

Take $0 \neq x \in S[\varphi, B]_+$. As $\theta(x) \neq 0$, one has that

$$\theta(x) = \sum_{(n', X)}^{\text{finite}} \beta_{(n', X)} s^{*n'} e_{[X]} s^{n'},$$

where $(n', X) \in \mathbb{N} \times C(B)$ and $\beta_{(n', X)} \in \mathbb{C}$. Define n to be the sum of all n' with

$$\beta_{(n', X)} s^{*n'} e_{[X]} s^{n'} \neq 0.$$

Then

$$\theta(x) = s^{*n} \Big(\sum_{(n', X)}^{\text{finite}} \beta_{(n', X)} e_{[\varphi^{n-n'}(X)]} \Big) s^n.$$

Moreover, using Lemma 1.4, it is possible to write

$$\theta(x) = s^{*n}\Big(\sum_{(g,H)}^{\text{finite}} \beta_{(g,H)} e_{[gH]}\Big) s^n, \tag{4.1}$$

where the sum is over finitely many $(g, H) \in G \times B'$ and $\beta_{(g,H)} \in \mathbb{C}$. Recall from Lemma 1.4 that $B' = \Big\{ \bigcap_{i=1}^{m} \varphi^{n_i}(H_i) : H_i \in B \cup \{G\}, n_i \in \mathbb{N} \Big\}$.

Note that

$$s^{*n} e_{[gH]} = s^{*n} s^n s^{*n} e_{[gH]} = s^{*n} e_{[\varphi^n(G)]} e_{[gH]} = s^{*n} e_{[\varphi^n(G) \cap gH]},$$

so we can assume that $gH \subseteq \varphi^n(G)$, for each $(g, H) \in G \times B'$.

LEMMA 4.5. *There exist finitely many pairwise orthogonal (nontrivial) projections p_i in $\mathbb{Z}$-span$(D[\varphi, B])$ such that $C^*(\{e_{[gH]} : \beta_{(g,H)} \neq 0\}) = C^*(\{p_i\})$.*

For the proof, just orthogonalize the $e_{[g'H']}$ and rearrange the coefficients to be in $\mathbb{Z}$. Thus, take some $p \in \{p_i\}$ among the p_i's obtained above. Then

$$p = \sum_j n_j e_{[g_j H_j]} - \sum_{j'} \widetilde{n}_{j'} e_{[\widetilde{g}_{j'} \widetilde{H}_{j'}]} \tag{4.2}$$

with finitely many $n_j, \widetilde{n}_{j'} \in \mathbb{Z}_{>0}$ and $(g_j, H_j), (\widetilde{g}_{j'}, \widetilde{H}_{j'}) \in G \times B'$.

LEMMA 4.6. *We can express p as in (4.2) so that $\forall K, \widetilde{K} \in \{H_j, \widetilde{H}_j\}$ the cardinality of $\frac{K}{K \cap \widetilde{K}}$ is 1 or ∞.*

Proof. By induction, enumerate $\{H_j, \widetilde{H}_j\}$ by $\{K_i\}$. Of course the lemma holds if there is just K_1.

Suppose that it holds for $\{K_1, \ldots, K_h\}$. Define $K_{h+1}^{(0)} := K_{h+1}$ and for $j = 1, \ldots, h$

$$K_{h+1}^{(j)} := \begin{cases} K_{h+1}^{(j-1)} & \text{if } \#[K_{h+1}^{(j-1)} / (K_{h+1}^{(j-1)} \cap K_j)] \in \{1, \infty\}, \\ K_{h+1}^{(j-1)} \cap K_j & \text{otherwise.} \end{cases} \tag{4.3}$$

We want to change K_{h+1} successively to $K_{h+1}^{(0)}, K_{h+1}^{(1)}, \ldots$, until $K_{h+1}^{(h)}$.

Suppose that $K_{h+1}^{(j)} = K_{h+1}^{(j-1)} \cap K_j$ as described above in (4.3). Therefore we have $1 < \#[K_{h+1}^{(j-1)} / (K_{h+1}^{(j-1)} \cap K_j)] = M < \infty$, and then $K_{h+1}^{(j-1)} = \bigcup_{i=1}^{M} g_i(K_{h+1}^{(j-1)} \cap K_j)$. So we can replace K_{h+1} by $K'_{h+1} := K_{h+1}^{(h)}$, because the projections will still be written using the initial $\{K_i\}$.

Now note that

$$\#\Big[\frac{K'_{h+1}}{K'_{h+1} \cap K}\Big] \in \{1, \infty\}, \quad \forall K \in \{K_1, \ldots, K_h\}.$$

Let us prove by induction on j that $\#\left[\frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)}\cap K}\right] \in \{1,\infty\}$, for every $K \in \{K_1,\ldots,K_j\}$. By construction it holds for $j=1$. Suppose it holds for $j-1$, that is

$$\#\left[\frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j-1)}\cap K}\right] \in \{1,\infty\}, \quad \forall K \in \{K_1,\ldots,K_{j-1}\},$$

and let us prove the assertion for j. Also by construction $\#\left[\frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)}\cap K_j}\right]$ belongs to $\{1,\infty\}$. Then, we need to show that

$$\#\left[\frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)}\cap K}\right] \in \{1,\infty\}, \quad \forall K \in \{K_1,\ldots,K_{j-1}\},$$

which holds by the induction hypothesis if $K_{h+1}^{(j)} = K_{h+1}^{(j-1)}$.

But, if $K_{h+1}^{(j)} = K_{h+1}^{(j-1)} \cap K_j$, then $K_{h+1}^{(j)} \subset K_{h+1}^{(j-1)}$ and therefore it follows that

$$1 < \#\left[\frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j-1)}\cap K_j}\right] < \infty.$$

Now, by our induction hypothesis, we have two possibilities for each $K \in \{K_1,\ldots,K_{j-1}\}$.

Case $\#\left[\frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j-1)}\cap K}\right] = 1$. In this case, as $K_{h+1}^{(j)} \subset K_{h+1}^{(j-1)} \subset K$, it follows that $\#\left[\frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)}\cap K}\right] = 1$.

Case $\#\left[\frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j-1)}\cap K}\right] = \infty$. Consider the exact sequence

$$\frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)}\cap K} \hookrightarrow \frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j)}\cap K} \twoheadrightarrow \frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j)}}.$$

The inclusion $\frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j-1)}\cap K} \subset \frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j)}\cap K}$ implies that the second term has size ∞. The third term has cardinality $< \infty$ because it is equal to $\frac{K_{h+1}^{(j-1)}}{K_{h+1}^{(j-1)}\cap K_j}$. As that sequence is exact, we must have $\#\left[\frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)}\cap K}\right] = \infty$. Thus we conclude that $\#\left[\frac{K_{h+1}^{(j)}}{K_{h+1}^{(j)}\cap K}\right] \in \{1,\infty\}$, $\forall K \in \{K_1,\ldots,K_j\}$.

Set

$$K_j' := \begin{cases} K_j \cap K_{h+1}' & \text{if } 1 < \#\left[\frac{K_j}{K_j\cap K_{h+1}'}\right] < \infty, \\ K_j & \text{otherwise,} \end{cases} \tag{4.4}$$

for $j = 1, \ldots, h$. This gives a new sequence $\{K'_1, \ldots, K'_{h+1}\}$. And then it only remains to prove that

$$\#\Big[\frac{K'_j}{K'_j \cap K'_{\widetilde{j}}}\Big] \in \{1, \infty\}.$$

Note that, if j or $\widetilde{j}$ is equal to $h+1$, this holds by the claim above. So, suppose that j and $\widetilde{j}$ are in $\{1, \ldots, h\}$. Then, by our induction hypothesis, we have two possibilities.

Case $\#\Big[\frac{K_j}{K_j \cap K_{\widetilde{j}}}\Big] = 1$; then $K_j \subseteq K_{\widetilde{j}}$. If $K'_j = K_j \cap K'_{h+1}$, then $K'_j \subseteq K'_{\widetilde{j}}$, and (4.4) holds. Otherwise, $K'_j = K_j \neq K_j \cap K'_{h+1}$, and therefore $\#\Big[\frac{K_j}{K_j \cap K'_{h+1}}\Big] = \infty$. Then (as $K_j \subseteq K_{\widetilde{j}}$) we have the inclusion

$$\frac{K_j}{K_j \cap K'_{h+1}} \subseteq \frac{K_{\widetilde{j}}}{K_{\widetilde{j}} \cap K'_{h+1}},$$

which implies $\Big[\frac{K_{\widetilde{j}}}{K_{\widetilde{j}} \cap K'_{h+1}}\Big] = \infty$. So $K'_{\widetilde{j}} = K_{\widetilde{j}}$ and our claim holds.

Case $\#\Big[\frac{K_j}{K_j \cap K_{\widetilde{j}}}\Big] = \infty$. As $\frac{K_j}{K_j \cap K_{\widetilde{j}}} \subseteq \frac{K_j}{K_j \cap K'_{\widetilde{j}}}$, if $K'_j = K_j$ the claim holds. Now, if $K'_j = K_j \cap K'_{h+1} \neq K_j$, then we have the exact sequence

$$\frac{K'_j}{K'_j \cap K_{\widetilde{j}}} \hookrightarrow \frac{K_j}{K'_j \cap K_{\widetilde{j}}} \twoheadrightarrow \frac{K_j}{K'_j}.$$

The set $\frac{K_j}{K_j \cap K_{\widetilde{j}}}$ has size ∞ and is contained in the second term, so it has size ∞ too. The third term has size $< \infty$ as $K_j \cap K'_{h+1} \neq K_j$ implies that $\#\Big[\frac{K_j}{K_j \cap K'_{h+1}}\Big] < \infty$.

Hence, we conclude that $\Big[\frac{K'_j}{K'_j \cap K_{\widetilde{j}}}\Big] = \infty$, proving the lemma. ∎

LEMMA 4.7. *There exist finitely many pairwise orthogonal projections $p_i \in \mathbb{U}[\varphi, B]$ such that*

$$C^*(\{p_i\}) \cong C^*(\{e_{[gH]} : \beta_{(g,H)} \neq 0\}),$$

where the (g, H)'s come from equation (4.1). *Moreover, if there exists $m \in \mathbb{N}$ such that $\varphi^m(G) \subseteq \bigcap_{H \in B} H$, then for all i, there exists $h_i \in G$ and $m_i \in \mathbb{N}$ so that*

$$e_{[h_i \varphi^{m_i}(G)]} \leqslant p_i.$$

Proof. We have

$$\theta(x) = s^{*n} \Big(\sum_{(g,H)}^{\text{finite}} \beta_{(g,H)} e_{[gH]} \Big) s^n, \quad \text{with } (g, H) \in G \times B',$$

where we recall that

$$B' = \Big\{ \bigcap_{i=1}^{m} \varphi^{n_i}(H_i) : H_i \in B \cup \{G\}, n_i \in \mathbb{N} \Big\}.$$

We can assume that $gH \subset \varphi^n(G)$ and, by Lemma 4.5, we have finitely many pairwise orthogonal projections p_i in $\mathbb{Z}$-span$(D[\varphi, B])$ with

$$C^*(\{e_{[gH]} : \beta_{(g,H)} \neq 0\}) = C^*(\{p_i\}).$$

Choose some $p \in \{p_i\}$ and write it as

$$p = \sum_j n_j e_{[g_j H_j]} - \sum_{j'} \widetilde{n}_{j'} e_{[\widetilde{g}_{j'} \widetilde{H}_{j'}]}$$

with finitely many $n_j, \widetilde{n}_{j'} \in \mathbb{Z}_{>0}$. We can write p such that each projection $e_{[g,H]}$ appears at most once and $\#\Big[\frac{K}{K \cap \widetilde{K}}\Big] \in \{1, \infty\}$ $\forall K, \widetilde{K} \in \{H_j, \widetilde{H}_{j'}\}$ by Lemma 4.6.

Choose some maximal $H \in \{H_j, \widetilde{H}_{j'}\}$. Take $g \in G$ and $n \in \mathbb{Z}_{>0}$ so that $ne_{[gH]}$ appears in p. Multiplying p with $e_{[gH]}$ gives us

$$e_{[gH]} p = ne_{[gH]} + \sum_k n_k e_{[c_k(H \cap H_k)]} - \sum_l \widetilde{n}_l e_{[\widetilde{c}_l(H \cap \widetilde{H}_l)]},$$

for (finitely many) $c_k, \widetilde{c}_l \in G$ and $n_k, \widetilde{n}_l \in \mathbb{Z}_{>0}$.

Note that we must have $\#\Big[\frac{H}{H \cap H_k}\Big] = \infty$ because if $\#\Big[\frac{H}{H \cap H_k}\Big] = 1$ then $H_k = H$ would imply $e_{[g_j H_j]} = e_{[\widetilde{g}_{j'} \widetilde{H}_{j'}]}$ for some j and j'. Then, by Lemma 4.4,

$$gH \nsubseteq \Big[\bigcup_k c_k(H \cap H_k)\Big] \cup \Big[\bigcup_l \widetilde{c}_l(H \cap \widetilde{H}_l)\Big],$$

which allows us to find $r \in gH \backslash \Big[\bigcup_k c_k(H \cap H_k)\Big] \cup \Big[\bigcup_l \widetilde{c}_l(H \cap \widetilde{H}_l)\Big]$. One can conclude that:

$$\begin{aligned} &e_{[r(\bigcap_k(H \cap H_k) \cap \bigcap_l (H \cap \widetilde{H}_l))]} \leqslant e_{[gH]}, \\ &e_{[r(\bigcap_k(H \cap H_k) \cap \bigcap_l (H \cap \widetilde{H}_l))]} \perp e_{[c_k(H \cap H_k)]}, \quad \forall k, \quad \text{and} \\ &e_{[r(\bigcap_k(H \cap H_k) \cap \bigcap_l (H \cap \widetilde{H}_l))]} \perp e_{[\widetilde{c}_l(H \cap \widetilde{H}_l)]}, \quad \forall l. \end{aligned}$$

Multiplying the equation above by $e_{[r(\bigcap_k(H \cap H_k) \cap \bigcap_l (H \cap \widetilde{H}_l))]}$ leads to

$$e_{[r(\bigcap_k(H \cap H_k) \cap \bigcap_l (H \cap \widetilde{H}_l))]} p = ne_{[r(\bigcap_k(H \cap H_k) \cap \bigcap_l (H \cap \widetilde{H}_l))]}.$$

As the first term is a projection (because it is the product of two commuting projections) we must have $n = 1$. So, $e_{[r(\bigcap_k(H \cap H_k) \cap \bigcap_l (H \cap \widetilde{H}_l))]} \leqslant p$. If our additional hypothesis is satisfied, we have $\widetilde{m} \in \mathbb{N}$ such that

$$\varphi^m(G) \subseteq \bigcap_{H \in B} H \Rightarrow \varphi^{\widetilde{m}}(G) \subseteq \bigcap_{H_i \in B', 0 \leqslant i \leqslant n} H_i$$

for some $\widetilde{m}$ bigger than m. Then

$$e_{[r\varphi^{\widetilde{m}}(G)]} \leqslant e_{[r(\bigcap_k (H\cap H_k)\cap\bigcap_l (H\cap \widetilde{H}_l)))]} \leqslant p.$$

Therefore we just need to denote $h_i = r$ and $m_i = \widetilde{m}$. The conclusion holds if this is done for every element of $\{p_i\}$. ∎

REMARK 4.8. Note that for every i in the last lemma we can choose m_i as big as we want, because $\varphi^{m+1}(G) \subset \varphi^m(G)$.

THEOREM 4.9. *Let G be an amenable group, B some family of subgroups in G containing G and φ a pure injective endomorphism of G. Also suppose that $\exists k \in \mathbb{N}$ such that $\varphi^k(G) \subseteq \bigcap_{H\in B} H$. Then, the C^*-algebra $\mathbb{U}[\varphi, B]$ is purely infinite and simple.*

Proof. We already have the candidates to use with Proposition 4.1, namely

$$\theta : \mathbb{U}[\varphi, B] \to \Phi^{-1}(\iota(1)\overline{D}[\varphi, B]\iota(1)), \quad \text{and}$$
$$S[G, B] = \operatorname{span}(\{s^{*n}u_{g^{-1}}e_{[I]}u_{g'}s^m : I \in B, g, g' \in G, n, m \in \mathbb{N}\}).$$

Take $0 \neq x \in S[G, B]_{\mathrm{sa}}$. Then

$$x = \sum_{(g,g',l,l',J)} \eta_{(g,g',l,l',J)} s^{*l} u_{g^{-1}} e_{[J]} u_{g'} s^{l'}$$

where $\eta_{(g,g',l,l',J)} \in \mathbb{C}$ for each multiindex (g, g', l, l', J). As in the previous Lemma 4.7,

$$\theta(x) = s^{*n}\Big(\sum_{(g,H)}^{\text{finite}} \beta_{(g,H)} e_{[gH]}\Big)s^n,$$

for some $n \in \mathbb{N}$ and $(g, H) \in G \times B'$ with $\beta_{(g,H)} \neq 0$, where $gH \subset \varphi^n(G)$.

By Lemma 4.7 we find finitely many pairwise orthogonal (nontrivial) projections $\{p_i\}$ with $C^*(\{e_{[gH]} : \beta_{(g,H)} \neq 0\}) = C^*(\{p_i\})$. Furthermore, there exist $m_i \in \mathbb{N}$ and $h_i \in G$ such that $e_{[h_i\varphi^{m_i}(G)]} \leqslant p_i \leqslant e_{[\varphi^n(G)]}$ $\forall i$. Using Remark 4.8, we can suppose that $m_i \geqslant n$ $\forall i$. Also note that $h_i \in \varphi^n(G)$. Thus the projections $F_i := s^{*n}e_{[h_i\varphi^{m_i}(G)]}s^n$ satisfy $F_i \leqslant s^{*n}p_is^n$ and

$$\begin{aligned} C^*(\{s^{*n}e_{[gH]}s^n : \beta_{(g,H)} \neq 0\}) = C^*(\{s^{*n}p_is^n\}) &\to C^*(\{F_i\}) \\ y &\mapsto \sum_i F_i y F_i \end{aligned} \tag{4.5}$$

is an isomorphism that maps $s^{*n}p_is^n$ to F_i. These projections F_i satisfy only (i) and (ii) of the conditions in Proposition 4.1.

Call (g, g', l, l', J) critical if $\eta_{(g,g',l,l',J)}s^{*l}u_{g^{-1}}e_{[J]}u_{g'}s^{l'} \neq 0$ and $\delta_{g,g'}\delta_{l,l'} = 0$. Note that

$$x - \theta(x) = \sum_{(g,g',l,l',J)\ \text{critical}} s^{l*}u_{g^{-1}}e_{[J]}u_{g'}s^{l'}.$$

But for each i, it is possible to take some $a_i \in \varphi^{-n}(h_i)\varphi^{m_i-n}(G)$ satisfying

$$\varphi^{l'}(a_i^{-1})g'^{-1}g\varphi^l(a_i) \neq e$$

for all critical (g, g', l, l', J). Otherwise, we would have $r_1 \neq r_2 \in \varphi^{m_i - n}(G)$ such that

$$\varphi^{l'}(r_1^{-1})g'^{-1}g\varphi^l(r_1) = e = \varphi^{l'}(r_2^{-1})g'^{-1}g\varphi^l(r_2).$$

If $l = l'$ we have $g \neq g'$ (as $\delta_{g,g'}\delta_{l,l'} = 0$) and then

$$\varphi^l(r_1^{-1})g'^{-1}g\varphi^l(r_1) = e \Rightarrow g'^{-1}g = e$$

which contradicts $g \neq g'$.

Suppose now that $l \neq l'$. As $r_1 = r_2 r_2^{-1} r_1$ we get

$$e = \varphi^{l'}((r_2 r_2^{-1} r_1)^{-1})g'^{-1}g\varphi^l(r_2 r_2^{-1} r_1) = \varphi^{l'}(r_1^{-1} r_2)\varphi^l(r_2^{-1} r_1),$$

which implies that $r_1 = r_2$ (because φ is pure). This contradicts our assumptions.

Now as our endomorphism φ is pure, for all critical (g, g', l, l', J) and for all i there exists $n_{(g,g',l,l',J,i)} \in \mathbb{N}$ (as big as we need) such that $\varphi^{l'}(a_i^{-1})g'^{-1}g\varphi^l(a_i) \notin \varphi^{n_{(g,g',l,l',J,i)}}(G)$. Let us call

$$b_i := (m_i - n) \prod_{(g,g',l,l',J) \text{ critical}} n_{(g,g',l,l',J,i)}.$$

Note that

$$\varphi^{l'}(a_i^{-1})g'^{-1}g\varphi^l(a_i) \notin \varphi^{b_i}(G). \tag{4.6}$$

Define $f_i := e_{[a_i \varphi^{b_i}(G)]}$. We want to prove that these projections satisfy the conditions of Proposition 4.1, which are:

(i) $f_i \perp f_j, \forall i \neq j$,

(ii) $f_i \sim_{z_i} 1$, via isometries $z_i \in A, \forall i$,

(iii) $\left\| \sum_i f_i \theta(x) f_i \right\| = \|\theta(x)\|$, and

(iv) $f_i x f_i = f_i \theta(x) f_i \in \mathbb{C} f_i$, $\forall i$.

As $b_i \geqslant m_i - n$ and $\varphi^n(a_i) \in h_i \varphi^{m_i}(G)$ it follows that $s^n e_{[a_i \varphi^{b_i}(G)]} s^{*n} \leqslant e_{[h_i \varphi^{m_i}(G)]}$ and then

$$f_i = s^{*n} s^n e_{[a_i \varphi^{b_i}(G)]} s^{*n} s^n \leqslant s^{*n} e_{[h_i \varphi^{m_i}(G)]} s^n = F_i.$$

This implies that $f_i \perp f_j \; \forall i \neq j$ and (i) is satisfied. Item (ii) is also easily satisfied, because

$$f_i = e_{[a_i \varphi^{b_i}(G)]} = (u_{a_i} s^{b_i})(u_{a_i} s^{b_i})^* \sim (u_{a_i} s^{b_i})^*(u_{a_i} s^{b_i}) = 1.$$

Since (4.5) is an isomorphism and $f_i \leqslant F_i$, the map

$$\begin{aligned} C^*(\{s^{*n} e_{[gH]} s^n : \beta_{(g,H)} \neq 0\}) &\to C^*(\{f_i\}) \\ y &\mapsto \sum_i f_i y f_i \end{aligned}$$

is an isomorphism as well. Therefore it is isometric and (iii) is satisfied. And finally, for the last condition, let us expand $f_i(x-\theta(x))f_i$:

$$\begin{aligned} & f_i(x-\theta(x))f_i \\ & = f_i\Big(\sum_{(g,g',l,l',J)\text{ critical}} \eta_{(g,g',l,l',J)} s^{*l} u_{g^{-1}} e_{[J]} u_{g'} s^{l'}\Big) f_i \\ & = \sum_{(g,g',l,l',J)\text{ critical}} \eta_{(g,g',l,l',J)} s^{*l} u_{g^{-1}} (u_g s^l f s^{*l} u_{g^{-1}}) e_{[J]} (u_{g'} s^{l'} f s^{*l'} u_{g'^{-1}}) u_{g'} s^{l'} \\ & = \sum_{(g,g',l,l',J)\text{ critical}} \eta_{(g,g',l,l',J)} s^{*l} u_{g^{-1}} e_{[g\varphi^l(a_i)\varphi^{l+b_i}(G)]} e_{[g'\varphi^{l'}(a_i)\varphi^{l'+b_i}(G)]} e_{[J]} u_{g'} s^{l'}. \end{aligned}$$

Now, note that

$$[g\varphi^l(a_i)\varphi^{l+b_i}(G)] \cap [g'\varphi^{l'}(a_i)\varphi^{l'+b_i}(G)] \neq \emptyset \Rightarrow \varphi^{l'}(a_i^{-1}) g'^{-1} g \varphi^l(a_i) \in \varphi^{b_i}(G),$$

which contradicts our choice of b_i by (4.6). So the intersection above must be empty and then $f_i x f_i = f_i\theta(x) f_i \in \mathbb{C} f_i$, $\forall i$. Therefore, by Proposition 4.1, our C^*-algebra is simple and purely infinite. ∎

COROLLARY 4.10. *When the conditions of Theorem* 4.9 *are satisfied, the concrete* C^**-algebra* $C^*_r[\varphi, B]$ *is isomorphic to the universal one* $\mathbb{U}[\varphi, B]$*, as defined in Definitions* 1.1 *and* 1.2 *respectively.*

THEOREM 4.11. *If the conditions of Theorem* 4.9 *are satisfied, the universal* C^**-algebra* $\mathbb{U}[\varphi, B]$ *is a Kirchberg algebra satisfying the* UCT *property (Propositions* 3.1*,* 3.2 *and* 3.3*).*

EXAMPLE 4.12. Take G to be the free group $\mathbb{F}_2$ with generators $\{a, b\}$ and define the injective endomorphism

$$\begin{aligned} \varphi : \mathbb{F}_2 &\to \mathbb{F}_2 \\ a &\mapsto a \\ b &\mapsto b^2, \quad \text{linearly extended.} \end{aligned}$$

Note that $a \neq ab \neq abab \neq ababab \neq \cdots$ in $\frac{G}{\varphi(G)}$ and thus we need to work with a family of subgroups of $\mathbb{F}_2$. So denote $H_1 := \varphi(G) = \langle a, b^2\rangle$ and choose $B = \{G, H_1\}$.

Since G is not amenable, we can only say that $\mathbb{U}[\varphi, B]$ is separable. Also by the previous results,

$$\mathbb{U}[\varphi, B] \cong D[\varphi, B] \rtimes_\omega \mathbb{F}_2 \rtimes_\tau \mathbb{N}$$

with $D[\varphi, B] = C^*(\{e_{[g\langle a,b^{2^n}\rangle]} : g \in G, n \in \mathbb{N}\})$ and

$$\begin{aligned} \omega : \mathbb{F}_2 &\to \mathrm{Aut}(D[\varphi, B]) \\ g &\mapsto u_g(\cdot)u_{g^{-1}}; \\ \tau : \mathbb{N} &\to \mathrm{End}(D[\varphi, B] \rtimes_\omega \mathbb{F}_2) \\ n &\mapsto s^n(\cdot)s^{*n}. \end{aligned}$$

Note that $D[\varphi, B]$ can be viewed as the direct limit of

$$D_n := C^*(\{e_{[g\langle a,b^{2^i}\rangle]} : \text{ with } 0 \leqslant i \leqslant n\}),$$

with the canonical inclusions $i_{m,n} : D_n \to D_m$ for $n < m$.

Define

$$A_n := \dot{\bigcup_{0\leqslant i\leqslant n}} \frac{G}{\langle a, b^{2^i}\rangle}.$$

For $g \in A_n$ there exists some $0 \leqslant i_g \leqslant n$ such that $g \in \frac{G}{\langle a,b^{2^{i_g}}\rangle} \subseteq A_n$. Denote $H_g := \langle a, b^{2^{i_g}}\rangle$ and define the following partial order on A_n: $g \leqslant h$ if $H_g \subseteq H_h$ and $gH_h = hH_h$. The topology of A_n is such that a sequence (x_m) converges to x if and only if x is the only minimal element of the set

$$\{x' \in A_n : x_m \leqslant x' \text{ for all but finitely many } m\}.$$

It is easy to see that $\widehat{D}_n$ is homeomorphic to A_n. Take $g \in A_n$ and define $e_{[g]} := e_{[gH_g]}$. For some $\chi \in \widehat{D}_n$, define $g_\chi \in A_n$ by (for $h \in A_n$):

$$\chi(e_{[h]}) = \begin{cases} 1 & \text{if } h \geqslant g_\chi, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that this correspondence is a bijection. So $\forall g \in A_n$, we have one and only one correspondent element $\chi_g \in \widehat{D}_n$.

Now, χ_{g_m} converges to χ_g if and only if $\lim_{m\to\infty} \chi_{g_m}(e_{g'}) = \chi_g(e_{g'})$. But this corresponds to saying that $g' \geqslant g_m$ for almost all m if and only if $g' \geqslant g$, that is, that g is the only minimal element of $\{g' \in A_n : g_m \leqslant g' \text{ for almost all } m\}$. So the topologies are equivalent. Then

$$\widehat{D[\varphi, B]} \cong \varprojlim(A_n, p_{n,m}) =: A$$

and note that

$$A = \left\{(g_m)_m \in \prod_{m\in\mathbb{N}} A_m : p_{n,m}(g_m) = g_n\right\}$$

with

$$\begin{aligned} p_{n,m} : A_m &\to A_n \\ g\langle a, b^{2^i}\rangle &\mapsto g\langle a, b^{2^{\min\{i,n\}}}\rangle, \end{aligned}$$

for $m \geqslant n$. With this we can also conclude that

$$\mathbb{U}[\varphi, B] \cong C_0(A) \rtimes_{\overline{\omega}} \mathbb{F}_2 \rtimes_{\overline{\tau}} \mathbb{N}$$

where for $f \in C_0(A)$ and $x \in A$,

$$\overline{\omega}_g(f)(x) = f(gxg^{-1})$$

using the pointwise product gxg^{-1} and

$$\overline{\tau}_n(f\delta_h) = \overline{f}\delta_{\varphi^n}$$

with $\overline{f}(x) = f(\varphi(x))$, where we consider $\varphi(x)$ pointwise.

In Example 5.4 we calculate the K-groups of $\mathbb{U}[\varphi, B]$.

5. THE CASE $B = \{G\}$

In this section we study the particular case when B contains, instead of subgroups, only subsets of the form $g\varphi^k(G)$ for $k \in \mathbb{N}$ and $g \in G$. By Proposition 1.7, the C^*-algebra $\mathbb{U}[\varphi, B]$ is isomorphic to the one obtained when $B = \{G\}$. Therefore, for the sake of simplicity, we omit B and use the notation $\mathbb{U}[\varphi]$.

Its K-theory will be calculated using a similar idea as presented in [11], i.e, using the continuity of the functors K_0 and K_1 and also the Khoshkam–Skandalis sequence [14]. We conclude that $K_*(\mathbb{U}[\varphi]) \cong K_*(C^*(G))$. We also conclude that, when G is amenable, we can use Kirchberg's classification theorem to $\mathbb{U}[\varphi]$.

Finally, we use the recently-introduced semigroup C^*-algebras from [19] and [18] and show that $\mathbb{U}[\varphi]$ is isomorphic to the full semigroup C^*-algebra of the semigroup $S = G \rtimes_\varphi \mathbb{N}$. This implies that when the group G is amenable and the endomorphism φ is pure, the three semigroup C^*-algebras defined by Li are isomorphic to $\mathbb{U}[\varphi]$. Furthermore we can apply Kirchberg's classification theorem to them.

PROPOSITION 5.1. *When B contains only subsets of the form $g\varphi^k(G)$, for some fixed φ, $k \in \mathbb{N}$ and $g \in G$, the C^*-algebra $\mathbb{U}[\varphi, B]$ is isomorphic to $\mathbb{U}[\varphi]$, and can be redefined as the universal C^*-algebra generated by unitaries $\{u_g : g \in G\}$ and one isometry $\{s\}$ satisfying:*

(i) $u_g s^n u_h s^m = u_{g\varphi^n(h)} s^{n+m}$;

(ii) *we have*

$$u_g s^n s^{*n} u_{g^{-1}} u_h s^m s^{*m} u_{h^{-1}} = u_h s^m s^{*m} u_{h^{-1}} u_g s^n s^{*n} u_{g^{-1}} = \begin{cases} u_g s^n s^{*n} u_{g^{-1}} & \text{if } h \in g\varphi^m(G), \\ 0 & \text{otherwise,} \end{cases}$$

for $n \geqslant m$.

A simple use of Propositions 3.1, 3.2 and 3.3, and Theorem 4.9 yields the following proposition.

PROPOSITION 5.2. *The C^*-algebra $\mathbb{U}[\varphi]$ is separable. When the group G is amenable, it is also nuclear and satisfies the* UCT. *Furthermore, if φ is pure, then $\mathbb{U}[\varphi]$ is also simple and purely infinite, therefore a Kirchberg algebra satisfying the* UCT.

To calculate the K-theory of $\mathbb{U}[\varphi]$, using Remark 2.2 we know that

$$\mathbb{U}[\varphi] \cong (D[\varphi] \rtimes_\omega G) \rtimes_\tau \mathbb{N}$$

with

$$\begin{aligned} \omega : G &\to \operatorname{Aut}(D[\varphi]) \\ g &\mapsto u_g(\cdot)u_{g^{-1}}, \\ \tau : \mathbb{N} &\to \operatorname{End}(D[\varphi] \rtimes_\omega G) \\ n &\mapsto s^n(\cdot)s^{*n} \end{aligned}$$

where for $a_g\delta_g \in D[\varphi] \rtimes_\omega G$, $\tau_n(a_g\delta_g) = s^n a_g s^{*n}\delta_{\varphi^n(g)}$. But note that

$$D[\varphi] \cong \varinjlim_n D_n$$

for $n \in \mathbb{N}$ with

$$D_n := C^*\left(\left\{u_g s^k s^{*k} u_{g^{-1}} : 0 \leqslant k \leqslant n, g \in \frac{G}{\varphi^k(G)}\right\}\right)$$

and the inclusion being the identity. Therefore

$$D[\varphi] \rtimes_\omega G \cong \varinjlim_n (D_n \rtimes_\omega G),$$

where

$$D_n \rtimes_\omega G \cong C^*(\{u_g s^k s^{*k} u_{h^{-1}} : 0 \leqslant k \leqslant n, g, h \in G\}).$$

Moreover, for $k \in \mathbb{N}$, $A_k := C^*(\{u_g s^k s^{*k} u_{h^{-1}} : g, h \in G\})$ is an ideal of $D_k \rtimes_\omega G$, because for $m \leqslant k$,

$$u_h s^k s^{*k} u_{h^{-1}} u_g s^m s^{*m} u_{g^{-1}} = \begin{cases} u_h s^k s^{*k} u_{h^{-1}} & \text{if } g \in h\varphi^m(G), \\ 0 & \text{otherwise.} \end{cases}$$

But note that every element $u_g s^k s^{*k} u_{h^{-1}}$ in A_k can be uniquely written as $u_{g_i} s^k s^{*k} u_{g_j^{-1}} u_{\varphi^k(t)}$, for $g_i, g_j \in \frac{G}{\varphi^k(G)}$ and $t \in G$. So, if one defines the correspondence

$$u_g s^k s^{*k} u_{h^{-1}} = u_{g_i} s^k s^{*k} u_{g_j^{-1}} u_{\varphi^k(t)} \mapsto E_{i,j} \otimes u_{\varphi^k(t)},$$

where $\{E_{i,j}\}$ is the family of unit matrices which give rise to the set $\mathbb{K}$ of compact operators, it follows that

$$A_k \cong \mathbb{K} \otimes C^*(\varphi^k(G)) \cong \mathbb{K} \otimes C^*(G).$$

Hence, starting with the case $n = 1$, we can build the exact sequence

$$0 \to A_1 \xrightarrow{\iota} D_1 \rtimes_\omega G \xrightarrow{\rho} C^*(G) \to 0$$

where ι and ρ are the canonical inclusion and projection maps, respectively. But the sequence above splits if we also consider the canonical inclusion

$$\gamma : C^*(G) \to D_1 \rtimes_\omega G.$$

This implies that the corresponding exact sequence of K-groups also splits, which means that (using the Künneth formula [22])

$$K_*(D_1 \rtimes_\omega G) \cong K_*(C^*(G)) \oplus K_*(\mathbb{K} \otimes C^*(\varphi(G))) \cong K_*(C^*(G)) \oplus K_*(C^*(G)).$$

Repeating the argument, it is easy to conclude that

$$K_*(D_n \rtimes_\omega G) = \bigoplus_{i=0}^{n} K_*(C^*(G))$$

and consequently

$$K_*(D[\varphi] \rtimes_\omega G) = \varinjlim_n \bigoplus_{i=0}^{n} K_*(C^*(G)) = \bigoplus_{\mathbb{N}} K_*(C^*(G)) \tag{5.1}$$

where the k-th group $K_*(C^*(G))$ of the direct sum above represents the K-group of A_k.

Applying the Khoshkam–Skandalis sequence for $\mathbb{N}$-crossed products [14], we have the sequence

$$\begin{array}{ccccc} \bigoplus_{\mathbb{N}} K_0(C^*(G)) & \xrightarrow{1-K_0(\tau)} & \bigoplus_{\mathbb{N}} K_0(C^*(G)) & \to & K_0(\mathbb{U}[\varphi]) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{U}[\varphi]) & \leftarrow & \bigoplus_{\mathbb{N}} K_1(C^*(G)) & \xleftarrow{1-K_1(\tau)} & \bigoplus_{\mathbb{N}} K_1(C^*(G)) \end{array}$$

where $\tau_n(u_g) = s^n u_g s^{*n}$. Since $K_0(\mathbb{K})$ is described only by matrices of the type $E_{i,i}$, consider some $u_{g_i} s^n s^{*n} u_{g_i^{-1}} u_{\varphi^n(t)} \in D[\varphi] \rtimes_\omega G$. Then

$$K_*(\tau)[u_{g_i} s^n s^{*n} u_{g_i^{-1}} u_{\varphi^n(t)}]_* = [u_{\varphi(g_i)} s^{n+1} s^{*n+1} u_{\varphi(g_i^{-1})} u_{\varphi^{n+1}(t)}]_*$$

which implies that $K_*(\tau)$ corresponds to a shift in $\bigoplus_{\mathbb{N}} K_*(C^*(G))$. So denote by σ the shift operator, to see that the six-term sequence above turns into

$$\begin{array}{ccccc} \bigoplus_{\mathbb{N}} K_0(C^*(G)) & \xrightarrow{1-\sigma} & \bigoplus_{\mathbb{N}} K_0(C^*(G)) & \to & K_0(\mathbb{U}[\varphi]) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{U}[\varphi]) & \leftarrow & \bigoplus_{\mathbb{N}} K_1(C^*(G)) & \xleftarrow{1-\sigma} & \bigoplus_{\mathbb{N}} K_1(C^*(G)). \end{array}$$

But the application $1-\sigma$ has null kernel and $\mathrm{Im}(1-\sigma)$ only contains vectors $(x_0, x_1, \ldots, x_n, 0, 0, \ldots)$ whose sum of coordinates equals zero. This together with the direct limit description (5.1) implies that

$$\frac{\bigoplus_{\mathbb{N}} K_*(C^*(G))}{\mathrm{Im}\,(1-\sigma)} \cong K_*(C^*(G))$$

via

$$\overline{(x_0, x_1, \ldots, x_n, 0, 0, \ldots)} \mapsto \sum_{i=0}^{n} x_i.$$

Solving the six-term sequence we get

$$K_*(\mathbb{U}[\varphi]) \cong K_*(C^*(G)).$$

Therefore, we have the following theorem.

THEOREM 5.3. *Consider φ an injective endomorphism with infinite cokernel of some discrete countable group G, and construct the C^*-algebra $\mathbb{U}[\varphi]$ as in Proposition* 5.1. *Then $K_*(\mathbb{U}[\varphi]) \cong K_*(C^*(G))$.*

EXAMPLE 5.4. Let us recall Example 4.12. The group G is the free group $\mathbb{F}_2$ with generators $\{a, b\}$,

$$\begin{aligned} \varphi : \mathbb{F}_2 &\to \mathbb{F}_2 \\ a &\mapsto a \\ b &\mapsto b^2, \quad \text{linearly extended} \end{aligned}$$

and $B = \{G, H_1\}$ where $H_1 = \varphi(G) = \langle a, b^2 \rangle$. By Proposition 5.1 we have $\mathbb{U}[\varphi, B] = \mathbb{U}[\varphi]$.

Using Theorem 5.3, we conclude that

$$K_0(\mathbb{U}[\varphi]) \cong K_0(C^*(\mathbb{F}_2)) = \mathbb{Z}$$

with generator $[1]_0$ and

$$K_1(\mathbb{U}[\varphi]) \cong K_1(C^*(\mathbb{F}_2)) = \mathbb{Z}^2$$

with generators $[u_a]_1$ and $[u_b]_1$.

THEOREM 5.5. *Consider φ a pure injective endomorphism with infinite cokernel of some discrete countable amenable group G. Construct the C^*-algebra $\mathbb{U}[\varphi]$ as in Proposition* 5.1. *Then it is classifiable by Kirchberg's classification theorem.*

Consider two diferent pure injective endomorphisms of some discrete countable amenable group G. Then, both C^*-algebras will be classifiable by Kirchberg's theorem and, in both objects, $K_0(\mathbb{U}[\varphi]) \ni [1]_0 \mapsto [1]_0 \in K_0(C^*(G))$. Thus, the respective C^*-algebras are isomorphic.

COROLLARY 5.6. *Assuming that the above conditions are satisfied, for a fixed group G, any choice of endomorphism φ generates the same C^*-algebra $\mathbb{U}[\varphi]$.*

6. SEMIGROUP C^*-ALGEBRA DESCRIPTION OF $\mathbb{U}[\varphi]$

In [18] and [19] Li introduced and developed the concept of a C^*-algebra associated with a semigroup. We prove that when the semigroup is $S = G \rtimes_\varphi \mathbb{N}$, i.e, a semidirect product of a group G with $\mathbb{N}$ implemented by an injective endomorphism, the C^*-algebra of this semigroup can be viewed as the C^*-algebra associated with the endomorphism φ. This isomorphism together with extra restrictions on our initial data will allow us to conclude similar results concerning the K-theory of $\mathbb{U}[\varphi]$ as the one obtained in Theorem 5.3.

We compare the set of projections used in both definitions and, for this purpose, we study the sets which index these projections, namely B' in our case (Lemma 1.4) and the set $\mathcal{J}$ of constructible right ideals in Li's case (before Definition 2.2 in [18]). Note that both are defined as a certain set of subsets of the given structure, and they are closed with respect to some set operations.

The problem is that here B' is a set of subsets of a group and Li defines $\mathcal{J}$ containing subsets of a semigroup. However the following holds.

PROPOSITION 6.1. $\mathcal{J} = \{(g,n)S : (g,n) \in S\}$.

Proof. Use the fact that sets of the type

$$(g,n)S \cap (h,m)S \quad \text{and} \quad (g,n)^{-1}(h,m)S$$

are both of the form $(k,l)S$ or $\emptyset$. This result is also proved in Lemma 6.3.3 of [9]. ∎

The result above will allow us to establish the isomorphism between the algebra $\mathbb{U}[\varphi]$ defined in this section and the full semigroup C^*-algebra $C^*(S)$ defined by Li in Definition 2.2 in [18].

Consider an endomorphism φ of a group G with B containing only subgroups of the form $\varphi^k(G)$. By Proposition 5.1, $\mathbb{U}[\varphi]$ is the universal C^*-algebra generated by unitaries $\{u_g : g \in G\}$ and one isometry s satisfying

(i) $u_g s^n u_h s^m = u_{g\varphi^n(h)} s^{n+m}$.

PROPOSITION 6.2. *We have*

$$\mathbb{U}[\varphi] \cong C^*(S),$$

with the latter defined as in [18].

Proof. The C^*-algebra $C^*(S)$ is generated by isometries $\{v_{(g,n)} : (g,n) \in S\}$ and projections $\{e_X : X \in \mathcal{J}\}$ with $\mathcal{J} = \{(g,n)S : (g,n) \in S\}$ (by the proposition above).

To prove that the isomorphism holds, first note that the unitaries $v_{(g,0)}$ and the isometries $v_{(e,n)}$ satisfy the relation generating $\mathbb{U}[\varphi]$ ((i) above). Therefore, there exists a $*$-homomorphism

$$\begin{aligned} \Phi : \mathbb{U}[\varphi] &\to C^*(S) \\ u_g &\mapsto v_{(g,0)}, \quad \text{and} \\ s^n &\mapsto v_{(e,n)}. \end{aligned}$$

For the inverse map, consider the set of isometries $\{u_g s^n : (g, n) \in S\}$ and the set of projections

$$\{u_h s^m s^{*m} u_{h^{-1}} : \text{ associated with } (h, m)S \in \mathcal{J}\}.$$

Some calculations show that these two sets satisfy the five conditions generating $C^*(S)$ ([18]). By the universality of this C^*-algebra we have a $*$-homomorphism

$$\begin{aligned}\Psi : C^*_{\mathrm{s}}(S) &\to \mathbb{U}[\varphi]\\ v_{(g,n)} &\mapsto u_g s^n, \quad \text{and}\\ e_{[(h,m)S]} &\mapsto u_h s^m s^{*m} u_{h^{-1}}.\end{aligned}$$

It is easy to see that Φ and Ψ are inverses of each other. ∎

COROLLARY 6.3. *Consider φ an injective endomorphism with infinite cokernel of some discrete countable group G. Construct the semidirect product semigroup $S = G \rtimes_\varphi \mathbb{N}$. Then*

$$K_*(C^*(S)) \cong K_*(C^*(G)),$$

with $C^*(S)$ as defined in [18].

There are two more C^*-algebras associated with a semigroup S. The first one is the concrete representation of S called the reduced semigroup C^*-algebra of S, denoted by $C^*_{\mathrm{r}}(S)$ and defined in Definition 2.1 in [18]. It is easy to check that there exists a surjective $*$-homomorphism

$$\lambda : C^*(S) \to C^*_{\mathrm{r}}(S).$$

For the second one, note that the semigroup S can be viewed as a subsemigroup of the group $\overline{S}$ (defined previously), and this allows us to define another C^*-algebra associated with S, namely $C^*_{\mathrm{s}}(S)$ ([18], Definition 3.2). It has the same generators as $C^*(S)$ with minor additional relations, so that there is a surjective $*$-homomorphism

$$\pi_{\mathrm{s}} : C^*(S) \to C^*_{\mathrm{s}}(S).$$

But remember that if φ is pure and G is amenable the C^*-algebra $\mathbb{U}[\varphi]$ is simple (and purely infinite) by Theorem 4.9, and thus so is $C^*(S)$. Therefore we have the following theorem.

THEOREM 6.4. *Consider G an amenable discrete countable group and φ a pure injective endomorphism of G. Construct the semigroup $S = G \rtimes_\varphi \mathbb{N}$. Then the C^*-algebras $C^*(S)$, $C^*_{\mathrm{s}}(S)$ and $C^*_{\mathrm{r}}(S)$ defined in* [18] *are isomorphic to $\mathbb{U}[\varphi]$. By Theorem* 5.3*, we also conclude that*

$$K_*(C^*(S)) \cong K_*(C^*(G)).$$

Moreover, by Theorem 5.5*, they are classifiable by Kirchberg's classification theorem* [15].

EXAMPLE 6.5. Consider the shift endomorphism of $\bigoplus_{\mathbb{N}} \mathbb{Z}$, i.e

$$\varphi : \bigoplus_{\mathbb{N}} \mathbb{Z} \to \bigoplus_{\mathbb{N}} \mathbb{Z}$$
$$(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots).$$

Denote by G the group $\bigoplus_{\mathbb{N}} \mathbb{Z}$ and let us choose $B = \{G\}$ to apply the theorem above. It is well-known that

$$C^*\Big(\bigoplus_{\mathbb{N}} \mathbb{Z}\Big) \cong \bigotimes_{\mathbb{N}} C^*(\mathbb{Z}) \cong \bigotimes_{\mathbb{N}} C(\mathbb{S}^1)$$

which together with the Künneth formula ([22]) implies

$$K_0(\mathbb{U}[\varphi]) = K_1(\mathbb{U}[\varphi]) = \bigoplus_{\mathbb{N}} \mathbb{Z}.$$

Consider φ the shift (to the right) on G. With $S = G \rtimes_\varphi \mathbb{N}$ the theorem above implies that

$$\mathbb{U}[\varphi] \cong C^*(S) \cong C^*_{\mathrm{s}}(S) \cong C^*_{\mathrm{r}}(S)$$

(as defined in [18]) and this C^*-algebra is nuclear, simple and purely infinite.

The theorem above provides another powerful tool to calculate the K-theory of $\mathbb{U}[\varphi]$, which agrees with Theorem 5.3, just using Theorem 6.3.4 in [9]. For this, note that G being amenable implies that $\overline{S}$ also is (in [9] this group is called the enveloping group of S). Therefore it satisfies the Baum–Connes conjecture with coefficients, and the following result applies.

THEOREM 6.6. *For an amenable group G and a pure injective endomorphism with infinite cokernel φ of G consider the semigroup $S = G \rtimes_\varphi \mathbb{N}$. Choose $B = \{\varphi^k(G)\}$ for some $k \in \mathbb{N}$. Then*

$$K_*(\mathbb{U}[\varphi]) \cong K_*(C^*(G)).$$

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