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SOME PROPERTIES OF THE SPHERICAL m-ISOMETRIES

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ABSTRACT. A commuting d-tuple $T = (T_1, \dots, T_d)$ is called a spherical m-isometry if $\sum\limits_{j=0}^{m} (-1)^j \binom{m}{j} Q_T^j(I) = 0$, where $Q_T(A) = \sum\limits_{i=1}^{d} T_i^* A T_i$ for every bounded linear operator A on a Hilbert space $\mathcal{H}$. Under some assumptions we prove that every power of T is a spherical m-isometry. Also, we study the products of spherical m-isometries when they remain spherical n-isometries, for a suitable n. Besides, we prove that the spherical m-isometries are power regular and for every proper spherical m-isometry there are linearly independent operators $A_0, \dots, A_{m-1}$ such that $Q_T^n(I) = \sum\limits_{i=0}^{m-1} A_i n^i$ for every $n \geqslant 0$.

KEYWORDS: *m-Isometry, power regularity, spherical m-isometry, d-tuple.*

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1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the space of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called an m-isometry ($m \in \mathbb{N}$), if it satisfies the following property:

$$(1.1) \qquad (yx-1)^m(T) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0.$$

Since $(yx-1)^m(T)$ is a self-adjoint operator, we observe that T is an m-isometry if and only if for each $x \in \mathcal{H}$,

$$(1.2) \qquad \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0.$$

It is clear that the notions of 1-isometry and isometry coincide.

These operators have been introduced by Agler in [3] and their applications to differential operators, disconjugacy, and Brownian motion have been obtained in [4], [5], [6]. In recent years, the m-isometric operators have received substantial attention. Patton and Robbins [44] studied which composition operators are m-isometries. Furthermore, m-isometry weighted shift operators have been discussed in [1], [19], [22], [36], [41]. In addition, it has been proved in [17], [21] that the powers of an m-isometry are m-isometries and some products of m-isometries are again m-isometries. On the other hand, the perturbation of m-isometries by nilpotent operators has been considered in [20], [23], [37] and the dynamics of m-isometries has been explored in [15], [16], [18], [35]. Furthermore, Duggal studied the tensor product of m-isometries in [31], [32]. Gleason and Richter in [38] extended the notion of m-isometric operator to the case of commuting d-tuple of bounded linear operators on a Hilbert space. Then Hoffmann and Mackey in [43] generalized the definition of m-isometric operators tuple on a normed space.

In this paper, we establish some properties of spherical m-isometries which are natural generalizations of m-isometries. Indeed, they are tuples that extend the notion of an isometry in two directions from a single operator T to a tuple of operators $(T_1, \ldots, T_d)$ and in the degree of a $*$-algebra identity satisfied, from $I - T^*T = 0$ in the single operator case where as usual I denotes the identity operator. When $m = 1$, it is called a spherical isometry, which is a commuting subnormal tuple, thus, the theory of commutative subnormal tuples is applicable to spherical isometries. In particular, any spherical isometry can be uniquely decomposed as a direct sum of a normal tuple with joint spectrum in the unit sphere and a completely non-normal or pure spherical isometry. For some more interesting facts on spherical isometries one can see [34]. The behavior of d-tuple of operators which are near to spherical isometries has been much studied. For some recent papers one can see [7], [8], [13], [27], [28], [29], [45]. We also wish to mention that the spherical m-isometries have attracted additional interest from their relation with a moment problem [7]. Thus, we have enough motivation to seek some basic and non-trivial properties of spherical m-isometries.

When we talk about a property related to operators, a natural question is whether it is stable under powers or not. For example, it is interesting to consider under what extra hypotheses the basic concepts such as reflexivity, hypercyclicity, or supercyclicity of tuple of operators are stable under powers. One of our motivations is to discuss this question for spherical m-isometries. Notice that to find sufficient conditions in order to give a positive answer to this question, we offer ways of obtaining new spherical m-isometries. Indeed, in Section 3 we give a sufficient condition under which every power of a spherical m-isometry is again a spherical m-isometry.

Recall that an operator S in $\mathcal{B}(\mathcal{H})$ is power regular if $\lim\limits_{n\to\infty} \|S^n x\|^{1/n}$ exists for every $x \in \mathcal{H}$. It is known that compact operators, hyponormal operators, decomposable operators and isometric N-Jordan operators are power regular. The backward shift operator is an example of an operator that is not power regular. Power regularity helps us to find a non-trivial invariant subspace of an operator. For example, if S is hyponormal and $\lim_{n\to\infty} \|S^n x\|^{1/n} < \|S\|$ for some non-zero vector x then x is not a cyclic vector for S; i.e., $\bigvee_{n\geqslant 0} \{S^n x\}$ is a non-trivial invariant subspace of S [25]. For some information on power regularity of operators one can see [14] and references therein. In Section 4, we define the concept of power regularity for commuting tuples of operators and prove that the spherical m-isometries are power regular. Moreover, we provide conditions on a right invertible spherical isometry making it a spherical unitary. Finally, in the last section we show that the d-tuple T is a spherical m-isometry but not a spherical $(m-1)$-isometry if and only if there are linearly independent operators $A_0, A_1, \ldots, A_{m-1}$ in $\mathcal{B}(\mathcal{H})$ such that $Q_T^n(I) = \sum_{i=0}^{m-1} A_i n^i$ for all $n \geqslant 0$. This extends and improves a recent result of [23].

2. PRELIMINARIES

In this section, we refer to some of the well known facts which will be used in the next sections. A few comments are in order. Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ we set $|\alpha| = \sum_{j=1}^{d} \alpha_j$, $\alpha! = \alpha_1! \cdots \alpha_d!$ and further $T^\alpha = T_1^{\alpha_1} \cdots T_d^{\alpha_d}$. For every d-tuple of commuting operators $T = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$, there is a function $Q_T : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ defined by $Q_T(A) = \sum_{i=1}^{d} T_i^* A T_i$. It is easy to see that $Q_T^j(I) = \sum_{|\alpha|=j} (j!/\alpha!) T^{*\alpha} T^\alpha$ $(j \geqslant 1)$ where $T^* = (T_1^*, \ldots, T_d^*)$. For each $m \geqslant 0$, let

$$P_m(T) = (I - Q_T)^m(I) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} Q_T^j(I).$$

A commuting tuple, $T = (T_1, \ldots, T_d)$ is said to be a spherical m-isometry, if $P_m(T) = 0$. An example of a spherical m-isometry is the Drury–Arverson m-shift which has played a role in the dilation of the row contraction. ([10], [30], Theorem 4.2 of [38]). The Drury–Arverson m-shift is the m-tuple operator $M_z = (M_{z_1}, \ldots, M_{z_m})$ in the reproducing kernel Hilbert space associated with the positive definite kernel

$$\frac{1}{1 - z_1\overline{w}_1 - \cdots - z_m\overline{w}_m}$$

where z and w are in the open unit ball of $\mathbb{C}^m$. For some more examples of spherical m-isometries one can see [28]. The d-tuple T is called a proper spherical m-isometry, if it is a spherical m-isometry but it is not a spherical $(m-1)$-isometry. Note that

$$P_{n+1}(T) = (I - Q_T)(P_n(T)) = P_n(T) - Q_T(P_n(T)) \tag{2.1}$$

for all $n \geqslant 0$. Observe that if T is a commuting tuple of operators on $\mathcal{H}$ and $P_m(T) = 0$, then $P_{m+n}(T) = 0$, for all $n \geqslant 0$. Hence if T is a spherical m-isometry, then T is a spherical $(m+n)$-isometry, for all $n \geqslant 0$. For a spherical m-isometry T, define

$$\Delta_{T,m} := (-1)^{m-1} P_{m-1}(T).$$

It is proved that if T is a spherical m-isometry for some $m \geqslant 0$, then $\Delta_{T,m}$ is a positive operator (see Proposition 2.3 of [38]). For $T \in \mathcal{B}(\mathcal{H})$, we define $\beta_\ell(T) = (1/\ell!)(yx-1)^\ell(T)$ for $\ell \geqslant 0$. Using the notion $\beta_\ell(T)$, if $T \in \mathcal{B}(\mathcal{H})$ is an m-isometry, $\|T^k x\|^2 = \sum_{\ell=0}^{m-1} k^{(\ell)} \langle \beta_\ell(T)x, x\rangle$, where $k^{(\ell)} = k \cdot (k-1) \cdots (k-\ell+1)$ for $\ell \geqslant 1, k \geqslant 0$ and $k^{(0)} = 1$.

LEMMA 2.1 ([38], Lemma 2.1). *If $T = (T_1, \ldots, T_d)$ is a commuting tuple of operators on a Hilbert space $\mathcal{H}$, then*

$$P_m(T) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{|\alpha|=j} \frac{j!}{\alpha!} (T^\alpha)^* T^\alpha$$

and for all $x \in \mathcal{H}$

$$\langle P_m(T)x, x\rangle = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{|\alpha|=j} \frac{j!}{\alpha!} \|T^\alpha x\|^2.$$

LEMMA 2.2 ([38], Lemma 2.2). *For $k \geqslant 0$,*

$$Q_T^k(I) = \sum_{j=0}^{\infty} \Big(\frac{(-1)^j}{j!} P_j(T) \Big) k^{(j)}.$$

In the next proposition we provide a condition on a spherical m-isometry which ensure it to be a spherical isometry.

PROPOSITION 2.3. *Suppose that $T = (T_1, \ldots, T_d)$ is a spherical m-isometric commuting tuple of operators on a Hilbert space $\mathcal{H}$ and $a_k(x) := \langle Q_T^k(I)x, x\rangle$, $x \in \mathcal{H}$. If for each $x \in \mathcal{H}$, there is a strictly increasing sequence $\{k_i\}_{i=0}^\infty$ of positive integers and a constant M such that $|a_{k_i}(x)| \leqslant M$ for $i = 1, 2, \ldots$, then T is a spherical isometry.*

Proof. Since $P_k(T) = 0$ for all $k \geqslant m$, Lemma 2.2 shows that

$$0 \leqslant \frac{(-1)^{m-1}}{(m-1)!} \langle P_{m-1}(T)x, x\rangle = \lim_{i\to\infty} \frac{\langle Q_T^{k_i}(I)x, x\rangle}{k_i^{m-1}} \leqslant \lim_{i\to\infty} \frac{M}{k_i^{m-1}} = 0.$$

Hence

$$\frac{1}{(m-1)!} \langle \Delta_{T,m} x, x\rangle = \Big\langle \frac{(-1)^{m-1}}{(m-1)!} P_{m-1}(T)x, x \Big\rangle = 0.$$

It follows that T is a spherical $(m-1)$-isometry. Continuing this process and applying the same argument, it can be seen that T is a spherical isometry. ∎

Recall that an operator A in $\mathcal{B}(\mathcal{H})$ is power bounded if $\sup_n \|A^n\| < \infty$.

COROLLARY 2.4. *Suppose that $T = (T_1, \dots, T_d)$ is a spherical m-isometric commuting tuple of operators on a Hilbert space $\mathcal{H}$. If one of the following conditions holds then T is a spherical isometry:*
(i) *A subsequence of $\{\langle Q_T^k(I)\cdot, \cdot\rangle\}_{k=1}^{\infty}$ is bounded.*
(ii) *Each T_i is idempotent.*
(iii) *Each T_i is power bounded and $T_iT_j = 0,\ i \neq j$.*

3. POWERS AND PRODUCTS

In this section, we consider the following question. Is the class of spherical m-isometries stable under products and powers? To achieve our goal we need two lemmas. We borrow the first one from [33].

LEMMA 3.1 ([33], Corollary 2.5). *If $(a_{i,j})_{i,j=0}^{\infty}$ is a double sequence of complex numbers satisfying*

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}a_{k+i,\ell} = 0 \quad \textit{and} \quad \sum_{j=0}^{n}(-1)^j\binom{n}{j}a_{k,\ell+j} = 0,$$

then

$$\sum_{s=0}^{m+n-1}(-1)^s\binom{m+n-1}{s}a_{s,s} = 0.$$

LEMMA 3.2. *If $P_m(k) = a_0 + a_1k + \cdots + a_mk^m$, $m \geqslant 1$ then*

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}P_m(k) = m!a_m. \tag{3.1}$$

Proof. An easy induction argument on m shows that

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^i = 0, \quad 0 \leqslant i \leqslant m-1,$$

so we can assume that $P_m(k) = k^m$, $m \geqslant 1$. We prove (3.1) by induction on m. Clearly the result is true for $m = 1$. Assume that the result holds for m. Therefore,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}q(k) = m!$$

where $q(k) = \sum\limits_{n=0}^{m}\binom{m}{n}k^n$. Thus

$$\begin{aligned}\sum_{k=0}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}P_{m+1}(k) &= \sum_{k=1}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}P_{m+1}(k)\\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m+1}{k+1}P_{m+1}(k+1)\end{aligned}$$

$$= (m+1)\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(k+1)^m$$
$$= (m+1)\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}q(k) = (m+1)!.$$

Hence the proof is completed. ∎

A sufficient condition that we use in this section, on a spherical m-isometry $T = (T_1, \dots T_d)$ is the orthogonality condition $T_iT_j = 0$ for $i \neq j$. For example, if $\mathcal{H} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_d$, then $T = (P_{\mathcal{M}_1}, \dots, P_{\mathcal{M}_d})$ is a spherical isometry on $\mathcal{H}$ and $P_{\mathcal{M}_i}P_{\mathcal{M}_j} = 0$, $i \neq j$ where $P_{\mathcal{M}_i}$ is the projection onto $\mathcal{M}_i$. To see that this additional assumption is not strong we will find a way to show that orthogonal spherical m-isometries are as many as m-isometries. Note that the orthogonality condition on the commuting d-tuple $T = (T_1, \dots, T_d)$ implies that

$$Q_T^k(A) = \sum_{j=1}^{d} T_j^{*k} A T_j^k \tag{3.2}$$

for every A in $\mathcal{B}(\mathcal{H})$. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K} = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{d-\text{times}}$. For m-isometric operators $S_i : \mathcal{H} \longrightarrow \mathcal{H}$, $i = 1, \dots, d$, it is straightforward to verify that $T = (T_1, \dots, T_d)$ is a spherical m-isometry on $\mathcal{K}$ where $T_i = \bigoplus_{j=1}^{d} \delta_{ij} S_j$ and δ_{ij} is the Kronecker delta; moreover, $T_iT_j = 0$ for $i \neq j$. Suppose that one of the S_is is a proper m-isometry, then T is also a proper spherical m-isometry. Indeed, assuming that T is a spherical $(m-1)$-isometry, for every $x_1 \oplus \cdots \oplus x_d \in \mathcal{K}$ we have

$$0 = \Big\langle \sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}Q_T^j(I)(x_1 \oplus \cdots \oplus x_d), x_1 \oplus \cdots \oplus x_d \Big\rangle$$
$$= \sum_{j=0}^{m-1}\sum_{i=1}^{d}(-1)^j\binom{m-1}{j}\|T_i^j(x_1 \oplus \cdots \oplus x_d)\|^2 = \sum_{i=1}^{d}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\|S_i^j x_i\|^2.$$

Since every S_i is an m-isometry,

$$\langle \Delta_{S_i,m} x_i, x_i \rangle = \sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\|S_i^j x_i\|^2 \geqslant 0, \quad (i = 1, \dots, d);$$

hence

$$\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\|S_i^j x_i\|^2 = 0,$$

for each $i = 1, \dots, d$ and each x_i in $\mathcal{H}$. But this contradicts the fact that one of the S_is is a proper m-isometry.

REMARK 3.3. Note that an infinite dimensional separable Hilbert space $\mathcal{H}$ is isometrically isomorphic to the Hilbert space $\mathcal{K} = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{d-\text{times}}$. Let $U : \mathcal{H} \to \mathcal{K}$ be an isometric isomorphism and $(T_1, \ldots, T_d)$ be an orthogonal spherical m-isometry on $\mathcal{K}$. It can easily be verified that $(U^{-1}T_1U, \ldots, U^{-1}T_dU)$ is an orthogonal spherical m-isometry on $\mathcal{H}$.

Roughly speaking we have found a way to get orthogonal proper spherical m-isometries from a single proper m-isometry. In what follows, we produce some non-trivial orthogonal spherical m-isometries.

EXAMPLE 3.4. Let $\mathcal{H}$ denote the Hilbert space obtained by taking the completion of the continuously differentiable functions on $[0,1]$ with the norm induced by the inner product

$$\langle f, g \rangle = \int_0^1 (f'\overline{g}' + f\overline{g})(t)\mathrm{d}t$$

and let A be the operator of the multiplication by t on $\mathcal{H}$, i.e., $(Af)(t) = tf(t)$. Therefore $A^{*3} - 3A^{*2}A + 3A^*A^2 - A^3 = 0$, and consequently the operator $S = \exp(\mathrm{i}rA)$ is a 3-isometry for every real number r. So the d-tuple $T = (T_1, \ldots, T_d)$ is an orthogonal spherical 3-isometry where $T_i = \bigoplus_{j=1}^{d} \delta_{ij}S$.

REMARK 3.5. We can replace the multiplication operator A in the preceding example by any operator A on a Hilbert space with $A^{*3} - 3A^{*2}A + 3A^*A^2 - A^3 = 0$. Note that this notion can be generalized in a natural way to m-self-adjoint operators by $\sum_{k=0}^{m} (-1)^k \binom{m}{k} A^{*k}A^{m-k} = 0$.

EXAMPLE 3.6. If $c \neq 0$ then the matrix $S = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ defines a proper 3-isometric operator on $\mathbb{C}^2$ [44]. Thus, the d-tuple $(T_1, \ldots, T_d)$ is an orthogonal proper spherical 3-isometry where $T_i = \bigoplus_{j=1}^{d} \delta_{ij}S$.

EXAMPLE 3.7. For z and ω in the open unit disc $\mathbb{D}$, define

$$\begin{cases} \kappa_{-1}(z,\omega) = -\log(1 - z\overline{\omega}), \\ \kappa_{-m}(z,\omega) = \sum_{n=1}^{\infty} \frac{z^{n+m-1}\overline{\omega}^{n+m-1}}{n(n+1)\cdots(n+m-1)} \quad (m \geqslant 2). \end{cases}$$

It is well-known that corresponding to any κ_{-m} there exists a functional Hilbert space $\mathcal{H}(\kappa_{-m})$ such that κ_{-m} serves as a reproducing kernel for $\mathcal{H}(\kappa_{-m})$ [9]. In fact, $\mathcal{H}(\kappa_{-m})$ may be identified with the space of all analytic functions $f(z) =$

$\sum\limits_{n=m}^{\infty} a_n z^n$ on $\mathbb{D}$ so that

$$\|f\|_{-m}^2 = \frac{1}{\pi(m-1)!} \int\limits_{\mathbb{D}} |f^{(m)}(z)|^2 (1-|z|^2)^{m-1} \mathrm{d}A(z) < \infty.$$

Note that $\{z^n\}_{n=m}^{\infty}$ generates $\mathcal{H}(\kappa_{-m})$ and we get

$$\begin{aligned}\|z^n\|_{-m}^2 &= \frac{1}{(m-1)!}(n(n-1)\cdots(n-m+1))^2 \frac{\Gamma(n-m+1)\Gamma(m)}{\Gamma(n+1)} \\ &= n(n-1)\cdots(n-m+1).\end{aligned}$$

Taking $e_n = z^{m+n-1}/\|z^{m+n-1}\|_{-m}$, for all $n \geqslant 1$, a simple computation shows that $\{e_n\}_{n=1}^{\infty}$ forms an orthonormal basis for $\mathcal{H}(\kappa_{-m})$. Let $M(\kappa_{-m})$ denote the operator of multiplication by z on $\mathcal{H}(\kappa_{-m})$. Then $M(\kappa_{-m})(e_n) = \sqrt{(n+m)/n}\, e_{n+1}$. Athavale in [12] showed that for every positive integer m, the operator $M(\kappa_{-m})$ is a proper $(m+1)$-isometry. Hence the d-tuple $(T_1 \ldots, T_d)$ is an orthogonal proper spherical $(m+1)$-isometry where $T_i = \bigoplus\limits_{j=1}^{d} \delta_{ij} M(\kappa_{-m})$.

REMARK 3.8. In the above example, $M(\kappa_{-1})$ is the Dirichlet shift which is a proper 2-isometry.

We are now in a position to get into the main subject of this section. For a positive integer r, a commuting tuple $T = (T_1, \ldots, T_d)$ is called a spherical m-isometry of order r, if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} Q_T^{rk}(I) = 0.$$

THEOREM 3.9. *Suppose that $T = (T_1, \ldots, T_d)$ and $S = (S_1, \ldots, S_d)$ are commuting tuples of operators on a Hilbert space $\mathcal{H}$ such that T is a spherical m-isometry, S is a spherical n-isometry and $TS = ST$. Then T is a spherical m-isometry of order r for every $r \geqslant 1$. Moreover, if $T_iT_j = S_iS_j = 0$ for $i \neq j$, then*

$$\sum_{t=0}^{m+n-1} (-1)^t \binom{m+n-1}{t} Q_T^t Q_S^t(I) = 0.$$

Proof. Since $P_j(T) = 0$ for $j \geqslant m$, Lemma 2.2 can be invoked to give

$$\langle Q_T^k(I)x, x\rangle = \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} \langle p_j(T)x, x\rangle k^{(j)}, \quad k = 1, 2, \ldots$$

for every $x \in \mathcal{H}$. This shows that $\langle Q_T^k(I)x, x\rangle$ is a polynomial in k of degree at most $m-1$. Therefore, Lemma 3.2 implies that

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \langle Q_T^{rk}(I)x, x\rangle = 0.$$

Now as $\sum\limits_{k=0}^{m}(-1)^k\binom{m}{k}Q_T^{rk}(I)$ is a self-adjoint operator, the result holds.

For the proof of the second part, let $x \in \mathcal{H}$ and put $a_{i,j} = \langle Q_T^i(Q_S^j(I))x, x\rangle$. Then

$$a_{i+k,\ell} = \langle Q_S^\ell Q_T^{i+k}(I)x, x\rangle = \Big\langle \sum_{p=1}^{d} S_p^{*\ell} Q_T^{i+k}(I)S_p^\ell x, x\Big\rangle = \sum_{p=1}^{d}\langle Q_T^{i+k}(I)S_p^\ell x, S_p^\ell x\rangle$$

$$= \sum_{p=1}^{d}\Big\langle\Big(\sum_{q=1}^{d} T_q^{*k}Q_T^i(I)T_q^k\Big)S_p^\ell x, S_p^\ell x\Big\rangle = \sum_{p=1}^{d}\sum_{q=1}^{d}\langle Q_T^i(I)T_q^kS_p^\ell x, T_q^kS_p^\ell x\rangle.$$

Since T is a spherical m-isometry, $\langle Q_T^i(I)T_q^kS_p^\ell x, T_q^kS_p^\ell x\rangle$ is a polynomial in i of degree at most $m-1$, and so is $a_{i+k,\ell}$. Therefore, by Lemma 3.2

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}a_{i+k,\ell} = 0.$$

On the other hand, since S is a spherical n-isometry, applying a similar argument one can show that

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}a_{k,\ell+j} = 0.$$

Putting these facts together, Lemma 3.1 implies that

$$\sum_{t=0}^{m+n-1}(-1)^t\binom{m+n-1}{t}a_{t,t} = 0;$$

hence we obtain the desired assertion. ∎

PROPOSITION 3.10. *Suppose that $T = (T_1, \dots, T_d)$ and $S = (S_1, \dots, S_d)$ are commuting tuples of operators on a Hilbert space $\mathcal{H}$ such that T is a spherical m-isometry, S is a spherical n-isometry, $TS = ST$ and $T_iT_j = S_iS_j = T_iS_j = 0$, for $i \neq j$. Then $T^r = (T_1^r, \dots, T_d^r)$ is a spherical m-isometry for each positive integer r. Moreover, $TS = (T_1S_1, \dots, T_dS_d)$ is a spherical $(m+n-1)$-isometry.*

Proof. The preceding theorem implies that

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}Q_{T^r}^k(I) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}Q_{T^{rk}}(I) = \sum_{k=0}^{m}(-1)^k\binom{m}{k}Q_T^{rk}(I) = 0,$$

hence T^r is a spherical m-isometry.

The second part holds because

$$Q_T^kQ_S^k(I) = Q_{T^k}Q_{S^k}(I) = Q_{T^kS^k}(I) = Q_{(TS)^k}(I) = Q_{TS}^k(I). \tag{3.3}$$

Now apply the second part of the previous theorem. ∎

The following example shows that in the previous proposition the orthogonality condition $T_iT_j = 0, i \neq j$ is essential.

EXAMPLE 3.11. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and $A \in \mathcal{B}(\mathcal{H})$ be the unilateral shift operator defined by $Ae_n = e_{n+1}$. Put $T_1 = T_2 = A/\sqrt{2}$. Then it is easily seen that $T = (T_1, T_2)$ is a spherical isometry but $T^2 = (T_1^2, T_2^2)$ is not.

An immediate corollary of Proposition 3.10 is the following result obtained in [17], [21].

COROLLARY 3.12. *Let $\mathcal{H}$ be a Hilbert space and S, T in $\mathcal{B}(\mathcal{H})$ commute with each other. Then*

(i) *if T is an m-isometry, then so is any power of T;*

(ii) *if T is an m-isometry and S is an n-isometry, then TS is an $(m+n-1)$-isometry.*

COROLLARY 3.13. *Let $T = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a spherical isometry such that $T_iT_j = 0$ for $i \neq j$, then $S = (T_1^{n_1}, \dots, T_d^{n_d})$ is also a spherical isometry for every d-tuple $(n_1, \dots, n_d)$ of positive integers.*

Proof. Suppose that $n_1 = \min\{n_1, n_2, \dots, n_d\}$. As $(T_1^{n_1}, \dots, T_d^{n_1})$ is a spherical isometry, $\sum\limits_{i=1}^{d} T_i^{*n_1} T_i^{n_1} = I$. Thus,

$$I - Q_S(I) = \sum_{i=2}^{d} T_i^{*n_1} T_i^{n_1} - \sum_{i=2}^{d} T_i^{*n_i} T_i^{n_i} = \sum_{\substack{i=2 \\ n_i \neq n_1}}^{d} T_i^{*n_1} (I - T_i^{*n_i - n_1} T_i^{n_i - n_1}) T_i^{n_1}$$

$$= \sum_{\substack{i=2 \\ n_i \neq n_1}}^{d} T_i^{*n_1} \Big(\sum_{\substack{j=1 \\ j \neq i}}^{d} T_j^{*n_i - n_1} T_j^{n_i - n_1} \Big) T_i^{n_1} = 0.$$

The next-to-last equality holds because $(T_1^{n_i - n_1}, \dots, T_d^{n_i - n_1})$ is a spherical isometry for every $n_i \neq n_1$. ∎

It is natural to suspect the correctness of the above corollary for a spherical m-isometry. To prove it let $p(\overline{z}_1, \dots, \overline{z}_d, z_1, \dots, z_d)$ be a $2d$-variable polynomial of the form

$$p(\overline{z}_1, \dots, \overline{z}_d, z_1, \dots, z_d) = \sum_{i=1}^{d} \sum_{k_i=0}^{m} \sum_{\ell_i=0}^{m} c_{k_i \ell_i} \overline{z}_i^{\ell_i} z_i^{k_i} \quad c_{k_i \ell_i} \in \mathbb{C},$$

and suppose that $T_1, \dots, T_d$ are in $\mathcal{B}(\mathcal{H})$. Define

$$p(T_1, \dots, T_d) = \sum_{i=1}^{d} \sum_{k_i=0}^{m} \sum_{\ell_i=0}^{m} c_{k_i \ell_i} T_i^{*\ell_i} T_i^{k_i} \quad c_{k_i \ell_i} \in \mathbb{C},$$

and note that if $p(\overline{z}_1, z_1)$ is a polynomial in two variables then the definition of $p(T_1)$ is the classical definition of hereditary functional calculus for operators in [2], [4] and [42]. Keeping this in mind, we have the following theorem. The idea of the proof is taken from Corollary 2.4 of [40].

THEOREM 3.14. *Suppose that $T = (T_1, \dots, T_d)$ is a spherical m-isometry on $\mathcal{H}$ such that $T_iT_j = 0, i \neq j$. Then the d-tuple $(T_1^{n_1}, \dots, T_d^{n_d})$ is a spherical m-isometry for every d-tuple $(n_1, \dots, n_d)$ of positive integers.*

Proof. Put $p(\overline{z}_1, \dots, \overline{z}_d, z_1, \dots, z_d) = \sum_{i=1}^{d} (1 - \overline{z}_i z_i)^m - (d-1)$. It is straightforward to verify that T is a spherical m-isometry if and only if $p(T_1, \dots, T_d) = 0$. On the other hand, since

$$(1 - \overline{z}_i^{n_i} z_i^{n_i})^m = \sum_{m_1+\cdots+m_{n_i}=m} \binom{m}{m_1, \dots, m_{n_i}} \overline{z}_i^{(0\cdot m_1+1\cdot m_2+\cdots+(n_i-1)m_{n_i})} \cdot (1 - \overline{z}_i z_i)^m z_i^{(0\cdot m_1+1\cdot m_2+\cdots+(n_i-1)m_{n_i})}$$

for $i = 1, \dots, d$ if we assume that $z_i z_j = 0, i \neq j$ then

$$\sum_{i=1}^{d}(1-\overline{z}_i^{n_i} z_i^{n_i})^m-(d-1)=\sum_{i=1}^{d} \sum_{m_1+\cdots+m_{n_i}=m} \binom{m}{m_1, \dots, m_{n_i}} \overline{z}_i^{(0\cdot m_1+1\cdot m_2+\cdots+(n_i-1)m_{n_i})} \cdot p(\overline{z}_1, \dots, \overline{z}_d, z_1, \dots, z_d) z_i^{(0\cdot m_1+1\cdot m_2+\cdots+(n_i-1)m_{n_i})} - (d-1) p(\overline{z}_1, \dots, \overline{z}_d, z_1, \dots, z_d).$$

Now, substituting T_i for z_i and T_i^* for $\overline{z}_i$ ultimately, we get $q(T_1, \dots, T_d) = 0$, where $q(\overline{z}_1, \dots, \overline{z}_d, z_1, \dots, z_d) = \sum_{i=1}^{d} (1 - \overline{z}_i^{n_i} z_i^{n_i})^m - (d-1)$. So $(T_1^{n_1}, \dots, T_d^{n_d})$ is a spherical m-isometry. ∎

Before stating the next result, we are first going to state an independently interesting lemma about complex numbers. The proof is taken from the proof of Theorem 3.6 of [17] with a small modification. For the benefit of the reader and the sake of completeness we include its proof.

LEMMA 3.15. *If $(a_n)_n$ is a sequence of complex numbers and r, s, m, ℓ are positive integers satisfying*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{rk+n} = 0, \quad \textit{and} \tag{3.4}$$

$$\sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} a_{sk+n} = 0, \tag{3.5}$$

for all $n \geqslant 0$, then

$$\sum_{k=0}^{h} (-1)^k \binom{h}{k} a_{tk} = 0 \tag{3.6}$$

where t is the greatest common divisor of r and s, and h is the minimum of m and ℓ.

Proof. Let V_1 and V_2 be the vector space of all sequences $(a_n)_n$ satisfying (3.4) and (3.5), respectively. Then $\dim V_1 = mr$ and $\dim V_2 = sl$. Indeed,

$$(n^k)_{n\geqslant 0},\ (n^k e^{i2\pi n/r})_{n\geqslant 0},\ (n^k e^{i4\pi n/r})_{n\geqslant 0}, \dots, (n^k e^{i2\pi(r-1)n/r})_{n\geqslant 0} \quad \text{for } 0\leqslant k\leqslant m-1$$

is a basis for V_1 and

$$(n^k)_{n\geqslant 0},\ (n^k e^{i2\pi n/s})_{n\geqslant 0},\ (n^k e^{i4\pi n/s})_{n\geqslant 0}, \dots, (n^k e^{i2\pi(s-1)n/s})_{n\geqslant 0} \quad \text{for } 0\leqslant k\leqslant \ell-1$$

is a basis for V_2. Therefore,

$$(n^k)_{n\geqslant 0},\ (n^k e^{i2\pi n/t})_{n\geqslant 0},\ (n^k e^{i4\pi n/t})_{n\geqslant 0}, \dots, (n^k e^{i2\pi(t-1)n/t})_{n\geqslant 0} \quad \text{for } 0\leqslant k\leqslant h-1$$

is a basis for $V_1 \cap V_2$. It is sufficient to prove that the basis of $V_1 \cap V_2$ satisfies (3.6). Suppose that $(n^k e^{i2\pi jn/t})_{n\geqslant 0}$, $0 \leqslant j \leqslant t-1$, $0 \leqslant k \leqslant h-1$ is an element of the basis for $V_1 \cap V_2$. Hence

$$\sum_{q=0}^{h} (-1)^q \binom{h}{q} (tq)^k e^{i2\pi jtq/t} = t^k \sum_{q=0}^{h} (-1)^q \binom{h}{q} q^k = 0,$$

by Lemma 3.2. ∎

PROPOSITION 3.16. *Suppose that $T = (T_1, \dots, T_d)$ is a commuting tuple of operators on a Hilbert space $\mathcal{H}$ and r, s, m, ℓ be positive integers. Moreover, let t be the greatest common divisor of r and s, and h be the minimum of m and ℓ.*

(i) *If T is a spherical m-isometry of order r and T is a spherical ℓ-isometry of order s, then T is a spherical h-isometry of order t.*

(ii) *Suppose that $T_iT_j = 0$, for $i \neq j$. If T^r is a spherical m-isometry and T^s is a spherical ℓ-isometry, then T^t is a spherical h-isometry.*

Proof. (i) Let $x \in \mathcal{H}$ and put $a_n := \langle Q_T^n(I)x, x\rangle$. Since

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} Q_T^{rk+n}(I) = Q_T^n\Big(\sum_{k=0}^{m} (-1)^k \binom{m}{k} Q_T^{rk}(I)\Big) = 0,$$

we observe that

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{rk+n} = 0,$$

for all $n \geqslant 0$. Similarly, we get

$$\sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} a_{sk+n} = 0$$

for all $n \geqslant 0$. Applying Lemma 3.15, we observe that

$$\sum_{k=0}^{h} (-1)^k \binom{h}{k} Q_T^{tk}(I) = 0.$$

(ii) The orthogonality condition implies that

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}Q_T^{rk}(I)=\sum_{k=0}^{m}(-1)^k\binom{m}{k}Q_{T^r}^{k}(I)=0,\quad\text{and}$$
$$\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}Q_T^{sk}(I)=\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}Q_{T^s}^{k}(I)=0.$$

Therefore, the result follows from the part (i). ∎

If $T=(T_1,\dots,T_d)$ is a spherical isometry of orders r and $r+1$, then it is a spherical isometry. Indeed,

$$I=Q_T^{r+1}(I)=Q_T(Q_T^r(I))=Q_T(I).$$

In the following, we give sufficient conditions ensuring that a spherical m-isometry of some order is a spherical m-isometry. The following corollaries are direct consequences of preceding proposition.

COROLLARY 3.17. *Suppose that $T=(T_1,\dots,T_d)$ is a commuting tuple of operators on a Hilbert space $\mathcal{H}$ and r,s,m are positive integers.*

(i) *If T is a spherical m-isometry and T is a spherical isometry of order s, then T is a spherical isometry.*

(ii) *If T is a spherical m-isometry of orders r and $r+1$, then T is a spherical m-isometry.*

(iii) *If T is a spherical m-isometry of order r and T is a spherical n-isometry of order $r+1$ with $m<n$, then T is a spherical m-isometry.*

The tuple T is called a proper spherical m-isometry of order r if it is a spherical m-isometry of order r but not a spherical $(m-1)$-isometry of order r. Note that every proper spherical m-isometry of order 1 is a proper spherical m-isometry.

COROLLARY 3.18. *If $T=(T_1,\dots,T_d)$ is a proper spherical m-isometry and r is a positive integer, then T is a proper spherical m-isometry of order r.*

Proof. By Theorem 3.9, T is a spherical m-isometry of order r. Assume that T is a spherical $(m-1)$-isometry of order r, then by Proposition 3.16, T is a spherical $(m-1)$-isometry which is a contradiction. ∎

4. POWER REGULARITY

Let $F(k,d)$ be the set of all functions from the set $\{1,2,\dots,k\}$ to the set $\{1,2,\dots,d\}$. For f in $F(k,d)$, let

$$A_f=A_{f(1)}A_{f(2)}\cdots A_{f(k)}.$$

For a commuting d-tuple $T = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$, the algebraic joint spectral radius is defined by

$$r(T) := \inf_k \Big\{ \Big\| \sum_{f\in F(k,d)} (T_f)^* T_f \Big\|^{1/2k} \Big\}.$$

Since $\sum\limits_{f\in F(k,d)} (T_f)^* T_f = Q_T^k(I)$, the above formula can be rewritten as

$$r(T) := \inf_k \|Q_T^k(I)\|^{1/2k}.$$

It is proved in [26] that

$$r(T) = \lim_{k\to\infty} \|Q_T^k(I)\|^{1/2k}.$$

For $d = 1$, $r(T)$ is the usual spectral radius of T. The joint approximate point spectrum of T denoted by $\sigma_\pi(T)$ is defined by

$$\sigma_\pi(T) = \left\{ \begin{array}{c} \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d : \lim\limits_{k\to\infty} \sum\limits_{j=1}^{d} \|(T_j - \lambda_j)x_k\| = 0, \\ \text{for some sequence of unit vectors } \{x_k\}_k \end{array} \right\}.$$

This is equivalent to $\lim\limits_{k\to\infty} (T_j - \lambda_j)x_k = 0$ for all $j = 1, \dots, d$.

In this section, we define the power regularity of the tuple of the operators and prove that spherical m-isometries are power regular.

The d-tuple $T = (T_1, \dots, T_d)$ is power regular if for each $x \in \mathcal{H}$,

$$r(x, T) := \lim_{k\to\infty} \Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha x\|^2 \Big)^{1/2k} = \lim_{k\to\infty} \|\langle Q_T^k(I)x, x\rangle\|^{1/2k}$$

exists.

PROPOSITION 4.1. *Let $T = (T_1, \dots, T_d)$ be in $\mathcal{B}(\mathcal{H})^d$ and suppose that $x, y \in \mathcal{H}$ such that $r(x, T), r(y, T), r(x + y, T)$ exist. Then $r(x + y, T) \leqslant \max\{r(x, T), r(y, T)\}$. Moreover, if $T_iT_j = 0$, for $i \neq j$, then $r(x, T^k) = r(x, T)^k$ for all positive integers k, where $T^k = (T_1^k, \dots, T_d^k)$.*

Proof. Note that for any positive operator $A \in \mathcal{B}(\mathcal{H})$,

$$|||x||| = |\langle Q_T(A)x, x\rangle|^{1/2}$$

is a seminorm on $\mathcal{H}$. Indeed, the fact that $A = B^*B$ for some $B \in \mathcal{B}(\mathcal{H})$ and the Cauchy–Schwarz inequality show that $|||x + y||| \leqslant |||x||| + |||y|||$. Also, clearly $|||\alpha x||| = |\alpha|\, |||x|||$ for every $\alpha \in \mathbb{C}$. Now, for any $\varepsilon > 0$ there is natural number k_0 such that $|\langle Q_T^k(I)x, x\rangle|^{1/2} \leqslant (r(x, T) + \varepsilon)^k$ and $|\langle Q_T^k(I)y, y\rangle|^{1/2} \leqslant (r(y, T) + \varepsilon)^k$ for all $k \geqslant k_0$. Then for $A = Q_T^{k-1}(I)$ we have

$$\Big|\Big\langle Q_T^k(I)\frac{x+y}{2}, \frac{x+y}{2}\Big\rangle\Big|^{1/2} = \Big|\Big|\Big|\frac{x+y}{2}\Big|\Big|\Big| \leqslant \frac{1}{2}|||x||| + \frac{1}{2}|||y|||$$

$$
\begin{aligned}
&= \frac{1}{2}|\langle Q_T^k(I)x, x\rangle|^{1/2} + \frac{1}{2}|\langle Q_T^k(I)y, y\rangle|^{1/2} \\
&\leqslant \frac{1}{2}(r(x,T)+\varepsilon)^k + \frac{1}{2}(r(y,T)+\varepsilon)^k \\
&\leqslant (\max\{(r(x,T)+\varepsilon)^k, (r(y,T)+\varepsilon)^k\}) \\
&\leqslant (\max\{r(x,T), r(y,T)\}+\varepsilon)^k
\end{aligned}
$$

for all $k \geqslant k_0$. Therefore, $r(x+y,T) = r((x+y)/2,T) \leqslant \max\{r(x,T), r(y,T)\} + \varepsilon$, for each $\varepsilon > 0$. Hence the result holds. For the next part observe that

$$
r(x,T^k) = \lim_{n\to\infty} |\langle Q_{T^k}^n(I)x,x\rangle|^{1/2n} = \Big(\lim_{n\to\infty} |\langle Q_T^{nk}(I)x,x\rangle\Big|^{1/2kn}\Big)^k = r(x,T)^k. \quad \blacksquare
$$

THEOREM 4.2. *Every spherical m-isometry $T = (T_1,\dots,T_d)$ is power regular. Moreover, for every non-zero vector $x \in \mathcal{H}$ the spectral radius of the restriction of the d-tuple T to the subspace $M := \bigvee\{T_1^{n_1}T_2^{n_2}\cdots T_d^{n_d}x : n_1 \geqslant 0, n_2 \geqslant 0, \dots, n_d \geqslant 0\}$ is one.*

Proof. By Lemma 3.2 of [38], $\sigma_\pi(T)$ is in the boundary of the unit ball. Therefore, for every sequence of unit vectors $(x_k)_k$, there is j with $1 \leqslant j \leqslant d$ such that

$$
\lim_{k\to\infty} T_j x_k \neq 0
$$

and this implies that there is a positive constant c such that

$$
|\langle Q_T(I)x,x\rangle| \geqslant c\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \tag{4.1}
$$

To prove the existence of $r(x,T)$ note that $r(0,T) = 0$. So let x be a non-zero element in $\mathcal{H}$. Since T is a spherical m-isometry, then $P_j(T) = 0$ for $j \geqslant m$. Thus Lemma 2.2 implies that

$$
\langle Q_T^k(I)x,x\rangle = \sum_{j=0}^{m-1} \frac{(-1)^j}{j!}\langle P_j(T)x,x\rangle k^{(j)} \quad k = 1,2,\dots.
$$

Note that $Q_T(I)$ is a positive operator. Moreover, $P_j(T)$s are self-adjoint operators; therefore, $\langle Q_T^k(I)x,x\rangle$ is a polynomial in k with real coefficients of degree at most $m-1$ with non-negative leading coefficient

$$
\frac{(-1)^{m-1}}{(m-1)!}\langle P_{m-1}(T)x,x\rangle.
$$

Suppose that $\langle Q_T^k(I)x,x\rangle = a_0 + a_1k + \cdots + a_{m-1}k^{m-1}$. Now, (4.1) implies that there exists $0 \leqslant i \leqslant m-1$ so that $a_i \neq 0$. Let a_r be the largest non-zero coefficient. Hence $\lim_{k\to\infty} |\langle Q_T^k(I)x,x\rangle|^{1/2k} = \lim_{k\to\infty}(a_rk^r)^{1/2k} = 1$. Therefore, T is power regular.

Moreover, for a non-zero vector x in $\mathcal{H}$ we have

$$
\begin{aligned}
1 = r(x,T) &= \lim_{k\to\infty}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha x\|^2\Big)^{1/2k} = \lim_{k\to\infty}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha|_M x\|^2\Big)^{1/2k} \\
&= \lim_{k\to\infty}(|\langle Q_{T|_M}^k(I)x,x\rangle|)^{1/2k} \leqslant \lim_{k\to\infty}\|Q_{T|_M}^k(I)\|^{1/2k}\|x\|^{1/k} = r(T|_M);
\end{aligned}
$$

hence the spectral radius of the restriction of the tuple T to the subspace M is in the closed interval $[r(x,T), r(T)]$. Now since for every spherical m-isometry T, $r(T) = 1$ (see Proposition 3.1 of [38]) we get the result. ∎

COROLLARY 4.3. *Every m-isometry is power regular. Moreover, for every non-zero vector $x \in \mathcal{H}$ the spectral radius of the restriction of the operator T to the subspace $M = \bigvee_{n\geqslant 0} \{T^n x\}$ is one.*

Recall that the d-tuple $T = (T_1, \dots, T_d)$ is right invertible if there are operators $A_1, \dots, A_d$ in $\mathcal{B}(\mathcal{H})$ such that $T_1A_1 + \cdots + T_dA_d = I$. Moreover, it is spherical unitary if T and T^* are spherical isometries. Since every spherical isometry is subnormal ([11], Proposition 2), T is a spherical unitary if and only if T is a normal spherical isometry. It is known that every spherical isometry on a finite-dimensional Hilbert space is necessarily a spherical unitary. On the other hand, there are examples of Taylor invertible spherical isometries which are not spherical unitaries ([34], Theorem 3.1). The question is under what conditions a Taylor invertible spherical isometry is a spherical unitary. In the next part of this section, we give sufficient conditions under which a right invertible spherical isometry is a spherical unitary.

PROPOSITION 4.4. *Let $T = (T_1, \dots, T_d)$ be a d-tuple of operators in $\mathcal{B}(\mathcal{H})$,*

(i) *If T is a spherical m-isometry then $Q_T(I)$ is invertible. Moreover, if $T_iT_j = 0$, for $i \neq j$ then $Q_{(T_1^{n_1},\dots,T_d^{n_d})}(I)$ is invertible for every d-tuple $(n_1, \dots, n_d)$ of positive integers. In particular, $Q_T^k(I)$ is invertible for every $k \geqslant 1$.*

(ii) *If T is a right invertible spherical isometry such that $T_i^*T_j = T_jT_i^* = 0$ for all $i \neq j$, then T is a spherical unitary.*

Proof. (i) As we have seen in the proof of Theorem 4.2, there is a positive constant c such that

$$|\langle Q_T(I)x, x\rangle| \geqslant c\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

Therefore,

$$c\|x\|^2 \leqslant |\langle Q_T(I)x, x\rangle| \leqslant \|Q_T(I)x\|\|x\|$$

and so $Q_T(I)$ is a bounded below operator. Now since every bounded below operator on the Hilbert space has closed range, we have

$$\mathrm{ran}Q_T(I) = \overline{\mathrm{ran}Q_T(I)} = \ker Q_T(I)^{*\perp} = \ker Q_T(I)^{\perp} = \mathcal{H}.$$

Hence $Q_T(I)$ is invertible. Moreover, if $T_iT_j = 0$, for $i \neq j$, then in light of Theorem 3.14, $(T_1^{n_1}, \dots, T_d^{n_d})$ is a spherical m-isometry. So the results hold.

(ii) Since T is right invertible, T^* is left invertible; hence there is a constant $c > 0$ such that

$$\langle Q_{T^*}(I)x, x\rangle = \|T_1^*x\|^2 + \cdots + \|T_d^*x\|^2 \geqslant c\|x\|^2$$

for all $x \in \mathcal{H}$. Therefore, $Q_{T^*}(I)$ is a self-adjoint bounded below operator which in turn implies that it is invertible. On the other hand,

$$\begin{aligned}(Q_{T^*}(I))^2 &= \Big(\sum_{i=1}^{d} T_iT_i^*\Big)\Big(\sum_{i=1}^{d} T_iT_i^*\Big) = \sum_{i=1}^{d} T_iT_i^*T_iT_i^* \\ &= \sum_{i=1}^{d} T_i\Big(I - \sum_{j=1, j\neq i}^{d} T_j^*T_j\Big)T_i^* = Q_{T^*}(I).\end{aligned}$$

Therefore, $Q_{T^*}(I) = I$ and the proof is completed. ∎

Note that the condition $T_i^*T_j = T_jT_i^* = 0, i \neq j$ is not a necessary condition in part (ii) of the above proposition. For example $(I/\sqrt{2}, I/\sqrt{2})$ is a spherical unitary.

5. PROPER SPHERICAL m-ISOMETRIES

Now we give a basic result about spherical m-isometries which is the multivariable analog of Theorem 1 in [23].

THEOREM 5.1. *Let $\mathcal{H}$ be a Hilbert space. Then the d-tuple $T = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is a proper spherical m-isometry if and only if there are $A_{m-1}, A_{m-2}, \dots, A_1, A_0$ in $\mathcal{B}(\mathcal{H})$ such that $A_{m-1} \neq 0$ and for every $n = 0, 1, 2, \dots$*

$$Q_T^n(I) = \sum_{i=0}^{m-1} A_i n^i.$$

Moreover, the sets $\{A_i : i = 0, \dots, m-1\}$ and $\{Q_T^{n-i}(A_i) : i = 0, \dots, m-1\}$ when $n \geqslant m$ are linearly independent.

Proof. If $T = (T_1, \dots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is a proper spherical m-isometry then from Lemma 2.2 we have

$$\begin{aligned}Q_T^n(I) &= \sum_{j=0}^{m-1} \Big(\frac{(-1)^j}{j!}P_j(T)\Big)n^{(j)} = \sum_{j=0}^{m-1}\sum_{i=0}^{j} \alpha_{(i,j)}\frac{(-1)^j}{j!}P_j(T)n^i \\ &= \sum_{i=0}^{m-1}\Big(\sum_{j=i}^{m-1} \alpha_{(i,j)}\frac{(-1)^j}{j!}P_j(T)\Big)n^i = \sum_{i=0}^{m-1} A_in^i\end{aligned}$$

where $A_i = \sum\limits_{j=i}^{m-1} \alpha_{(i,j)}((-1)^j/j!)P_j(T)$ for suitable $\alpha_{(i,j)}$s when $\alpha_{(m-1,m-1)} = 1$.

Conversely, assume that there are $A_{m-1} \neq 0, A_{m-2}, \dots, A_0$ so that

$$Q_T^n(I) = \sum_{i=0}^{m-1} A_in^i \quad \text{for all } n \geqslant 0.$$

By applying Lemma 3.2 we observe that

$$P_m(T) = \sum_{n=0}^{m}(-1)^n\binom{m}{n}Q_T^n(I) = \sum_{n=0}^{m}(-1)^n\binom{m}{n}\sum_{i=0}^{m-1}A_i n^i$$
$$= \sum_{i=0}^{m-1}A_i\sum_{n=0}^{m}(-1)^n\binom{m}{n}n^i = 0.$$

Moreover,

$$P_{m-1}(T) = \sum_{i=0}^{m-1}A_i\sum_{n=0}^{m-1}(-1)^n\binom{m-1}{n}n^i = (-1)^{m-1}(m-1)!A_{m-1} \neq 0.$$

To prove the next part, suppose that $\sum\limits_{i=0}^{m-1} x_iA_i = 0$ for some scalars $x_1, \ldots, x_{m-1}$. Therefore

$$0 = \sum_{i=0}^{m-1}x_iA_i = \sum_{i=0}^{m-1}x_i\sum_{j=i}^{m-1}\alpha_{(i,j)}\frac{(-1)^j}{j!}P_j(T)$$
$$= \sum_{j=0}^{m-1}\sum_{i=0}^{j}x_i\alpha_{(i,j)}\frac{(-1)^j}{j!}P_j(T) = \sum_{j=0}^{m-1}\Big[\frac{(-1)^j}{j!}\sum_{i=0}^{j}x_i\alpha_{(i,j)}\Big]P_j(T).$$

Now it is sufficient to show that $\{P_j(T) : j = 0, \ldots, m-1\}$ is a linearly independent set. Indeed, in this case $\sum\limits_{i=0}^{j} x_i\alpha_{(i,j)} = 0$ for $j = 0, \ldots, m-1$, and since the matrix of the coefficients of this system is lower triangular with diagonal components $\alpha_{(j,j)} = 1$, $j = 0, \ldots, m-1$, we get $x_i = 0$ for $i = 0, \ldots, m-1$. To finish the proof of this part suppose that $\sum\limits_{k=0}^{m-1} \alpha_kP_k(T) = 0$ for some complex numbers $\alpha_1, \ldots, \alpha_{m-1}$. Then $\sum\limits_{k=0}^{m-1} \alpha_kQ_T(P_k(T)) = 0$. On the other hand, by (2.1) we have

$$\sum_{k=0}^{m-1}(P_k(T) - Q_T(P_k(T))) = \sum_{k=0}^{m-1}P_{k+1}(T);$$

consequently,

$$\sum_{k=0}^{m-1}\alpha_kP_{k+1}(T) = 0.$$

By continuing this way we get

$$\sum_{k=0}^{m-1}\alpha_kP_k(T) = \sum_{k=0}^{m-1}\alpha_kP_{k+1}(T) = \sum_{k=0}^{m-1}\alpha_kP_{k+2}(T) = \cdots = \sum_{k=0}^{m-1}\alpha_kP_{k+m-1}(T) = 0.$$

Moreover, since $P_{m-1}(T) \neq 0$ and

$$0 = \sum_{k=0}^{m-1}\alpha_kP_{k+m-1}(T) = \alpha_0P_{m-1}(T),$$

we conclude that $\alpha_0 = 0$. In the next step we have

$$0 = \sum_{k=0}^{m-1} \alpha_k P_{k+m-2}(T) = \alpha_1 P_{m-1}(T);$$

thus $\alpha_1 = 0$. Continuing this process we obtain $\alpha_k = 0$ for all $k = 0, 1, \dots, m-1$.

To prove the last part of the theorem, suppose that there are $x_1, \dots, x_{m-1}$ such that

$$0 = \sum_{i=0}^{m-1} x_i Q_T^{n-i}(A_i) = \sum_{i=0}^{m-1} x_i \sum_{j=i}^{m-1} \alpha_{(i,j)} \frac{(-1)^j}{j!} Q_T^{n-i}(P_j(T)).$$

Since T is a proper spherical m-isometry, there is $x \in \mathcal{H}$ such that $P_{m-1}(T)x \neq 0$. But

$$\begin{aligned}
&\langle Q_T^{n-i}(P_{j+1}(T))x, P_{m-1}(T)x\rangle \\
&\qquad = \langle Q_T^{n-i}(P_j(T))x, P_{m-1}(T)x\rangle - \langle Q_T^{n-i+1}(P_j(T))x, P_{m-1}(T)x\rangle.
\end{aligned}$$

So

$$\begin{aligned}
&\sum_{i=0}^{m-1} x_i \sum_{j=i}^{m-1} \alpha_{(i,j)} \frac{(-1)^j}{j!} \langle Q_T^{n-i}(P_{j+1}(T))x, P_{m-1}(T)x\rangle \\
&\quad = -\sum_{i=0}^{m-1} x_i \sum_{j=i}^{m-1} \alpha_{(i,j)} \frac{(-1)^j}{j!} \langle Q_T(Q_T^{n-i}(P_j(T)))x, P_{m-1}(T)x\rangle \\
&\quad = -\sum_{k=1}^{d} \sum_{i=0}^{m-1} x_i \sum_{j=i}^{m-1} \alpha_{(i,j)} \frac{(-1)^j}{j!} \langle Q_T^{n-i}(P_j(T)T_k)x, T_k P_{m-1}(T)x\rangle = 0.
\end{aligned}$$

Applying this process, we will have

$$\sum_{i=0}^{m-1} x_i \sum_{j=i}^{m-1} \alpha_{(i,j)} \frac{(-1)^j}{j!} \langle Q_T^{n-i} P_{j+k}(T)x, P_{m-1}(T)x\rangle = 0$$

for all $0 \leqslant k \leqslant m-1$. Take $k = m-1$ and note that by (2.1)

$$Q_T^n(P_{m-1}(T)) = P_{m-1}(T)$$

for all $n \geqslant m$; thus $x_0 = 0$. In the next step taking $k = m-2$ we get

$$x_1 \alpha_{(1,1)} \|P_{m-1}(T)x\|^2 = 0$$

then $x_1 = 0$. Continuing this process we will have $x_i = 0$ for all $i = 0, \dots, m-1$. ∎

As an immediate consequence of Theorem 5.1, we have the following corollary.

COROLLARY 5.2 ([38], Proposition 2.3). *If $T = (T_1, \dots, T_d)$ is a proper spherical m-isometry then*

$$0 \leqslant \lim_{n \to \infty} \frac{Q_T^n(I)}{n^{m-1}} = \frac{(-1)^{1-m} P_{m-1}(T)}{(m-1)!}$$

The next result is the multivariable setting of Proposition 3 in [39].

PROPOSITION 5.3. *Let $T = (T_1, \ldots, T_d)$ be a proper spherical m-isometry. Then the following sets are linearly independent:*
(i) $\{P_k(T) : k = 0, 1, \ldots, m-1\}$;
(ii) $\{Q_T^k(I) : k = 0, 1, \ldots, m-1\}$;
(iii) $\{Q_T^{n-k}(P_k(T)) : k = 0, 1, \ldots, m-1\}$ *when* $n \geqslant m$.

Proof. From the proof of the preceding theorem, it follows that (i) is linearly independent. Moreover, the proof of the part (iii) is on the same lines as the last part of the above theorem.

(ii) Assume that $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ are complex numbers such that

$$\sum_{k=0}^{m-1} \alpha_k Q_T^k(I) = 0.$$

Thus in light of Lemma 2.2, we obtain

$$\sum_{k=0}^{m-1}\sum_{j=0}^{m-1} \Big(\alpha_k \frac{(-1)^j}{j!} k^{(j)}\Big) P_j(T) = \sum_{j=0}^{m-1} \Big(\sum_{k=0}^{m-1} \alpha_k \frac{(-1)^j}{j!} k^{(j)}\Big) P_j(T) = 0.$$

On the other hand, (i) implies that

$$\sum_{k=0}^{m-1} \alpha_k k^{(j)} = 0, \quad j = 0, 1, \ldots, m-1.$$

Now, since the matrix of coefficients of the above system is an upper triangular matrix with determinant $\prod_{i=0}^{m-1} i!$, we conclude that $\alpha_k = 0$, $k = 0, \ldots, m-1$. ∎

Now, the following result due to Botelho, Jamison and Zheng is a consequence of the preceding proposition.

COROLLARY 5.4 ([24], Theorem 3.1). *If $A \in \mathcal{B}(\mathcal{H})$ is a proper m-isometry then the set $\{I, A^*A, \ldots, A^{*m-1}A^{m-1}\}$ is linearly independent.*

For the proof put $d = 1$ in part (ii) of the previous proposition.

COROLLARY 5.5. *If $T = (T_1, \ldots, T_d)$ is a proper spherical m-isometry such that*

$$\sum_{k=0}^{m} c_k Q_T^k(I) = 0,$$

then $c_k = c_m(-1)^{m-k}\binom{m}{k}$, $k = 0, 1, \ldots, m$.

Proof. Note that

$$\sum_{k=0}^{m-1} c_k Q_T^k(I) = -c_m Q_T^m(I) = \sum_{k=0}^{m-1} c_m (-1)^{m-k} \binom{m}{k} Q_T^k(I).$$

As $\{Q_T^k(I) : k = 0, 1, \dots, m-1\}$ is a linearly independent set, $c_k = c_m(-1)^{m-k}\binom{m}{k}$ for $k = 0, \dots, m-1$. ∎

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