

CROSSED-PRODUCTS BY LOCALLY COMPACT GROUPS: INTERMEDIATE SUBFACTORS

RÉMI BOUTONNET and ARNAUD BROTHIER

Communicated by Stefaan Vaes

ABSTRACT. We study actions of locally compact groups on von Neumann factors and the associated crossed-product von Neumann algebras. In the setting of totally disconnected groups we provide sufficient conditions on an action $G \curvearrowright Q$ ensuring that the inclusion $Q \subset Q \rtimes G$ is irreducible and that every intermediate subfactor is of the form $Q \rtimes H$ for a closed subgroup $H < G$. This partially generalizes previous results obtained by Izumi–Longo–Popa and Choda. We moreover show that one can not hope to use their strategy for non-discrete groups.

KEYWORDS: *Von Neumann algebras, subfactors, totally disconnected groups, Galois correspondence, Hecke pairs of groups.*

MSC (2010): 46L10, 46L37.

1. INTRODUCTION

In the theory of von Neumann algebras, the *crossed-product construction* associates to an action of a locally compact group G on a von Neumann algebra Q a new von Neumann algebra, denoted by $Q \rtimes G$, which encodes the action (to some extent). This construction goes back to Murray and von Neumann [28] in the case of state preserving actions of countable groups on abelian von Neumann algebras, and was called the *group measure-space construction*. Thus crossed-product algebras appear as one of the most basic examples of von Neumann algebras. In the case of actions of discrete groups, elementary properties of these crossed-product algebras are quite well understood. For instance:

(i) If Q is abelian, $Q = L^\infty(X, \mu)$, and if the corresponding non-singular G -action on (X, μ) is essentially free then $Q' \cap (Q \rtimes G) \subset Q$, so that the crossed-product $Q \rtimes G$ is a factor if and only if the action is ergodic.

(ii) If Q is a factor, then $Q' \cap (Q \rtimes G) = \mathbb{C}$ if and only if the action is properly outer, meaning that for all $g \in G$, the corresponding automorphism of Q is not inner.

Moreover, in both settings one can completely describe all the intermediate subalgebras between Q and $Q \rtimes G$. Hybrid cases combining aspects of cases (i) and (ii) above have been considered recently in [10].

In the case of non-discrete groups the picture is not as nice. The main difference is that there is no “Fourier decomposition” of elements $Q \rtimes G$. Namely, not every element of $x \in Q \rtimes G$ can be represented as an L^2 -element $x \in L^2(Q) \otimes L^2(G)$.

Nevertheless, in the setting (i) above of actions on abelian algebras there are some satisfying results. Sauvageot showed in Section 2 of [32] that the equivalence between $Q' \cap (Q \rtimes G) \subset Q$ and the action $G \curvearrowright (X, \mu)$ being essentially free still holds. Moreover, in the case of state preserving actions of unimodular groups, a powerful tool is available: the so-called *crossed-section equivalence relation*. It relies on the observation that appropriate II_1 corners of $Q \rtimes G$ can be described by an explicit equivalence relation. We refer to [25] for details and references.

In this article we will be interested in the setting (ii) of actions on factors. An action $G \curvearrowright Q$ of a locally compact group G on a factor Q is called *strictly outer* if $Q' \cap (Q \rtimes G) = \mathbb{C}$. It is known in this case that properly outer actions need not be strictly outer. As we will see, assuming that the action is strictly outer allows to deduce more conclusions. For instance we will see that it implies that the normalizer of Q inside $Q \rtimes G$ is the semi-direct product $\mathcal{U}(Q) \rtimes G$, see Corollary 3.11. In particular we will derive as in the discrete case that for strictly outer actions the pairs $Q \subset Q \rtimes G$ completely characterize the actions up to cocycle conjugacy. We refer to Theorem 5.1 of [38] for examples of strictly outer actions. We will also give a new criterion providing more examples in Proposition 4.5.

Our main goal is to prove an intermediate subfactor theorem (a Galois correspondence). Namely we will provide examples of strictly outer actions of non-discrete groups that satisfy the following property.

DEFINITION 1.1. We will say that a strictly outer action $G \curvearrowright Q$ of a locally compact group on a factor Q satisfies the *intermediate subfactor property* if any subfactor of $Q \rtimes G$, containing Q is of the form $Q \rtimes H$ for some closed subgroup $H < G$.

In the case of outer actions of discrete groups, Choda [11] proved such an intermediate subfactor property under the extra assumption that the intermediate subfactor is the range of a normal conditional expectation on $Q \rtimes G$. Then Izumi–Longo–Popa [22] were able to show that the existence of such a conditional expectation is automatic for crossed-products by discrete groups. So the property is known to hold for actions of discrete groups. Unfortunately, as we discuss below, there is no hope to adapt this strategy for general locally compact groups. In fact, the general situation is analytically much harder to handle precisely because conditional expectations need not exist in general.

The proof of our main result relies on a very different approach inspired from II_1 -factor techniques. We will restrict our attention to actions of totally disconnected groups. This will allow us to use Fourier decomposition arguments. Here is the statement.

THEOREM 1.2. *Consider an action $\sigma : G \curvearrowright Q$ of a totally disconnected locally compact group G on a semi-finite factor Q . Assume that σ is properly outer relative to a compact open subgroup $K_0 < G$ whose action is minimal.*

Then σ satisfies the intermediate subfactor property.

We refer to Subsection 2.5 for the definition of a minimal action and to Definition 4.6 for the notion of relatively properly outer action. Theorem 1.2 applies to Bernoulli shifts $G \curvearrowright (\overline{\otimes}_{G/K} Q_0)$ where Q_0 is an arbitrary II_1 -factor, and K is any compact open subgroup of G , see Subsection 4.4. For instance if G is a closed subgroup of automorphisms of a locally finite tree T , and if G acts transitively on T , then the Bernoulli shift action $G \curvearrowright (\overline{\otimes}_T Q_0)$ satisfies the intermediate subfactor property. We refer to Subsection 4.4 for more examples.

As we mentioned above this result is not a simple generalization of the discrete case, even if Q is tracial, because in general an intermediate subalgebra $Q \subset N \subset Q \rtimes G$ does not behave well from the Hilbert theory perspective. Nevertheless we still manage to use Hilbert techniques to perform the proof.

Our approach relies on two ingredients. The first one is an averaging argument. This is where the semi-finiteness assumption on Q will be used. As we will see in Remark 4.14, this technique also allows to deal with some actions on type III factors Q , but we need to make an assumption that there is a G -invariant state on Q that has a *large centralizer*.

Our second ingredient is an extension of the notion of *support* defined by Eymard [14]. In the language of quantum groups, the support of an element of $Q \rtimes G$ is *the spectrum of the dual action*. In the setting of totally disconnected groups, this notion is particularly well suited, see for instance the proof of Proposition 4.5. The key fact that we will use in the proof of Theorem 1.2 is that an element whose support is contained in some closed subgroup $H < G$ actually belongs to the subalgebra $Q \rtimes H \subset Q \rtimes G$. This result is certainly known to experts in quantum groups, but as we were not able to find an explicit reference, we will provide a self contained proof in Section 3.

In view of Theorem 1.2 and Remark 4.14 we make the following general conjecture.

CONJECTURE 1.3. *Any strictly outer action $G \curvearrowright Q$ on a factor (of any type) satisfies the intermediate subfactor property.*

REMARK 1.4. The conjecture holds in full generality for compact groups. Indeed, if K is a compact group and $K \curvearrowright Q$ is a strictly outer action on an arbitrary von Neumann algebra, then the pair $Q \subset Q \rtimes K$ can be identified with the

basic construction of the inclusion $Q^K \subset Q$. This result was showed more generally for *integrable* (strictly outer) actions of locally compact groups by Vaes ([37], Theorem 5.3). Moreover, by Theorem 3.15 of [22], every intermediate subfactor of the inclusion $Q^K \subset Q$ is of the form Q^L for some closed subgroup $L < K$. Combining these two facts indeed yields that $K \curvearrowright Q$ satisfies the intermediate subfactor property.

Going back to general (not necessarily compact) groups, can one prove the conjecture even when Q is tracial and G is an arbitrary locally compact group? As we mentioned, Remark 4.14 allows to produce actions on type III factors. But the condition on large centralizers that we need is never fulfilled in the case of actions on factors of type III₀. This raises the following question.

QUESTION 1.5. Can one provide an explicit example of a strictly outer action (of a non-discrete group) on a type III₀-factor that satisfies the intermediate subfactor property?

We discuss in Section 5 the (im)possibility to solve the above conjecture by generalizing the work of Izumi–Longo–Popa. Namely, we completely characterize in the case of arbitrary strictly outer actions $G \curvearrowright Q$, the existence of normal conditional expectation and operator valued weight for an inclusion $Q \rtimes H \subset Q \rtimes G$, where H is a closed subgroup of G .

THEOREM 1.6. *Consider a strictly outer action $G \curvearrowright Q$ on an arbitrary factor Q and take a closed subgroup $H < G$. Then we have the following characterizations:*

- (i) *There exists a normal faithful semi-finite operator valued weight $T \in \mathcal{P}(Q \rtimes G, Q \rtimes H)$ if and only if the modular functions δ_G and δ_H coincide on H .*
- (ii) *The inclusion $Q \rtimes H \subset Q \rtimes G$ is with expectation if and only if H is open inside G .*

For both parts in the above theorem, the only if parts follow easily from modular theory. Namely both conditions are easily seen to imply the existence of an operator valued weight and conditional expectation, respectively. Our main contribution is to show that they are actually necessary.

Statement (ii) above is in sharp contrast with the case of group algebras, since there can exist conditional expectations from LG onto LH although H is not open inside G . For instance, consider a product $G = G_1 \times G_2$, with the trivial action $G \curvearrowright \mathbb{C}$. Assume that G_2 is second countable and not discrete. Then G_1 is not open inside G but any faithful normal state on LG_2 gives rise to a conditional expectation from $LG \simeq LG_1 \overline{\otimes} LG_2$ onto LG_1 . In particular, the strictly outer assumption may not be removed. On the other hand we do not know whether (i) holds for arbitrary actions (not necessarily strictly outer).

Next, we will show that nevertheless the strategy of Izumi–Longo–Popa and Choda can be applied to some intermediate subfactors, yielding the following result. We will also mention applications to crossed-products by Hecke pairs of groups, see Corollary 5.7.

THEOREM 1.7. *Consider an arbitrary strictly outer action $G \curvearrowright Q$ on a factor with a separable predual and a compact open subgroup $K < G$. Then any subfactor of $Q \rtimes G$ containing $Q \rtimes K$ is of the form $Q \rtimes H$ for some intermediate subgroup H ; $K < H < G$.*

Going back to the conjecture, let us finally mention another partial result. It is about intermediate subfactors that are globally invariant under the dual action $\Delta_Q : Q \rtimes G \rightarrow (Q \rtimes G) \overline{\otimes} LG$. Namely if Q is a factor and if N is an intermediate subfactor such that $\Delta_Q(N) \subset N \overline{\otimes} LG$, then N is of the form $Q \rtimes H$ for some closed subgroup $H < G$, see for instance Chapter VII.2 of [29].

QUESTION 1.8. Consider a strictly outer action α of a locally compact quantum group (M, Δ) on a factor N . Can one show that any von Neumann subalgebra $Q \subset M \rtimes_{\alpha} N$ containing N is globally invariant under the dual action? We refer to [38] for definitions.

We mention that Remark 1.4 also applies in the quantum setting. Namely, Vaes' result ([37], Theorem 5.3) is valid in the general quantum setting, while the correspondence result of Izumi–Longo–Popa ([22], Theorem 3.15) was generalized by Tomatsu to the quantum setting in [35].

2. PRELIMINARIES

2.1. GENERAL NOTATIONS. In all the article, the letter G refers to a locally compact group; m_G denotes a left Haar measure on G , and δ_G the modular function of G . Note that we follow the French convention according to which the locally compact assumption also contains the Hausdorff axiom. When we consider L^2 -spaces it is always meant with respect to the measure m_G . When integrating functions on G , we will sometimes use the notation ds instead of $dm_G(s)$, $s \in G$. The left regular representation of G on $L^2(G)$ is denoted by λ_G .

The letter Q refers to an arbitrary von Neumann algebra on which G acts. The action will be denoted by $G \curvearrowright Q$, and called generically σ . By an action, we mean an ultraweakly continuous homomorphism from G into the automorphism group of Q .

Given a von Neumann algebra M represented on a Hilbert space H , M' denotes its commutant, $(M)_1$ its unit ball in the operator norm, $\mathcal{U}(M)$ its unitary group and $\text{Aut}(M)$ its automorphism group. We denote by $\mathcal{S}(M)$ and $\mathcal{P}(M)$ the set of normal faithful states on M and the set of normal faithful semi-finite weights (*nfs* weights for short), respectively. For any weight $\Phi \in \mathcal{P}(Q)$, consider the left ideal $\mathfrak{n}_{\Phi}(M) = \{x \in M : \Phi(x^*x) < \infty\}$, on which $x \mapsto \sqrt{\Phi(x^*x)}$ defines a norm $\|\cdot\|_{\Phi}$. We denote by $L^2(M, \Phi)$ the Hilbert space completion of $\mathfrak{n}_{\Phi}(M)$ and we write Λ_{Φ} for the inclusion map $\Lambda_{\Phi} : \mathfrak{n}_{\Phi}(M) \rightarrow L^2(M, \Phi)$.

2.2. GROUP ACTIONS AND CROSSED-PRODUCT VON NEUMANN ALGEBRAS. Let us give the precise definition of our main object of study. We refer to Chapter X of [34] and [16], [17] for details about the facts below.

DEFINITION 2.1. Fix an action $G \curvearrowright Q$ and represent Q on a Hilbert space \mathcal{H} . The *crossed-product von Neumann algebra*, denoted by $Q \rtimes G$, is the von Neumann algebra on $L^2(G, \mathcal{H})$ generated by the operators $\{\pi(x) : x \in Q\}$ and $\{u_s : s \in G\}$ defined by the formulae:

$$(\pi(x)\xi)(t) = \sigma_t^{-1}(x)\xi(t) \quad \text{and} \quad (u_s\xi)(t) = \xi(s^{-1}t),$$

for all $\xi \in L^2(G, \mathcal{H})$.

For notational simplicity we will often omit the π and identify $\pi(Q)$ with Q in the above definition.

Throughout the article we will always assume that Q is standardly represented on \mathcal{H} , with conjugation operator J and positive cone \mathcal{P} . In this case, we abuse with notations and denote again by $\sigma : g \in G \rightarrow \sigma_g \in \mathcal{U}(\mathcal{H})$ the canonical implementation of the action $G \curvearrowright Q$, see [15]. Then for all $g \in G$ the operator $\rho_G(g)$ on $L^2(G, \mathcal{H})$ defined as follows lies in the commutant of $Q \rtimes G$:

$$(2.1) \quad \rho_G(g)(f)(s) = \delta_G(g)^{1/2} \sigma_g(f(sg)), \quad \text{for all } s \in G, f \in L^2(G, \mathcal{H}).$$

Denote by $\mathcal{K}(G, Q)$ the $*$ -algebra of compactly supported, $*$ -ultrastrongly continuous functions, endowed with product and involution given by the formulae

$$(F_1 \cdot F_2)(t) = \int_G \sigma_s(F_1(ts))F_2(s^{-1}) \, ds \quad \text{and} \quad F_1^*(t) = \delta_G(t)^{-1} \sigma_{t^{-1}}(F_1(t^{-1})^*),$$

for all $F_1, F_2 \in \mathcal{K}(G, Q)$, $t \in G$. The algebra $\mathcal{K}(G, Q)$ is also a two sided Q -module with actions

$$(F \cdot x)(t) = F(t)x \quad \text{and} \quad (x \cdot F)(t) = \sigma_t^{-1}(x)F(t),$$

for all $F \in \mathcal{K}(G, Q)$, $x \in Q$, $t \in G$. The map $F \mapsto \int_G u_s F(s) \, ds$ defines a Q -bimodular embedding of $\mathcal{K}(G, Q)$ into the crossed-product $Q \rtimes G$. In this way $\mathcal{K}(G, Q)$ is viewed as an ultraweakly dense $*$ -subalgebra of $Q \rtimes G$.

2.3. CROSSED-PRODUCTS BY SUBGROUPS. Given a subgroup H of G , one can restrict any action $G \curvearrowright Q$ to an action $H \curvearrowright Q$. In the case where $H < G$ is closed inside G , then the von Neumann subalgebra of $Q \rtimes G$ generated by Q and by the unitaries u_h , $h \in H$, is isomorphic to $Q \rtimes H$. This can be seen using induced representations, see Chapter X.4 of [34]. With the same tools one can also compute the commutant of $Q \rtimes H$ inside $B(L^2(G, \mathcal{H}))$.

THEOREM 2.2 ([29], Theorem VII.1.1). *The commutant of $Q \rtimes H$ inside $B(L^2(G, \mathcal{H}))$ is the von Neumann algebra generated by the commutant of $Q \rtimes G$ and the*

subalgebra $\mathcal{L}^\infty(H \setminus G) \subset L^\infty(G)$ consisting of left- H -invariant functions on G . Equivalently,

$$Q \rtimes H = (Q \rtimes G) \cap \mathcal{L}^\infty(H \setminus G)'.$$

REMARK 2.3. Using the J -map, the above theorem also allows to compute the basic construction $\langle Q \rtimes G, Q \rtimes H \rangle$ of the inclusion $Q \rtimes H \subset Q \rtimes G$. In fact this description gives the isomorphism

$$\langle Q \rtimes G, Q \rtimes H \rangle \simeq (\mathcal{L}^\infty(G/H) \overline{\otimes} Q) \rtimes G,$$

where G acts diagonally.

2.4. MODULAR THEORY AND OPERATOR VALUED WEIGHTS. Given a von Neumann algebra M , the modular flow of a weight $\Phi \in \mathcal{P}(M)$ is denoted by $\sigma_t^\Phi \in \text{Aut}(M)$, $t \in \mathbb{R}$. The centralizer of Φ in M is the subalgebra of elements fixed by the flow $(\sigma_t^\Phi)_t$, it is denoted by Q^Φ . If $\psi \in \mathcal{P}(M)$ is another weight, $(D\psi : D\Phi)_t$, $t \in \mathbb{R}$, denotes the Connes–Radon–Nikodym derivate as defined in Subsection 1.2 of [13]. We will need the following simple lemma.

LEMMA 2.4. Consider a von Neumann algebra M , two weights $\Phi, \Psi \in \mathcal{P}(M)$, and an automorphism $\theta \in \text{Aut}(M)$. Then $(D\Psi \circ \theta : D\Phi \circ \theta)_t = \theta^{-1}((D\Psi : D\Phi)_t)$, for all $t \in \mathbb{R}$.

Proof. Denote by $(e_{i,j})_{1 \leq i,j \leq 2}$ the canonical basis of $M_2(\mathbb{C})$. Put $\tilde{M} := M \otimes M_2(\mathbb{C})$, and define $\varphi \in \mathcal{P}(\tilde{M})$ by the formula

$$\varphi\left(\sum_{i,j=1}^2 x_{i,j} \otimes e_{i,j}\right) = \Phi(x_{1,1}) + \Psi(x_{2,2}), \quad \text{for all } x_{i,j} \in M.$$

Note that the weight on \tilde{M} associated in a similar manner to $\Phi \circ \theta$ and $\Psi \circ \theta$ is $\varphi \circ (\theta \otimes \text{id})$. By the definition of Connes–Radon–Nikodym derivative ([13], Lemme 1.2.2), we have for all $t \in \mathbb{R}$,

$$(D\Psi : D\Phi)_t \otimes e_{2,1} = \sigma_t^\varphi(1 \otimes e_{2,1}) \quad \text{and} \quad (D\Psi \circ \theta : D\Phi \circ \theta)_t = \sigma_t^{\varphi \circ (\theta \otimes \text{id})}(1 \otimes e_{2,1}).$$

The KMS condition, see Chapter VIII.1 of [34], implies $\sigma_t^{\varphi \circ (\theta \otimes \text{id})} = (\theta \otimes \text{id})^{-1} \circ \sigma_t^\varphi \circ (\theta \otimes \text{id})$ for all $t \in \mathbb{R}$. The lemma easily follows. ■

The notions of modular group and Connes–Radon–Nikodym derivative have been defined for normal faithful conditional expectations in [12]. We will need the extended definition for operator valued weights defined by Haagerup [19]. Let us fix some notations and recall known facts about operator valued weights. We refer to [18], [19] for precise definitions and proofs.

If $N \subset M$ is an inclusion of von Neumann algebras, then we denote by $\mathcal{P}(M, N)$ the set of nfs operator valued weights from M to N . If T is in $\mathcal{P}(M, N)$, then we set $n_T(M) := \{x \in M : T(x^*x) \in N\}$. One can define the composition $\Phi \circ T$ of a weight $T \in \mathcal{P}(M, N)$, with an nfs weight $\Phi \in \mathcal{P}(N)$; the resulting weight $\Phi \circ T$ is again normal faithful semi-finite. More generally, it

makes sense to compose operator valued weights: if $M_1 \subset M_2 \subset M_3$ are von Neumann algebras and $S \in \mathcal{P}(M_3, M_2)$, $T \in \mathcal{P}(M_2, M_1)$ then one can naturally define $T \circ S \in \mathcal{P}(M_3, M_1)$ in such a way that for all $\Phi \in \mathcal{P}(M_1)$, we have $\Phi \circ (T \circ S) = (\Phi \circ T) \circ S$.

If $T \in \mathcal{P}(M, N)$ and $\Phi \in \mathcal{P}(N)$, then for all $t \in \mathbb{R}$, the modular automorphism $\sigma_t^{\Phi \circ T}$ associated to the weight $\Phi \circ T$ leaves the von Neumann subalgebra $N^c := N' \cap M$ globally invariant, and its restriction to N^c does not depend on the choice of Φ . This restriction is called the *modular flow of T* , denoted by σ_t^T . Moreover, if $S \in \mathcal{P}(M, N)$ is another operator valued weight then for all $t \in \mathbb{R}$, the Connes–Radon–Nikodym derivative $(D(\Phi \circ S) : D(\Phi \circ T))_t$ is an element of N^c that does not depend on Φ . It is then denoted by $(DS : DT)_t$ and called the *Connes–Radon–Nikodym derivative* at time t .

If $\mathcal{P}(M, N)$ is non-empty then $\mathcal{P}(N', M')$ is non-empty. In particular in this case $\mathcal{P}(M_1, M)$ is non-empty as well, where M_1 denotes the Jones basic construction of the inclusion $N \subset M$, see [23]. Better, if $N \subset M$ is with expectation E there exists $\hat{E} \in \mathcal{P}(M_1, M)$ such that $\hat{E}(e) = 1$, where e is the Jones projection associated with E , see Lemma 3.1 of [24]. \hat{E} is called the *dual operator valued weight*.

Finally, consider an action $G \curvearrowright Q$. By Theorem 3.1 of [17], there exists a unique operator valued weight $T_Q \in \mathcal{P}(Q \rtimes G, Q)$ such that for all $F \in \mathcal{K}(G, Q)$, $g \in G$ and $x \in Q \rtimes G_+$,

$$(2.2) \quad T_Q(F^*F) = (F^*F)(e) \quad \text{and} \quad T_Q(u_g x u_g^*) = \delta_G(g) u_g T_Q(x) u_g^*.$$

We call it the *Plancherel operator valued weight*. For any weight $\Phi \in \mathcal{P}(Q)$, one defines the *dual weight* $\Phi \circ T_Q \in \mathcal{P}(Q \rtimes G)$. See [16] for a different construction.

2.5. OUTER ACTIONS ON VON NEUMANN ALGEBRAS.

DEFINITION 2.5. We say that a group action $G \curvearrowright Q$ on a von Neumann factor is

- (i) *properly outer* if no non-trivial element of G acts on Q by inner automorphism;
- (ii) *strictly outer* if the relative commutant $Q' \cap Q \rtimes G$ is trivial;
- (iii) *minimal* if it is faithful and the fixed point subalgebra Q^G is an irreducible subfactor of Q : $(Q^G)' \cap Q = \mathbb{C}$.

As observed in [37], [38], a strictly outer action has to be properly outer, but the converse is not true in general.

LEMMA 2.6 ([37], Proposition 6.2). *An action of a compact group is strictly outer if and only if it is minimal.*

Let us record a basic example for later use.

EXAMPLE 2.7. Consider a diffuse factor Q_0 . For any faithful action of a group H on a finite set X , the corresponding Bernoulli action $H \curvearrowright Q_0^{\bar{\otimes} X}$ is strictly outer, hence minimal.

In the setting of totally disconnected groups the following lemma will be useful. Its proof is very much inspired from that of Proposition 6.2 in [37], with another input, the Galois correspondence theorem ([22], Theorem 3.15).

LEMMA 2.8. *Consider a minimal action $K_0 \curvearrowright Q$ of a compact group K_0 . For any open subgroup $K < K_0$, we have $(Q^K)' \cap (Q \rtimes K_0) = LK \subset Q \rtimes K$.*

Proof. Recall that Q is represented on the Hilbert space \mathcal{H} and that $Q \rtimes K_0$ is the subalgebra of $B(L^2(K_0)) \overline{\otimes} Q$ generated by $LK_0 \otimes 1$ and by the operators $\pi(a) \in L^\infty(K_0) \overline{\otimes} Q$, $a \in Q$, defined by $\pi(a)(g) = \sigma_g^{-1}(a)$, $g \in G$. In this picture, we have $\pi(Q^{K_0}) = 1 \otimes Q^{K_0}$.

Take $x \in (Q^K)' \cap Q \rtimes K_0$. Since x commutes with Q^{K_0} and K_0 acts minimally, we deduce that $x \in B(L^2(K_0)) \otimes 1$. Note also that $(Q \rtimes K_0) \cap B(L^2(K_0)) \otimes 1 = LK_0 \otimes 1$, so we can write $x = z \otimes 1$ for some $z \in LK_0$.

For all $a \in Q^K$ and all linear functional $\mu \in Q_*$, we have

$$z(\text{id} \otimes \mu)(\pi(a)) = (\text{id} \otimes \mu)(\pi(a))z.$$

Note that the functions $(\text{id} \otimes \mu)(\pi(a)) \in L^\infty(K_0)$ are left K -invariant. Let us prove that these functions generate $\ell^\infty(K \backslash K_0)$ as a von Neumann algebra as $a \in Q^K$ and $\mu \in Q_*$. To see that it is sufficient to check that they separate points of $K \backslash K_0$. Take $g, h \in K_0$ such that $gh^{-1} \notin K$. Then by Theorem 3.15 of [22], there exists $a \in Q^K$ such that $\sigma_{gh^{-1}}(a) \neq a$. In particular $\pi(a)(g) = \sigma_g^{-1}(a)$ and $\pi(a)(h) = \sigma_h^{-1}(a)$ are distinct elements of Q . So we can find a linear functional $\mu \in Q_*$ that separates them.

So we arrived at the conclusion that z commutes with $\ell^\infty(K \backslash K_0)$, and in particular with $P(K)$, the orthogonal projection onto $L^2(K)$. This means that z leaves $L^2(K)$ invariant : $z \in LK$. ■

2.6. FOURIER ALGEBRA, DUAL ACTION, MULTIPLIERS. Given a locally compact group G , we denote by $A(G)$ its *Fourier algebra* as introduced by Eymard [14]. By definition $A(G)$ is the set of functions on G of the form $\xi * \tilde{\eta}$, $\xi, \eta \in L^2(G)$, where $*$ denotes the convolution product and $\tilde{\eta}$ is the function $g \mapsto \overline{\eta(g^{-1})}$. All functions of $A(G)$ are continuous and note that given $g \in G$, we have the equality $(\xi * \tilde{\eta})(g) = \langle \lambda(g)\xi, \eta \rangle$. This set is an algebra under the pointwise multiplication.

The norm of $\phi \in A(G)$ is defined to be the minimal value of $\|\xi\| \|\eta\|$ as the functions $\xi, \eta \in L^2(G)$ satisfy $\phi = \xi * \tilde{\eta}$. With this norm, $A(G)$ is a Banach algebra. The linear span of positive definite, compactly supported functions on G is contained (densely) in $A(G)$. We record in the following lemma some facts we will use in the sequel.

LEMMA 2.9. *Fix an open set $U \subset G$ and a compact subset $K \subset U$.*

- (i) *There exists $\phi \in A(G)$ such that $\phi|_K = 1$ and $\phi|_{U^c} = 0$.*
- (ii) *Given a function $\psi \in A(G)$ that does not vanish on K , the closed ideal $\mathcal{I}(\psi)$ in $A(G)$ generated by ψ contains a function ϕ as above.*

Proof. The first part is Lemme 3.2 of [14]. The second item follows from abstract harmonic analysis considerations. First, Théorème 3.34 of [14] shows that the spectrum of $A(G)$ is homeomorphic to G and that the Gelfand transform is the identity map. In combination with Lemme 3.2 of [14], it follows that $A(G)$ is regular in the sense of Section 25 in [27]. Denote by $F \subset G$ the closed subset consisting of zeroes of ψ . Then the hull of $\mathcal{I}(\psi)$ is contained in F , so the statement follows from Lemma 25C of [27]. ■

As a Banach space, $A(G)$ is isometric to the predual $(LG)_*$ of LG . The duality pairing is given by the well defined formula $\langle x, \phi \rangle := \langle x\xi, \eta \rangle$, for all $x \in LG$, and $\phi = \xi * \tilde{\eta} \in A(G)$. We will abuse with notations and write $\phi(x)$ for $x \in LG$ to mean $\langle x, \phi \rangle$. This notation is somewhat consistent with the fact that ϕ is a function on G , namely we have $\phi(u_g) = \phi(g)$ for all $g \in G$.

Using the product on $A(G)$ we can define multipliers on LG . More precisely, any $\phi \in A(G)$ gives rise to a normal completely bounded map $m_\phi : LG \rightarrow LG$, defined by the formula

$$\langle m_\phi(x), \psi \rangle := \langle x, \psi\phi \rangle \quad \text{for all } x \in LG, \psi \in A(G).$$

More generally, for any action $G \curvearrowright Q$, from an element $\phi \in A(G)$ one can construct a multiplier m_ϕ on $M := Q \rtimes G$ in the following way. Consider the unitary operator W on $L^2(G \times G, \mathcal{H})$ such that $(W\xi)(g, h) = \xi(g, gh)$ for all $\xi \in L^2(G \times G, \mathcal{H})$, $g, h \in G$. Denote by $\Delta := \text{Ad}(W^*)$ the associated automorphism of $B(L^2(G \times G, \mathcal{H}))$. We can identify $L^2(G \times G, \mathcal{H})$ with $L^2(G, \mathcal{H}) \otimes L^2(G)$ in such a way that

$$\Delta(M \otimes 1) \subset M \overline{\otimes} LG, \text{ and the restriction } \Delta|_{L^\infty(G) \otimes 1} = \text{id}.$$

DEFINITION 2.10. With the above notations, the *Fourier multiplier* associated with an element $\phi \in A(G)$ is the normal completely bounded map $m_\phi : Q \rtimes G \rightarrow Q \rtimes G$ defined by the formula

$$m_\phi(x) = (\text{id} \otimes \phi) \circ \Delta(x \otimes 1), \quad x \in Q \rtimes G.$$

In practice, the multiplier m_ϕ is characterized by the formula $m_\phi(au_g) = \phi(g)au_g$ for all $a \in Q$, $g \in G$. In this way, one easily checks that in the case where $Q = \mathbb{C}$, the two constructions of multipliers coincide.

3. SUPPORT AND APPLICATIONS

In this section we give generalities about the spectrum of the dual action, defined for instance in Chapter IV.1 of [29]. We adopt the point of view of Eymard [14], and rather talk about “support” because we believe it is more transparent for the reader who is familiar with actions of discrete groups, and not so much with the quantum group language. Our goal is to prove Theorem 3.7, regarding elements whose support is contained in a subgroup. This is certainly known to

experts but we were not able to find an explicit reference, although it is used in Theorem VII.2.1 of [29]. For convenience we tried to keep this section self-contained.

3.1. DEFINITION AND FIRST PROPERTIES. Let us fix an arbitrary action $G \curvearrowright Q$ on a von Neumann algebra.

DEFINITION 3.1. The *support* of an element $x \in Q \rtimes G$, denoted by $\text{supp}(x)$, is the set of elements $g \in G$ such that for all $\phi \in A(G)$ satisfying $\phi(g) \neq 0$, we have $m_\phi(x) \neq 0$.

We also describe the support explicitly in terms of interactions between $Q \rtimes G$ and the copy of $L^\infty(G)$ inside $B(L^2(G, \mathcal{H}))$. If $\Omega \subset G$ is a measurable subset, we write $P(\Omega) \in L^\infty(G) \otimes 1_{\mathcal{H}}$ for the orthogonal projection from $L^2(G, \mathcal{H})$ onto $L^2(\Omega, \mathcal{H})$.

PROPOSITION 3.2. Take $x \in Q \rtimes G$ and $g \in G$. The following are equivalent:

- (i) $g \in \text{supp}(x)$;
- (ii) for any non-empty open set $\Omega \subset G$ we have $P(g\Omega)xP(\Omega) \neq 0$.

Proof. (i) \Rightarrow (ii) Consider a non-empty open set $\Omega \subset G$. Take a non-empty open subset $\Omega_0 \subset \Omega$ and an open neighborhood V of g in G such that $V\Omega_0 \subset g\Omega$. Fix a function $\phi \in A(G)$ supported on V such that $\phi(g) \neq 0$. Then for all $y \in Q \rtimes G$, we have $m_\phi(y)P(\Omega_0) = P(g\Omega)m_\phi(y)P(\Omega_0)$. Indeed the formula is clear for all y of the form au_h , $a \in Q$, $h \in G$, and follows for arbitrary y by linearity and density.

Note that $(Q \rtimes G)'L^2(\Omega_0, \mathcal{H})$ spans a dense subset of $L^2(G, \mathcal{H})$. Indeed, for all $g \in G$, we see that $\rho_G(g)L^2(\Omega_0, \mathcal{H}) = L^2(\Omega_0g, \mathcal{H})$, where $\rho_G(g)$ is defined in (2.1). It follows that $m_\phi(x)P(\Omega_0) \neq 0$.

Extend arbitrarily $\phi \in A(G) = (LG)_*$ to a linear functional $\tilde{\phi}$ on $B(L^2(G))$. With the notations from Section 2.6, since W commutes with $L^\infty(G) \otimes 1$, we have

$$\begin{aligned} A &:= (\text{id} \otimes \tilde{\phi}) \circ \text{Ad}(W^*)(P(g\Omega)xP(\Omega_0) \otimes 1) \\ &= P(g\Omega)(\text{id} \otimes \phi) \circ \text{Ad}(W^*)(x \otimes 1)P(\Omega_0) \\ &= P(g\Omega)m_\phi(x)P(\Omega_0) = m_\phi(x)P(\Omega_0) \neq 0. \end{aligned}$$

We deduce that $P(g\Omega)xP(\Omega_0) \neq 0$, and in particular $P(g\Omega)xP(\Omega) \neq 0$.

(ii) \Rightarrow (i) Fix $\phi \in A(G)$ such that $\phi(g) \neq 0$. We want to show that $m_\phi(x) \neq 0$. For this it is sufficient to find $\phi' \in \mathcal{I}(\phi)$ such that $m_{\phi'}(x) \neq 0$ (recall that $\mathcal{I}(\phi)$ denotes the closed ideal in $A(G)$ generated by ϕ). By Lemma 2.9, there exists $\phi' \in \mathcal{I}(\phi)$ which is equal to 1 on a neighborhood V of g .

Pick a non-empty open set $\Omega \subset G$ such that $g\Omega \cdot \Omega^{-1} \subset V$. We claim that for all $y \in Q \rtimes G$

$$P(g\Omega)m_{\phi'}(y)P(\Omega) = P(g\Omega)yP(\Omega).$$

By linearity and density, it suffices to check this formula for all y of the form au_h , $a \in Q$, $h \in G$. If $h \in V$, then $m_{\phi'}(au_h) = au_h$ for all $a \in Q$, so the formula is

obvious. If $h \notin V$, and $y = au_h$ for some $a \in Q$, then both sides of the formula are equal to 0. Indeed, since $m_{\phi'}$ is a multiplier, the two terms are scalar multiple of each other, and it suffices to check vanishing of the right-hand side. Since $g\Omega \cdot \Omega^{-1} \subset V$ and $h \notin V$, we deduce that $g\Omega \cap h\Omega = \emptyset$. This leads to

$$P(g\Omega)au_hP(\Omega) = P(g\Omega)P(h\Omega)au_h = 0,$$

as wanted. This proves the claimed equality and hence $m_{\phi'}(x) \neq 0$. ■

In the sequel it will sometimes appear that one of the above two descriptions will be better suited to work with than the other. We will freely switch between these two points of view to reach the simplest arguments.

Let us record a few properties of the support.

LEMMA 3.3. *Take $x \in Q \rtimes G$. The following assertions are true:*

- (i) *the support of x is a closed subset of G ;*
- (ii) *if x belongs to $\mathcal{K}(G, Q)$, then $\text{supp}(x)$ coincides with its support as a function on G ;*
- (iii) *if $x = u_g$ for some $g \in G$, then $\text{supp}(x) = \{g\}$.*

Proof. (i) Consider a net $(g_i)_i$ of elements in $\text{supp}(x)$ that converges to some $g \in G$, and take $\phi \in A(G)$ such that $\phi(g) \neq 0$. Since ϕ is continuous, for i large enough we also have $\phi(g_i) \neq 0$. Hence $m_{\phi}(x) \neq 0$, as desired.

(ii) If x is a continuous function in $\mathcal{K}(G, Q)$ and $\psi \in A(G)$ observe that $m_{\psi}(x)$ is the function in $\mathcal{K}(G, Q)$ defined by $g \mapsto \psi(g)x(g)$. The statement easily follows.

(iii) If $x = u_g$, for all $\phi \in A(G)$ we have $m_{\phi}(x) = \phi(g)x$. The result is then obvious. ■

LEMMA 3.4. *Consider $x \in Q \rtimes G$ and some open subsets $\Omega_1, \Omega_2 \subset G$ satisfying the relation $\Omega_1 \cdot \Omega_2^{-1} \cap \text{supp}(x) = \emptyset$. Then*

$$P(\Omega_1)xP(\Omega_2) = 0.$$

In particular, for any open set $\Omega \subset G$, we have $xP(\Omega) = P(\overline{\text{supp}(x)\Omega})xP(\Omega)$.

Proof. We first treat the special case where $\Omega_1 \cdot \overline{\Omega_2}^{-1} \cap \text{supp}(x) = \emptyset$ and $\overline{\Omega_2}$ is compact.

For any $(g, h) \in \Omega_1 \times \overline{\Omega_2}$, there exists a non-empty open set $V_{g,h}$ such that $P(gh^{-1}V_{g,h})xP(V_{g,h}) = 0$. Conjugating this equation with $\rho_G(k)$ for some $k \in G$ we can assume that $V_{g,h}$ is an open neighborhood of h . Here, recall that ρ_G is the right action defined in (2.1), which satisfies $\rho_G(k)P(V_{g,h})\rho_G(k^{-1}) = P(V_{g,h} \cdot k^{-1})$.

By compactness, for a fixed $g \in G$ there exists $n \geq 1, h_1, \dots, h_n \in \overline{\Omega_2}$ such that $\{V_{g,h_i}\}_{i=1}^n$ is a finite open cover of Ω_2 . We can then define an open neighborhood W_g of $e \in G$:

$$W_g := \bigcap_{i=1}^n h_i^{-1}V_{g,h_i}.$$

We get that $P(gW_g)xP(V_{g,h_i}) = 0$ for any $1 \leq i \leq n, g \in \Omega_1$. Since $P(\overline{\Omega_2})$ is smaller than the supremum of the projections $\bigvee_{i=1}^n P(V_{g,h_i})$, we obtain that $P(gW_g)xP(\overline{\Omega_2}) = 0$ for any $g \in \Omega_1$. We deduce that $P(\Omega_1)xP(\overline{\Omega_2}) = 0$ since $\bigvee_{g \in \Omega_1} P(gW_g) \geq P(\Omega_1)$.

Suppose now that Ω_2 is open and non-necessarily relatively compact. For any $g \in \Omega_2$, there exists a compact neighborhood K_g of g such that $\Omega_1 \cdot K_g^{-1} \cap \text{supp}(x) = \emptyset$. We have that $P(\Omega_1)xP(K_g) = 0$ for any $g \in G$, by the proof of above. This proves the desired equality $P(\Omega_1)xP(\Omega_2) = 0$ because $P(\Omega_2) \leq \bigvee_{g \in \Omega_2} P(K_g)$.

The second part of the statement follows from taking $\Omega_2 = \Omega, \Omega_1 = G \setminus \text{supp}(x)\Omega$. ■

Before mentioning interesting consequences of this lemma, let us give an essentially equivalent form involving the multipliers.

LEMMA 3.5. Consider $x \in Q \rtimes G$ with compact support, and take a function $\phi \in A(G)$ which is equal to 1 on a neighborhood of $\text{supp}(x)$. Then $x = m_\phi(x)$.

Proof. Take a non-empty open set $\Omega \subset G$ such that ϕ is equal to 1 on $\text{supp}(x)\Omega \cdot \Omega^{-1}$. Proceeding as in the proof of Proposition 3.2, (ii) \Rightarrow (i), one checks that

$$P(\overline{\text{supp}(x)\Omega})xP(\Omega) = P(\overline{\text{supp}(x)\Omega})m_\phi(x)P(\Omega).$$

Since $\text{supp}(m_\phi(x)) \subset \text{supp}(x)$, Lemma 3.4 and the above equality imply that

$$xP(\Omega) = m_\phi(x)P(\Omega).$$

Moreover, for all $g \in G$, the set Ωg satisfies the same condition as Ω . So the same equality holds with Ωg in the place of Ω , for all $g \in G$: $xP(\Omega g) = m_\phi(x)P(\Omega g)$. Since Ω is open inside G , we get that $1 = \bigvee_g P(\Omega g)$. The equality $x = m_\phi(x)$ follows. ■

COROLLARY 3.6. Consider $x, y \in Q \rtimes G$. The following assertions are true:

- (i) *adjoint*: $\text{supp}(x^*) = \text{supp}(x)^{-1}$;
- (ii) *sum*: $\text{supp}(x + y) \subset \text{supp}(x) \cup \text{supp}(y)$;
- (iii) *product*: $\text{supp}(xy) \subset \text{supp}(x) \cdot \text{supp}(y)$.
- (iv) *vanishing criterion*: if $\text{supp}(x) = \emptyset$, then $x = 0$.

Proof. (i) This results from the fact that for all $\phi \in A(G)$ we have that $(m_\phi(x))^* = m_{\phi^*}(x^*)$, where $\phi^* \in A(G)$ is defined by $\phi^*(g) = \phi(g^{-1}), g \in G$.

(ii) Consider g in the complementary of $\text{supp}(x) \cup \text{supp}(y)$. Then there exists ϕ_1 and ϕ_2 such that $\phi_i(g) \neq 0, i = 1, 2$, while $m_{\phi_1}(x) = 0$ and $m_{\phi_2}(y) = 0$. In particular the product $\phi := \phi_1\phi_2$ satisfies $\phi(g) \neq 0$, while $m_\phi(x) = m_{\phi_2} \circ$

$m_{\phi_1}(x) = 0$ and $m_{\phi}(y) = m_{\phi_1} \circ m_{\phi_2}(y) = 0$. In summary $m_{\phi}(x + y) = 0$ and hence $g \notin \text{supp}(x + y)$.

(iii) Consider an element $g \in G$ which is not in $\overline{\text{supp}(x) \cdot \text{supp}(y)}$ and take open neighborhoods of the identity $e \in G, \Omega$ and V , such that $\text{supp}(x) \cdot \text{supp}(y) \cap g\Omega V\Omega^{-1} = \emptyset$. In particular, for all $h \in \overline{\text{supp}(x)^{-1}g\Omega}$ and $k \in \overline{\text{supp}(y)\Omega}$ we have $h^{-1}k \notin V$. Further, the closures $X := \overline{\text{supp}(x^*)g\Omega}$ and $Y := \overline{\text{supp}(y)\Omega}$ do not intersect. Besides, Lemma 3.4 implies that:

- (•) $yP(\Omega) = P(Y)yP(\Omega)$;
- (•) $x^*P(g\Omega) = P(X)x^*P(g\Omega)$.

Altogether, the following equality shows that $g \notin \text{supp}(xy)$:

$$P(g\Omega)xyP(\Omega) = P(g\Omega)xP(X)P(Y)yP(\Omega) = P(g\Omega)xP(X \cap Y)yP(\Omega) = 0.$$

(iv) If $\text{supp}(x) = \emptyset$, then Lemma 3.4 implies that $x = P(G)xP(G) = 0$. ■

THEOREM 3.7. *Consider an action of a locally compact group $G \curvearrowright Q$ on an arbitrary von Neumann algebra and take a closed subgroup $H < G$.*

Then an element $x \in Q \rtimes G$ belongs to the subalgebra $Q \rtimes H$ if and only if its support is contained in H .

Proof. First, assume that $x \in Q \rtimes H$. For all $g \in G \setminus H$ we can find a non-empty open set $\Omega \subset G$ such that $g\Omega \cdot \Omega^{-1} \cap H = \emptyset$. By Lemma 3.4, we deduce that $P(g\Omega)xP(\Omega) = 0$, so that $g \notin \text{supp}(x)$. Hence $\text{supp}(x) \subset H$.

Conversely assume that $x \in Q \rtimes G$ is an element with support contained in H . In order to show that $x \in Q \rtimes H$, we will use Theorem 2.2. This reduces our task to check that x commutes inside $B(L^2(G, \mathcal{H}))$ with the subalgebra $\mathcal{L}^\infty(H \setminus G) \subset L^\infty(G)$ consisting of left H -invariant functions on G .

Given a left H -invariant open set Ω whose boundary has measure 0, we have equality of the projections $P(\Omega) = P(H\Omega)$. By Lemma 3.4, we get $xP(\Omega) = P(\Omega)xP(\Omega)$. Since x^* is supported in H we have $x^*P(\Omega) = P(\Omega)x^*P(\Omega)$. This shows that x commutes with $P(\Omega) \in \mathcal{L}^\infty(H \setminus G)$.

Claim. The set of functions $P(\Omega)$ with Ω as above generates $\mathcal{L}^\infty(H \setminus G)$.

Denote by $q : G \rightarrow H \setminus G$ the canonical projection and by μ a quasi-invariant measure on the coset space $H \setminus G$. The map $f \in L^\infty(H \setminus G, \mu) \mapsto f \circ q \in \mathcal{L}^\infty(H \setminus G)$ is a normal isomorphism. With this identification, the indicator function 1_U of a set $U \subset H \setminus G$ is identified with the indicator function $1_{q^{-1}(U)} = P(q^{-1}(U)) \in \mathcal{L}^\infty(H \setminus G)$.

Moreover, for all open subset $U \subset H \setminus G$ such that $\mu(\partial U) = 0$, the set $\Omega := q^{-1}(U) \subset G$ is open, H -invariant and its boundary has Haar measure 0. So we are left to check that the span of functions 1_U for $U \subset H \setminus G$ such that $\mu(\partial U) = 0$, is ultraweakly dense in $L^\infty(H \setminus G, \mu)$. This is a classical fact about Borel measures on locally compact spaces (see e.g. Proof of (3.33) in [14] and the references therein).

We deduce from the claim that x commutes with $\mathcal{L}^\infty(H \setminus G)$. So as wanted, we can use Theorem 2.2 to deduce that $x \in Q \rtimes H$. ■

In the special case where H is the trivial subgroup of G , the above theorem yields Beurling’s theorem ([14], Théorème 4.9). In particular we stress the following corollary that we will use several times.

COROLLARY 3.8. *Given an element x in a crossed product von Neumann algebra $Q \rtimes G$, if the support of x is a singleton $\{g\}$, then there exists $a \in Q$ such that $x = au_g$.*

Proof. If $\text{supp}(x) = \{g\}$, then $\text{supp}(u_g^*x) = \{e\}$. In particular $u_g^*x \in Q$, by Theorem 3.7. ■

3.2. APPLICATIONS. Before moving on to the proof of our main theorems, let us mention a few classical results that follow easily from the above properties of the support.

Our first application concerns a generalization of the so-called Eymard–Stinespring–Tatsuuma’s theorem ([34], Theorem VII.3.9). We start with analyzing how the support behaves under the co-product map.

Given an action $\sigma : G \curvearrowright Q$, put $M = Q \rtimes G$ and consider the notations $W \in B(L^2(G \times G, \mathcal{H}))$ and $\Delta = \text{Ad}(W^*) : M \otimes 1 \rightarrow M \overline{\otimes} LG$ introduced in Section 2.6. We will view the algebra $M \overline{\otimes} LG$ as the crossed-product von Neumann algebra associated with the action $\sigma \times \text{id} : G \times G \curvearrowright Q$:

$$M \overline{\otimes} LG \simeq Q \rtimes (G \times G).$$

In this way it makes sense to talk about the support of an element inside $M \overline{\otimes} LG$.

LEMMA 3.9. *With the above notations, the following facts hold:*

- (i) *for all $x \in M$, the support $\Delta(x \otimes 1)$ is equal to $\{(g, g) : g \in \text{supp}(x)\}$;*
- (ii) *for all $a \in M$ and $b \in LG$, $\text{supp}(a \otimes b) = \text{supp}(a) \times \text{supp}(b)$.*

Proof. (i) For all $\phi, \psi \in A(G)$ the function $\phi \times \psi : (g, h) \mapsto \phi(g)\psi(h)$ belongs to the Fourier algebra $A(G \times G)$ and one easily checks the formula

$$(3.1) \quad m_{\phi \times \psi}(\Delta(x \otimes 1)) = \Delta(m_{\phi, \psi}(x) \otimes 1).$$

In particular if $g \neq h \in G$, we can find $\phi, \psi \in A(G)$ such that $(\phi \times \psi)(g, h) = \phi(g)\psi(h) \neq 0$, but such that $\phi \cdot \psi = 0$ (just take functions with disjoint supports). By (3.1), we get $m_{\phi \times \psi}(\Delta(x \otimes 1)) \neq 0$ and hence (g, h) does not belong to the support of $\Delta(x \otimes 1)$. We thus deduce that the support of $\Delta(x \otimes 1)$ is contained in the diagonal of $G \times G$. Further, (3.1) easily implies that if (g, g) belongs to the support of $\Delta(x \otimes 1)$, then $g \in \text{supp}(x)$.

Conversely, take $g \in \text{supp}(x)$, and $\rho \in A(G \times G)$ such that $\rho(g, g) \neq 0$. We show that $m_\rho(\Delta(x \otimes 1)) \neq 0$. Denote by $\mathcal{I}(\rho)$ the closed ideal in $A(G \times G)$ generated by ρ . By Lemma 2.9, there exists a function ρ' in the closed ideal $\mathcal{I}(\rho)$ generated by ρ which is equal to 1 on some neighborhood $U \times U$ of (g, g) . If we prove that $m_{\rho'}(\Delta(x \otimes 1)) \neq 0$, then it will follow that $m_\rho(\Delta(x \otimes 1)) \neq 0$. Hence, we may replace ρ with ρ' and assume that ρ is equal to 1 on $U \times U$.

Pick any function $\phi \in A(G)$ supported on U and such that $\phi(g) \neq 0$. We deduce from (3.1) that

$$m_{\phi \times \phi}(m_\rho(\Delta(x \otimes 1))) = m_{(\phi \times \phi), \rho}(\Delta(x \otimes 1)) = m_{\phi \times \phi}(\Delta(x \otimes 1)) = \Delta(m_{\phi^2}(x) \otimes 1).$$

Since $g \in \text{supp}(x)$ and $\phi^2(g) \neq 0$, the above term is non-zero and hence $m_\rho(\Delta(x \otimes 1)) \neq 0$.

(ii) Take $(g, h) \in \text{supp}(a \otimes b)$, and $\phi, \psi \in A(G)$ such that $\phi(g), \psi(h) \neq 0$. Then $(\phi \times \psi)(g, h) \neq 0$, and we get

$$m_\phi(a) \otimes m_\psi(b) = m_{\phi \times \psi}(a \otimes b) \neq 0.$$

We deduce that $g \in \text{supp}(a)$ and $h \in \text{supp}(b)$.

Conversely, take $g \in \text{supp}(a), h \in \text{supp}(b)$ and take a non-empty open set $\Omega \subset G \times G$. We find non-empty open sets $U, V \subset G$ such that $U \times V \subset \Omega$. By assumption we know that $P(gU)aP(U) \neq 0$ and $P(hV)bP(V) \neq 0$, and hence $P(gU \times hV)(a \otimes b)P(U \times V) \neq 0$. It then clearly follows that $P((g, h) \cdot \Omega)(a \otimes b)P(\Omega) \neq 0$. This proves that $(g, h) \in \text{supp}(a \otimes b)$. ■

We can now prove the following well known generalization of Theorem VII.3.9 in [34]. This theorem initially deals with the case of group algebras LG , that is, with the case of trivial actions $G \curvearrowright \mathbb{C}$. The proof given in [34] is rather involved, while an elementary proof relying (implicitly) on the support already appears in Théorème 3.34 of [14]. In the case of general actions $G \curvearrowright Q$, the predual $(Q \rtimes G)_*$ is not identified with an algebra in general, and hence the notion of character does not apply anymore. Nevertheless we can still provide an easy proof relying on the notion of support.

COROLLARY 3.10. *Given any action $G \curvearrowright Q$, put $M = Q \rtimes G$ and denote by $\Delta : M \otimes 1 \rightarrow M \overline{\otimes} LG$ the co-product map defined above. Assume that $x \in M$ is an element such that $\Delta(x \otimes 1)$ is of the form $a \otimes b$ for some elements $a \in M$ and $b \in LG$.*

Then there exist $g \in G$ and $y \in Q$ such that $x = yu_g$.

Proof. We may assume that $x \neq 0$. By Lemma 3.9, the equality $\Delta(x \otimes 1) = a \otimes b$ implies that $\text{supp}(a) \times \text{supp}(b)$ is contained in the diagonal of $G \times G$. The only way this can happen is if $\text{supp}(\Delta(x \otimes 1))$ is a singleton $\{(g, g)\}$. In this case, $\text{supp}(x) = \{g\}$ and the result follows from Corollary 3.8. ■

As an immediate corollary, we deduce the following result.

COROLLARY 3.11. *Given any strictly outer action $G \curvearrowright Q$, the normalizer of Q inside the crossed-product $M := Q \rtimes G$ is equal to $\{au_g : g \in G, a \in \mathcal{U}(Q)\}$.*

In particular, two strictly outer actions $\alpha_i : G_i \curvearrowright Q_i, i = 1, 2$ of locally compact groups are cocycle conjugate if and only if the pairs $Q_1 \subset Q_1 \rtimes G_1$ and $Q_2 \subset Q_2 \rtimes G_2$ are isomorphic.

Proof. Take $u \in \mathcal{N}_M(Q)$. Use the notation $\Delta : M \otimes 1 \rightarrow M \overline{\otimes} LG$ as above. We have that Δ is the identity map on $Q \otimes 1$. Hence one easily checks that $(u^* \otimes$

1) $\Delta(u \otimes 1)$ commutes with $Q \otimes 1$. In other words,

$$(u^* \otimes 1)\Delta(u \otimes 1) \in (Q' \cap M) \overline{\otimes} LG = 1 \otimes LG.$$

So there exists $b \in LG$ such that $\Delta(u \otimes 1) = u \otimes b$, and we conclude by Corollary 3.10 that $u = au_g$ for some $a \in \mathcal{U}(Q)$ and $g \in G$.

The second part of the statement is routine. The only if part is always true, even when the actions are not strictly outer, see Corollary 3.6 of [33]. The if part follows from adapting Proposition 1.7 of [5] to the case of actions of general locally compact groups. ■

4. CROSSED-PRODUCTS BY ACTIONS OF TOTALLY DISCONNECTED GROUPS

4.1. NOTATIONS AND TOOLS. There are several advantages in working with totally disconnected groups. Fix a locally compact group G and an action $G \curvearrowright Q$ on a von Neumann algebra and put $M = Q \rtimes G$.

Firstly, given a compact open subgroup $K < G$, one can define a projection $p_K := \mathbf{1}_K/m_G(K)$. Note that the net of projections $(p_K)_K$ increases to 1 as the compact open subgroups K decrease to $\{e\}$. Let us mention two elementary properties of these projections.

LEMMA 4.1. *Fix a weight $\Phi \in \mathcal{P}(Q)$ and denote by $\Psi \in \mathcal{P}(M)$ the corresponding dual weight. Consider a compact open subgroup $K < G$. Then for all $a \in Q^K$ we have $\|ap_K\|_\Psi = \|a\|_\Phi / \sqrt{m_G(K)}$.*

The proof is an immediate consequence of (2.2).

LEMMA 4.2. *Given a compact open subgroup $K < G$, the map $a \in Q^K \mapsto ap_K \in p_K(Q \rtimes K)p_K$ is an onto isomorphism of von Neumann algebras.*

Proof. This is a particular case of Proposition 5.6. We give a complete proof of this simpler case for the convenience of the reader. Since p_K commutes with Q^K , it is clear that the map is a normal $*$ -morphism. By the computations given in Lemma 4.1 above, we see that it is moreover injective. To check that it is onto, we only need to prove that it has dense image.

For all $a \in Q$ and $g \in K$, we have $p_K(au_g)p_K = p_Kap_K = E(a)p_K$, where $E : Q \rightarrow Q^K$ is the conditional expectation. Hence $p_K(au_g)p_K$ belongs to the range of our map, proving the lemma. ■

Secondly, for all open subgroup $K < G$ and any action $G \curvearrowright Q$, there always exists a faithful normal conditional expectation $E_K : Q \rtimes G \rightarrow Q \rtimes K$ such that $E_K(u_g) = \mathbf{1}_K(g)$ for all $g \in G$. In the case where K is compact open, one sees that the multiplier m_ϕ associated with the function $\phi := \mathbf{1}_K \in A(G)$ gives the desired expectation. In the general case of open subgroups ϕ does not necessarily belong to $A(G)$ but it is still positive definite and one can use Theorem 3.1.a of [17] to construct the associated multiplier m_ϕ .

An alternative way to construct the expectation E_K is by considering modular flows as in [21]. With this point of view, it becomes obvious that E_K preserves the Plancherel operator valued weight: $T_Q \circ E_K = T_Q$.

When G is totally disconnected, one checks that the support of an element $x \in M$ is described as follows:

$$(4.1) \quad \text{supp}(x) = \{g \in G : E_K(u_g^* x) \neq 0 \text{ for all compact open subgroups } K < G\}.$$

NOTATION 4.3. Given a compact open subgroup K of a group G and a set $S \subset G/K$, we denote by $\text{lift}(S)$ a set of representatives of S inside G . This means that for all $g \in \text{lift}(S)$ we have $gK \in S$ and for all $s \in S$ there exists a unique element $g \in \text{lift}(S)$ such that $gK = s$.

LEMMA 4.4. Consider a compact open subgroup $K < G$. Given an element $x \in M$ with compact support, the map $g \mapsto u_g E_K(u_g^* x)$ is right K -invariant on G and compactly supported. Moreover,

$$x = \sum_{g \in \text{lift}(G/K)} u_g E_K(u_g^* x).$$

Proof. Fix $S \subset G/K$ a finite set such that the support of x is contained in $\text{lift}(S) \cdot K$. The function $\phi = \mathbf{1}_{\text{lift}(S)K} \in A(G)$ is equal to 1 on a neighborhood of the support of x , so by Lemma 3.5, we have that $x = m_\phi(x)$. Moreover, ϕ can be decomposed as $\phi = \sum_{g \in \text{lift}(S)} \mathbf{1}_K(g^{-1} \cdot)$. One easily checks that the corresponding multiplier satisfies

$$m_\phi = \sum_{g \in \text{lift}(S)} u_g E_K(u_g^* \cdot).$$

We leave the rest of the proof to the reader. ■

Although we will not use this fact, let us mention that for a general element $x \in M$, a K -decomposition as above still makes sense. In this case the sum that appears is infinite but it converges in the Bures topology associated with the inclusion $Q \rtimes K \subset Q \rtimes G$, with expectation E_K . We refer to Section 2 of [10] or to the original book [9] for the definition of the Bures topology.

4.2. STRICTLY OUTER ACTIONS. The following proposition is proved by combining the “Fourier coefficient approach” used in the setting of discrete groups with Lemma 2.6 about actions of compact groups.

PROPOSITION 4.5. Consider a properly outer action of a totally disconnected locally compact group $\sigma : G \curvearrowright Q$. The action σ is strictly outer if and only if G admits a compact open subgroup that acts minimally.

In that case any unitary $u \in Q \rtimes G$ normalizing Q is of the form au_g for some $a \in Q, g \in G$.

Proof. Assume that $\sigma : G \curvearrowright Q$ is strictly outer. Then one easily checks from the definitions that σ is properly outer. Moreover, the restriction of σ to any compact subgroup is strictly outer and thus minimal by Proposition 6.2 of [37].

Conversely, assume that $\sigma : G \curvearrowright Q$ is properly outer and admits a compact open subgroup K_0 that acts minimally. Note that any subgroup $K < K_0$ acts minimally as well, and hence in a strictly outer way.

Put $M = Q \rtimes G$ and take a non-zero element $x \in Q' \cap M$. By Proposition 3.8 we only need to show that the support of x is the singleton $\{e\}$. The support of x is non-empty by Corollary 3.6(iv). Take $g \in \text{supp}(x)$. For all compact open subgroups $K < K_0$, put $a_K := E_K(u_g^*x) \in Q \rtimes K$. Since $g \in \text{supp}(x)$, these elements a_K are non-zero.

By minimality, for all $K' < K < K_0$, we have

$$(4.2) \quad a_{K'}^* a_K \in Q' \cap (Q \rtimes K) = \mathbb{C}.$$

When $K = K'$ this relation tells us that a_K is a (non-zero) multiple of a unitary element. Then (4.2) further implies that all these unitaries are proportional to each other. In particular, for all $K < K_0$, a_{K_0} is proportional to $a_K \in Q \rtimes K$.

We conclude that a_{K_0} is contained in $Q \rtimes K$, for all $K < K_0$. Hence its support is equal to $\{e\}$, implying that $a_{K_0} \in Q$. Moreover, a_{K_0} satisfies

$$a_{K_0} x = \sigma_{g^{-1}}(x) a_{K_0}, \quad \text{for all } x \in Q.$$

Since the G -action is properly outer, this gives $g = e$, as desired. The statement on the normalizer then follows from Corollary 3.11 (although this could be checked directly by similar computations on the support). ■

4.3. INTERMEDIATE SUBFACTORS. We now turn to the question of determining all intermediate subfactors $Q \subset N \subset Q \rtimes G$. In order to establish our main result Theorem 1.2, we will need to be able to compute relative commutants of the form $(Q^K)' \cap Q \rtimes G$ for small compact open subgroups $K < G$. This forces us to strengthen our assumptions on the action.

DEFINITION 4.6. Given a subgroup K of G we will say that the action $G \curvearrowright Q$ is *properly outer relative to K* if the following holds: the only elements $g \in G$ for which there exists a non-zero $a \in Q$ such that $\sigma_g(x)a = ax$ for all $x \in Q^K$ are the elements of K : $g \in K$.

Note that an action is properly outer if and only if it is properly outer relative to the trivial subgroup. We will provide examples of relatively properly outer actions in the next section.

LEMMA 4.7. Consider an action $\sigma : G \curvearrowright Q$ of a totally disconnected group G which is properly outer relative to a compact open subgroup $K < G$. Put $M = Q \rtimes G$. We have

$$(Q^K)' \cap M \subset Q \rtimes K.$$

Proof. Take $x \in (Q^K)' \cap M$. We show that $\text{supp}(x) \subset K$. Fix $g \in \text{supp}(x)$, so that $E_K(u_g^*x) \neq 0$. Since the net $(p_L)_L$ converges strongly to 1 when $L \searrow \{e\}$, we can find a compact open subgroup $L_0 < G$ such that $p_{L_0}E_K(u_g^*x)p_{L_0} \neq 0$.

Define $L_1 = L_0 \cap K \cap gKg^{-1}$ and $L := \bigcap_{h \in K} hL_1h^{-1}$. Since L_1 is open, it has finite index inside K , and hence the intersection defining L is in fact finite. We conclude that $L < K$ is an open subgroup which is moreover contained in $L_0 \cap K \cap gKg^{-1}$. In particular $p_L \geq p_{L_0}$, so that $p_LE_K(u_g^*x)p_L \neq 0$. Since L is normal inside K , the projection p_L commutes with u_h for all $h \in K$. By Lemma 4.4, we have

$$p_LE_K(u_g^*x)p_L = \sum_{h \in \text{lift}(K/L)} p_Lu_hE_L(u_{gh}^*x)p_L = \sum_{h \in \text{lift}(K/L)} u_hp_LE_L(u_{gh}^*x)p_L.$$

The non-vanishing of this sum implies that there exists $h \in K$ such that $X := p_LE_L(u_{gh}^*x)p_L$ is non-zero.

Since $L < K \cap gKg^{-1}$, we have that $Q^K \subset Q^L$ and $\sigma_g(Q^K) = Q^{gKg^{-1}} \subset Q^L$. Hence for all $b \in Q^K$, the two elements b and $\sigma_g(b)$ commute with p_L . It follows

$$Xb = p_LE_L(u_{gh}^*xb)p_L = p_LE_L(u_{gh}^*bx)p_L = p_L\sigma_g(b)E_L(u_{gh}^*x)p_L = \sigma_g(b)X.$$

By Lemma 4.2, $p_L(Q \rtimes L)p_L = Q^Lp_L$, so there exists a unique (non-zero) $a \in Q^L$ such that $X = ap_L$. By uniqueness, we see that a satisfies $ab = \sigma_g(b)a$ for all $a \in Q^K$. We conclude that $g \in K$, and hence $\text{supp}(x) \subset K$. By Theorem 3.7 we have that $x \in Q \rtimes K$. ■

PROPOSITION 4.8. *Consider an action $\sigma : G \curvearrowright Q$ of a totally disconnected group G which is properly outer relative to a compact open subgroup $K_0 < G$ whose action is minimal. For all open subgroup $K < K_0$, we have $(Q^K)' \cap M = LK \subset Q \rtimes K$.*

Proof. Fix $x \in (Q^K)' \cap M$. By Lemma 4.7 applied to K_0 , we see that $x \in Q \rtimes K_0$. Since K_0 acts minimally, the result follows from Lemma 2.8. ■

In order to prove Theorem 1.2, we will use a convex combination argument. The following lemma will be needed.

LEMMA 4.9 ([20], Lemma 4.4). *Consider a von Neumann algebra \mathcal{M} , a weight $\Theta \in \mathcal{P}(\mathcal{M})$, and a σ -weakly closed convex subset \mathcal{C} of \mathcal{M} .*

If \mathcal{C} is bounded both in the operator norm and the $\|\cdot\|_{\Theta}$ -norm, then $\Lambda_{\Theta}(\mathcal{C})$ is $\|\cdot\|_{\Theta}$ -closed in $L^2(\mathcal{M}, \Theta)$.

Proof of Theorem 1.2. Fix an intermediate subfactor $Q \subset N \subset M$. Set $H := \{g \in G : u_g \in N\}$, so that $Q \rtimes H \subset N$. Take $x_0 \in N$. We will show that the support of x_0 is contained in H . This will conclude by Theorem 3.7.

Denote by Tr a normal faithful semi-finite trace on Q , and denote by $\Psi \in \mathcal{P}(M)$ the corresponding dual weight.

Step 1. For all compact open subgroups $K < K_0$, all $g \in G$ and all finite trace projection $q \in Q^K$, if there exists $x \in N$ such that $qp_K E_K(u_g^* x) q p_K \neq 0$, then $u_g q p_K \in (N)_1 p_K$.

Fix $K < K_0$, $q \in Q^K$, $x \in N$ and $g \in G$ such that $qp_K E_K(u_g^* x) q p_K \neq 0$. Put $x' := \sigma_g(q)xq \in N$. We have $p_K E_K(u_g^* x') p_K = qp_K E_K(u_g^* x) q p_K \neq 0$. By Lemma 4.2, there exists a unique $a \in Q^K$ such that $p_K E_K(u_g^* x') p_K = a p_K$. Note that $a \in qQq$. The element $y := \sigma_g(a^*)x' \in \sigma(q)Nq$ satisfies $p_K E_K(u_g^* y) p_K = a^* a p_K \neq 0$. Lemma 4.1, implies that

$$\Psi(p_K u_g^* y p_K) = \Psi(p_K E_K(u_g^* y) p_K) = \frac{1}{m_{\mathbb{C}}(K)} \|a\|_{\text{Tr}}^2 \neq 0.$$

Denote by $\mathcal{C} \subset N$ the ultraweak closure of the convex hull

$$\mathcal{C}_0 := \text{conv}(\{\sigma_g(u)yu^* : u \in \mathcal{U}(qQ^Kq)\}).$$

We proceed as in the proof of Theorem 4.3 (5) \Rightarrow (6) in [20] to show that $\mathcal{C} p_K$ is bounded in $\|\cdot\|_{\Psi}$ -norm, except that we have slightly weaker assumptions. Since Q is contained in the centralizer of Ψ , it is clear from the triangle inequality that $\|Xp_K\|_{\Psi} \leq \|yp_K\|_{\Psi}$ for all $X \in \mathcal{C}_0$. Now if $X \in \mathcal{C}$ is arbitrary, take a net $(X_j)_{j \in J} \subset \mathcal{C}_0$ that converges ultrastrongly to X . Then $(X_j^* X_j)_{j \in J}$ converges ultraweakly to $X^* X$, and since Ψ is lower ultraweakly semi-continuous (see Theorem VII.1.11.(iii) of [34]) we get that

$$\|Xp_K\|_{\Psi}^2 = \Psi(p_K X^* X p_K) \leq \liminf_j \Psi(p_K X_j^* X_j p_K) \leq \|yp_K\|_{\Psi}^2.$$

Note moreover that $\|yp_K\|_{\Psi} < +\infty$ since $y = yq$ and $qp_K \in \mathfrak{n}_{\Psi}(M)$. Thus the ultraweakly closed convex set $\mathcal{C} p_K \subset M$ is bounded both in the operator norm and in the $\|\cdot\|_{\Psi}$ -norm and we can apply Lemma 4.9 to it. In particular we can find $z \in \mathcal{C}$, such that zp_K is the unique element of $\mathcal{C} p_K$ with minimal $\|\cdot\|_{\Psi}$ -norm. Note however that z itself needs not be unique.

Let us check that such an element $z \in N$ satisfies $zp_K \neq 0$. First, since $qp_K \in \mathfrak{n}_{\Psi}(M)$, $\Psi(qp_K \cdot p_K q)$ is a bounded normal linear functional on M , which is constant on $u_g^* \mathcal{C}_0$. Hence it is constant on $u_g^* \mathcal{C}$. It follows:

$$\Psi(qp_K u_g^* zp_K q) = \Psi(qp_K u_g^* yp_K q) = \Psi(p_K u_g^* (\sigma_g(q) y q) p_K) = \Psi(p_K u_g^* yp_K) \neq 0.$$

So we indeed find that $zp_K \neq 0$.

Note also that $\mathcal{C} p_K$ is globally invariant under the affine action $\mathcal{U}(qQ^Kq) \curvearrowright \sigma(q)Mq$ given by $u \cdot X := \sigma_g(u)Xu^*$. Since Ψ centralizes Q , this action is $\|\cdot\|_{\Psi}$ -isometric, so it fixes zp_K . Equivalently, we have $u_g^* zp_K \in (qQ^Kq)' \cap M$. Moreover $u_g^* zp_K$ belongs to qMq . By Proposition 4.8, we have that $(qQ^Kq)' \cap qMq \subset qLK$, so $u_g^* zp_K \in qLKp_K$. Since the projection p_K is both central and minimal inside LK , there exists a non-zero scalar $\lambda \in \mathbb{C}$ such that

$$(4.3) \quad u_g^* zp_K = \lambda p_K.$$

We obtain that $u_g q p_K = (z/\lambda) p_K \in N p_K$. Unfortunately we do not have, a priori, any control on how small $|\lambda|$ is, so z/λ could have a very large operator norm. To get around this issue, we would like to identify polar parts, but we need extra commutations properties. We apply the convex combination argument a second time.

Arguing as above, one can find an element $z_1 \in \mathcal{C}'$, where $\mathcal{C}' \subset \mathcal{C}$ is the ultraweak closure of $\text{conv}(\{\sigma_g(u) z u^* : u \in \mathcal{U}(q Q^K q)\}) \subset N$, such that $z_1^* z p_K$ is the unique element in $(\mathcal{C}')^* z p_K$ with minimal $\|\cdot\|_\Psi$ -norm. Note that $(\mathcal{C}')^* z p_K$ is nothing but the ultraweakly closed convex hull of $\{u z^* z p_K u^* : u \in \mathcal{U}(q Q^K q)\}$. Then z_1 enjoys the following properties:

(i) $z_1 \in N$ and $z_1 p_K = z p_K$, because for all $u \in \mathcal{U}(q Q^K q)$, $\sigma_g(u) z u^* p_K = \sigma_g(u) z p_K u^* = z p_K$; in particular $z_1 p_K \neq 0$.

(ii) $z_1^* z p_K \in (q Q^K q)' \cap q M q$, by uniqueness of a $\|\cdot\|_\Psi$ -norm minimizer inside $\mathcal{C}' z p_K$. Hence Proposition 4.8 gives that $z_1^* z p_K = \lambda' q p_K$ for some non-zero scalar $\lambda' \in \mathbb{C}$.

In particular the above facts give $p_K z_1^* z_1 p_K = p_K z_1^* z p_K = \lambda' q p_K$ and λ' follows positive. Hence the equality $z_1^* z_1 p_K = \lambda' q p_K = (\lambda' q p_K)^* = p_K z_1^* z_1$ shows that $z_1^* z_1$ commutes with p_K . Write the polar decomposition $z_1 = u |z_1|$, with $u \in N$ a partial isometry and $|z_1| = (z_1^* z_1)^{1/2}$. Note that $u q p_K$ is a partial isometry since $q p_K$ commutes with $z_1^* z_1$ (recall that $z_1 = z_1 q$).

We have: $z_1 p_K = u |z_1| p_K = \sqrt{\lambda'} u q p_K$. Combining this with (4.3), we get

$$\lambda u_g q p_K = z p_K = z_1 p_K = \sqrt{\lambda'} u q p_K.$$

Hence $u_g q p_K$ and $u q p_K$ are proportional partial isometries; they have to coincide,

$$u_g q p_K = (u q) p_K \in (N)_1 p_K.$$

This proves Step 1.

Step 2. For all compact open subgroups $K < K_0$ and $g \in G$, if there exists $x \in N$ such that $p_K E_K(u_g^* x) p_K \neq 0$, then $u_g p_K \in (N)_1 p_K$.

Fix a subgroup $K < K_0$, $g \in G$ and $x \in N$ such that $p_K E_K(u_g^* x) p_K \neq 0$.

Since K acts minimally on Q and Q^K is with expectation inside Q , Lemma 5.1 of [7] implies that the trace Tr is still semi-finite on Q^K . Denote by $(q_i)_{i \in I}$ an increasing net of projections in Q^K with finite trace that converges to 1. Then there exists $i_0 \in I$ such that for all $i \geq i_0$, we have that $q_i p_K E_K(u_g^* x) p_K q_i \neq 0$.

By Step 1, we deduce that for all $i \geq i_0$, there exists $y_i \in (N)_1$ such that

$$(4.4) \quad u_g q_i p_K = y_i p_K.$$

Denote by $y \in (N)_1$ an ultraweak limit of the net $(y_i)_{i \in I}$. Then taking the corresponding limit in (4.4) gives $u_g p_K = y p_K$, as desired.

Step 3. The support of x_0 is contained in H .

Take $g \in \text{supp}(x_0)$. Fix a compact open subgroup $K < K_0$. Then $E_K(u_g^*x_0) \neq 0$. As the net of projections $(p_L)_{L < G}$, converges strongly to the identity when $L \searrow \{e\}$, we have $p_L E_K(u_g^*x_0) p_L \neq 0$ for all small enough compact open subgroups L .

Take a normal open subgroup $L < K$, so that p_L commutes with u_g for all $g \in K$. By Lemma 4.4, we can write

$$\begin{aligned} p_L E_K(u_g^*x_0) p_L &= \sum_{h \in \text{lift}(K/L)} p_L u_h E_L(u_h^* E_K(u_g^*x_0)) p_L \\ &= \sum_{h \in \text{lift}(K/L)} u_h p_L E_L(u_{gh}^*x_0) p_L. \end{aligned}$$

For all small enough compact open subgroup $L < G$ that is normal inside K , since the above sum is non-zero, we can find $h_L \in K$ such that $p_L E_L(u_{gh_L}^*x_0) p_L \neq 0$. Applying Step 2 to L , gh_L and x_0 , we find an element $z_L \in (N)_1$ such that $u_{gh_L} p_L = z_L p_L$.

Note that there exists a net $(L_i)_i$ of compact open subgroups of G which are all normal in K and form a neighborhood basis of $e \in G$. This comes from the fact that any open subgroup $L < K$ has finite index inside K , so that $\bigcap_{h \in K} hLh^{-1}$ is an open normal subgroup of K , contained in L . By compactness of K and $(N)_1$ (for the weak operator topology), there exist subnets of $(h_{L_i})_i$ and $(z_{L_i})_i$ that converge to elements $h \in K$ and $z \in (N)_1$, respectively. Taking ultraweak limits, we get that

$$u_{gh} = \lim_i u_{gh_{L_i}} p_{L_i} = \lim_i z_{L_i} p_{L_i} = z \in N,$$

because the net of projections $(p_L)_L$ converges ultrastrongly to the identity. Hence $gh \in H$, and we conclude that $gK \cap H \neq \emptyset$.

As K can be arbitrarily small and H is closed, we conclude that $g \in H$. This finishes the proof of Step 3. Now the theorem follows from Theorem 3.7. ■

4.4. EXAMPLES OF ACTIONS.

PROPOSITION 4.10. *Let G be a totally disconnected group with a compact open subgroup $K < G$ such that $\bigcap_{g \in G} gKg^{-1} = \{e\}$ (let us call such a subgroup K eventually malnormal). Take a II_1 -factor Q_0 , with trace τ_0 . The following G -actions satisfy the assumptions of Theorem 1.2, so they are strictly outer and satisfy the intermediate subfactor property.*

- (i) *The Bernoulli action $G \curvearrowright (Q_0, \tau_0) \overline{\otimes}^{G/K}$ obtained by shifting indices.*
- (ii) *The free Bernoulli action $G \curvearrowright *_{G/K}(Q_0, \tau_0)$;*

Proof. Let us first check separately the minimality condition on the K -action for each case.

(i) Put $Q = Q_0 \overline{\otimes}^{G/K}$. Since K is eventually malnormal inside G , it acts faithfully on G/K and hence on Q . Since K is compact open in G it is commensurated, and hence it acts on G/K with finite orbits. Let us denote by $\mathcal{O}_j, j \in J$ these orbits

and by $Q_j := Q_0^{\overline{\otimes} \mathcal{O}_j}$. Example 2.7 shows that for all j , the fixed point algebra $Q_j^K \subset Q_j$ is an irreducible subfactor. Since Q^K contains $\overline{\otimes}_{j \in J} Q_j^K$, it is irreducible inside $Q = \overline{\otimes}_{j \in J} Q_j$. So K acts minimally.

(ii) Put $Q = *_G/K Q_0$. Again, since K is not normal inside G , it acts faithfully on Q . Because of the free product situation, it is clear that the copy of Q_0 located at the label $K \in G/K$ is irreducible inside Q . Moreover this algebra is contained in Q^K , so K acts faithfully.

We now check the relatively properly outer condition simultaneously for both situations. Take $g \in G \setminus K$ and decompose Q as a product

$$Q \simeq Q_0^{(K)} \overline{\otimes} Q_0^{(gK)} \overline{\otimes} P \quad (\text{or } Q \simeq Q_0^{(K)} * Q_0^{(gK)} * P),$$

where $Q_0^{(K)}$ and $Q_0^{(gK)}$ are the copies of Q_0 in position $K \in G/K$ and $gK \in G/K$ and P is the tensor product (or free product) of all the remaining copies of Q_0 .

In both the tensor situation and the free situation, one easily checks that for all nets $(u_n)_n \subset \mathcal{U}(Q_0^{(K)})$ and $(v_n)_n \subset \mathcal{U}(Q_0^{(gK)})$ that converge weakly to 0, and for all $a, b \in Q$, one has

$$\lim_n \tau(u_n a v_n b) = 0, \quad \text{where } \tau \text{ is the trace of } Q.$$

Assuming that there exists $a \in Q_1$ such that $\sigma_g(x)a = ax$ for all $x \in Q_1^K$, we take for $(u_n)_n \subset \mathcal{U}(Q_0^{(K)}) \subset \mathcal{U}(Q^K)$ any net of unitaries that converges weakly to 0, and we set $v_n = \sigma_g(u_n)$. We get

$$\|a\|_2 = \lim_n \tau(u_n^* a^* a u_n) = \lim_n \tau(u_n a^* v_n a) = 0.$$

Hence $a = 0$, as desired. \blacksquare

More generally, one can easily check the relative outerness condition in Theorem 1.2 when the action has a large commutant thanks to the following fact.

LEMMA 4.11. *Consider an action $G \curvearrowright Q$ and a closed subgroup $K < G$ whose action is minimal. Assume that the centralizer Γ of G in $\text{Aut}(Q)$ satisfies: for all $x \in Q \setminus \mathbb{C}$, there exists $\gamma \in \Gamma$ such that $\gamma(x) \notin \mathbb{C}x$.*

Then the action is properly outer relative to K if and only if the only elements of G acting trivially on Q^K are the elements of K .

Proof. The only if part is trivial. Conversely assume that K is precisely the set of elements of G that act trivially on Q^K . Take $g \in G$ such that there exists a non-zero $a \in Q$ satisfying $\sigma_g(x)a = ax$ for all $x \in Q^K$. Since K acts minimally, we can assume that a is a unitary. For all automorphism $\gamma \in \Gamma$, we have

$$\sigma_g(\gamma(x))\gamma(a) = \gamma(a)\gamma(x), \quad \text{for all } x \in Q^K.$$

Moreover, using again that γ commutes with the G -action, we see that $\gamma(Q^K) = Q^K$, and the previous equation reads as

$$\sigma_g(x)\gamma(a) = \gamma(a)x, \quad \text{for all } x \in Q^K.$$

In particular, we find that $a^*\gamma(a) \in (Q^K)' \cap Q = \mathbb{C}$. Hence for all $\gamma \in \Gamma$, we have $\gamma(a) \in \mathbb{C}a$. By our assumption on Γ , this leads to $a \in \mathbb{C}$, and hence g fixes Q^K pointwise. Thus $g \in K$ by assumption. ■

Note that the condition on the centralizer Γ in the above lemma is fulfilled as soon as Γ admits a subgroup Γ_0 that preserves a state on Q , and such that $Q^{\Gamma_0} = \mathbb{C}$. Moreover, if Q is a II_1 -factor the trace is invariant under any automorphism, and hence only the second condition needs to be verified.

We deduce the following result in the spirit of Vaes' examples ([38], Theorem 5.1).

COROLLARY 4.12. *Fix a totally disconnected group and a compact open subgroup $K < G$. Consider any faithful action $G \curvearrowright Q_0$ on a II_1 -factor such that*

$$\{g \in G : g|_{Q^K} = \text{id}\} = K.$$

Then the diagonal action $G \curvearrowright (Q, \tau) := (Q_0, \tau_0)^{\overline{\otimes}^{\mathbb{N}}}$ satisfies the assumptions of Theorem 1.2.

Proof. The K -action on Q_0 being faithful, Theorem 5.1 of [38] implies that it is strictly outer, and hence minimal. Moreover the centralizer Γ of such a diagonal action contains all shift automorphisms obtained by permuting indices. Hence the condition on Γ appearing in Lemma 4.11 is satisfied (see the comment after Lemma 4.11).

Thus the result follows from Lemma 4.11. ■

REMARK 4.13. Note that the Bernoulli shift action $G \curvearrowright Q_0^{\overline{\otimes}^{G/K}}$ as in Lemma 4.10 is sometimes a special case of diagonal action as in Corollary 4.12. For instance this happens if Q_0 is the hyperfinite factor. However it is not clear why this should be the case when Q_0 is a prime factor. Hence even the strict outerness for such actions does not follow from Theorem 5.1 of [38].

Before moving to the next section, let us briefly explain how to adapt our argument to cover some actions on type III factors.

REMARK 4.14. Let G be a totally disconnected group and let $K < G$ be an eventually malnormal compact open subgroup, see Lemma 4.10. Let Q_0 be an arbitrary diffuse factor admitting a faithful normal state ϕ_0 with large centralizer (meaning that $(Q_0^{\phi_0})' \cap Q_0 = \mathbb{C}$). Then the Bernoulli shift $G \curvearrowright (Q_0, \phi_0)^{\overline{\otimes}^{G/K}}$ satisfies the intermediate subfactor property.

Let us briefly explain. Denote by $(Q, \phi) := (Q_0, \phi_0)^{\overline{\otimes}^{G/K}}$. Then the centralizer Q^ϕ of ϕ is irreducible inside Q , and it is invariant under the G -action. One can show that for all compact open subgroup $L < G$ which is a finite intersection

of conjugates of K , the fixed point algebra $Q_L := (Q^\phi)^L$ satisfies $Q'_L \cap (Q \rtimes G) \subset L(L)$. This is done by following the proof of Lemma 4.7, and by noting that the action is properly outer relative to L .

Therefore one can use the averaging argument from the proof of Theorem 1.2 but instead we average with elements in $\mathcal{U}((Q^\phi)^L)$ for small groups L as above. Note moreover that in this case since ϕ is a state (and not a weight), one does not need to bother with the projection q appearing in Step 1 of the proof of Theorem 1.2.

In fact the above remark also applies for free Bernoulli actions with weaker assumptions on (Q_0, ϕ_0) , but we will not elaborate on this.

5. EXISTENCE OF CONDITIONAL EXPECTATIONS AND OPERATOR VALUED WEIGHTS

In this section we discuss various results about existence (or non-existence) of conditional expectations and operator valued weights in connection with the article of Izumi, Longo, and Popa [22].

Let us start our discussion by investigating the existence of conditional expectations/operator valued weights for pairs of the form $Q \rtimes H \subset Q \rtimes G$ associated with closed subgroups $H < G$.

5.1. PROOF OF THEOREM 1.6. Let us start with two lemmas which rely on the notion of support.

LEMMA 5.1. *Consider two actions $G \curvearrowright Q$ and $G \curvearrowright P$ and the diagonal action of G on $Q \overline{\otimes} P$. If the action $G \curvearrowright Q$ is strictly outer, then*

$$Q' \cap ((Q \overline{\otimes} P) \rtimes G) = P.$$

Proof. Embed G into $G \times G$ diagonally. Then $M := (Q \overline{\otimes} P) \rtimes G$ is identified with a subalgebra of $\tilde{M} := (Q \overline{\otimes} P) \rtimes (G \times G)$. Note that

$$Q' \cap \tilde{M} \simeq (Q' \cap (Q \rtimes G)) \overline{\otimes} (P \rtimes G) = \mathbb{C} \otimes (P \rtimes G).$$

In particular any element $x \in Q' \cap M$, viewed as an element of \tilde{M} has its support contained in the diagonal subgroup of $G \times G$ (because $x \in M$) and in $\{e\} \times G$ (because $x \in Q' \cap \tilde{M}$). So such an element x has its support contained in the trivial group. By Corollary 3.8, we get $x \in Q' \cap (Q \overline{\otimes} P) = P$, as wanted. ■

LEMMA 5.2. *Consider an action $G \curvearrowright Q$ on an arbitrary von Neumann algebra and put $M = Q \rtimes G$. Take a weight $\Phi \in \mathcal{P}(Q)$ and denote by $\Psi \in \mathcal{P}(M)$ the corresponding dual weight.*

For any non-zero $x \in \mathfrak{n}_\Psi(M)$, the support $\text{supp}(x)$ has positive Haar measure inside G .

Proof. We will use the description of the dual weight relying on the Hilbert algebra approach, [16]. According to this approach, given the weight Φ , there exists a left Hilbert algebra A_Φ satisfying the following properties:

- (i) the Hilbert completion of A_Φ is isomorphic to $L^2(G, \mathcal{H})$ and the left von Neumann algebra $L(A_\Phi)$ is identified with $Q \rtimes G$;
- (ii) the dual weight Ψ corresponds to the canonical weight on $L(A_\Phi)$ associated with the Hilbert algebra A_Φ ;
- (iii) for any Φ -right bounded vector $\eta \in \mathcal{H}$ and any $f \in \mathcal{K}(G)$, the function $f\eta \in \mathcal{K}(G, \mathcal{H})$ defined by $f\eta : g \mapsto f(g)\eta$ is a right bounded vector for A_Φ . The corresponding operator $\pi_r(f\eta)$ is given by

$$(5.1) \quad \pi_r(f\eta)\xi(g) = \int_G (\pi_r(\eta)\sigma_s(\xi(gs)))f(s^{-1}) ds, \quad \text{for all } \xi \in L^2(G, \mathcal{H}), g \in G,$$

where $\pi_r(\eta)$ denotes the operator on $\mathcal{H} = L^2(Q, \Phi)$ associated with the right bounded vector η . Here we follow [17] and use the uniqueness of the standard form to identify canonically \mathcal{H} with $L^2(Q, \Phi)$.

Now take a non-zero $x \in \mathfrak{n}_\Psi(M)$. By the above facts and Chapter VII.2 of [34], there exists a non-zero left bounded vector $\xi \in L^2(G, \mathcal{H})$ such that $x = \pi_l(\xi)$ (that is, x is the operator extending the left multiplication by ξ). We claim that the equality holds, but we clearly only need this inclusion to deduce the lemma.

Take $g \in G$ in the function support of ξ . Take an open neighborhood $\Omega \subset G$ of the identity element e . We have to show that $P(g\Omega)\pi_l(\xi)P(\Omega)$ is non-zero. For all $f \in \mathcal{K}(\Omega)$ and all Φ -right bounded vector $\eta \in \mathcal{H}$, we have

$$P(g\Omega)\pi_l(\xi)P(\Omega)(f\eta) = P(g\Omega)\pi_l(\xi)(f\eta) = P(g\Omega)\pi_r(f\eta)\xi.$$

We now check that for a suitable choice of f and η the above quantity is non-zero.

Since g is in the function support of ξ , there exists $\eta_0 \in \mathcal{H}$ such that $g\Omega \cap \{h \in G : \langle \xi(h), \eta_0 \rangle \neq 0\}$ has positive measure. Since the set $\{\pi_r(\eta_1)^*\eta_2 : \eta_1, \eta_2 \in \mathcal{H}, \Phi\text{-right bounded}\}$ is dense inside \mathcal{H} , we may find two Φ -right bounded vectors $\eta_1, \eta_2 \in \mathcal{H}$ such that $\pi_r(\eta_1)^*\eta_2$ is sufficiently close to η_0 so that the set $g\Omega \cap \{h \in G : \langle \xi(h), \pi_r(\eta_1)^*\eta_2 \rangle \neq 0\}$ has positive Haar measure. In particular, the function $\zeta \in L^2(G, \mathcal{H})$ defined by $h \mapsto \pi_r(\eta_1)\xi(h)$ satisfies $P(g\Omega)\zeta \neq 0$. Put $C := \|P(g\Omega)\zeta\|_2 > 0, D := \|\pi_r(\eta_1)\|$.

For $f \in \mathcal{K}(G)$, denote by \check{f} the function $h \mapsto f(h^{-1})$. For all non-negative function $f \in \mathcal{K}(G)$ such that $\int_G f = 1$ we have, by (5.1),

$$\begin{aligned} A &:= \|\pi_r(\check{f}\eta_1)\xi - \zeta\|_2^2 = \int_G \|(\pi_r(\check{f}\eta_1)\xi)(h) - \zeta(h)\|^2 dh \\ &= \int_G \left\| \int_G f(s)\pi_r(\eta_1)(\sigma_s(\xi(hs))) - \zeta(h) \right\|^2 dh \end{aligned}$$

$$\begin{aligned}
 &\leq \int_G \left(\int_G f(s) \|\pi_r(\eta_1)\| \|\sigma_s(\xi(hs)) - \xi(h)\| ds \right)^2 dh \\
 &= D^2 \int_{G \times G \times G} f(s)f(t) \|\sigma_s(\xi(hs)) - \xi(h)\| \|\sigma_t(\xi(ht)) - \xi(h)\| ds dt dh \\
 &\leq D^2 \int_{G \times G} f(s)f(t) \|\delta_G(s)^{-1/2} \rho_G(s)(\xi) - \xi\| \|\delta_G(t)^{-1/2} \rho_G(t)(\xi) - \xi\| ds dt \\
 &= \left(D \int_G f(s) \|\delta_G(s)^{-1/2} \rho_G(s)(\xi) - \xi\| ds \right)^2,
 \end{aligned}$$

where ρ_G is the representation defined in (2.1). Since ρ_G is a continuous representation and δ_G is a continuous function on G , we get that if f is supported on a small enough neighborhood of e , then $\|\pi_r(\check{f}\eta_1)\xi - \xi\|_2 < C/2$. By definition of C we get:

$$\|P(g\Omega)\pi_r(\check{f}\eta_1)\xi\| \geq \|P(g\Omega)\xi\| - \|\pi_r(\check{f}\eta_1)\xi - \xi\| \geq \frac{C}{2}.$$

So there indeed exists a Φ -right bounded vector η_1 and a function $f_1 = \check{f}$ which is supported on Ω such that

$$P(g\Omega)xP(\Omega)(f_1\eta_1) = P(g\Omega)\pi_1(\xi)(f_1\eta_1) \neq 0. \quad \blacksquare$$

Proof of Theorem 1.6. We prove the two facts separately.

(i) First assume that the modular functions δ_G and δ_H coincide on H . Then by Theorem 3.2 of [16], for any weight $\Phi \in \mathcal{P}(Q)$, the dual weight $\Psi_G \in \mathcal{P}(Q \rtimes G)$ and $\Psi_H \in \mathcal{P}(Q \rtimes H)$ satisfy

$$\sigma_t^{\Psi_G}(x) = \sigma_t^{\Psi_H}(x), \quad \text{for all } x \in Q \rtimes H, t \in \mathbb{R}.$$

Therefore there exists a nfs operator valued weight $T \in \mathcal{P}(Q \rtimes G, Q \rtimes H)$ by Theorem 5.1 of [19].

Conversely, assume that $\mathcal{P}(Q \rtimes G, Q \rtimes H)$ is non-trivial. By Theorem 5.9 of [19], and Remark 2.3 we deduce that there exists an operator valued weight $T \in \mathcal{P}(\tilde{M}, M)$, where $M = Q \rtimes G$ and $\tilde{M} = (\mathcal{L}^\infty(G/H) \overline{\otimes} Q) \rtimes G$ (because \tilde{M} is isomorphic to the basic construction of $Q \rtimes H \subset M$).

Our intermediate goal is to deduce that there exists a G -invariant nfs weight on $A := \mathcal{L}^\infty(G/H)$. Unfortunately, we do not know a priori that T is semi-finite on A . To get around this issue we will exploit the fact that the action is strictly outer and use modular theory.

Let us consider the following operator valued weights in $\mathcal{P}(\tilde{M}, Q)$:

(•) $T_1 := T_G \circ T$, where $T_G \in \mathcal{P}(Q \rtimes G, Q)$ is the Plancherel operator valued weight;

(•) $T_\phi := (\phi \otimes \text{id}) \circ \tilde{T}_G$, where $\tilde{T}_G \in \mathcal{P}(\tilde{M}, A \overline{\otimes} Q)$ is the Plancherel operator valued weight and $\phi \otimes \text{id} \in \mathcal{P}(A \overline{\otimes} Q, Q)$ is the tensor product operator valued

weight associated to some weight $\phi \in \mathcal{P}(A)$ and the identity map on Q (see Theorem 5.5 of [19]).

Fix $\phi \in \mathcal{P}(A)$. By Lemma 5.1, we have that $Q' \cap \tilde{M} = A$, hence the Connes–Radon–Nikodym cocycle $(DT_1 : DT_\phi)_t$ in the sense of Definition 6.2 in [19] takes values into A , and it is a 1-cocycle for the flow $\sigma_t^{T_\phi}$. By construction, for any weight $\psi \in \mathcal{P}(Q)$, the weight $\psi \circ T_\phi$ is simply the dual weight associated to $\phi \otimes \psi \in \mathcal{P}(A \overline{\otimes} Q)$. Hence $(\sigma_t^{T_\phi})|_A = \sigma_t^\phi = \text{id}$, since A is abelian. We conclude that $(v_t)_t := (DT_1 : DT_\phi)_t$ is a one parameter subgroup of unitaries of A .

By Théorème 1.2.4 of [13], there exists a nfs weight $\phi' \in \mathcal{P}(A)$ such that $(D\phi' : D\phi)_t = v_t$ for all $t \in \mathbb{R}$.

Claim 1. The weight ϕ' is G -invariant.

We will denote generically by the letter σ all the G actions. We fix $g \in G$ and show that the Connes–Radon–Nikodym derivative $(D\phi' \circ \sigma_g : D\phi')_t$ is equal to 1 for all $t \in \mathbb{R}$. We have

$$(5.2) \quad (D\phi' \circ \sigma_g : D\phi')_t = (D\phi' \circ \sigma_g : D\phi \circ \sigma_g)_t (D\phi \circ \sigma_g : D\phi)_t (D\phi : D\phi')_t.$$

By Lemma 2.4, we have that $(D\phi' \circ \sigma_g : D\phi \circ \sigma_g)_t = \sigma_g^{-1}((D\phi' : D\phi)_t) = \sigma_g^{-1}(v_t)$. Hence (5.2) becomes

$$(5.3) \quad (D\phi' \circ \sigma_g : D\phi')_t = \sigma_g^{-1}(v_t) (D\phi \circ \sigma_g : D\phi)_t v_t^*.$$

We will show that the right hand side above is equal to 1 by computing all the terms in the equality

$$(5.4) \quad (DT_1^g : DT_1)_t = (DT_1^g : DT_\phi^g)_t (DT_\phi^g : DT_\phi)_t (DT_\phi : DT_1)_t.$$

Here we denoted by T_1^g (respectively T_ϕ^g) the operator valued weight $\sigma_g^{-1} \circ T_1 \circ \text{Ad}(u_g) \in \mathcal{P}(\tilde{M}, Q)$ (respectively $\sigma_g^{-1} \circ T_\phi \circ \text{Ad}(u_g)$).

Take a weight $\psi \in \mathcal{P}(Q)$. By definition of the Connes–Radon–Nikodym derivative for operator valued weights, we have

$$\begin{aligned} (DT_1^g : DT_\phi^g)_t &= (D(\psi \circ T_1^g) : D(\psi \circ T_\phi^g))_t \\ &= (D(\psi \circ \sigma_g^{-1} \circ T_1) \circ \text{Ad}(u_g) : D(\psi \circ \sigma_g^{-1} \circ T_\phi) \circ \text{Ad}(u_g))_t \\ &= \sigma_g^{-1}((D(\psi \circ \sigma_g^{-1} \circ T_1) : D(\psi \circ \sigma_g^{-1} \circ T_\phi))_t) = \sigma_g^{-1}((DT_1 : DT_\phi)_t). \end{aligned}$$

where the third equality follows from Lemma 2.4. Hence (5.4) becomes

$$(5.5) \quad (DT_1^g : DT_1)_t = \sigma_g^{-1}(v_t) (DT_\phi^g : DT_\phi)_t v_t^*.$$

Now ([17], Theorem 3.1) is easily seen to imply that $T_1^g = \delta_G(g)T_1$ while $T_\phi^g = \delta_G(g)T_{\phi \circ \sigma_g}$. So $(DT_1^g : DT_1)_t = \delta_G(g)^{\text{it}}$ and

$$(DT_\phi^g : DT_\phi)_t = \delta_G(g)^{\text{it}} (DT_{\phi \circ \sigma_g} : DT_\phi)_t = \delta_G(g)^{\text{it}} (D\phi \circ \sigma_g : D\phi)_t.$$

Altogether, (5.5) can be rewritten

$$(5.6) \quad \delta_G(g)^{it} = \sigma_g^{-1}(v_t)\delta_G(g)^{it}(D\phi \circ \sigma_g : D\phi)_t v_t^*.$$

And we see that the right hand side of (5.3) is equal to 1, proving Claim 1.

Recall that $A = \mathcal{L}^\infty(G/H)$ is the subalgebra of right H -invariant functions inside $L^\infty(G)$. Denote by $q : G \rightarrow G/H$ the quotient map. The formula $B \subset G/H \mapsto \phi'(\mathbf{1}_{q^{-1}(B)})$ defines a measure ν on the Borel σ -algebra of G/H . This measure is G -invariant and non-zero (because ϕ' is faithful).

Claim 2. The G -invariant measure ν is finite on every compact set of G/H .

Since ϕ' is semi-finite (and faithful), there exists a Borel set $B \subset G/H$ such that $0 < \nu(B) < \infty$. Take a compactly supported non-negative function on G , $f \in \mathcal{K}(G)$, $f \neq 0$. Then the function $f * 1_B$ defined as follows is continuous:

$$(f * 1_B)(xH) := \int_G f(g)1_B(g^{-1}xH) dm_G(g), \quad xH \in G/H.$$

By Fubini–Tonelli’s theorem, we have the key equation

$$\begin{aligned} \int_{G/H} (f * 1_B) d\nu &= \int_{G/H} \int_G f(g)1_B(g^{-1}xH) dm_G(g) d\nu(xH) \\ &= \int_G f(g)\nu(gB) dm_G(g) = \nu(B) \int_G f dm_G. \end{aligned}$$

This key equation tells us first that the continuous non-negative function $f * 1_B$ is non-zero. So there exists an open set $U \subset G/H$ and $\alpha > 0$ such that $1_U \leq \alpha(f * 1_B)$. The key equation also tells us that $f * 1_B$ is ν -integrable, and in particular, $\nu(U) < \infty$. Since any compact set $K \subset G/H$ can be covered by finitely many translates of U , Claim 2 follows.

By Corollary B.1.7 of [4] we deduce from the existence of ν that the modular functions δ_G and δ_H must coincide on H . This proves (i).

(ii) As mentioned earlier in the paper, if H is open inside G , then the indicator function 1_H is continuous and positive definite on G . Then the associated multiplier (see Theorem 3.1.a of [17]) gives the desired conditional expectation from $Q \rtimes G$ onto $Q \rtimes H$.

Conversely, assume that H is not open inside G . If the modular functions δ_G and δ_H do not coincide on H , then part (i) ensures that there is no nfs operator valued weight from $Q \rtimes G$ onto $Q \rtimes H$, and in particular, no conditional expectation.

Assume now that the modular functions do coincide. Fix a nfs weight $\Phi \in \mathcal{P}(Q)$, and denote by $\Psi_H \in \mathcal{P}(Q \rtimes H)$ and $\Psi_G \in \mathcal{P}(Q \rtimes G)$ the associated dual weights. As we saw in the proof of (i), Theorem 5.1 of [19] implies that there exists an operator valued weight $T \in \mathcal{P}(Q \rtimes G, Q \rtimes H)$ such that $\Psi_H \circ T = \Psi_G$. As the inclusion $Q \rtimes H \subset Q \rtimes G$ is irreducible, it suffices to show that T is unbounded, by Theorem 6.6 of [19].

Fix an element $f \in \mathcal{K}(H, Q)$. Note that since H is not open inside G it has measure 0 inside G . Indeed for any positive measure Borel set A inside G , the product $A^{-1}A$ contains a neighborhood of the identity. In particular, since $\text{supp}(f) \subset H$, Lemma 5.2 implies that $f \notin \mathfrak{n}_{\Psi_G}(Q \rtimes G)$. In contrast, $f \in \mathfrak{n}_{\Psi_H}(Q \rtimes H)$, and hence the expression $\psi_f := \Psi_H(f^* \cdot f)$ defines a normal positive linear functional on $Q \rtimes H$. We have

$$T(1)(\psi_f) = \psi_f \circ T(1) = \Psi_H \circ T(f^* f) = \Psi_G(f^* f) = +\infty.$$

Hence $T(1) \in \widehat{Q \rtimes H_+} \setminus Q \rtimes H$, and so T is not bounded. The proof is complete. ■

5.2. WHEN CONDITIONAL EXPECTATIONS DO EXIST. APPLICATIONS TO HECKE PAIRS. Before proving Theorem 1.7, let us mention that the argument of Choda applies beyond the setting of discrete groups.

THEOREM 5.3 ([11], Theorem 3). *Consider a strictly outer action $G \curvearrowright Q$ of a locally compact group and take a von Neumann subalgebra $N \subset Q \rtimes G$ that contains Q and that is the range of a faithful normal conditional expectation $E_N : Q \rtimes G \rightarrow N$.*

Then N is of the form $Q \rtimes H$ for some open subgroup H of G .

Proof. As usual, consider the closed subgroup H of G defined by $H = \{g \in G : u_g \in N\}$, so that $Q \rtimes H \subset N$. Let us show that the converse inclusion also holds.

Since N contains Q , we have $u_g^* E_N(u_g) \in Q' \cap (Q \rtimes G)$, for all $g \in G$. As the action is strictly outer, we deduce that for any $g \in G$, $E_N(u_g)$ is a scalar multiple of u_g . If this scalar multiple is non-zero, this means that $u_g \in N$, and hence the scalar in question must be 1. So we obtain the following computation:

$$E_N(au_g) = \mathbf{1}_{\{g \in H\}} au_g \in Q \rtimes H, \quad \text{for all } a \in Q \text{ and } g \in G.$$

Since E_N is normal we deduce by linearity and density that $E_N(Q \rtimes G) \subset Q \rtimes H$. Thus $N \subset Q \rtimes H$, and we have equality. The fact that H is open follows from Theorem 1.6. ■

We now mention a lemma that provides existence of conditional expectations. It follows from the main technical result of [22]. Note that we need a separability assumption.

LEMMA 5.4. *Consider a compact open subgroup $K < G$ in a locally compact group. Let $\sigma : G \curvearrowright Q$ be a strictly outer action on a factor with a separable predual and put $M := Q \rtimes G, N := Q \rtimes K$. Then any intermediate von Neumann algebra $N \subset L \subset M$ is the range of a normal faithful conditional expectation $E_L : M \rightarrow L$.*

Proof. We show that $N \subset M$ satisfy the assumptions of Corollary 3.11 in [22]. Observe that the inclusion $N \subset M$ is irreducible since the action of G is strictly outer. As mentioned earlier in the paper, since K is open inside G , there exists a conditional expectation $E = E_K$ from $M = Q \rtimes G$ onto $N = Q \rtimes K$.

By Remark 2.3, the basic construction M_1 of the $N \subset M$ is isomorphic to $(Q \overline{\otimes} \ell^\infty(G/K)) \rtimes G$, where G acts diagonally. The embedding $M \subset M_1$ is given by $au_g \mapsto (a \otimes 1)u_g$ for all $a \in Q, g \in G$. Note that G/K is discrete since K is open. By Lemma 5.1, we have $Q' \cap M_1 = \ell^\infty(G/K)$. Hence, the relative commutant $N^c := N' \cap M_1$ is isomorphic to the von Neumann algebra of K -bi-invariant maps $\ell^\infty(K \backslash G / K)$.

We represent faithfully M_1 on $\mathcal{H} \otimes \ell^2(G/K) \otimes L^2(G)$ in the obvious way. In this picture, the Jones projection e_N is the orthogonal projection onto $\mathcal{H} \otimes \mathbb{C}\delta_K \otimes L^2(G)$ where δ_K is the Dirac mass of the coset $K \in G/K$. Consider the dual operator valued weight $\widehat{E} \in \mathcal{P}(M_1, M)$ of E . We have that $\widehat{E}(e_N) = 1$ by Lemma 3.1 of [24]. This implies that $\widehat{E}(1_{KgK}) = [K : K \cap gKg^{-1}]$, where 1_{KgK} is the characteristic function of $KgK, g \in G$. Since $K < G$ is a compact open subgroup, the index $[K : K \cap gKg^{-1}]$ is finite for any $g \in G$ and thus the operator valued weight $T := E \circ \widehat{E}$ is semi-finite on N^c . Given any weight $\theta \in \mathcal{P}(N)$, Theorem 6.6 of [19] states that the restriction $\sigma^{\theta \circ T}|_{N^c}$ of the modular flow associated to $\theta \circ T$ to N^c is equal to the modular flow of the restriction of $\theta \circ T$ to N^c . Therefore, $\sigma_t^T(x) = x$ for any $t \in \mathbb{R}, x \in N^c$ since N^c is commutative. Hence, the pair $N \subset M$ indeed satisfies the assumptions of Corollary 3.11 in [22], which implies the lemma. ■

Proof of Theorem 1.7. This follows immediately by combining Theorem 5.3 and Lemma 5.4. ■

Let us now derive applications to the setting of crossed-products by Hecke pairs of groups. By definition, a *Hecke pair* is a pair of groups (G, H) such that H is a subgroup of G which is commensurated (or almost normal) in G , in the sense that $H^g := H \cap gHg^{-1}$ has finite index in H and gHg^{-1} for all $g \in G$. We refer to [2] and [8], [30] for more details on facts below. See also [3] in the context of equivalence relations.

A typical example of a Hecke pair (G, H) arises when G is a subgroup of the automorphism group of a locally finite connected graph Γ and H is the subgroup of elements of G that stabilize a given vertex of Γ . In fact, this example is somewhat generic, see Theorem 2.15 of [1].

A Hecke pair (G, H) admits a *Schlichting completion* $(\widetilde{G}, \widetilde{H})$ that is a new Hecke pair, for which \widetilde{G} is a totally disconnected group and \widetilde{H} is a compact open subgroup of \widetilde{G} . The precise construction goes as follows: view G as a subgroup of $S_{G/H}$, the permutation group of G/H . Endow $S_{G/H}$ with the topology of pointwise converge (where G/H is viewed as a discrete space), and define \widetilde{G} (respectively \widetilde{H}) to be the closure of G (respectively H) inside $S_{G/H}$. The idea of using the Schlichting completion to study operator algebras of Hecke pairs goes back to Tzanev [36]. The key observation is Proposition 5.6 below.

We will say that an action $G \curvearrowright Q$ is an *action of the Hecke pair* (G, H) if it extends continuously to an action of the Schlichting completion \widetilde{G} . The action is *strictly outer* if its extension to the Schlichting completion is strictly outer. Let

us define the crossed-product von Neumann algebra associated with such an action. The construction generalizes that of C^* -algebras associated with Hecke pairs (which correspond to the trivial action $G \curvearrowright \mathbb{C}$), which originate in [6]. We also refer to [2] for a general treatment. Crossed-product C^* -algebras for actions of Hecke pairs were then defined in [30] and the von Neumann algebraic version in Section 4 of [8].

Let $\mathbb{C}[Q; G, H]$ be the space of continuous functions $f : G \rightarrow Q$ such that $f(hgk) = \sigma_h(f(g))$ for any $g \in G, h, k \in H$, and such that the induced functions $\bar{f} : G/H \rightarrow Q$ is finitely supported. Note that if $f \in \mathbb{C}[Q; G, H]$ and $g \in G$, then $f(g)$ is fixed by $\sigma(H \cap gHg^{-1})$. We define a multiplication and an involution $*$ on $\mathbb{C}[Q; G, H]$ as follows:

$$f_1 f_2(g) = \sum_{s \in \text{lift}(G/H)} f_1(s) \sigma_s(f_2(s^{-1}g)), \quad \text{for any } f_1, f_2 \in \mathbb{C}[Q; G, H] \text{ and } g \in G,$$

where $\text{lift}(G/H)$ is a system of representatives of G/H ,

$$f^*(g) = \sigma_g(f(g^{-1})^*), \quad \text{for any } f \in \mathbb{C}[Q; G, H] \text{ and } g \in G.$$

The space $\mathbb{C}[Q; G, H]$ endowed with those operations is a unital $*$ -algebra. It contains a copy of the fixed point von Neumann algebra Q^H via the map $j(a)(g) = \chi_H(g)a, a \in Q^H, g \in G$, where χ_H is the characteristic function of H .

Assume that Q is standardly represented on a Hilbert space \mathcal{H} and denote again by $\sigma : G \rightarrow \mathcal{U}(\mathcal{H})$ the canonical implementation of the action, see [15]. Let $L^2(G/H, \mathcal{H})$ be the Hilbert space of L^2 -functions from G/H to \mathcal{H} , where G/H is viewed as a discrete space (and endowed with the counting measure). Consider the subspace

$$\mathcal{K} := \{ \xi \in L^2(G/H, \mathcal{H}) : \xi(hgH) = \sigma_h(\xi(gH)), \forall h \in H, gH \in G/H \}.$$

A similar proof to that of Proposition 5.1 in [8] gives us the following lemma.

LEMMA 5.5. *The map $\pi : \mathbb{C}[Q; G, H] \rightarrow B(\mathcal{K})$ defined by*

$$\pi(f)\xi(gH) = \sum_{s \in \text{lift}(G/H)} f(s) \sigma_s(\xi(s^{-1}gH)), \quad f \in \mathbb{C}[Q; G, H], \xi \in \mathcal{K}, gH \in G/H,$$

is a bounded representation of the $$ -algebra $\mathbb{C}[Q; G, H]$. We call it the standard representation of the Hecke algebra.*

Denote by $\text{vN}[Q; G, H]$ the bicommutant of $\pi(\mathbb{C}[Q; G, H])$. We call it the crossed-product von Neumann algebra of Q by (G, H) .

The next proposition relates crossed-products by Hecke pairs to the group crossed-product by the Schlichting completion, it generalizes Lemma 4.2. As mentioned above, this idea goes back to Tzanev [36].

PROPOSITION 5.6. *Consider an action of a Hecke pair (G, H) on a von Neumann algebra Q . Denote by (\tilde{G}, \tilde{H}) the Schlichting completion of (G, H) . Then the pairs*

$$(Q^H \subset \text{vN}[Q; G, H]) \quad \text{and} \quad (p_{\tilde{H}}(Q \rtimes \tilde{H})p_{\tilde{H}} \subset p_{\tilde{H}}(Q \rtimes \tilde{G})p_{\tilde{H}})$$

are isomorphic. Here $p_{\tilde{H}}$ is the averaging projection associated with the compact open subgroup $\tilde{H} < \tilde{G}$ as defined above.

Proof. Note that the coset spaces G/H and \tilde{G}/\tilde{H} are naturally isomorphic. Since the action $G \curvearrowright Q$ extends continuously to an action of \tilde{G} , the pairs

$$(Q^H \subset \text{vN}[Q; G, H]) \quad \text{and} \quad (Q^{\tilde{H}} \subset \text{vN}[Q; \tilde{G}, \tilde{H}])$$

are isomorphic. Moreover, the Schlichting completion of (\tilde{G}, \tilde{H}) is equal to itself. Replacing (G, H) with (\tilde{G}, \tilde{H}) if necessary, we may assume that H is compact open inside G , and $\tilde{G} = G, \tilde{H} = H$.

Consider the isometry $u : \mathcal{K} \rightarrow L^2(G, \mathcal{H})$ given by the formula

$$u\tilde{\zeta}(g) = \frac{1}{m_G(H)^{1/2}} \sigma_g^{-1}(\tilde{\zeta}(gH)), \quad g \in G.$$

Put $M := Q \rtimes G$. Observe that the map $x \in p_H M p_H \mapsto u^* x u \in B(\mathcal{K})$ is a representation that is equivalent to $p_H M p_H$ acting on $p_H J p_H L^2(G, \mathcal{H})$, where J is the conjugation operator on the standard representation $L^2(G, \mathcal{H})$ of M . In particular, $x \in p_H M p_H \mapsto u^* x u \in B(\mathcal{K})$ is a faithful representation.

Consider the map $\phi : \mathbb{C}[Q; G, H] \rightarrow \mathcal{K}(G, Q)$ defined by the formula

$$\phi(f)(g) = \frac{1}{m_G(H)} \sigma_g^{-1}(f(g)), \quad \text{for any } f \in \mathbb{C}[Q; G, H], g \in G.$$

Observe that the range of ϕ is the space of functions $F \in \mathcal{K}(G, Q)$ such that $F(kgl) = \sigma_l^{-1}(F(g))$ for any $g \in G, k, l \in H$. This range is precisely the corner $p_H \mathcal{K}(G, Q) p_H$ and we have that

$$(5.7) \quad u^* \phi(f) u = f \quad \text{for any } f \in \mathbb{C}[Q; G, H].$$

Therefore, $x \in p_H M p_H \mapsto u^* x u \in B(\mathcal{K})$ is an isomorphism of von Neumann algebras onto $\text{vN}[Q; G, H]$. By Lemma 4.2, we have that $p_H(Q \rtimes H)p_H = Q^H p_H$ and one easily checks that $u^*(Q^H p_H)u = Q^H$. ■

Observe that if $H < L < G$ is an intermediate closed group, then there is a natural identification of $\mathbb{C}[Q; L, H]$ as a subalgebra of $\mathbb{C}[Q; G, H]$ of all functions that are supported on L . This identification extends to the respective crossed-product von Neumann algebras. To see this with little effort we can use the above proposition as follows. By definition of the Schlichting completion we have that \tilde{L} is isomorphic to the closure of L inside \tilde{G} , where \tilde{L} and \tilde{G} are the Schlichting completions of L and G with respect to the subgroup H . As mentioned in Section 2.3, the crossed-product $Q \rtimes \tilde{L}$ is isomorphic to the weak closure of the algebraic crossed-product $Q \rtimes_{\text{alg}} \tilde{L}$ inside $Q \rtimes \tilde{G}$. Proposition 5.6 and this later fact imply that there exists an injective morphism of von Neumann algebras $\phi : \text{vN}[Q; L, H] \rightarrow \text{vN}[Q; G, H]$ that sends Q^H to itself and such that $\phi(\text{vN}[Q; L, H])$ is the weak closure of $\mathbb{C}[Q; L, H]$ inside $\text{vN}[Q; G, H]$.

Theorem 1.7 together with Proposition 5.6 implies the following result for crossed-products by Hecke pairs.

COROLLARY 5.7. *Consider a strictly outer action of a Hecke pair (G, H) on a von Neumann algebra Q that has a separable predual. If $Q^H \subset L \subset \mathfrak{vN}[Q; G, H]$ is an intermediate von Neumann algebra, then there exists an intermediate group $H < H' < G$ such that L is isomorphic to the crossed-product $\mathfrak{vN}[Q; H', H]$, which we identified as a subalgebra of $\mathfrak{vN}[Q; G, H]$.*

Recently, it has been shown that Hecke pairs appear in subfactor theory. Consider a finite index subfactor $N \subset M$ and its symmetric enveloping inclusion $T \subset S$, see [31] (or [26] in the type III finite depth setting). Then in some cases, there exists a Hecke pair (G, K) and actions $G \curvearrowright M \overline{\otimes} M^{\text{op}}, K \times K \curvearrowright M \overline{\otimes} M^{\text{op}}$ such that $T \subset S$ is isomorphic with $(M \overline{\otimes} M^{\text{op}})^{K \times K} \subset \mathfrak{vN}[M \overline{\otimes} M^{\text{op}}; G, K]$, see Theorem 5.5 of [8]. Hence, our last corollary gives us information about the lattice of intermediate subfactors in the symmetric enveloping inclusion of $N \subset M$.

Acknowledgements. This work was initiated during the trimester *von Neumann algebras* held at the Hausdorff Institute in Spring 2016. Parts of this work are also due to discussions during the Conference in the memory of Uffe Haagerup held in Copenhagen in June 2016 and during the Oberwolfach meeting on C^* -algebras in August 2016. We warmly thank the organizers of all these events. The second author was invited at the Mathematical Institute of Bordeaux in September 2016. He gratefully acknowledges the kind hospitality he received.

Mathematically, we are grateful to Sven Raum for interesting discussions and for pointing out [14] to us regarding the notion of support. We are also indebted to Amaury Freslon for explaining to us part of the history on quantum groups related to this work and for providing references. Theorem 1.6 was motivated by discussions of the first author with Hiroshi Ando and Cyril Houdayer. Many thanks to them. We thank Narutaka Ozawa, Yoshimichi Ueda, Stefaan Vaes and Amine Marrakchi for helpful comments on a previous version of this article.

The first author was partially supported by a PEPS grant from INSMI and the second author was partially supported by NSF Grant DMS-1362138 and by the ERC Advanced Grant 669240.

REFERENCES

- [1] C. ANANTHARAMAN-DELAROCHE, Invariant proper metrics on coset spaces, *Topology Appl.* **160**(2013), 546–552.
- [2] C. ANANTHARAMAN-DELAROCHE, Approximation properties for coset spaces and their operator algebras, in *Proceedings of the Operator Theory*, Timisoara, 2014.
- [3] H. AOI, T. YAMANOUCI, Hecke von Neumann algebra of ergodic discrete measured equivalence relations, *Publ. Res. Inst. Math. Sci.* **46**(2010), 607–667.

- [4] B. BEKKA, P. DE LA HARPE, A. VALETTE, *Kazhdan's Property (T)*, New Math. Monogr., vol. 11, Cambridge Univ. Press, Cambridge 2008.
- [5] D. BISCH, R. NICOARA, S. POPA, Continuous families of hyperfinite subfactors with the same standard invariant, *Internat. J. Math.* **18**(2007), 255–267.
- [6] J.-B. BOST, A. CONNES, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, *Selecta Math. (N.S.)* **1**(1995), 411–457.
- [7] R. BOUTONNET, C. HOUDAYER, S. VAES, Strong solidity of free Araki-Woods factors, preprint 2015, arXiv:1512.04820 [math.OA].
- [8] A. BROTHIER, On fixed point planar algebras, *J. Math. Phys.* **57**(2016), 111703.
- [9] D. BURES, Abelian subalgebras of von Neumann algebras, *Mem. Amer. Math. Soc.* **110**(1971).
- [10] J. CAMERON, R.R. SMITH, Intermediate subalgebras and bimodules for crossed products of general von Neumann algebras, *Int. J. Math.* **27**(2016), ID 1650091.
- [11] H. CHODA, A Galois correspondence in a von Neumann algebra, *Tohoku Math. J.* **30**(1978), 491–504.
- [12] F. COMBES, C. DELAROCHE, Groupe modulaire d'une espérance conditionnelle dans une algèbre de von Neumann, *Bull. Soc. Math. France* **103**(1975), 385–426.
- [13] A. CONNES, Une classification des facteurs de type III, *Ann. Sci. École Norm. Sup.* **6**(1973), 133–252.
- [14] P. EYMARD, L'algèbre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France* **92**(1964), 181–236.
- [15] U. HAAGERUP, The standard form of a von Neumann algebra, *Math. Scand.* **37**(1975), 271–283.
- [16] U. HAAGERUP, On the dual weight for crossed-product of von Neumann algebras. I, *Math. Scand.* **43**(1978), 99–118.
- [17] U. HAAGERUP, On the dual weight for crossed-product of von Neumann algebras. II, *Math. Scand.* **43**(1978), 119–140.
- [18] U. HAAGERUP, Operator valued weights in von Neumann algebras. I, *J. Funct. Anal.* **32**(1979), 175–206.
- [19] U. HAAGERUP, Operator valued weights in von Neumann algebras. II, *J. Funct. Anal.* **33**(1979), 339–361.
- [20] C. HOUDAYER, Y. ISONO, Unique prime factorization and bicentralizer problem for a class of type III factors, *Adv. Math.* **305**(2017), 402–455.
- [21] C. HOUDAYER, S. RAUM, Locally compact groups acting on trees, the type I conjecture and non-amenable von Neumann algebras, arXiv:1610.00884 [math.GR].
- [22] M. IZUMI, R. LONGO, S. POPA, A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras, *J. Funct. Anal.* **155**(1998), 25–63.
- [23] V.F.R. JONES, Index for subfactors, *Invent. Math.* **72**(1983), 1–25.

- [24] H. KOSAKI, Extension of Jones theory on index to arbitrary factors, *J. Funct. Anal.* **66**(1986), 123–140.
- [25] D. KYED, H. DENSING-PEDERSEN, S. VAES, L^2 -Betti numbers of locally compact groups and their cross section equivalence relations, *Trans. Amer. Math. Soc.* **367**(2015), 4917–4956.
- [26] R. LONGO, K. REHREN, Nets of subfactors, *Rev. Math. Phys.* **7**(1995), 567–597.
- [27] L.H. LOOMIS, *An introduction to Abstract Harmonic Analysis*, Van Nostrand, New-York 1953.
- [28] F. MURRAY, J. VON NEUMANN, Rings of operators. I, *Ann. of Math.* **37**(1936), 116–229.
- [29] Y. NAKAGAMI, M. TAKESAKI, *Duality for Crossed Products of von Neumann Algebras*, Lecture Notes in Math., vol. 731, Springer, Berlin 1979.
- [30] R. PALMA, Crossed products by Hecke pairs. I. $*$ -completions, *Mem. Amer. Math. Soc.*, to appear, arXiv:1212.5756 [math.OA].
- [31] S. POFA, Symmetric enveloping algebras, amenability and AFD properties for subfactors, *Math. Res. Lett.* **1**(1994), 409–425.
- [32] J.-L. SAUVAGEOT, Sur le type du produit croisé d’une algèbre de von Neumann par un groupe localement compact, *Bull. Soc. Math. France* **105**(1977), 349–368.
- [33] M. TAKESAKI, Duality for crossed products and the structure of von Neumann algebras of type III, *Acta Math.* **131**(1973), 249–310.
- [34] M. TAKESAKI, *Theory of Operator Algebras*. II, Encyclopaedia Math. Sci., vol. 125, Oper. Algebras Non-commutative Geometry, vol. 6, Springer-Verlag, Berlin 2003.
- [35] R. TOMATSU, A Galois correspondence for compact quantum group actions, *J. Reine Angew. Math.* **633**(2009), 165–182.
- [36] K. TZANEV, Hecke C^* -algebras and amenability, *J. Operator Theory* **50**(2003), 169–178.
- [37] S. VAES, The unitary implementation of a locally compact quantum group action, *J. Funct. Anal.* **180**(2001), 426–480.
- [38] S. VAES, Strictly outer actions of groups and quantum groups, *J. Reine Angew. Math.* **578**(2005), 147–184.

RÉMI BOUTONNET, INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UNIVERSITÉ DE BORDEAUX, 351 COURS DE LA LIBÉRATION, 33 405 TALENCE CEDEX, FRANCE
E-mail address: remi.boutonnet@math.u-bordeaux.fr

ARNAUD BROTHIER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROME TOR VERGATA, VIA DELLA RICERCA SCIENTIFICA 00133 ROMA, ITALY
E-mail address: arnaud.brothier@gmail.com

Received January 19, 2017.