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CONTINUOUS PERTURBATIONS OF NONCOMMUTATIVE EUCLIDEAN SPACES AND TORI

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ABSTRACT. We prove the existence of the $\mathrm{Lip}^{1/2}$ continuous Moyal deformation of Euclidean plane. It is a noncompact version of Haagerup and Rørdam's result about continuous paths of the rotation C^*-algebras. Moveover, our construction is generalized to noncommutative Euclidean spaces of dimension $d \geqslant 2$. As a corollary, we extend Haagerup and Rørdam's result to noncommutative d-tori.

KEYWORDS: *Moyal deformation, noncommutative Euclidean space, noncommutative tori.*

MSC (2010): 46L05.

1. INTRODUCTION

The celebrated Heisenberg commutation relation,

$$PQ - QP = -\mathrm{i}I,$$

where I is the identity operator, plays an important role in quantum mechanics and the related mathematics. This commutation relation affiliates to the Moyal deformation of Euclidean plane. Let $d \geqslant 2$ and $\theta = (\theta_{jk})_{j,k=1}^{d}$ be a real skew-symmetric $d \times d$-matrix. The associated noncommutative Euclidean space (for a nonsingular θ) is given by d one-parameter unitary groups $u_1(t), u_2(t), \ldots, u_d(t)$ satisfying the following commutation relations

$$u_j(s)u_k(t) = \mathrm{e}^{\mathrm{i}st\theta_{jk}}u_k(t)u_j(s), \quad \forall s, t \in \mathbb{R},$$

for $j, k = 1, 2, \ldots, d$. The noncommutative Euclidean space, also called Moyal plane, is a prototype of noncompact noncommutative manifolds (see e.g. [10]). Moreover, interesting objects and structures from quantum physics have been studied on the noncommutative plane and noncommutative $\mathbb{R}^4$ (see e.g. [17], [18], [24]).

Another class of fundamental examples in noncommutative geometry are the noncommutative tori. They have been extensively studied over decades (we refer to the survey paper by Rieffel [21] for the study before 90s, and [3], [5], [8] for more recent development). Recall that the noncommutative d-torus A_θ^d associated to θ is the universal C^*-algebra generated by d unitaries $u_1, u_2, \dots, u_d$ subject to the commutation relations

$$u_j u_k = \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} u_k u_j, \quad j,k = 1,2,\dots,d.$$

It is clear from the definition that A_θ^d is a noncommutative deformation of $C(\mathbb{T}^d)$, the C^*-algebra of continuous functions on a usual d-torus ($\theta = 0$). When $d = 2$, the commutation relations reduce to two unitaries u, v satisfying

$$uv = \mathrm{e}^{2\pi \mathrm{i}\theta} vu$$

for a real number θ. The noncommutative 2-tori are also called rotation C^*-algebras (cf. [6]).

In this paper, we will consider generalizations of the following result by Haagerup and Rørdam in [14].

THEOREM A. *Let H be an infinite dimensional Hilbert space and $U(H)$ be its unitary group. There exist two continuous paths $u, v : [0,1] \to U(H)$ and a universal constant $C > 0$ such that $u(0) = u(1), v(0) = v(1)$, and*

(i) $u(\theta)v(\theta) = \mathrm{e}^{2\pi \mathrm{i}\theta} v(\theta)u(\theta)$;

(ii) $\max\{\|u(\theta) - u(\theta')\|, \|v(\theta) - v(\theta')\|\} \leqslant C|\theta - \theta'|^{1/2}$;

for all $\theta, \theta' \in [0,1]$.

It was shown by Elliott in [7] that the family of rotation algebras forms a continuous field of C^*-algebra (see e.g. [9] for the definition). The above theorem gives this family a continuous embedding in $B(H)$. Kirchberg and Phillips in [16] also obtained a $\mathrm{Lip}^{1/2}$-continuous embedding of rotation algebras into the Cuntz algebra $\mathcal{O}_2$. The existence of $\mathrm{Lip}^{1/2}$-continuous paths has applications in estimating the spectrum of almost Mathieu operators (see [2]).

It is natural to expect that the similar $\mathrm{Lip}^{1/2}$-continuous embedding exists for high dimensional noncommutative tori. Both Theorem 5.7 of [16] and Theorem 3.2 of [1] prove the existence of the embedding with continuity in norm, but with little information about the concrete continuity. We confirm that the $\mathrm{Lip}^{1/2}$-continuous embedding also exist for noncommutative d-tori of dimension $d > 2$. Let us denote $\mathcal{A}[d] \equiv [0,1]^{(d-1)d/2}$ as the space of all skew-symmetric $d \times d$ matrices with entries in the unit interval.

THEOREM 1.1. *There exist d continuous maps $u_1, u_2, \dots, u_d : \mathcal{A}[d] \to U(H)$ and a universal constant $C > 0$ such that:*

(i) $u_j u_k = \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}} u_k u_j, j,k = 1,2,\dots,d$;

(ii) $\|u_j(\theta) - u_j(\theta')\| \leqslant C\Big(\sum\limits_{1 \leqslant k \leqslant d} |\theta_{jk} - \theta'_{jk}|^{1/2}\Big), j = 1,2,\dots,d$;

for all $\theta, \theta' \in \mathcal{A}[d]$.

The proof is based on an explicit construction, which we illustrate here for the case $d = 3$. Given the two continuous paths u, v from Theorem A, we define the following maps $u_1, u_2, u_3 : \mathcal{A}[3] \to U(H^{\otimes 3})$,

$$u_1(\theta) = u(\theta_{12}) \otimes u(\theta_{13}) \otimes I, \quad u_2(\theta) = v(\theta_{12}) \otimes I \otimes u(\theta_{23}),$$

$$u_3(\theta) = I \otimes v(\theta_{13}) \otimes v(\theta_{23}). \tag{1.1}$$

Because each pair of operators only shares one nontrivial tensor component (other than the identity), u_1, u_2 and u_3 satisfy the commutation relations

$$u_1(\theta)u_2(\theta) = e^{2\pi i \theta_{12}} u_2(\theta)u_1(\theta), \quad u_1(\theta)u_3(\theta) = e^{2\pi i \theta_{13}} u_3(\theta)u_1(\theta),$$
$$u_2(\theta)u_3(\theta) = e^{2\pi i \theta_{23}} u_3(\theta)u_2(\theta).$$

By induction, this construction can be generalized to higher dimension and the $\mathrm{Lip}^{1/2}$-continuity follows from the triangle inequality. It also works for the paths in the Cuntz algebra $\mathcal{O}_2$ and implies a continuous embedding into $\mathcal{O}_2$ because $\mathcal{O}_2 \otimes \mathcal{O}_2 = \mathcal{O}_2$ (see [16]).

We also prove the "noncompact" analog of the above results. It is proved in [14] that for an infinite multiplicity representation (P, Q) of the Heisenberg relation, there exists a commuting pair of self-adjoint operators (P_0, Q_0) on H such that $P - P_0$ and $Q - Q_0$ are bounded. This bounded perturbation of unbounded operators is used in the construction of continuous path of rotation algebras. We find that the methods of Haagerup and Rørdam, with careful modifications, also applies to the Heisenberg relation, to construct a continuous Moyal deformation of $\mathbb{R}^2$. Moreover, using the same idea of (1.1), this can be generalized to noncommutative Euclidean space of dimension $d > 2$.

Denote $\mathcal{A}(d) \equiv \mathbb{R}^{d(d-1)/2}$ as the space of all real skew-symmetric $d \times d$-matrices. Our main result can be stated as follows.

THEOREM 1.2. *There exist continuous maps* $u_1, u_2, \dots, u_d : \mathcal{A}(d) \times \mathbb{R} \to U(H)$ *and a universal constant* $C > 0$ *such that:*

(i) *for each* θ, $u_1(\theta, \cdot), u_2(\theta, \cdot), \dots, u_d(\theta, \cdot)$ *are strongly continuous one-parameter unitary groups satisfying*

$$u_j(\theta, s)u_k(\theta, t) = e^{ist\theta_{jk}} u_k(\theta, t)u_j(\theta, s), \quad \forall s, t \in \mathbb{R}, j, k = 1, \dots, d;$$

(ii) *for any* $t \in \mathbb{R}$ *and* $\theta, \theta' \in \mathcal{A}(d)$,

$$\|u_j(\theta, t) - u_j(\theta', t)\| \leqslant C|t|\Big(\sum_k |\theta_{kj} - \theta'_{kj}|^{1/2}\Big), \quad j = 1, \dots, d.$$

The present work is organized as follows. In Section 2, we apply the method of Haagerup and Rørdam to construct a continuous deformation of the Heisenberg relations. Section 3 extends both the bounded perturbation and the continuous deformation to noncommutative Euclidean space of higher dimensions. Section 4 is devoted to corresponding results for noncommutative d-tori.

2. CONTINUOUS PERTURBATION OF HEISENBERG RELATIONS

We first discuss Theorem 1.2 for $d = 2$ by the method of Haagerup and Rørdam in [14]. In this section, θ always denotes a real number. Let P and Q be two (unbounded) self-adjoint operators on a Hilbert space H and $u(s) = \mathrm{e}^{\mathrm{i}Ps}, v(t) = \mathrm{e}^{\mathrm{i}Qt}$ be their associated one-parameter groups. For a nonzero θ, we say P and Q satisfy *the Heisenberg relation with parameter* θ

$$[P,Q] = PQ - QP = -\mathrm{i}\theta I, \tag{2.1}$$

if $u(s), v(t)$ satisfy the Weyl relation

$$u(s)v(t) = \mathrm{e}^{\mathrm{i}st\theta} v(t)u(s). \tag{2.2}$$

When $\theta = 1$, we call (2.1) *the standard Heisenberg relation*. Thanks to the well-known Stone–von Neumann theorem (cf. pp. 285–287 of [15]), any representation of the standard Heisenberg relation is unitarily equivalent to a (finite or infinite) multiple of the Schrödinger picture. More precisely, the only irreducible representation, up to a unitary equivalence, is given by the momentum operator and position operator from quantum mechanics

$$Pf = -\mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}x}, \quad (Qf)(x) = xf(x), \quad f \in C_c^1(\mathbb{R}).$$

Both P, Q are unbounded self-adjoint operators on the Hilbert space $L_2(\mathbb{R})$ and they have a common core of $C_c^1(\mathbb{R})$ (continuously differentiable compactly supported functions). The associated one-parameter unitary groups are given by

$$(u(s)f)(x) = f(x+s), \quad (v(t)f)(x) = \mathrm{e}^{\mathrm{i}xt} f(x), \tag{2.3}$$

which satisfy (2.2). Note that (2.1) implies that $(\frac{1}{\theta}P, Q)$ satisfies the standard Heisenberg relation, so the Stone–von Neumann theorem is easily generalized for any nonzero θ. When $\theta = 0$, the one-parameter groups commute

$$\mathrm{e}^{\mathrm{i}Ps}\mathrm{e}^{\mathrm{i}Qt} = \mathrm{e}^{\mathrm{i}Qt}\mathrm{e}^{\mathrm{i}Ps},$$

and we say P and Q commute *strongly*. Strongly commuting pairs (P, Q) are one-to-one corresponding to unitary representations of $\mathbb{R}^2$. In particular, the left regular group representation of $\mathbb{R}^2$ is given by

$$Pf = -\mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}x}, \quad Qf = -\mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}y} \tag{2.4}$$

as unbounded self-adjoint operators on $L_2(\mathbb{R}^2)$ and the unitaries are translations,

$$(u(t)f)(x,y) = f(x+t,y), \quad (v(t)f)(x,y) = f(x,y+t), \quad f \in L_2(\mathbb{R}^2).$$

We will combine the discussions of Section 3 and 5 from [14], working on the unbounded operators (P, Q) instead of unitaries. Let us begin with a modification of Theorem 3.1 of [14].

THEOREM 2.1. *Let $\theta \neq 0$. Let (P, Q) be a representation of $[P, Q] = -\mathrm{i}\theta I$ with infinite multiplicity on a separable Hilbert space H. Then for any $\theta' \in \mathbb{R}$, there exist self-adjoint operators P' and Q' on H satisfying $[P', Q'] = -\mathrm{i}\theta' I$ such that $P - P'$ and $Q - Q'$ are bounded and moreover,*

$$\max\{\|P - P'\|, \|Q - Q'\|\} \leqslant 9|\theta - \theta'|^{1/2}.$$

Proof. We may first assume $\theta' > \theta$ and denote $\delta = |\theta' - \theta|^{1/2}$. Let K be an infinite dimensional separable Hilbert space. Because all infinite multiplicity representations of the Heisenberg relation on a separable Hilbert space are unitarily equivalent, we may assume that (P, Q) is given by

$$P = -\mathrm{i}\delta\frac{\partial}{\partial x}, \quad Q = -\mathrm{i}\frac{\partial}{\partial y} + \frac{\theta}{\delta}x,$$

on $L_2(\mathbb{R}^2, K)$. The associated one-parameter groups $u(t) = \mathrm{e}^{\mathrm{i}Pt}, v(t) = \mathrm{e}^{\mathrm{i}Qt}$ are

$$(u(t)f)(x, y) = f(x + \delta t, y), \quad (v(t)f)(x, y) = \mathrm{e}^{\mathrm{i}(\theta/\delta)xt} f(x, y + t).$$

Let $w : \mathbb{R}^2 \to U(K)$ be a C^1-function with values in the unitary group $U(K)$ of K. It can be regarded as a unitary on $L_2(\mathbb{R}^2, K)$ via pointwise action

$$(wf)(x, y) = w(x, y)f(x, y).$$

The subspace $C^1_{\mathrm{c}}(\mathbb{R}^2, K)$ is a common core of P, Q and also invariant under w. Then for $f \in C^1_{\mathrm{c}}(\mathbb{R}^2, K)$,

$$(w^*Pwf)(x, y) = -\mathrm{i}\delta\Big(\frac{\partial f}{\partial x}(x, y) + \mathrm{i}w(x, y)^*\frac{\partial w}{\partial x}(x, y)f(x, y)\Big),$$

$$(w^*Qwf)(x, y) = -\mathrm{i}\Big(\frac{\partial f}{\partial y}(x, y) + \mathrm{i}w(x, y)^*\frac{\partial w}{\partial y}(x, y)f(x, y)\Big) + \frac{\theta}{\delta}xf(x, y). \tag{2.5}$$

It is proved in Theorem 3.1 of [14] that there exists a C^1-function $w : \mathbb{R}^2 \to U(K)$ such that

$$\sup_{(x,y)\in\mathbb{R}^2}\Big\|\frac{\partial w}{\partial x}(x, y)\Big\| \leqslant 9, \qquad \sup_{(x,y)\in\mathbb{R}^2}\Big\|\frac{\partial w}{\partial y}(x, y) - \mathrm{i}xw(x, y)\Big\| \leqslant 9. \tag{2.6}$$

Set $\overline{w}(x, y) = w(x, \delta y)$, and choose the self-adjoint operators $P' = \overline{w}P\overline{w}^*, Q' = \overline{w}(Q + \frac{\theta'-\theta}{\delta}x)\overline{w}^*$. The pair (P', Q') satisfies $[P', Q'] = -\mathrm{i}\theta' I$ and also shares the common core $C^1_{\mathrm{c}}(\mathbb{R}^2, K)$. On this dense domain $C^1_{\mathrm{c}}(\mathbb{R}^2, K)$

$$P - P' = \overline{w}(\overline{w}^*P\overline{w} - P)\overline{w}^* = \mathrm{i}\overline{w}\Big(\delta\overline{w}^*\frac{\partial\overline{w}}{\partial x}\Big)\overline{w}^* = \mathrm{i}\delta\frac{\partial\overline{w}}{\partial x}\overline{w}^*,$$

$$Q - Q' = \overline{w}(\overline{w}^*Q\overline{w} - Q - \delta x)\overline{w}^* = \mathrm{i}\overline{w}\Big(\overline{w}^*\frac{\partial\overline{w}}{\partial y} - \delta x\Big)\overline{w}^* = \mathrm{i}\Big(\frac{\partial\overline{w}}{\partial y}\overline{w}^* - \delta x\Big).$$

Both are bounded because for any $(x,y) \in \mathbb{R}^2$,

$$\Big\| -\mathrm{i}\delta \frac{\partial \overline{w}}{\partial x}(x,y)\overline{w}(x,y)^* \Big\| = \Big\| \delta \frac{\partial w}{\partial x}(x,\delta y) \Big\| \leqslant 9\delta,$$

$$\Big\| -\mathrm{i} \frac{\partial \overline{w}}{\partial y}(x,y)\overline{w}^*(x,y) - \delta x \Big\| = \Big\| \delta \Big(\frac{\partial w}{\partial y}(x,\delta y) - \mathrm{i}xw(x,\delta y) \Big) \Big\| \leqslant 9\delta.$$

For $\theta > \theta'$, the estimates follow similarly by taking $\overline{w}(x,y) = \overline{w}(x,-\delta y)$. ∎

REMARK 2.2. The pair (P', Q') gives a representation of infinite multiplicity. In particular when $\theta' = 0$, P' and Q' strongly commute and are unitarily equivalent to an infinite multiple of regular representations (2.4). Conversely, the above theorem remains valid for $\theta = 0$ if in addition (P, Q) is unitarily equivalent to the regular representation.

Stone's theorem states that self-adjoint operators on a Hilbert space H are one-to-one correspondent to one-parameter unitary groups in $B(H)$. The next proposition shows that this correspondence is of a certain continuity.

PROPOSITION 2.3. *Let P and P' be (possibly unbounded) self-adjoint operators on a Hilbert space H. Then their domains $D(P)$ and $D(P')$ coincide and $P - P'$ is bounded with its norm less than a constant $C > 0$ if and only if $\|\mathrm{e}^{\mathrm{i}Pt} - \mathrm{e}^{\mathrm{i}P't}\| \leqslant C|t|$ for any $t \in \mathbb{R}$.*

Proof. The necessity is Lemma 4.3 of [14]. Here we prove the sufficiency. For $\xi \in D(P)$ and $\eta \in D(P')$, it follows by Stone's theorem that

$$\lim_{t\to 0} \frac{\mathrm{e}^{\mathrm{i}Pt}\xi - \xi}{t} = \mathrm{i}P\xi, \quad \lim_{t\to 0} \frac{\mathrm{e}^{\mathrm{i}P't}\eta - \eta}{t} = \mathrm{i}P'\eta$$

converge strongly. Then the derivative of the inner product $\langle \mathrm{e}^{\mathrm{i}Pt}\xi, \mathrm{e}^{\mathrm{i}P't}\eta \rangle$ at $t = 0$ is given by

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t}(\langle \mathrm{e}^{\mathrm{i}Pt}\xi, \mathrm{e}^{\mathrm{i}P't}\eta\rangle - \langle \xi,\eta\rangle) &= \lim_{t\to 0} \frac{1}{t}(\langle \mathrm{e}^{\mathrm{i}Pt}\xi, \mathrm{e}^{\mathrm{i}P't}\eta\rangle - \langle \mathrm{e}^{\mathrm{i}Pt}\xi,\eta\rangle) + \frac{1}{t}(\langle \mathrm{e}^{\mathrm{i}Pt}\xi,\eta\rangle - \langle \xi,\eta\rangle) \\ &= \langle \xi, \mathrm{i}P'\eta\rangle + \langle \mathrm{i}P\xi, \eta\rangle.\end{aligned}$$

On the other hand,

$$\|\mathrm{e}^{-\mathrm{i}P't}\mathrm{e}^{\mathrm{i}Pt} - 1\| = \|\mathrm{e}^{\mathrm{i}Pt} - \mathrm{e}^{\mathrm{i}P't}\| \leqslant C|t|, \quad t \in \mathbb{R}$$

by assumptions. This implies

$$\langle \mathrm{e}^{\mathrm{i}Pt}\xi, \mathrm{e}^{\mathrm{i}P't}\eta\rangle - \langle \xi,\eta\rangle = \langle (\mathrm{e}^{-\mathrm{i}P't}\mathrm{e}^{\mathrm{i}Pt} - 1)\xi, \eta\rangle \leqslant Ct\|\xi\|\,\|\eta\|.$$

Therefore

$$|\langle \xi, \mathrm{i}P'\eta\rangle + \langle \mathrm{i}P\xi,\eta\rangle| \leqslant C\|\xi\|\,\|\eta\|, \quad |\langle \xi, P'\eta\rangle| \leqslant (C\|\xi\| + \|P\xi\|)\|\eta\|. \tag{2.7}$$

Since P' is self-adjoint, we have $\xi \in D(P'^*) = D(P')$. Now we are able to rewrite (2.7) to obtain

$$|\langle (P - P')\xi, \eta\rangle| \leqslant C\|\xi\|\,\|\eta\|$$

for all $\xi \in D(P), \eta \in D(P')$. Since $D(P)$ and $D(P')$ are dense in H, $\|P - P'\| \leqslant C$ and P, P' have the same domain. Note that for sufficiency we only use

$$\|e^{iPt} - e^{iP't}\| \leqslant C|t|, \quad t \in [0, \varepsilon]$$

for some $\varepsilon > 0$. ∎

The following is Lemma 5.1 of [14], which is used as a key tool in the construction of continuous paths. We omit its proof here.

LEMMA 2.4. *Let $M \subset B(H)$ be a von-Neumann algebra with properly infinite commutant M'. For a unitary $u \in M$, there exists a smooth path $u(t), t \in [0,1]$ of unitary, such that:*

(i) $u(0) = 1$ *and* $u(1) = u$;
(ii) $\|u'(t)\| \leqslant 9$;
(iii) $\|[u(t), a]\| \leqslant 4\|[u, a]\|$;
(iv) $\|[u'(t), a]\| \leqslant 9\|[u, a]\|$;
(v) $\|\frac{d}{dt}u(t)au(t)^*\| \leqslant 45\|[u, a]\|$;

for all $t \in [0,1]$ and $a \in M$.

The next lemma is an analog of Lemma 5.2 of [14].

LEMMA 2.5. *Let $\theta \neq \theta'$ both be nonzero and $k \in \mathbb{N}$ be given. Let $(P(\theta), Q(\theta))$ (respectively $(P(\theta'), Q(\theta'))$) be a representation of $[P, Q] = -i\theta I$ (respectively $[P, Q] = -i\theta' I$) of infinite multiplicity on a separable Hilbert space H. Denote the associated one-parameter unitary groups as*

$$u_0(t) = e^{iP(\theta)t}, \quad v_0(t) = e^{iQ(\theta)t}, \quad u_1(t) = e^{iP(\theta')t}, \quad v_1(t) = e^{iQ(\theta')t}.$$

Assume that the commutant of $\{u_0(t), u_1(t), v_0(t), v_1(t) : t \in \mathbb{R}\}$ is properly infinite and $P(\theta) - P(\theta')$ and $Q(\theta) - Q(\theta')$ are bounded. Denote

$$d = \max\{\|P(\theta) - P(\theta')\|, \|Q(\theta) - Q(\theta')\|\},$$

and set

$$s_j = \theta + \frac{j}{k}(\theta' - \theta), \quad j = 0, 1, \ldots, k$$

so that $s_0 = \theta, s_k = \theta'$. Then there exist pairs $(P(s_j), Q(s_j)), j = 1, 2, \ldots, k-1$, of self-adjoint operators on H such that:

(i) *$(P(s_j), Q(s_j))$ satisfies the Heisenberg relation $[P(s_j), Q(s_j)] = -is_j I$;*
(ii) *$P(s_j) - P(s_{j+1})$ and $Q(s_j) - Q(s_{j+1})$ are bounded and*

$$\max\{\|P(s_j) - P(s_{j+1})\|, \|Q(s_j) - Q(s_{j+1})\|\} \leqslant 1224\Big(\frac{|\theta - \theta'|}{k}\Big)^{1/2} + 45\frac{d}{k};$$

(iii) *the commutant of the one-paramter groups $\{e^{iP(s_0)t}, \ldots, e^{iP(s_k)t}, e^{iQ(s_0)t}, \ldots, e^{iQ(s_k)t}\}$ is properly infinite.*

Proof. We decompose $H = H_1 \otimes H_2 \otimes H_3$ as a tensor product of three infinite dimensional Hilbert spaces. Moreover we may assume the four one-parameter groups $\{u_0(t), u_1(t), v_0(t), v_1(t)\}$ are in the subalgebra $B(H_1) \otimes \mathbb{C}I_{B(H_2)} \otimes \mathbb{C}I_{B(H_3)}$, since the commutant of $\{u_0(t), u_1(t), v_0(t), v_1(t)\}$ is properly infinite. Also, $P(\theta), Q(\theta), P(\theta')$ and $Q(\theta')$ can be regarded as operators on H_1 by identifying operators P with $P \otimes 1_{B(H_2)} \otimes 1_{B(H_3)}$.

Denote $\delta = (\frac{|\theta'-\theta|}{k})^{1/2}$ and set $\overline{P}(s_0) = P(\theta), \overline{Q}(s_0) = Q(\theta)$. We can apply Theorem 2.1 inductively to obtain k pairs of self-adjoint operators $(\overline{P}(s_j), \overline{Q}(s_j))$ on H_1 satisfying (i) and

$$\max\{\|\overline{P}(s_j) - \overline{P}(s_j+1)\|, \|\overline{Q}(s_j) - \overline{Q}(s_j+1)\|\} \leqslant 9\delta. \tag{2.8}$$

By the assumption on d and the triangle inequality, we have

$$\max\{\|\overline{P}(s_k) - P(\theta')\|, \|\overline{Q}(s_k) - Q(\theta')\|\} \leqslant 9k\delta + d. \tag{2.9}$$

Note that for $s_k = \theta'$, both pairs $(\overline{P}(s_k), \overline{Q}(s_k))$ and $(P(\theta'), Q(\theta'))$ are infinite multiplicity representations of the Heisenberg relation with the nonzero parameter θ'. Then $P(\theta') = WP(s_k)W^*, Q(\theta') = WQ(s_k)W^*$ for some unitary $W \in B(H_1) \otimes \mathbb{C}1 \otimes \mathbb{C}1$. Thus (2.9) implies

$$\max\{\|\overline{P}(s_k) - W\overline{P}(s_k)W^*\|, \|\overline{Q}(s_k) - W\overline{Q}(s_k)W^*\|\} \leqslant 9k\delta + d.$$

Denote $\overline{u}_{s_j}(t) = \mathrm{e}^{t\overline{P}(s_k)\mathrm{i}}, \overline{v}_{s_j}(t) = \mathrm{e}^{t\overline{Q}(s_k)\mathrm{i}}$. By Proposition 2.3,

$$\max\{\|[w, \overline{u}_{s_k}(t)]\|, \|[w, \overline{v}_{s_k}(t)]\|\} \leqslant (9k\delta + d)|t|, \quad t \in \mathbb{R}.$$

For any $j = 0, 1, \ldots, k$, by (2.8) and the triangle inequality we have

$$\max\{\|[w, \overline{u}_{s_j}(t)]\|, \|[w, \overline{v}_{s_j}(t)]\|\} \leqslant (27k\delta + d)|t|, \quad t \in \mathbb{R}.$$

All of the operators above are in the subalgebra $B(H_1) \otimes \mathbb{C}I \otimes \mathbb{C}I$, which is of properly infinite commutant inside $B(H_1) \otimes B(H_2) \otimes \mathbb{C}I$. Hence we can apply Lemma 2.4 for W, to obtain a path of unitary $W : [0,1] \to B(H_1) \otimes B(H_2) \otimes \mathbb{C}1$ such that $W(0) = I, W(1) = W$, and

$$\max\left\{\left\|\frac{\mathrm{d}}{\mathrm{d}s} W(s) u_{s_j}(t) W^*(s)\right\|, \left\|\frac{\mathrm{d}}{\mathrm{d}s} W(s) v_{s_j}(t) W^*(s)\right\|\right\} \leqslant 45(27k\delta + d)|t|,$$

for all $j = 0, 1, \ldots, k$ and $t \in \mathbb{R}$. Now for each j, set

$$u_{s_j}(t) = W\Big(\frac{j}{k}\Big)\overline{u}_{s_j}(t)W^*\Big(\frac{j}{k}\Big), \quad v_{s_j}(t) = W\Big(\frac{j}{k}\Big)\overline{v}_{s_j}(t)W^*\Big(\frac{j}{k}\Big),$$

and self-adjoint operators $P(s_j), Q(s_j)$ for the associated infinitesimal generators. We claim that this gives the desired construction.

First, $P(s_0) = P(\theta), Q(s_0) = Q(\theta)$ and $P(s_k) = P(\theta'), Q(s_k) = Q(\theta')$. Each pair $(P(s_j), Q(s_j))$ satisfies the commutation relations (i). Moreover,

$$\begin{aligned}\|u_{s_j}(t) - u_{s_{j+1}}(t)\| &\leqslant \Big\|W\Big(\frac{j+1}{k}\Big)(\overline{u}_{s_{j+1}}(t) - \overline{u}_{s_j}(t))W^*\Big(\frac{j+1}{k}\Big)\Big\| \\ &\quad + \Big\|\int_{j/k}^{(j+1)/k} \frac{\mathrm{d}}{\mathrm{d}s}(W(s)\overline{u}_{s_j}(t)W^*(s))\mathrm{d}s\Big\| \\ &\leqslant 9\delta|t| + 45\Big(27\delta + \frac{d}{k}\Big)|t| = \Big(1224\delta + 45\frac{d}{k}\Big)|t|,\end{aligned}$$

and the same bound holds for $\|v_{s_{j+1}}(t) - v_{s_j}(t)\|$. By Lemma 2.3, we obtain that

$$\max\{\|P(s_j) - P(s_{j+1})\|, \|Q(s_j) - Q(s_{j+1})\|\} \leqslant 1224\delta + 45\frac{d}{k}.$$

Finally, all unitary groups $u_{s_j}(t), v_{s_j}(t)$ belong to $B(H_1) \otimes B(H_2) \otimes \mathbb{C}1$ and hence the commutant is properly infinite. ∎

REMARK 2.6. Note that for $(P(\theta), Q(\theta))$, we only use the fact that Theorem 2.1 applies. Hence the above theorem remains valid if θ is 0 and $(P(\theta), Q(\theta))$ is unitarily equivalent to an infinite multiple of left regular representation. This point is used in the next theorem.

Let us denote by $S(H)$ the set of all self-adjoint operators on the Hilbert space H. Based on the above lemma, we construct maps $P, Q : \mathbb{R} \to S(H)$ with continuously bounded perturbation. The next theorem is an analog of Lemma 5.3 and Theorem 5.4 of [14] for Heisenberg relations.

THEOREM 2.7. *Let H be an infinite dimensional Hilbert space. Then there exist maps $P, Q : \mathbb{R} \to S(H)$ and a universal constant $C > 0$ such that for all $\theta, \theta' \in \mathbb{R}$,*

(i) $[P(\theta), Q(\theta)] = -\mathrm{i}\theta I$;

(ii) *$P(\theta) - P(\theta')$ and $Q(\theta) - Q(\theta')$ are bounded on H and moreover,*

$$\max\{\|P(\theta) - P(\theta')\|, \|Q(\theta) - Q(\theta')\|\} \leqslant C|\theta - \theta'|^{1/2}. \tag{2.10}$$

Proof. Set $k = 8100$ and $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$, where

$$\Gamma_n = \Big\{\frac{j}{k^n} : j \in \mathbb{Z}, |j| \leqslant (n+1)k^n\Big\}.$$

Write $H = H_1 \otimes H_2$ where both H_1 and H_2 are infinite dimensional. Let K be a separable infinite dimensional Hilbert space. We may assume $H_1 = L_2(\mathbb{R}^2, K)$ and define the map P, Q in $S(L_2(\mathbb{R}^2, K))$ for all integers $j \in \mathbb{Z}$ as follows

$$(P(0)f)(x,y) = -\mathrm{i}\frac{\partial f}{\partial x}(x,y), \quad (Q(0)f)(x,y) = -\mathrm{i}\frac{\partial f}{\partial y}(x,y), \quad f \in C_{\mathrm{c}}^1(\mathbb{R}^2, K),$$

$$P(j+1) = w^*P(j)w, \quad Q(j+1) = w^*(Q(j) + x)w,$$

where $w \in U(L_2(\mathbb{R}^2, K))$ is the unitary operator described in (2.6). Theorem 2.1 implies

$$\max\{\|P(j) - P(j+1)\|, \|Q(j) - Q(j+1)\|\} \leqslant 9.$$

Now identify $P(j)$ and $Q(j)$ with their amplifications $P(j) \otimes I$ and $Q(j) \otimes I$ on $H_1 \otimes H_2$. Denote that $u(\theta, t) = \mathrm{e}^{\mathrm{i}P(\theta)t}$ and $v(\theta, t) = \mathrm{e}^{\mathrm{i}Q(\theta)t}$. Then $\{(P(j), Q(j)), j = -1, 0, 1\}$ defines the map on Γ_0 satisfying condition (i) and (ii) for constant $C' = 2500$, and

(iii) *the commutant of* $\{u(\theta, t), v(\theta, t) : \theta \in \Gamma_n\}$ *is properly infinite.*

Now assume that the maps P, Q are defined on Γ_n with conditions (i), (ii) and (iii) satisfied. For the induction step, we first add two integer points $\theta = \pm(n+2)$, and then apply the Lemma 2.5 to the subintervals $[\frac{j}{k^n}, \frac{j+1}{k^n}]$ (note that at $j = 0$, $(P(0), Q(0))$ is the left regular representation) and unit intervals $[-(n+2), -(n+1)]$, $[n+1, n+2]$. In particular, for these two intervals of unit length, we apply the Lemma 2.5 of k-division $n+1$ times. Thus we extend the maps P, Q to Γ_{n+1} with (i), (ii) and (iii) still satisfied. Indeed, for (ii),

$$\begin{aligned}
\max\Big\{\Big\|P\Big(\frac{j}{k^{n+1}}\Big) - P\Big(\frac{j+1}{k^{n+1}}\Big)\Big\|, \Big\|Q\Big(\frac{j}{k^{n+1}}\Big) - Q\Big(\frac{j+1}{k^{n+1}}\Big)\Big\|\Big\}& \\
\leqslant 1224 k^{-(n+1)/2} + 45\frac{2500}{k}k^{-n/2}& \\
= \Big(1224 + 2500\frac{45}{\sqrt{k}}\Big)k^{-(n+1)/2} \leqslant 2500 k^{-(n+1)/2},&
\end{aligned}$$

where the last inequality follows from that $\sqrt{k} = 90$. Thus by induction, we construct the maps P, Q on Γ.

Finally, we extend P, Q from the dense subset Γ to $\mathbb{R}$. By Lemma 2.3, the one-parameter unitary groups $u(\theta, t) = \mathrm{e}^{\mathrm{i}P(\theta)t}, v(\theta, t) = \mathrm{e}^{\mathrm{i}Q(\theta)t}$ satisfy that

$$\max\{\|u(\theta, t) - u(\theta', t)\|, \|v(\theta, t) - v(\theta', t)\|\} \leqslant 2500|t| \cdot |\theta - \theta'|^{1/2}, \quad t \in \mathbb{R}, \theta \in \Gamma,$$

For a fixed t, we can continuously extend $u(\cdot, t), v(\cdot, t)$ for all $\theta \in \mathbb{R}$. It can be proved by the same argument of Theorem 5.4 in [14] that this extension is again $\mathrm{Lip}^{1/2}$-continuous, but with a larger constant $C = 320 \cdot 2500|t| = 800{,}000\ |t|$. Namely, for all $t, \theta, \theta' \in \mathbb{R}$, our extension satisfies

$$\max\{\|u(\theta, t) - u(\theta', t)\|, \|v(\theta, t) - v(\theta', t)\|\} \leqslant 800{,}000|t| \cdot |\theta - \theta'|^{1/2}. \tag{2.11}$$

The continuity implies that for each θ, $u(\theta, t)$ and $v(\theta, t)$ are strongly continuous one-parameter unitary groups such that

$$u(\theta, s)v(\theta, t) = \mathrm{e}^{\mathrm{i}st\theta} v(\theta, t)v(\theta, s), \quad \forall s, t \in \mathbb{R}. \tag{2.12}$$

Finally we choose the self-adjoint operators $P(\theta)$ and $Q(\theta)$ as the infinitesimal generators of $u(\theta, t)$ and $v(\theta, t)$. By Proposition 2.3, (ii) is satisfied with constant $C = 800{,}000$. ∎

REMARK 2.8. Proposition 3.9 of [14] proves that if (P, Q) is a representation of the Heisenberg relation of finite multiplicity on a Hilbert space H, there exists no strongly commuting pair (P_0, Q_0) on H such that $P - P_0$ and $Q - Q_0$ are bounded. The argument works for all $\theta \neq 0$, which implies that $(P(\theta), Q(\theta))$ constructed above at each θ is a representation of infinite multiplicity.

The above theorem can be reformulated with one-parameter unitary groups.

COROLLARY 2.9. *Let H be an infinite dimensional Hilbert space. There exist two maps $u, v : \mathbb{R} \times \mathbb{R} \to U(H)$ and a universal constant $C > 0$ such that:*

(i) *for each $\theta \in \mathbb{R}$, $u(\theta, \cdot)$ and $v(\theta, \cdot)$ are strongly continuous one-parameter unitary groups satisfying*

$$u(\theta, s)v(\theta, t) = \mathrm{e}^{\mathrm{i}st\theta} v(\theta, t)u(\theta, s), \quad s, t \in \mathbb{R}, \tag{2.13}$$

(ii) *for each $t \in \mathbb{R}$, $u(\cdot, t)$ and $v(\cdot, t)$ are* $\mathrm{Lip}^{1/2}$*-continuous:*

$$\max\{\|u(\theta, t) - u(\theta', t)\|, \|v(\theta, t) - v(\theta', t)\|\} \leqslant C|t||\theta - \theta'|^{1/2}, \quad \forall \theta, \theta' \in \mathbb{R}.$$

Moreover, for all θ, $u(\theta, \cdot)$ and $v(\theta, \cdot)$ are a representation of (2.13) *of infinite multiplicity.*

3. PERTURBATIONS OF NONCOMMUTATIVE EUCLIDEAN SPACE

We now consider the case of dimension $d > 2$. From this section on, we denote by $\theta = (\theta_{jk})_{j,k=1}^{d}$ a real skew-symmetric $d \times d$-matrix. Let $(P_1, P_2, \ldots, P_d)$ be a d-tuple of self-adjoint operators on a Hilbert space H. We say $(P_1, P_2, \ldots, P_d)$ satisfy the *Heisenberg relations with parameter θ*

$$[P_j, P_k] = -\theta_{jk} I, \quad j, k = 1, 2, \ldots, d$$

if the one-parameter unitary groups $u_1(s) = \mathrm{e}^{\mathrm{i}P_1 s}, u_2(s) = \mathrm{e}^{\mathrm{i}P_2 s}, \ldots, u_d(s) = \mathrm{e}^{\mathrm{i}P_d s}$ satisfy

$$u_j(s)u_k(t) = \mathrm{e}^{\mathrm{i}st\theta_{jk}} u_k(t)u_j(s), \quad s, t \in \mathbb{R}, j, k = 1, 2, \ldots, d. \tag{3.1}$$

When θ is the zero matrix, $(P_1, P_2, \ldots, P_j)$ gives a unitary representation of $\mathbb{R}^d$. The left regular representation of $\mathbb{R}^d$ is the translation action on $L_2(\mathbb{R}^d)$ as follows,

$$P_j f = -\mathrm{i}\frac{\partial f}{\partial x_j}, \quad (u_j(t)f)(x_1, x_2, \ldots, x_d) = f(x_1, x_2, \ldots, x_j + t, \ldots, x_d).$$

The standard noncommutative case is that $d = 2n$ and $\theta = \begin{bmatrix} 0 & I_n \\ -\mathrm{i}_n & 0 \end{bmatrix}$, where I_n is the n-dimensional identity matrix. This gives the canonical commutation relations (CCR) which consist of n pairs of Heisenberg relations that mutually commute, i.e.

$$\begin{cases} [P_j, P_{j+n}] = -\mathrm{i}I & \forall 1 \leqslant j \leqslant n, \\ \text{otherwise} & [P_j, P_k] = 0. \end{cases} \tag{3.2}$$

The Stone–von Neumann theorem applies here (cf. Theorem 14.8 of [15]): any irreducible representation of (3.2) is unitarily equivalent to n-dimensional quantum mechanics model on $L_2(\mathbb{R}^n)$,

$$P_j f = -\mathrm{i}\frac{\partial f}{\partial x_j}, \quad P_{j+n} f(x_1, \ldots, x_n) = x_j f(x_1, \ldots, x_n), \quad j = 1, \ldots, n; \tag{3.3}$$

any representation of (3.2) is unitarily equivalent to a finite or infinite multiple of the irreducible representation. It is known that a similar property holds for all nonsingular θ, which we briefly discuss in the following.

We use boldface letters for real vectors such as $\mathbf{s} = (s_1, s_2, \ldots, s_d)$. Given a d-tuple $(u_1, u_2, \ldots, u_d)$ of one-parameter unitary groups satisfying (3.1), we introduce the following strongly continuous map:

$$u : \mathbb{R}^d \to U(H), \quad u(\mathbf{s}) = \exp\Big(-\frac{\mathrm{i}}{2} \sum_{j<k} \theta_{jk} s_j s_k \Big) u_1(s_1) u_2(s_2) \cdots u_d(s_d).$$

It satisfies the commutation relation

$$u(\mathbf{s})u(\mathbf{t}) = \mathrm{e}^{(\mathrm{i}/2)\theta(\mathbf{s},\mathbf{t})} u(\mathbf{s}+\mathbf{t}) = \mathrm{e}^{\mathrm{i}\theta(\mathbf{s},\mathbf{t})} u(\mathbf{t})u(\mathbf{s}), \quad \mathbf{s}, \mathbf{t} \in \mathbb{R}^d, \tag{3.4}$$

where $\theta(\mathbf{s}, \mathbf{t}) = \sum_{jk} \theta_{jk} s_j t_k$ is the symplectic bilinear form associated with θ. (3.4) is called a projective unitary representation of $\mathbb{R}^d$ (see Appendix for the definition), and it is an equivalent formulation of (3.1). Let $T = (T_{jk})_{j,k=1}^d$ be a real invertible matrix and T^{t} be its transpose. Then $\widetilde{\theta} = T\theta T^{\mathrm{t}}$ is also a skew-symmetric real matrix, and its associated Heisenberg relation admits the following representation,

$$u(T\mathbf{s})u(T\mathbf{t}) = \mathrm{e}^{\mathrm{i}\theta(T\mathbf{s},T\mathbf{t})} u(T\mathbf{t})u(T\mathbf{s}) = \mathrm{e}^{\widetilde{\theta}(\mathbf{s},\mathbf{t})} u(T\mathbf{t})u(T\mathbf{s}).$$

Since T is invertible, the Heisenberg relations associated with θ and associated with $\widetilde{\theta}$ generate each other and there is an one-to-one correspondence between their representations. Let us fix $S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ for the antipode matrix. For a nonsingular θ, there exists an invertible T such that $T\theta T^{\mathrm{t}} = S$. Hence the Stone–von Neumann theorem concludes the case for all nonsingular θ.

PROPOSITION 3.1. *Let $d = 2n$. Suppose θ is nonsingular and $T = (T_{jk})_{j,k=1}^d$ is a real invertible matrix such that $T\theta T^{\mathrm{t}} = S$. Then any irreducible representation of*

$$[P_j, P_k] = -\mathrm{i}\theta_{jk} I, \quad j, k = 1, 2, \ldots, d \tag{3.5}$$

is unitarily equivalent to the following representation on $L_2(\mathbb{R}^n)$,

$$P_j = \sum_{1 \leqslant k \leqslant n} \Big(T_{j,k}\Big(-\mathrm{i}\frac{\partial}{\partial x_k}\Big) + T_{j,k+n} x_k \Big), \quad j = 1,2,\dots,d. \tag{3.6}$$

Moreover, any representation of (3.5) *is a (finite or infinite) multiple of* (3.6).

For a general θ the representation always generates a tensor product of a type I factor and a commutative algebra (see [22]). The next theorem is the generalization of Theorem 2.1 for dimension $d \geqslant 2$.

THEOREM 3.2. *Let θ be nonsingular. Let $(P_1, P_2, \dots, P_d)$ be a representation of*

$$[P_j, P_k] = -\mathrm{i}\theta_{jk} I, \quad j,k = 1,2,\dots,d$$

on a separable Hilbert space H of infinite multiplicity. Then for any real skew-symmetric θ', there exist self-adjoint operators $P_1', P_2', \dots, P_d'$ on H such that for all j,k,

(i) $[P_j', P_k'] = -\mathrm{i}\theta_{jk}' I$;

(ii) $P_j - P_j'$ *is bounded on H and*

$$\|P_j - P_j'\| < 9(d-1) \max_k |\theta_{jk} - \theta_{jk}'|^{1/2}.$$

Proof. Let K be a separable infinite dimensional Hilbert space. Set $\delta_j = \max_k |\theta_{jk} - \theta_{jk}'|^{1/2}$. Let us first assume that $\delta_j > 0$ for every j. By Proposition 3.1, up to a unitary equivalence we may assume that $H = L_2(\mathbb{R}^d, K)$ and $(P_1, P_2, \dots, P_d)$ are given by

$$P_j = \Big(-\mathrm{i}\delta_j \frac{\partial}{\partial x_j} + \sum_{k<j} \frac{\theta_{kj}}{\delta_k} x_k \Big) \otimes 1_K, \tag{3.7}$$

where $(-\mathrm{i}\frac{\partial}{\partial x_j})$'s and x_j's are given in (3.3). Let $W : \mathbb{R}^d \to U(K)$ be a C^1-function with values in the unitary group $U(K)$. W can be viewed as a unitary on $L_2(\mathbb{R}^d, K)$ via pointwise action. A calculation similar to (2.5) yields

$$W^* P_j W = -\mathrm{i}\delta_j \Big(\frac{\partial}{\partial x_j} + W^* \frac{\partial W}{\partial x_j} \Big) + \sum_{k<j} \frac{\theta_{kj}}{\delta_k} x_k.$$

Let us recall the two-variable C^1-function $w : \mathbb{R}^2 \to U(K)$ in Theorem 2.1. Write $\delta_{jk} = \frac{\theta_{jk}' - \theta_{jk}}{\delta_j \delta_k}$ and define the following functions,

$$\begin{aligned}
w_2(x_1, x_2, \dots, x_d) &= w(x_1, \delta_{12} x_2),\\
w_3(x_1, x_2, \dots, x_d) &= w(x_1, \delta_{13} x_3) w(x_2, \delta_{23} x_3),\\
&\vdots\\
w_d(x_1, x_2, \dots, x_d) &= w(x_1, \delta_{1d} x_d) w(x_2, \delta_{2d} x_d) \cdots w(x_{d-1}, \delta_{d-1,d} x_d).
\end{aligned}$$

When $1 \leqslant j < k$, for any $(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$

$$\left\| \frac{\partial w_k}{\partial x_j}(x_1, x_2, \dots, x_d) \right\| = \left\| \frac{\partial w}{\partial x}(x_j, \delta_{jk} x_k) \right\| < 9, \tag{3.8}$$

and when $k < j \leqslant d$, $\frac{\partial w_k}{\partial x_j} = 0$. Because the pointwise unitaries $w(x_k, \delta_{kj} x_j)$ commute with the multipliers x_j, we have

$$\begin{aligned}
&\frac{\partial w_j}{\partial x_j}(x_1, x_2, \dots, x_d) - \sum_{1 \leqslant k < j} \delta_{kj} x_k w_j(x_1, x_2, \dots, x_d) \\
&\quad = \sum_{1 \leqslant k < j} \delta_{kj} w(x_1, \delta_{1j} x_j) \cdots \frac{\partial w}{\partial y}(x_k, \delta_{kj} x_j) \cdots w(x_{j-1}, \delta_{j-1,j} x_j) \\
&\qquad - \sum_{1 \leqslant k < j} \delta_{kj} x_k w_j(x_1, x_2, \dots, x_d) \\
&\quad = \sum_{1 \leqslant k < j} \delta_{kj} w(x_1, \delta_{1j} x_j) \cdots \Big(\frac{\partial w}{\partial y}(x_k, \delta_{kj} x_j) - \mathrm{i} x_k w(x_k, \delta_{kj} x_j) \Big) \qquad (3.9) \\
&\qquad \cdots w(x_{j-1}, \delta_{j-1,j} x_j).
\end{aligned}$$

Thus the norm estimate follows:

$$\left\| \frac{\partial w_j}{\partial x_j}(x_1, x_2, \dots, x_d) - \sum_{k<j} \delta_{kj} x_k w_j(x_1, x_2, \dots, x_d) \right\| \leqslant 9 \sum_{k<j} \delta_{kj}. \tag{3.10}$$

Now set

$$W(x_1, x_2, \dots x_d) = w_2(x_1, x_2, \dots, x_d) w_3(x_1, x_2, \dots, x_d) \cdots w_d(x_1, x_2, \dots, x_d),$$

and define $P_j' = W^* \Big(P_j + \sum_{1 \leqslant k < j} \frac{\theta_{kj}' - \theta_{kj}}{\delta_k} x_k \Big) W$. Then $(P_1', P_2', \dots, P_d')$ satisfies $[P_j', P_k'] = -\mathrm{i}\theta_{jk} I$ and for each j

$$P_j - P_j' = \mathrm{i}\Big(\delta_j W^* \frac{\partial W}{\partial x_j} - \mathrm{i} \sum_{1 \leqslant k < j} \frac{\theta_{kj}' - \theta_{kj}}{\delta_k} x_k \Big) = \mathrm{i}\delta_j \Big(W^* \frac{\partial W}{\partial x_j} - \mathrm{i} \sum_{1 \leqslant k < j} \delta_{kj} x \Big).$$

Note that for all $(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$

$$\begin{aligned}
&\left\| \frac{\partial W}{\partial x_j}(x_1, x_2, \dots, x_d) - \sum_{1 \leqslant k < j} \delta_{kj} x_k W(x_1, x_2, \dots, x_d) \right\| \\
&\leqslant \left\| \frac{\partial w_j}{\partial x_j}(x_1, x_2, \dots, x_d) - \sum_{1 \leqslant k < j} \delta_{kj} x_k w_j(x_1, x_2, \dots, x_d) \right\| + \sum_{j < k \leqslant d} \left\| \frac{\partial w_k}{\partial x_j}(x_1, x_2, \dots, x_d) \right\| \\
&\leqslant 9 \sum_{1 \leqslant k < j} \delta_{kj} + 9(j-1) = 9(j-1) + 9(d-j) = 9(d-1).
\end{aligned}$$

Therefore, $P_j - P_j'$ is bounded on H and $\|P_j - P_j'\| \leqslant 9(d-1)\delta_j$.

For a general θ', we may assume that $\delta_j > 0$ for $j \leqslant s$ and $\delta_j = 0$ for $s < j \leqslant d$. Then we take the representation (3.7) for $j \leqslant s$ and use

$$P_j = \Big(- \mathrm{i}\frac{\partial}{\partial x_j} + \sum_{k \leqslant s} \frac{\theta_{kj}}{\delta_k} x_k + \sum_{s<k<j} \theta_{jk} x_k \Big) \otimes 1_K,$$

for $s < j \leqslant d$. Applying the above argument to $P_1, \dots, P_s$, we obtain

$$P'_j = W^* \Big(P_j + \sum_{k<j} \frac{\theta'_{kj} - \theta_{kj}}{\delta_k} x_k \Big) W, \quad j \leqslant s.$$

Note that now the pointwise unitary W is independent of coordinates $x_{s+1}, \dots, x_d$, and hence it commutes with the newly defined $P_{s+1}, \dots, P_d$. One can verify that the d-tuple $(P'_1, \dots, P'_s, P_{s+1}, \dots, P_d)$ satisfies the desired conditions. ∎

The next proposition is a partial converse of the above theorem. The proof is a natural generalization of the Proposition 3.7 in [14].

PROPOSITION 3.3. *Let θ be nonsingular and $(P_1, P_2, \dots, P_d)$ be a representation of*

$$[P_j, P_k] = -\mathrm{i}\theta_{jk} I, \quad j, k = 1, 2, \dots, d$$

on a Hilbert space H. If $(P_1, P_2, \dots, P_d)$ is of finite multiplicity, then there exist no strongly commuting self-adjoint operators $(P'_1, P'_2, \dots, P'_d)$ on H such that $P_j - P'_j$ is bounded on H for all j.

Proof. Let us first assume that $(P_1, \dots, P_d)$ is irreducible. It is sufficient to consider the standard representation on $L_2(\mathbb{R}^n)$,

$$P_j = -\mathrm{i}\frac{\partial}{\partial x_j}, \quad P_{j+n} = x_j, \quad j = 1, \dots, n = \frac{d}{2}.$$

Other θ's follow by a linear transformation T as in Proposition 3.1. Consider the creation and annihilation operators of the n-dimensional harmonic oscillator,

$$a_j = \frac{1}{\sqrt{2}}(P_j - \mathrm{i}P_{j+n}), \quad a_j^* = \frac{1}{\sqrt{2}}(P_j + \mathrm{i}P_{j+n}).$$

We use the usual notations $m = (m_1, m_2, \dots, m_n) \in \mathbb{N}^n, m! = m_1! \cdots m_n!$ and $|m| = \sum_j m_j$. Denote $\phi_0(x) = \pi^{-n/4} \mathrm{e}^{-|x|^2/2}$ as the Gaussian density for $\mathbb{R}^n$. There is a natural orthonormal basis of harmonic oscillator

$$\phi_m = \frac{1}{c_m}(a^*)^m \phi_0, \quad m \in \mathbb{N}^n$$

where $(a^*)^m = (a_1^*)^{m_1} \cdots (a_n^*)^{m_n}$ and c_m is the normalization constant. The creation and annihilation actions are

$$a_j^* \phi_m = \sqrt{m_j + 1}\,\phi_{(m_1, \dots, m_j+1, \dots, m_n)}, \quad a_j \phi_m = \sqrt{m_j}\,\phi_{(m_1, \dots, m_j-1, \dots, m_n)},$$
$$a_j \phi_{(m_1, \dots, 0, \dots, m_n)} = 0.$$

Let $c_1, \dots, c_n$ be the $N \times N$ matrices which are self-adjoint generators of the complex Clifford algebra $\mathbb{C}l^n$ ($N = 2^{n/2}$ or $N = 2^{(n+1)/2}$). They satisfy the commutation relations:

$$\begin{cases} c_j c_k + c_k c_j = 2 & \text{if } j = k, \\ c_j c_k + c_k c_j = 0 & \text{otherwise.} \end{cases}$$

Set $A = \sum_j c_j \otimes a_j^*$. One calculates that

$$A^* A = 1 \otimes \sum_j a_j a_j^*, AA^* = 1 \otimes \sum_j a_j^* a_j.$$

Note that

$$\Big(\sum_j a_j a_j^*\Big)\phi_m = (|m| + n)\phi_m, \quad \Big(\sum_j a_j^* a_j\Big)\phi_m = |m|\phi_m.$$

Thus $|A| = (A^*A)^{1/2}$ is invertible with compact inverse $|A|^{-1}$ and $\ker(|A^*|) = \{\mathbb{C}\phi_0\} \otimes \mathbb{C}^N$. The polar $V = A|A|^{-1}$ of A is a partial isometry, $\ker(V) = \ker(|A|)$, and $\ker V^* = \ker(|A^*|)$. Hence V is a Fredholm operator with $\text{index}(V) = -N$.

Assume that $P'_1, \dots, P'_d$ on H commute strongly and $P'_j - P_j$ is bounded for all j. Then

$$A' = \frac{1}{\sqrt{2}} \sum_{j=1}^{n} c_j \otimes (P'_j + \mathrm{i}P'_{j+n})$$

is normal, and $A' - A$ is bounded on H. Let $V' = A'|A|^{-1}$. V' is everywhere defined and bounded since $V' - V = (A' - A)|A|^{-1}$ is compact. Hence V' is also a Fredholm operator with $\text{index}(V') = -N$. Nevertheless, since $|A|^{-1}$ is one-to-one and onto the domain $D(A)(= D(A'))$,

$$\dim(\ker(V')) = \dim(\ker(A')) = \dim(\ker(A'^*)) = \dim(\ker(V'^*)),$$

which leads to a contradiction.

When $(P_1, \dots, P_d)$ has finite multiplicity M, V is of Fredholm index $-MN$. The proof remains the same as above. ∎

We now prove our main Theorem 1.2. Recall that we denote by $\mathcal{A}(d)$ the space of real d-dimensional skew-symmetric matrices.

Proof of Theorem 1.2. The proof is by induction. The continuous maps u, v from Theorem 2.7 give the initial step $d = 2$. For the induction step, we may write $H = H_1 \otimes H_2^{\otimes(d-1)}$, where both H_1 and H_2 are infinite dimensional. We assume $d-1$ maps $U_1, U_2, \dots, U_{d-1}$ on H_1 satisfying the desired poperty for dimension $d-1$. Also we have the two maps u, v for $d = 2$ on each copy of H_2. Denote $\widehat{\theta}$ for the submatrix $(\theta_{jk})_{j,k=1}^{d-1}$, I_1 the identity on H_1 and I_2 the identity on H_2. We construct d maps from $\mathcal{A}(d)$ to $U(H_1 \otimes H_2^{\otimes(d-1)})$ as follows,

$$\begin{aligned}
u_1(\theta,t) &= U_1(\widehat{\theta},t) \otimes u(\theta_{1d},t) \otimes I_2 \otimes \cdots \otimes I_2,\\
u_2(\theta,t) &= U_2(\widehat{\theta},t) \otimes I_2 \otimes u(\theta_{2d},t) \otimes I_2 \otimes \cdots \otimes I_2,\\
u_3(\theta,t) &= U_3(\widehat{\theta},t) \otimes I_2 \otimes I_2 \otimes u(\theta_{3d},t) \otimes I_2 \otimes \cdots \otimes I_2,\\
&\ \ \vdots\\
u_{(d-1)}(\theta,t) &= U_{(d-1)}(\widehat{\theta},t) \otimes I_2 \otimes \cdots \otimes I_2 \otimes u(\theta_{(d-1),d},t),\\
u_d(\theta,t) &= I_1 \otimes v(\theta_{1d},t) \otimes v(\theta_{2d},t) \otimes v(\theta_{3d},t) \otimes \cdots \otimes v(\theta_{(d-1),d},t).
\end{aligned}$$

One can check that $(u_1, u_2, \ldots, u_d)$ satisfies the desired commutation relations. By the triangle inequality, for $j \leqslant d-1$,

$$\begin{aligned}
\|u_j(\theta,t) - u_j(\theta',t)\| &\leqslant \|U_j(\widehat{\theta},t) - U_j(\widehat{\theta'},t)\| + \|u(\theta_{jd},t) - u(\theta'_{jd},t)\|\\
&\leqslant C|t|\Big(\sum_{1\leqslant k\leqslant d-1} |\theta_{jk} - \theta'_{jk}|^{1/2}\Big) + C|t||\theta_{jd} - \theta'_{jd}|^{1/2}\\
&= C|t|\Big(\sum_{1\leqslant k\leqslant d} |\theta_{jk} - \theta'_{jk}|^{1/2}\Big),
\end{aligned} \tag{3.11}$$

and the estimate of u_d follows similarly. Here we used the inductive assumption on $d-1$ and the initial step on $d=2$. The constant C is independent of dimension d and it can be 800,000 as in Theorem 2.9. ∎

Denote $|\mathbf{s}| = \Big(\sum_j |s_j|^2\Big)^{1/2}$ as the Euclidean metric for vector $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ and recall the symplectic bilinear form $\theta(\mathbf{s},\mathbf{t}) = \sum_{jk} \theta_{jk} s_j t_k$. A strongly continuous map $u : \mathbb{R}^d \to U(H)$ is called a θ-projective unitary representation (or shortly θ-representation) of $\mathbb{R}^d$ if it satisfies

$$u(\mathbf{s})u(\mathbf{t}) = \mathrm{e}^{(\mathrm{i}/2)\theta(\mathbf{s},\mathbf{t})} u(\mathbf{s}+\mathbf{t}). \tag{3.12}$$

The above theorem can be reformulated as a continuous family of projective unitary representations.

COROLLARY 3.4. *Let H be an infinite dimensional Hilbert space. There exist a map $u : \mathcal{A}(d) \times \mathbb{R}^d \to U(H)$ and a universal constant $C > 0$ such that:*

(i) *for each $\theta \in \mathcal{A}(d)$, $u(\theta, \cdot)$ is a strongly continuous θ-representation of $\mathbb{R}^d$;*

(ii) *for any $\mathbf{s} \in \mathbb{R}^d$ and $\theta, \theta' \in \mathcal{A}(d)$,*

$$\|u(\theta,\mathbf{s}) - u(\theta',\mathbf{s})\| \leqslant C|\mathbf{s}|\Big(\sum_{k,j} |\theta_{kj} - \theta'_{kj}|^{1/2}\Big). \tag{3.13}$$

Proof. Let $u_1(\theta,\cdot), u_2(\theta,\cdot),\dots,u_d(\theta,\cdot)$ be one-parameter unitary groups from Theorem 1.2 and $P_1(\theta), P_2(\theta),\dots,P_d(\theta)$ be the corresponding infinitesimal generators. Then

$$[P_j(\theta), P_k(\theta)] = -\mathrm{i}\theta_{jk} I, \quad j,k = 1,\dots,d,$$

and by Lemma 2.3, $P_j(\theta) - P_j(\theta')$ is bounded on H:

$$\|P_j(\theta) - P_j(\theta')\| \leqslant C\Big(\sum_k |\theta_{jk} - \theta'_{jk}|^{1/2}\Big).$$

We first consider that θ and θ' are nonsingular. It is clear from Proposition 3.1 that $P_1(\theta), P_2(\theta),\dots,P_d(\theta)$ share a common core. For any vector $\mathbf{s}\in\mathbb{R}^d$, we obtain by the Baker–Campbell–Hausdorff formula that

$$u(\theta,\mathbf{s}) := \mathrm{e}^{\mathrm{i}(\sum_j s_j P_j(\theta))} = \exp\Big(-\frac{\mathrm{i}}{2}\sum_{j<k}\theta_{jk}s_j s_k\Big) u_1(\theta,s_1)u_2(\theta,s_2)\cdots u_d(\theta,s_d).$$

Therefore,

$$\begin{aligned}\|u(\theta,\mathbf{s}) - u(\theta',\mathbf{s})\| &= \|\mathrm{e}^{\mathrm{i}(\sum_j s_j P_j(\theta))} - \mathrm{e}^{\mathrm{i}(\sum_j s_j P_j(\theta'))}\| \leqslant \Big\|\sum_j s_j(P_j(\theta) - P_j(\theta'))\Big\| \\ &\leqslant C\sum_j |s_j|\Big(\sum_k |\theta_{jk}-\theta'_{jk}|^{1/2}\Big) \leqslant C|\mathbf{s}|\Big(\sum_j\Big(\sum_k|\theta_{jk}-\theta'_{jk}|^{1/2}\Big)^2\Big)^{1/2} \\ &\leqslant C|\mathbf{s}|\Big(\sum_{j,k}|\theta_{jk}-\theta'_{jk}|^{1/2}\Big).\end{aligned}$$

Note that when $\mathbf{s}$ is fixed, $u(\cdot,\mathbf{s})$ is continuous in norm. Then the estimates for general θ and θ' follows. ∎

We now explore applications on noncommutative Euclidean spaces. Let $\mathcal{S}(\mathbb{R}^d)$ be the complex Schwartz functions on $\mathbb{R}^d$, and $\mathbf{s}\cdot\mathbf{t}$ denote the Euclidean inner product. Fix a nonzero $\theta\in\mathcal{A}(d)$. The associated Moyal product, for $f,g\in\mathcal{S}(\mathbb{R}^d)$, is defined as

$$f\star_\theta g(\mathbf{x}) = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\Big(\mathbf{x}-\frac{1}{2}\theta\mathbf{s}\Big)g(\mathbf{x}+\mathbf{t})\mathrm{e}^{-\mathrm{i}\mathbf{s}\cdot\mathbf{t}}\mathrm{d}\mathbf{s}\,d\mathbf{t}, \quad f,g\in\mathcal{S}(\mathbb{R}^d) \tag{3.14}$$

(see [22]). The noncommutative Euclidean space E_θ associated to θ is the C^*-algebra generated by $\{\lambda_\theta(f) : f\in\mathcal{S}(\mathbb{R}^d)\}$, where $\lambda_\theta : (S(\mathbb{R}^d),\star_\theta)\to B(L_2(\mathbb{R}^d))$ is the left Moyal multiplication,

$$\lambda_\theta(f)g = f\star_\theta g, \quad f\in\mathcal{S}(\mathbb{R}^d), g\in L_2(\mathbb{R}^d).$$

E_θ is the Moyal deformation of $C_0(\mathbb{R}^d)$, the algebra of continuous functions vanishing at infinity. An equivalent formulation of Moyal product via Fourier transform is given by

$$\lambda_\theta(f) = \int_{\mathbb{R}^d}\widehat{f}(\mathbf{s})\lambda_\theta(\mathbf{s})\mathrm{d}\mathbf{s},$$

where $\widehat{f}$ is the Fourier transform of f and

$$\widehat{\lambda_\theta(\mathbf{s})g}(\mathbf{t}) = \mathrm{e}^{\mathrm{i}\theta(\mathbf{s},\mathbf{t}-\mathbf{s})}\widehat{g}(\mathbf{t}-\mathbf{s}), \tag{3.15}$$

is called the left regular θ-representation of $\mathbb{R}^d$ (see [11] for more information on Moyal analysis).

Let u be a θ-representation on a Hilbert space H. The associated quantization map

$$u(f) = \int_{\mathbb{R}^d} \widehat{f}(\mathbf{s})u(\mathbf{s})\mathrm{d}\mathbf{s},$$

gives a representation of Moyal product, i.e. $u(f)u(g) = u(f \star_\theta g)$. Actually, representations of E_θ are in one-to-one correspondence to θ-representations of $\mathbb{R}^d$ (see [20]). There is a (canonical) $*$-homomorphism from E_θ onto the C^*-algebra generated by $\{u(f) : f \in \mathcal{S}(\mathbb{R}^d)\}$:

$$\pi_u(\lambda_\theta(f)) = u(f).$$

If π_u is an isomorphism, we say u is *canonical*.

LEMMA 3.5. *Let $\theta \in \mathcal{A}(d)$ be nonsingular and $\widehat{\theta} = (\theta_{jk})_{j,k=1}^{d-1}$ be its principal submatrix. For a vector $\mathbf{s} \in \mathbb{R}^{d-1}$, write $(\mathbf{s}, 0) = (s_1, \dots, s_{(d-1)}, 0) \in \mathbb{R}^d$. Let $u : \mathbb{R}^d \to U(H)$ be a θ-representation on H. Then the following $\widehat{\theta}$-representation of $\mathbb{R}^{d-1}$*

$$\widehat{u} : \mathbb{R}^{d-1} \to U(H), \quad \widehat{u}(\mathbf{s}) = u(\mathbf{s}, 0),$$

is canonical.

Proof. For $f \in \mathcal{S}(\mathbb{R}^{d-1})$, denote $v(f) = \int \widehat{f}(\mathbf{s})\lambda_\theta(\mathbf{s},0)\mathrm{d}\mathbf{s}$, and $\lambda_{\widehat{\theta}}(f)$ for the left $\widehat{\theta}$-Moyal multiplication on $L_2(\mathbb{R}^{d-1})$. It is sufficient to show that

$$\|v(f)\| = \|\lambda_{\widehat{\theta}}(f)\|$$

holds for functions f which are L_1-norm dense in $\mathcal{S}(\mathbb{R}^{d-1})$. By Proposition 3.1, we may just consider that u is the left regular θ-representation (3.15) on $L_2(\mathbb{R}^d)$. For any $g \in S(\mathbb{R}^{d-1})$, define $g_n \in S(\mathbb{R}^d)$ as follows:

$$\widehat{g}_n(\mathbf{t}, t_d) = \widehat{g}(\mathbf{t})\phi_n(t_d), \quad (\mathbf{t}, t_d) \in \mathbb{R}^d,$$

where $\phi_n \in S(\mathbb{R})$ is a sequence of smooth function supported in $[-\varepsilon_n, \varepsilon_n]$ such that $\varepsilon_n \to 0$ and the L_2-norm $\|\phi_n\|_2 = 1$. For $f \in \mathcal{S}(\mathbb{R}^{d-1})$,

$$\begin{aligned}
&\widehat{v(f)g_n}(\mathbf{t}, t_d) - \widehat{\lambda_{\widehat{\theta}}(f)g}(\mathbf{t})\phi_n(t_d) \\
&\qquad = \phi_n(t_d)\int \widehat{f}(\mathbf{s})\widehat{g}(\mathbf{t}-\mathbf{s})\mathrm{e}^{(\mathrm{i}/2)\widehat{\theta}(\mathbf{s},\mathbf{t}-\mathbf{s})}\exp\Big(\frac{\mathrm{i}}{2}\sum_{j=1}^{d-1}\theta_{jd}s_j t_d\Big)\mathrm{d}\mathbf{s} \\
&\qquad\quad - \phi_n(t_d)\int \widehat{f}(\mathbf{s})\widehat{g}(\mathbf{t}-\mathbf{s})\mathrm{e}^{(\mathrm{i}/2)\widehat{\theta}(\mathbf{s},\mathbf{t}-\mathbf{s})}\mathrm{d}\mathbf{s}
\end{aligned}$$

$$= \phi_n(t_d)\int \widehat{f}(\mathbf{s})\widehat{g}(\mathbf{t}-\mathbf{s})\mathrm{e}^{\mathrm{i}/2\widehat{\theta}(\mathbf{s},\mathbf{t}-\mathbf{s})}\Big(\exp\Big(\frac{\mathrm{i}}{2}\sum_{j=1}^{d-1}\theta_{jd}s_jt_d\Big)-1\Big)\mathrm{d}\mathbf{s}.$$

Now assume that $\widehat{f}$ is compactly supported. Then the sequence

$$\beta_n := \sup_{t_d\in\mathrm{supp}(\phi_n)}\ \sup_{\mathbf{s}\in\mathrm{supp}(f)}\Big|\exp\Big(\frac{\mathrm{i}}{2}\sum_{j=1}^{d-1}\theta_{jd}s_jt_d\Big)-1\Big|$$

converges to 0 as $n\to\infty$. Hence

$$\|\widehat{v(f)g_n}-\widehat{\lambda_{\widehat{\theta}}(f)g}\phi_n\|_2 \leqslant \beta_n\|\phi_n\|_2\|f\|_2\|g\|_2\to 0.$$

Thus for compactly supported f, and any $g\in\mathcal{S}(\mathbb{R}^{d-1})$

$$\lim_n \|v(f)g_n\|_2 = \|\lambda_{\widehat{\theta}}(f)g\|_2,$$

which implies

$$\|v(f)\| = \|\lambda_{\widehat{\theta}}(f)\|. \quad ∎$$

Let $\alpha>0$ and Δ be the Laplacian in $\mathbb{R}^d$. Recall that the Sobolev space $H^\alpha(\mathbb{R}^d)$ is the Hilbert space $H^\alpha(\mathbb{R}^d):=\{f\in L_2(\mathbb{R}^d):(1+|\Delta|)^{\alpha/2}f\in L_2(\mathbb{R}^d)\}$ equipped with the norm $\|f\|_{H^\alpha}=\|(1+|\Delta|)^{\alpha/2}f\|_2$.

COROLLARY 3.6. *Let H be an infinite dimensional Hilbert space. There exists a map*

$$u:\mathcal{A}(d)\times\mathcal{S}(\mathbb{R}^d)\to B(H),\quad (\theta,f)\mapsto u_\theta(f)$$

such that:

(i) *for each $\theta\in\mathcal{A}(d)$,*

$$u_\theta(f\star_\theta g)=u_\theta(f)u_\theta(g),\quad \forall f,g\in\mathcal{S}(\mathbb{R}^d);$$

(ii) *for $\alpha>\frac{d}{2}+1$, there exists a constant $C_{\alpha,d}$ such that for all $f\in\mathcal{S}(\mathbb{R}^d)$ and $\theta,\theta'\in\mathcal{A}(d)$,*

$$\|u_\theta(f)-u_{\theta'}(f)\|\leqslant C_{\alpha,d}\Big(\sum_{j,k}|\theta_{jk}-\theta'_{jk}|^{1/2}\Big)\|f\|_{H^\alpha};$$

(iii) *for each θ, u_θ is a canonical representation.*

Proof. We first consider the case $d=2m$ is even. Let $u(\theta,\mathbf{s})$ be the continuous family of projective unitary representations from Corollary 3.4. Define

$$u_\theta(f)=\int_{\mathbb{R}^d}\widehat{f}(\mathbf{s})u(\theta,\mathbf{s})\mathrm{d}\mathbf{s}.$$

The first assertion follows from that $u(\theta,\cdot)$ is a θ-representation of $\mathbb{R}^d$. For (ii), we use the estimate (3.13):

$$\|u_\theta(f)-u_{\theta'}(f)\|\leqslant C\Big(\sum_{j,k}|\theta_{jk}-\theta'_{jk}|^{1/2}\Big)\int_{\mathbb{R}^d}|\widehat{f}(\mathbf{s})||\mathbf{s}|\mathrm{d}\mathbf{s}$$

$$\leqslant C\Big(\sum_{j,k}|\theta_{jk}-\theta'_{jk}|^{1/2}\Big)\||\widehat{f}||\mathbf{s}|(1+4\pi|\mathbf{s}|^2)^{\alpha/2-1/2}\|_2$$
$$\Big(\int_{\mathbb{R}^d}(1+4\pi|\mathbf{s}|^2)^{-\alpha+1}\mathrm{d}\mathbf{s}\Big)^{1/2}$$
$$\leqslant C_{\alpha,d}\Big(\sum_{j,k}|\theta_{jk}-\theta'_{jk}|^{1/2}\Big)\|f\|_{H^\alpha}.$$

The second integral converges when $\alpha-1>\frac{d}{2}$ and the constant

$$C_{\alpha,d}=\Big(\frac{V_d}{2\alpha-d-2}\Big)^{1/2}C$$

where V_d is the volume of unit $(d-1)$-sphere. For (iii), given a θ, it is sufficient to show that for any $f\in\mathcal{S}(R^d)$, $\|u_\theta(f)\|=\|\lambda_\theta(f)\|$. This is clear for all nonsingular θ. Given a singular θ in even dimensions, we choose a sequence of nonsingular skew-symmetric θ_n converging to θ. With the continuity in (ii), we obtain that for all $f\in H^\alpha(\mathbb{R}^d)$,

$$\|u_\theta(f)\|=\lim_n\|u_{\theta_n}(f)\|=\lim_n\|\lambda_{\theta_n}(f)\|\geqslant\|\lambda_\theta(f)\|.$$

The last inequality follows from the fact that $\lambda_{\theta_n}(f)\to\lambda_\theta(f)$ in strong operator topology, and it is actually an equality (see [22]). Thus we finish the proof for the even case. When $d=2m-1$ is odd, we set

$$u_\theta(f)=\int_{\mathbb{R}^{d-1}}\widehat{f}(\mathbf{s})u(\widetilde{\theta},(\mathbf{s},0))\mathrm{d}\mathbf{s},\quad f\in\mathcal{S}(\mathbb{R}^{d-1}),\theta\in\mathcal{A}(d-1)$$

where $\widetilde{\theta}=\begin{bmatrix}\theta&0\\0&0\end{bmatrix}$ is an embedding of $\mathcal{A}(d-1)$ into $\mathcal{A}(d)$. (i) and (ii) follows similarly. For (iii), again we choose a sequence of nonsingular $\widetilde{\theta}_n$ approximating $\widetilde{\theta}$. Denote by θ_n the corresponding $(d-1)\times(d-1)$ principal submatrix of $\widetilde{\theta}_n$. Then θ_n converges to θ, and by Lemma 3.5 we obtain that for all $f\in H^\alpha(\mathbb{R}^{d-1})$,

$$\|u_\theta(f)\|=\lim_n\Big\|\int\widehat{f}(\mathbf{s})u(\widetilde{\theta}_n,(\mathbf{s},0))\mathrm{d}\mathbf{s}\Big\|=\lim_n\|\lambda_{\theta_n}(f)\|\geqslant\|\lambda_\theta(f)\|,$$

which completes the proof. ∎

4. CONTINUOUS PERTURBATION OF NONCOMMUTATIVE TORI

Let $(u_1,u_2,\ldots,u_d)$ be a d-tuple of unitaries satisfy

$$u_ju_k=\sigma_{jk}u_ku_j,\quad j,k=1,2,\ldots,d, \tag{4.1}$$

where $\sigma_{jk}=\mathrm{e}^{2\pi\mathrm{i}\theta_{jk}}$. We say $(u_1,u_2,\ldots,u_d)$ is a canonical d-tuple of generators for A_θ if the canonical map from A_θ to $C^*(u_1,u_2,\ldots,u_d)$ is a $*$-isomorphism. We denote $\mathbb{T}(d)\cong\mathbb{T}^{(d-1)d/2}$ as the space of all Hermitian $d\times d$ matrices with unit entries. In this section u will denote a d-tuple of unitaries $(u_1, u_2, \dots, u_d)$ and m denote a d-tuple of integers $(m_1, m_2, \dots, m_d)$. We use the standard notation of multiple Fourier series as follows,

$$u^m = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}.$$

A polynomial in u with a finite number of nonzero coefficients is

$$a = \sum_{m \in \mathbb{Z}^d} \alpha_m u^m.$$

Denote $\mathcal{P}_\theta$ the $*$-algebra of all polynomials of $(u_1, u_2, \dots, u_d)$. A_θ is the enveloping C^*-algebra of $\mathcal{P}_\theta$. One can define a faithful tracial state τ on $\mathcal{P}_\theta$,

$$\tau\Big(\sum_{m \in \mathbb{Z}^d} \alpha_m u^m \Big) = \alpha_0.$$

The GNS-representation of τ is given as follows:

$$\pi(u^m)|m'\rangle = \exp\Big(2\pi \mathrm{i} \Big(- \sum_{1 \leqslant j < k \leqslant d} \theta_{jk} m_k m_j' \Big) \Big) |m + m'\rangle, \quad \forall m, m' \in \mathbb{Z}^d, \tag{4.2}$$

where we use "kets" $|m\rangle$ for the GNS-vector of u^m. $\{|m\rangle : m \in \mathbb{Z}^d\}$ forms an orthonormal basis and the Hilbert space is isomorphic to $l_2(\mathbb{Z}^d)$. The trace τ is implemented by the cyclic vector $|\mathbf{0}\rangle$,

$$\tau\Big(\sum_{m \in \mathbb{Z}^d} \alpha_m u^m \Big) = \langle \mathbf{0} | \sum_{m \in \mathbb{Z}^d} \alpha_m \pi(u^m) |\mathbf{0}\rangle = \alpha_0.$$

By universality, π extends to a $*$-representation of A_θ and so does the tracial state τ. To see that both τ and π are faithful, we recall the following reformulation of τ by the transference automorphisms of A_θ. Let $\mathbb{T}^d$ be the d-torus

$$\mathbb{T}^d = \{(z_1, z_2, \dots, z_d) \in \mathbb{C}^d : |z_j| = 1, \ \forall j\}.$$

For a d-tuple $z = (z_1, z_2, \dots, z_d) \in \mathbb{T}^d$, the transference automorphism associated to z is given by

$$\alpha_z(u^m) = z^m u^m \equiv z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d} u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}.$$

For each j, we introduce the following map

$$\Phi_j(a) = \int_{\mathbb{T}} \alpha_{(1,\dots,z_j,\dots,1)}(a) \mathrm{d}z_j.$$

As an averaging of automorphisms, Φ_j is faithful, completely positive and contractive. Note that

$$\Phi_j(u^m) = \begin{cases} u^m & \text{if } m_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Φ_j is the conditional expectation onto the subalgebra generated by all unitary generators except u_j. One can see that $\Phi_j \Phi_k = \Phi_k \Phi_j$, and this composition is the conditional expectation onto the subalgebra generated by all generators except u_j

and u_k. Inductively, the map $\Phi_1\Phi_2\cdots\Phi_d$ is the conditional expectation onto the scalars, which coincides with the canonical state τ:

$$\Phi_1\Phi_2\cdots\Phi_d(u^m) = \tau(u^m)I = \begin{cases} I & \text{if } m = (0,0,\ldots,0), \\ 0 & \text{otherwise.} \end{cases}$$

This justifies that τ is faithful and so is the representation π.

The following lemma is an analog of Lemma 4.3 in [14].

LEMMA 4.1. *Let $(u_1, u_2, \ldots, u_d)$ be a d-tuple of unitaries satisfying*

$$u_j u_k = \mathrm{e}^{2\pi\mathrm{i}\theta_{jk}} u_k u_j, \quad j,k = 1,2,\ldots,d.$$

Then $(u_1, u_2, \ldots, u_d)$ is a canonical d-tuple of generators for A_θ if and only if there exists a state τ on the C^*-algebra $C^*(u_1, u_2, \ldots, u_d)$ such that,

$$\tau(u^m) = \begin{cases} 1 & \textit{if } m = (0,0,\ldots,0), \\ 0 & \textit{otherwise.} \end{cases} \tag{4.3}$$

Proof. The necessity follows from the above discussion. Let us identify A_θ with the representation $\pi(A_\theta) \subset B(l_2(\mathbb{Z}^d))$ in (4.2). Given a state τ as (4.3), the GNS-representation π_τ maps $C^*(u_1, u_2, \ldots, u_d)$ into $B(l_2(\mathbb{Z}^d))$ and sends each u_j to the canonical unitary $\pi(\widetilde{u}_j) \in \pi(A_\theta)$. Denote π_u for the canonical map from A_θ onto $C^*(u_1, u_2, \ldots, u_d)$. Both compositions $\pi_u\pi_\tau$ and $\pi_\tau\pi_u$ are the identity maps, since they send generators to generators. Therefore the canonical π_u is a $*$-isomorphism. ∎

The next theorem is a refinement of Theorem 1.1 with periodicity.

THEOREM 4.2. *Let H be an infinite dimensional Hilbert space. There exist d continuous maps $u_1, u_2, \ldots, u_d$ from $\mathbb{T}(d)$ to $U(H)$ and a universal constant $C > 0$ such that:*

(i) *for σ such that $\sigma_{jk} = \mathrm{e}^{2\pi\mathrm{i}\theta_{jk}}$, $(u_1(\sigma), u_2(\sigma), \ldots, u_d(\sigma))$ is a canonical d-tuple of generators for A_θ;*

(iii) *for each j,*

$$\frac{1}{2}\max_k |\sigma_{jk} - \sigma'_{jk}|^{1/2} \leqslant \|u_j(\theta) - u_j(\theta')\| \leqslant C\Big(\sum_k |\sigma_{jk} - \sigma'_{jk}|^{1/2}\Big).$$

Proof. The continuous maps and the upper estimates of (ii) can be proved with the same construction as in Theorem 1.2. The lower estimates follows from Proposition 4.6 in [14], for each pair of indices (j,k). To show that

$$(u_1(\sigma), u_2(\sigma), \ldots, u_d(\sigma))$$

is canonical, we recall the fact that A_θ is simple when $\theta\mathbb{Z}^d \cap \mathbb{Z}^d = \{\mathbf{0}\}$ (see [13], [19], [23]). Such θ's are dense in all skew-symmetric $d \times d$ matrices. Then the conclusion can be derived by combining the argument of Remark 5.6 in [14] with Lemma 4.1. ∎

REMARK 4.3. For $\alpha > 0$, let us recall the Sobolev space on d-torus

$$H_\alpha(\mathbb{T}^d) = \Big\{ f \in L_2(\mathbb{T}^d) : \sum_{m\in\mathbb{Z}^d} (1+|m|^2)^\alpha |\widehat{f}(m)|^2 < \infty \Big\},$$
$$\|f\|_{H^\alpha} = \Big(\sum_{m\in\mathbb{Z}^d} (1+|m|^2)^\alpha |\widehat{f}(m)|^2 \Big)^{1/2},$$

where $\widehat{f}$ is the Fourier series of f. Given the d continuous maps $u_1, u_2, \dots, u_d$ above, we have the following quantization:

$$u_\sigma(f) := \sum_m \widehat{f}(m_1, m_2, \dots, m_d) u_1(\sigma)^{m_1} u_2(\sigma)^{m_2} \cdots u_d(\sigma)^{m_d}, \quad \sigma \in \mathbb{T}(d).$$

The series is well defined if $f \in H^\alpha(\mathbb{T}^d)$ for some $\alpha > \frac{d}{2}$. We have an analog of Corollary 3.6 as follows: for $\alpha > \frac{d}{2} + 1$ there exists constant $C_{\alpha,d}$ depending on α, d such that for any $f \in H^\alpha(\mathbb{T}^d)$ and $\sigma, \sigma' \in \mathbb{T}(d)$,

$$\|u_\sigma(f) - u_{\sigma'}(f)\| \leqslant C_{\alpha,d} \|f\|_{H^\alpha} \sum_{j,k} |\sigma_{jk} - \sigma'_{jk}|^{1/2}.$$

Let us define that for each pair $\sigma, \sigma' \in \mathbb{T}(d)$,

$$\rho(\sigma, \sigma') := \inf \max_j \|u_j - u'_j\|,$$

where the infimum runs over all d-tuple of unitaries $(u_1, u_2, \dots, u_d)$ on the seperable infinite dimensional Hilbert space H satisfying the commutation relation (4.1) for σ, and respectively $(u'_1, u'_2, \dots, u'_d)$ for σ'. It is proved in [14] that for $d = 2$, ρ is a translation invariant metric on $\mathbb{T}$ and

$$\frac{1}{2}|\sigma - \sigma'|^{1/2} \leqslant \rho(\sigma, \sigma') \leqslant 24|\sigma - \sigma'|^{1/2}, \quad \sigma, \sigma' \in \mathbb{T}.$$

Their argument generalizes to $d > 2$.

PROPOSITION 4.4. *ρ is a translation-invariant metric on $\mathbb{T}(d)$ and for any $\sigma, \sigma' \in \mathbb{T}(d)$,*

$$\frac{1}{2} \max_{j,k} |\sigma_{jk} - \sigma'_{jk}|^{1/2} \leqslant \rho(\sigma, \sigma') \leqslant 24(d-1) \max_{j,k} |\sigma_{jk} - \sigma'_{jk}|^{1/2}. \tag{4.4}$$

Proof. We first show the translation-invariance. Given $\sigma, \sigma', \sigma'' \in \mathbb{T}(d)$, let $(u_1, u_2, \dots u_d)$, $(u'_1, u'_2, \dots u'_d)$ and $(u''_1, u''_2, \dots u''_d)$ be d tuples of unitaries on H satisfying

$$u_j u_k = \sigma_{jk} u_k u_j, \quad u'_j u'_k = \sigma'_{jk} u'_k u'_j, \quad u''_j u''_k = \sigma''_{jk} u''_k u''_j, \quad j,k = 1, \dots, d.$$

Define the new unitaries on $H \otimes_2 H \cong H$,

$$v_j = u_j \otimes u''_j, \quad v'_j = u_j \otimes u''_j, \quad j = 1, \dots, k.$$

They satisfy

$$v_j v_k = \sigma_{j,k} \sigma''_{j,k} v_k v_j, \quad v'_j v'_k = \sigma'_{j,k} \sigma''_{j,k} v'_k v'_j.$$

Since $\|v_j - v_j'\| = \|u_j - u_j'\|$ for all j, we have $\rho(\sigma,\sigma') \leqslant \rho(\sigma\sigma'',\sigma'\sigma'')$ where $\sigma\sigma''$ is the Hadamard (entrywise) product. Thus the translation invariance follows by symmetry.

With the translation-invariance, it is sufficient to prove the triangle inequality

$$\rho(\sigma',\sigma'') \leqslant \rho(\sigma',\sigma) + \rho(\sigma,\sigma'') \tag{4.5}$$

for all triple $(\sigma,\sigma',\sigma'')$ with a fixed σ. Indeed for any $\eta,\sigma,\sigma',\sigma'' \in \mathbb{T}(d)$, the triangle inequalities (4.5) for $(\sigma',\sigma,\sigma'')$ and $(\sigma'\eta,\sigma\eta,\sigma''\eta)$ are equivalent. Choosing $\theta \in \mathcal{A}(d)$ such that $\theta\mathbb{Z}^d \cup \mathbb{Z}^d = \{\mathbf{0}\}$, then A_θ is simple. We claim that any two d-tuples of unitaries $(u_1,u_2,\ldots,u_d)$ and $(v_1,v_2,\ldots,v_d)$ on H satisfying the commutation relations of A_θ are approximately unitarily equivalent, i.e. there exists a sequence $\{w_n\}$ of unitaries on H such that for all j

$$\|w_n u_j w_n^* - v_j\| \to 0.$$

This can be shown, as in Proposition 4.2 of [14], by Voiculescu's noncommutative Weyl–von Neumann theorem [26]. Consider the two canonical $*$-homomorphisms $\pi_u,\pi_v : A_\theta \to B(H)$,

$$\pi_u(\widetilde{u}_j) = u_j, \quad \pi_v(\widetilde{u}_j) = v_j, \quad j = 1,\ldots,d,$$

where $\widetilde{u}_j$'s represent the generators of A_θ. Denote by $\mathcal{K}$ the ideal of compact operators on H. We need to verify that $\pi_u^{-1}(\mathcal{K}) \subset \ker \pi_u$ and $\pi_u^{-1}(\mathcal{K}) \subset \ker \pi_v$. $\pi_u^{-1}(\mathcal{K})$ and $\pi_u^{-1}(\mathcal{K})$ are proper ideals in A_θ, and hence both are trivial because A_θ is simple.

Now choose σ such that $\mathrm{e}^{2\pi\mathrm{i}\theta_{jk}} = \sigma_{jk}$. For any σ' and σ'', find d-tuples $(u_1,\ldots,u_d)$ and $(u_1',\ldots,u_d')$ of unitaries on H such that

$$u_j u_k = \sigma_{jk} u_k u_j, \quad u_j' u_k' = \sigma_{jk}' u_k' u_j', \quad \text{and} \quad \max_j \|u_j - u_j'\| \leqslant \rho(\sigma,\sigma') + \frac{\varepsilon}{2},$$

and also $(v_1,\ldots,v_d)$ and $(v_1'',\ldots,v_d'')$ such that

$$v_j v_k = \sigma_{jk} v_k v_j, \quad v_j'' v_k'' = \sigma_{jk}'' v_k'' v_j'', \quad \text{and} \quad \max_j \|v_j - v_j''\| \leqslant \rho(\sigma,\sigma'') + \frac{\varepsilon}{2}.$$

Since $(u_1,\ldots,u_d)$ and $(v_1,\ldots,v_d)$ are approximately unitarily equivalent, there exists a unitary w on H such that

$$\max_j \|w u_j w^* - v_j\| \leqslant \varepsilon.$$

Then take $\overline{u}_j = w u_j' w^*$, we have:

$$\begin{aligned}\rho(\sigma',\sigma'') &\leqslant \max_j \|\overline{u}_j - v_j''\| \leqslant \max_j (\|w u_j' w^* - w u_j w^*\| + \|w u_j w^* - v_j\| + \|v_j - v_j''\|) \\ &\leqslant \max_j \|w u_j' w^* - w u_j w^*\| + \max_j \|w u_j w^* - v_j\| + \max_j \|v_j - v_j''\| \\ &\leqslant \rho(\sigma',\sigma) + \rho(\sigma,\sigma'') + 2\varepsilon.\end{aligned}$$

Therefore we have proved the triangle inequality.

Finally, the left inequality of (4.4) is a consequence of Theorem 4.2. On the other hand, let $\theta, \theta' \in \mathcal{A}(d)$ (we may assume θ nonsingular by translation invariance) such that $\sigma_{jk} = \mathrm{e}^{2\pi \mathrm{i} \theta_{jk}}$ and $\sigma'_{jk} = \mathrm{e}^{2\pi \mathrm{i} \theta'_{jk}}$. Take $P_1, P_2, \ldots, P_d$ and $P'_1, P'_2, \ldots, P'_d$ to be the self-adjoint operators from Theorem 3.2. The second inequality follows from choosing

$$u_j(t) = \mathrm{e}^{\sqrt{2\pi} \mathrm{i} P_j t}, \quad u'_j(t) = \mathrm{e}^{\sqrt{2\pi} \mathrm{i} P'_j t}, \quad j = 1, \ldots, d. \quad \blacksquare$$

5. APPENDIX

In this appendix, we provide an argument for the universality of the noncommutative Euclidean space E_θ defined in Section 3. One can identify E_θ as a *twisted group C^*-algebra* and recall its natural connection to projective unitary representations. We refer to the survey [20] for more information about this topic.

Let G be a locally compact Hausdorff group and e be the identity of G. A strongly continuous map $u : G \to U(H)$ is a *projective unitary representation* if there exists a (continuous) function $\sigma : G \times G \to \mathbb{T}$ such that

$$u(g)u(h) = \sigma(g,h)u(gh), \quad g, h \in G.$$

The function σ is called the multiplier associated to u and u is called a σ-representation. It follows from the group structure that for all $g, g_1, g_2 \in G$,

(i) $\sigma(g,e) = \sigma(e,g) = 1$;
(ii) $\sigma(g,g_1)\sigma(gg_1,g_2) = \sigma(g,g_1g_2)\sigma(g_1,g_2)$.

A function $\sigma : G \times G \to \mathbb{T}$ satisfying (i) and (ii) is called a 2-cocycle of G with values in $\mathbb{T}$.

Given a $\mathbb{T}$-valued 2-cocycle σ of G, the Banach $*$-algebra $L_1(G,\sigma)$ is defined as the set $L_1(G)$ equipped with the σ-twisted convolution and involution given by

$$f_1 *_\sigma f_2(g) = \int_G f_1(g_1) f_2(g_1^{-1}g) \sigma(g_1, g_1^{-1}g) \mathrm{d}\mu(g_1)$$

where $\mathrm{d}\mu$ is the (left) Haar measure on G, and $f^*(g) = \overline{\sigma(g,g^{-1})} f(g^{-1})$. $L_1(G,\sigma)$ can be represented on $L_2(G,\mu)$ as follows:

$$\lambda_\sigma(f)(h) = f *_\sigma h, \quad f \in L_1(G), h \in L_2(G).$$

This is called the left σ-regular representation of G. The reduced σ-twisted group C^*-algebra, denoted by $C^*_{\mathrm{r}}(G,\sigma)$, is the norm closure of $L_1(G,\sigma)$ in $B(L_2(G))$. The full σ-twisted group C^*-algebra $C^*(G,\sigma)$ is defined as the enveloping C^*-algebra of $L_1(G,\sigma)$. There is a one-to-one correspondence between σ-representations of G and representations of $C^*(G,\sigma)$. If G is amenable, $C^*(G,\sigma)$ is isomorphic to $C^*_{\mathrm{r}}(G,\sigma)$ and the left σ-regular representation of $C^*(G,\sigma)$ on $L_2(G,\sigma)$ is faithful.

Back to the noncommutative Euclidean space E_θ, a symplectic bilinear form θ introduces a 2-cocycle of $\mathbb{R}^d$ as follows:

$$\sigma_\theta(\mathbf{s},\mathbf{t}) = \exp\Big(\frac{\mathrm{i}}{2}\theta(\mathbf{s},\mathbf{t})\Big), \quad \mathbf{s},\mathbf{t} \in \mathbb{R}^d.$$

The Moyal product $\star_\theta$ is the Fourier transform of σ-twisted convolution. One identifies $E_\theta = C^*_\mathrm{r}(\mathbb{R}^d, \sigma_\theta)$ and it is further isomorphic to $C^*(\mathbb{R}^d, \sigma_\theta)$ because $\mathbb{R}^d$ is amenable. Thus there is a one-to-one correspondence between $*$-homomorphism from E_θ and θ-representation of $\mathbb{R}^d$. One can use an alternative argument by identifying E_θ with an iterated crossed product $C_0(\mathbb{R}) \rtimes \mathbb{R} \rtimes \cdots \rtimes \mathbb{R}$, which uses the amenablity of $\mathbb{R}$.

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REFERENCES

[1] E. BLANCHARD, Subtriviality of continuous fields of nuclear C^*-algebras, *J. Reine Angew. Math.* **489**(1997), 133–150.

[2] F.P. BOCA, *Rotation C^*-Algebras and Almost Mathieu Operators*, Theta Ser. Adv. Math., vol. 1, The Theta Foundation, Bucharest 2001.

[3] Z. CHEN, Q. XU, Z. YIN, Harmonic analysis on quantum tori, *Comm. Math. Phys.* **322**(2013), 755–805.

[4] A. CONNES, G. LANDI, Noncommutative manifolds, the instanton algebra and isospectral deformations, *Comm. Math. Phys.* **221**(2001), 141–159.

[5] A. CONNES, H. MOSCOVICI, Modular curvature for noncommutative two-tori, *J. Amer. Math. Soc.* **27**(2014), 639–684.

[6] K.R. DAVIDSON, *C^*-Algebras by Example*, Amer. Math. Soc., Providence, RI 1996.

[7] G.A. ELLIOTT, Gaps in the spectrum of an almost periodic Schrödinger operator, *C.R. Math. Rep. Acad. Sci. Canada* **4**(1982), 255–259.

[8] G.A. ELLIOTT, H. LI, Morita equivalence of smooth noncommutative tori, *Acta Math.* **199**(2007), 1–27.

[9] P.A. FILLMORE, *A User's Guide to Operator Algebras*, Wiley-interscience, New York 1996.

[10] V. GAYRAL, J.M. GRACIA-BONDÍA, B. IOCHUM, T. SCHÜCKER, J.C. JOSEPH, Moyal planes are spectral triples, *Comm. Math. Phys.* **246**(2004), 569–623.

[11] J.M. GRACIA-BONDÍA, J.C. VÁRILLY, Algebras of distributions suitable for phase-space quantum mechanics. I, *J. Math. Phys.* **29**(1988), 869–879.

[12] J.M. GRACIA-BONDÍA, J.C. VÁRILLY, H. FIGUEROA, *Elements of Noncommutative Geometry*, Springer Sci. Business Media, Boston 2013.

[13] P. GREEN, The local structure of twisted covariance algebras, *Acta Math.* **140**(1978), 191–250.

[14] U. HAAGERUP, M. RØRDAM, Perturbations of the rotation C^*-algebras and of the Heisenberg commutation relation, *Duke Math. J.* **77**(1995), 627–656.

[15] B.C. HALL, *Quantum Theory for Mathematicians*, Grad. Texts Math., vol. 267, Springer, New York 2013.

[16] E. KIRCHBERG, N.C. PHILLIPS., Embedding of continuous fields of C^*-algebras in the Cuntz algebra $\mathcal{O}_2$, *J. Reine Angew. Math.* **525**(2000), 55–94.

[17] V. NAIR, A. POLYCHRONAKOS, Quantum mechanics on the noncommutative plane and sphere, *Phys. Lett. B* **505**(2001), 267–274.

[18] N. NEKRASOV, A. SCHWARZ, Instantons on noncommutative $\mathbb{R}^4$, and $(2,0)$ superconformal six dimensional theory, *Comm. Math. Phys.* **198**(1998), 689–703.

[19] D. OLESEN, G.K. PEDERSEN, M. TAKESAKI, Ergodic actions of compact abelian groups, *J. Operator Theory* **3**(1980), 237–269.

[20] J.A. PACKER, Projective representations and the Mackey obstruction - a survey, in *Group Representations, Ergodic Theory, and Mathematical Physics. A Tribute to George W. Mackey*, Contemp. Math., vol./ 449, Amer. Math. Soc., Providence, RI 2008, pp. 345–378.

[21] M.A. RIEFFEL, Non-commutative tori — A case study of non-commutative differentiable manifolds, *Contemp. Math.* **105**(1990), 191–211.

[22] M.A. RIEFFEL, *Deformation Quantization for Actions of* $\mathbb{R}^d$, Amer. Math. Soc., Providence, RI 1993.

[23] J. SLAWNY, On factor representations and the C^*-algebra of canonical commutation relations, *Comm. Math. Phys.* **24**(1972), 151–170.

[24] A. SMAILAGIC, E. SPALLUCCI, UV divergence-free QFT on noncommutative plane, *J. Phys. A* **36**(2003), L517–L521.

[25] J.C. VÁRILLY, *An Introduction to Noncommutative Geometry*, EMS Ser. Lect. Math., European Math. Soc., Zürich 2006.

[26] D. VOICULESCU, A non-commutative Weyl–von Neumann theorem, *Rev. Roumaine Math. Pures Appl.* **21**(1976), 97–113.

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