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SCHUR MULTIPLIERS ON $\mathcal{B}(L^p, L^q)$

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ABSTRACT. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces and $1 \leqslant p, q \leqslant +\infty$. We give a definition of Schur multipliers on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ which extends the definition of classical Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$. Our main result is a characterization of Schur multipliers in the case $1 \leqslant q \leqslant p \leqslant +\infty$. When $1 < q \leqslant p < +\infty$, $\phi \in L^\infty(\Omega_1 \times \Omega_2)$ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if and only if there are a measure space (a probability space when $p \neq q$) (Ω, μ), $a \in L^\infty(\mu_1, L^p(\mu))$ and $b \in L^\infty(\mu_2, L^{q'}(\mu))$ such that, for almost every $(s,t) \in \Omega_1 \times \Omega_2$,

$$\phi(s,t) = \langle a(s), b(t) \rangle.$$

This result is new, even in the classical case. As a consequence, we give new inclusion relationships among the spaces of Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$.

KEYWORDS: *Multiplier, tensor product.*

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1. INTRODUCTION

If $1 \leqslant r < +\infty$, we denote by ℓ_r the Banach space of r-summable sequences $(x_i)_{i \geqslant 1} \subset \mathbb{C}$ (that is, $\sum\limits_i |x_i|^r < +\infty$) endowed with the norm $\|x\|_{\ell_r} = \Big(\sum\limits_i |x_i|^r\Big)^{1/r}$. Let ℓ_∞ be the Banach space of bounded sequences $(y_i)_{i \geqslant 1} \subset \mathbb{C}$ with the norm $\|y\|_{\ell_\infty} = \sup\limits_i |y_i|$. If $n \in \mathbb{N}$, we denote by ℓ_r^n the n-dimensional versions of the spaces introduced before.

Let $m = (m_{ij})_{i,j \geqslant 1}$ be a bounded family of complex numbers and let $1 \leqslant p, q \leqslant +\infty$. We say that m is a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$ if for any matrix $[a_{ij}]_{i,j \geqslant 1}$ in $\mathcal{B}(\ell_p, \ell_q)$, the matrix $[m_{ij} a_{ij}]_{i,j \geqslant 1}$ defines an element of $\mathcal{B}(\ell_p, \ell_q)$. An application of the closed graph theorem shows that m is a Schur multiplier if and only if the mapping

$$\begin{aligned} T_m : \mathcal{B}(\ell_p, \ell_q) &\to \mathcal{B}(\ell_p, \ell_q) \\ [a_{ij}]_{i,j \geqslant 1} &\mapsto [m_{ij} a_{ij}]_{i,j \geqslant 1} \end{aligned} \tag{1.1}$$

is bounded. By definition, the norm of the Schur multiplier m is the norm of T_m.

There is a well-known characterization of Schur multipliers on $\mathcal{B}(\ell_2)$ (see for instance Theorem 5.1 in [11]) which can be extended to the case $\mathcal{B}(\ell_p)$ as follows.

THEOREM 1.1 ([11], Theorem 5.10). *Let $\phi = (c_{ij})_{i,j\in\mathbb{N}} \subset \mathbb{C}$, $C \geqslant 0$ be a constant and let $1 \leqslant p < \infty$. The following are equivalent:*

(i) *ϕ is a Schur multiplier on $\mathcal{B}(\ell_p)$ with norm $\leqslant C$;*

(ii) *there is a measure space (Ω, μ) and elements $(x_j)_{j\in\mathbb{N}}$ in $L^p(\mu)$ and $(y_i)_{i\in\mathbb{N}}$ in $L^{p'}(\mu)$ such that*

$$\forall i,j \in \mathbb{N}, \quad c_{ij} = \langle x_j, y_i\rangle \quad \text{and} \quad \sup_i \|y_i\|_{p'} \sup_j \|x_j\|_p \leqslant C.$$

Denote by $\mathcal{M}(p,q)$ the space of Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$. In [3], Bennett gives some results about the inclusions between the spaces $\mathcal{M}(p,q)$. In the same paper, he also gives a necessary and sufficient condition for a family m to belong to $\mathcal{M}(p,q)$, using the theory of absolutely summing operators. Theorem 1.1 provides a different type of characterization, which is more explicit and useful.

Let (Ω_1, μ_1) and (Ω_2, μ_2) be two σ-finite measure spaces. We will identify $L^2(\Omega_1 \times \Omega_2)$ with the space $S^2(L^2(\Omega_1), L^2(\Omega_2))$ of Hilbert–Schmidt operators. If $J \in L^2(\Omega_1 \times \Omega_2)$, the operator

$$\begin{aligned} X_J : L^2(\Omega_1) &\to L^2(\Omega_2) \\ f &\mapsto \int_{\Omega_1} J(t,\cdot) f(t) \mathrm{d}\mu_1(t) \end{aligned}$$

is a Hilbert–Schmidt operator and $\|X_J\|_2 = \|J\|_{L^2}$. Moreover, any element of $S^2(L^2(\Omega_1), L^2(\Omega_2))$ has this form.

Let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. We may associate the operator

$$\begin{aligned} R_\phi : S^2(L^2(\Omega_1), L^2(\Omega_2)) &\to S^2(L^2(\Omega_1), L^2(\Omega_2)) \\ X_J &\mapsto X_{\phi J} \end{aligned}$$

whose norm is equal to $\|\phi\|_\infty$.

We say that ϕ is a Schur multiplier on $\mathcal{B}(L^2(\Omega_1), L^2(\Omega_2))$ if R_ψ extends to a (necessarily unique) bounded operator still denoted by

$$R_\phi \colon \mathcal{K}(L^2(\Omega_1), L^2(\Omega_2)) \to \mathcal{K}(L^2(\Omega_1), L^2(\Omega_2)),$$

where $\mathcal{K}(L^2(\Omega_1), L^2(\Omega_2))$ denotes the space of compact operators from $L^2(\Omega_1)$ into $L^2(\Omega_2)$. When ϕ is a Schur multiplier, the norm of ϕ is by definition the norm of R_ϕ as an operator from $\mathcal{K}(L^2(\Omega_1), L^2(\Omega_2))$ into itself.

A characterization similar to the one in Theorem 1.1 holds in this setting. The following result was established by Peller [9].

THEOREM 1.2. *Let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$ and $C > 0$. The following are equivalent:*

(i) *ϕ is a Schur multiplier and $\|R_\phi\| < C$;*

(ii) *there exist families* $(a_i)_{i\geqslant 1} \subset L^\infty(\Omega_1)$ *and* $(b_i)_{i\geqslant 1} \subset L^\infty(\Omega_2)$ *such that*

$$\operatorname*{essup}_{s\in\Omega_1} \sum_{i=1}^{+\infty} |a_i(s)|^2 < C, \quad \operatorname*{essup}_{t\in\Omega_2} \sum_{i=1}^{+\infty} |b_i(t)|^2 < C,$$

and for almost every $(s,t) \in \Omega_1 \times \Omega_2$,

$$\phi(s,t) = \sum_{i=1}^{+\infty} a_i(s) b_i(t).$$

See also [12] for another formulation of this theorem and results about Schur multipliers in the measurable case.

In this paper, we define more generally Schur multipliers on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ for some measure spaces (Ω_1, μ_1) and (Ω_2, μ_2). To any $\phi \in L^\infty(\Omega_1, \Omega_2)$, we associate a linear mapping

$$T_\phi : L^{p'}(\Omega_1) \otimes L^q(\Omega_2) \to L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$$

and we say that ϕ is a Schur multiplier if T_ϕ is bounded. When $\Omega_1 = \Omega_2 = \mathbb{N}$ with the counting measures, T_ϕ corresponds to (1.1).

In the case $1 \leqslant q \leqslant p \leqslant +\infty$, we characterize the elements of $L^\infty(\Omega_1 \times \Omega_2)$ which are Schur multipliers on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$. We prove that if $1 < q \leqslant p < +\infty$, ϕ is a Schur multiplier if and only if there are a measure space (a probability space when $p \neq q$) (Ω, μ), $a \in L^\infty(\mu_1, L^p(\mu))$ and $b \in L^\infty(\mu_2, L^{q'}(\mu))$ such that, for almost every $(s,t) \in \Omega_1 \times \Omega_2$,

$$\phi(s,t) = \langle a(s), b(t) \rangle,$$

where $L^\infty(\mu_1, L^r(\mu))$ is the Bochner space valued in $L^r(\mu)$.

This result is new, even in the setting of classical Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$, and is of different nature than the characterization of Bennett. As a consequence, we give in the last section of this article new results of comparisons for the spaces $\mathcal{M}(p,q)$.

1.1. NOTATIONS. Let X and Y be Banach spaces.

If $z \in X \otimes Y$, the projective tensor norm of z is defined by

$$\|z\|_\wedge := \inf \Big\{ \sum \|x_i\| \|y_i\| \Big\},$$

where the infimum runs over all finite families $(x_i)_i$ in X and $(y_i)_i$ in Y such that

$$z = \sum_i x_i \otimes y_i.$$

The completion $X \overset{\wedge}{\otimes} Y$ of $(X \otimes Y, \|\cdot\|_\wedge)$ is called the projective tensor product of X and Y. Note that the projective tensor product is commutative, that is $X \overset{\wedge}{\otimes} Y = Y \overset{\wedge}{\otimes} X$.

The mapping taking any functional $\omega\colon X \otimes Y \to \mathbb{C}$ to the operator $u\colon X \to Y^*$ defined by $\langle u(x), y\rangle = \omega(x \otimes y)$ for any $x \in X, y \in Y$, induces an isometric identification

$$(X \overset{\wedge}{\otimes} Y)^* = \mathcal{B}(X, Y^*). \tag{1.2}$$

We refer to Chapter 8, Corollary 2 in [7] for this fact.

Let (Ω, μ) be a localizable measure space. Denote by $L^p(\Omega, Y)$ the Bochner space of p-integrable functions from Ω into Y. By Chapter 8, Example 10 in [7], the natural embedding $L^1(\Omega) \otimes Y \subset L^1(\Omega; Y)$ extends to an isometric isomorphism

$$L^1(\Omega, Y) = L^1(\Omega) \overset{\wedge}{\otimes} Y. \tag{1.3}$$

By (1.2), this implies

$$L^1(\Omega, Y)^* = \mathcal{B}(L^1(\Omega), Y^*). \tag{1.4}$$

Assume that Y^* has the Radon–Nikodym property (in short, Y^* has RNP). In this case,

$$L^1(\Omega, Y)^* = L^\infty(\Omega, Y^*).$$

The latter implies that

$$L^\infty(\Omega, Y^*) = \mathcal{B}(L^1(\Omega), Y^*), \tag{1.5}$$

and the isometric isomorphism is given by

$$\begin{aligned} L^\infty(\Omega, Y^*) &\to \mathcal{B}(L^1(\Omega), Y^*) \\ g &\mapsto \Big[f \in L^1(\Omega) \mapsto \int_\Omega f(t)g(t)\mathrm{d}\mu(t)\Big]. \end{aligned}$$

Assume now that $Y = L^1(\Omega')$ where (Ω', μ') is a localizable measure space. Then, an application of Fubini theorem gives

$$L^1(\Omega, L^1(\Omega')) = L^1(\Omega \times \Omega').$$

Using equality (1.3), we deduce that

$$\mathcal{B}(L^1(\Omega), L^\infty(\Omega')) = L^\infty(\Omega \times \Omega'), \tag{1.6}$$

and the correspondence is given by

$$\begin{aligned} L^\infty(\Omega \times \Omega') &\to \mathcal{B}(L^1(\Omega), L^\infty(\Omega')) \\ \psi &\mapsto \Big[f \in L^1(\Omega) \mapsto \int_\Omega f(t)\psi(t, \cdot)\mathrm{d}\mu(t)\Big]. \end{aligned}$$

We denote by u_ψ the corresponding element of $\mathcal{B}(L^1(\Omega), L^\infty(\Omega'))$.

If $z = \sum_i x_i \otimes y_i \in X \otimes Y$, $x^* \in X^*$ and $y^* \in Y^*$, we write

$$\langle z, x^* \otimes y^*\rangle = \sum_i x^*(x_i)y^*(y_i).$$

Then, the injective tensor norm of $z \in X \otimes Y$ is given by

$$\|z\|_\vee = \sup_{\|x^*\| \leqslant 1, \|y^*\| \leqslant 1} |\langle z, x^* \otimes y^* \rangle|.$$

The completion $X \overset{\vee}{\otimes} Y$ of $(X \otimes Y, \|\cdot\|_\vee)$ is called the injective tensor product of X and Y.

In this paper, we will often identify $X^* \otimes Y$ with the finite rank operators from X into Y as follow. If $u = \sum_i x_i^* \otimes y_i \in X^* \otimes Y$, we define $\widetilde{u} : X \to Y$ by

$$\widetilde{u}(x) = \sum_i x_i^*(x) y_i, \quad \forall x \in X. \tag{1.7}$$

Then, it is easy to check that $\|u\|_\vee = \|\widetilde{u}\|_{\mathcal{B}(X,Y)}$.

Moreover, if Y has the approximation property (see e.g. [6] for the definition), Theorem 1.4.21 in [6] gives the isometric identification

$$X^* \overset{\vee}{\otimes} Y = \mathcal{K}(X, Y)$$

where $\mathcal{K}(X, Y)$ denotes the space of compact operators from X into Y.

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two localizable measure spaces. Let $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$. Then $L^q(\Omega_2)$ has the approximation property so that we have

$$L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) = \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)). \tag{1.8}$$

Finally, if we assume that $1 < p, q < +\infty$, then by Theorem 2.5 in [5] and (1.2),

$$(L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2))^{**} = (L^p(\Omega_1) \overset{\wedge}{\otimes} L^{q'}(\Omega_2))^* = \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2)). \tag{1.9}$$

2. DEFINITION OF SCHUR MULTIPLIERS ON $\mathcal{B}(L^p, L^q)$

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two localizable measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. Let $1 \leqslant p, q \leqslant \infty$ and denote by p' and q' their conjugate exponents.

Let

$$T_\phi : L^{p'}(\Omega_1) \otimes L^q(\Omega_2) \to \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$$

be defined for any elementary tensor $f \otimes g \in L^{p'}(\Omega_1) \otimes L^q(\Omega_2)$ by

$$[T_\phi(f \otimes g)](h) = \Big(\int_{\Omega_1} \phi(s, \cdot) f(s) h(s) \mathrm{d}\mu_1(s) \Big) g(\cdot) \in L^q(\Omega_2),$$

for all $h \in L^p(\Omega_1)$.

We have an inclusion

$$L^{p'}(\Omega_1) \otimes L^q(\Omega_2) \subset L^{p'}(\Omega_1, L^q(\Omega_2))$$

given by $f \otimes g \mapsto [s \in \Omega_1 \mapsto f(s)g]$. Under this identification, T_ϕ is the multiplication by ϕ. Note that $L^{p'}(\Omega_1, L^q(\Omega_2))$ is invariant by multiplication by an element of $L^\infty(\Omega_1 \times \Omega_2)$ and that we have a contractive inclusion

$$L^{p'}(\Omega_1, L^q(\Omega_2)) \subset L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2).$$

Therefore, T_ϕ is valued in $L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$. Using the identification

$$L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) \subset \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$$

given by (1.7), we deduce that the elements of $L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$ are compact operators as limits of finite rank operators for the operator norm.

DEFINITION 2.1. We say that ϕ is a *Schur multiplier* on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if there exists a constant $C \geqslant 0$ such that for all $u \in L^{p'}(\Omega_1) \otimes L^q(\Omega_2)$,

$$\|T_\phi(u)\|_{\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))} \leqslant \|u\|_\vee,$$

that is, if T_ϕ extends to a bounded operator

$$T_\phi : L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) \to L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2).$$

In this case, the norm of ϕ is by definition the norm of T_ϕ.

REMARK 2.2. By $\mathcal{E}_1$ (respectively $\mathcal{E}_2$) we denote the space of simple functions on Ω_1 (respectively Ω_2). By density of $\mathcal{E}_1 \otimes \mathcal{E}_2$ in $L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$, T_ϕ extends to a bounded operator from $L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$ into itself if and only if it is bounded on $\mathcal{E}_1 \otimes \mathcal{E}_2$ equipped with the injective tensor norm.

Assume that $1 < p, q < +\infty$. By (1.8) we have

$$L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) = \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)),$$

so that ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if and only if T_ϕ extends to a bounded operator

$$T_\phi : \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)) \to \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)).$$

In this case, considering the bi-adjoint of T_ϕ, we obtain by (1.9) a w*-continuous mapping

$$\widetilde{T}_\phi : \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2)) \to \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$$

which extends T_ϕ. This explains the terminology "ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$".

2.1. CLASSICAL SCHUR MULTIPLIERS. Assume that $\Omega_1 = \Omega_2 = \mathbb{N}$ and that μ_1 and μ_2 are the counting measures. An element $\phi \in L^\infty(\mathbb{N}^2)$ is given by a family $c = (c_{ij})_{i,j\in\mathbb{N}}$ of complex numbers, where $c_{ij} = \phi(j,i)$. In this situation, the mapping T_ϕ is nothing but the classical Schur multiplier

$$A = [a_{ij}]_{i,j\geqslant 1} \in \mathcal{B}(\ell_p, \ell_q) \mapsto [c_{ij}a_{ij}]_{i,j\geqslant 1}.$$

When this mapping is bounded from $\mathcal{B}(\ell_p, \ell_q)$ into itself, we will denote it by T_c.

2.1.1. NOTATIONS. If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $n \in \mathbb{N}^*$, we denote by $\mathcal{A}_{n,\Omega}$ the collection of n-tuples $(A_1, \ldots, A_n)$ of pairwise disjoint elements of $\mathcal{F}$ such that

$$\text{for all } 1 \leqslant i \leqslant n, \quad 0 < \mu(A_i) < +\infty.$$

If $A = (A_1, \ldots, A_n) \in \mathcal{A}_{n,\Omega}$ and $1 \leqslant p \leqslant +\infty$, denote by $S_{A,p}$ the subspace of $L^p(\Omega)$ generated by $\chi_{A_1}, \ldots, \chi_{A_n}$. Then $S_{A,p}$ is 1-complemented in $L^p(\Omega)$, and a norm one projection from $L^p(\Omega)$ into $S_{A,p}$ is given by the conditional expectation

$$\begin{aligned} P_{A,p} : L^p(\Omega) &\to L^p(\Omega) \\ f &\mapsto \sum_{i=1}^{n} \frac{1}{\mu(A_i)} \Big(\int_{A_i} f \Big) \chi_{A_i}. \end{aligned} \tag{2.1}$$

Note that the mapping

$$\begin{aligned} \varphi_{A,p} : S_{A,p} &\to \ell_p^n \\ f = \sum_i a_i \chi_{A_i} &\mapsto (a_i(\mu_1(A_i))^{1/p})_{i=1}^n \end{aligned} \tag{2.2}$$

is an isometric isomorphism between $S_{A,p}$ and ℓ_p^n.

PROPOSITION 2.3. *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. The following are equivalent:*

(i) *ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$;*

(ii) *for all $n, m \in \mathbb{N}^*$, for all $A = (A_1, \ldots, A_n) \in \mathcal{A}_{n,\Omega_1}, B = (B_1, \ldots, B_m) \in \mathcal{A}_{m,\Omega_2}$, write*

$$\phi_{ij} = \frac{1}{\mu_1(A_j)\mu_2(B_i)} \int_{A_j \times B_i} \phi \mathrm{d}\mu_1 \mathrm{d}\mu_2.$$

Then the Schur multipliers on $\mathcal{B}(\ell_p^n, \ell_q^m)$ associated with the families $\phi_{A,B} = (\phi_{ij})$ are uniformly bounded with respect to n, m, A and B.

In this case,

$$\|T_\phi\| = \sup_{n,m,A,B} \|T_{\phi_{A,B}}\| < +\infty.$$

Proof. (i) $\Rightarrow$ (ii) Assume that ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ with $\|T_\phi\| \leqslant 1$. Let $n, m \in \mathbb{N}^*$, $A = (A_1, \ldots, A_n) \in \mathcal{A}_{n,\Omega_1}$ and $B = (B_1, \ldots, B_m) \in \mathcal{A}_{m,\Omega_2}$. Let $c = \sum_{i,j} c(i,j) e_j \otimes e_i \in \ell_{p'}^n \otimes \ell_q^m \simeq \mathcal{B}(\ell_p^n, \ell_q^m)$.

Let $\varphi_{A,p} : S_{A,p} \to \ell_p^n$ and $\psi_{B,q} : S_{B,q} \to \ell_q^m$ be the isometries defined in (2.2). Then $\widetilde{c} := \psi_{B,q}^{-1} \circ c \circ \varphi_{A,p} : S_{A,p} \to S_{B,q}$ satisfies $\|\widetilde{c}\| = \|c\|$ and we have

$$\widetilde{c} = \sum_{i,j} \frac{c(i,j)}{\mu_1(A_j)^{1/p'}\mu_2(B_i)^{1/q}} \chi_{A_j} \otimes \chi_{B_i} := \sum_{i,j} \widetilde{c}(i,j) \chi_{A_j} \otimes \chi_{B_i},$$

where $\widetilde{c}(i,j) = \frac{c(i,j)}{\mu_1(A_j)^{1/p'}\mu_2(B_i)^{1/q}}$.

The operator $u := \psi_{B,q} \circ P_{B,q} \circ T_\phi(\widetilde{c})_{|S_{A,p}} \circ \varphi_{A,p}^{-1} : \ell_p^n \to \ell_q^m$ satisfies

$$\|u\| \leqslant \|T_\phi(\widetilde{c})\|$$

and by assumption

$$\|T_\phi(\widetilde{c})\| \leqslant \|\widetilde{c}\|$$

so that

$$\|u\| \leqslant \|\widetilde{c}\| = \|c\|. \tag{2.3}$$

Let us prove that $u = T_{\phi_{A,B}}(c)$ where $T_{\phi_{A,B}}$ is the Schur multiplier associated with the family (ϕ_{ij}).

Write $u(i,j) := \psi_{B,q} \circ P_{B,q} \circ T_\phi(\chi_{A_j} \otimes \chi_{B_i})_{|S_{A,p}} \circ \varphi_{A,p}^{-1}$. We have

$$u = \sum_{i,j} \widetilde{c}(i,j) u(i,j).$$

Let $1 \leqslant k \leqslant n$.

$$\begin{aligned}
[u(i,j)](e_k) &= [\psi_{B,q} \circ P_{B,q} \circ T_\phi(\chi_{A_j} \otimes \chi_{B_i})_{|S_{A,p}}]\Big(\frac{1}{\mu_1(A_k)^{1/p}} \chi_{A_k}\Big) \\
&= \frac{1}{\mu_1(A_k)^{1/p}} [\psi_{B,q} \circ P_{B,q}]\Big(\chi_{B_i}(\cdot) \int_{\Omega_1} \phi(s,\cdot) \chi_{A_j}(s) \chi_{A_k}(s) \mathrm{d}\mu_1(s)\Big)
\end{aligned}$$

so that $[u(i,j)](e_k) = 0$ if $k \neq j$ and if $k = j$ then

$$\begin{aligned}
[u(i,j)](e_k) &= \frac{1}{\mu_1(A_k)^{1/p}} [\psi_{B,q} \circ P_{B,q}]\Big(\chi_{B_i}(\cdot) \int_{A_j} \phi(s,\cdot) \mathrm{d}\mu_1(s)\Big) \\
&= \frac{1}{\mu_1(A_k)^{1/p}\mu_2(B_i)} \Big(\int_{A_j \times B_i} \phi\Big) \psi_q(\chi_{B_i}) = \frac{1}{\mu_1(A_k)^{1/p}\mu_2(B_i)^{1/q'}} \Big(\int_{A_j \times B_i} \phi\Big) e_i.
\end{aligned}$$

It follows that

$$\begin{aligned}
u &= \sum_{i,j} \frac{c(i,j)}{\mu_1(A_j)^{1/p'}\mu_2(B_i)^{1/q}} \frac{1}{\mu_1(A_j)^{1/p}\mu_2(B_i)^{1/q'}} \Big(\int_{A_j \times B_i} \phi\Big) e_j \otimes e_i \\
&= \sum_{i,j} \frac{c(i,j)}{\mu_1(A_j)\mu_2(B_i)} \Big(\int_{A_j \times B_i} \phi\Big) e_j \otimes e_i = \sum_{i,j} \phi_{ij} c(i,j) e_j \otimes e_i
\end{aligned}$$

that is, $u = T_{\phi_{A,B}}(c)$. We conclude thanks to the inequality (2.3).

(ii) $\Rightarrow$ (i) Assume now that the assertion (ii) is satisfied: we show that ϕ is a Schur multiplier. By Remark 2.2, we just need to show that T_ϕ is bounded on $\mathcal{E}_1 \otimes \mathcal{E}_2$. Let $v \in \mathcal{E}_1 \otimes \mathcal{E}_2$ and write $\alpha = \sup_{n,m,A,B} \|T_c\|$. We will show that $\|T_\phi(v)\| \leqslant \alpha\|v\|$. By density, it is enough to prove that for any $h_1 \in \mathcal{E}_1, h_2 \in \mathcal{E}_2$,

$$|\langle [T_\phi(v)](h_1), h_2\rangle_{L^q,L^{q'}}| \leqslant \alpha \|v\|_{\mathcal{B}(L^p(\Omega_1),L^q(\Omega_2))} \|h_1\|_{L^p(\Omega_1)} \|h_2\|_{L^{q'}(\Omega_2)}. \tag{2.4}$$

Hence, there exist $n, m \in \mathbb{N}^*, A = (A_1, \ldots, A_n) \in \mathcal{A}_{n,\Omega_1}, B = (B_1, \ldots, B_m) \in \mathcal{A}_{m,\Omega_2}$ and complex numbers $v(i,j), a_i, b_j$ such that

$$v = \sum_{i,j} v(i,j)\chi_{A_j} \otimes \chi_{B_i}, h_1 = \sum_j a_j \chi_{A_j} \quad \text{and} \quad h_2 = \sum_i b_i \chi_{B_i}.$$

Equation (2.4) can be rewritten as

$$\Big|\sum_{i,j} v(i,j) a_j b_i \Big(\int_{A_j \times B_i} \phi \Big)\Big| \leqslant \alpha \|v\| \|h_1\|_{L^p(\Omega_1)} \|h_2\|_{L^{q'}(\Omega_2)}. \tag{2.5}$$

Consider $\widetilde{v} := \psi_{B,q} \circ v \circ \varphi_{A,p}^{-1} : \ell_p^n \to \ell_q^m$ and $z := \psi_{B,q} \circ P_{B,q} \circ T_\phi(v)_{|S_{A,p}} \circ \phi_{A,p}^{-1} : \ell_p^n \to \ell_q^m$. The computations made in the first part of the proof show that $z = T_m(\widetilde{v})$ where m is the family (ϕ_{ij}).

Now, let $x := \varphi_{A,p}(h_1)$ and $y := \psi_{B,q'}(h_2)$. Since T_m is bounded with norm smaller than α we have

$$|\langle [T_m(\widetilde{c})](x), y\rangle_{\ell_q^m, \ell_{q'}^m}| \leqslant \alpha \|\widetilde{c}\|_{\mathcal{B}(\ell_p^n, \ell_q^m)} \|x\|_{\ell_p^n} \|y\|_{\ell_{q'}^m}. \tag{2.6}$$

An easy computation shows that the left-hand side on this equality is nothing but the left-hand side of the inequality (2.5). Finally, the right-hand side of the inequalities (2.5) and (2.6) are equal, which concludes the proof. ∎

3. (p,q)-FACTORABLE OPERATORS

Let X and Y be Banach spaces.

3.1. DUAL NORM. Let $M \subset X$ and $N \subset Y$ be finite dimensional subspaces (in short, f.d.s.). If $u = \sum_{i=1}^n x_i \otimes y_i \in M \otimes N$ and $v = \sum_{j=1}^m x_j^* \otimes y_j^* \in M^* \otimes N^*$ we set

$$\langle v, u\rangle = \sum_{i,j} \langle x_j^*, x_i\rangle \langle y_j^*, y_i\rangle.$$

Let α be a tensor norm on tensor products of finite dimensional spaces. We define, for $z \in M \otimes N$,

$$\alpha'(z, M, N) = \sup\{|\langle v, u\rangle| : v \in M^* \otimes N^*, \alpha(v) \leqslant 1\}.$$

Now, for $z \in X \otimes Y$, we set

$$\alpha'(z, X, Y) = \inf\{\alpha'(z, M, N) : M \subset X, N \subset Y \text{ f.d.s., } z \in M \otimes N\}.$$

By Chapter 15 in [4], α' defines a tensor norm on $X \otimes Y$, called the dual norm of α.

In the sequel, we will write $\alpha'(z)$ instead of $\alpha'(z, X, Y)$ for the norm of an element $z \in X \otimes Y$ when there is no possible confusion.

3.2. LAPRESTÉ NORMS. Let $s \in [1, \infty]$. If $x_1, x_2, \dots, x_n \in X$, we define

$$w_s(x_i, X) := \sup_{x^* \in B_{X^*}} \Big(\sum_{i=1}^{n} |\langle x^*, x_i \rangle|^s \Big)^{1/s}.$$

Let $p, q \in [1, \infty]$ with $1/p + 1/q \geqslant 1$ and take $r \in [1, \infty]$ such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Denote by p' and q' the conjugate of p and q. For $z \in X \otimes Y$, we define

$$\alpha_{p,q}(z) = \inf \Big\{ \|(\lambda_i)_i\|_{\ell_r} w_{q'}(x_i, X) w_{p'}(y_i, Y) : z = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \Big\}.$$

Then, by Proposition 12.5 in [4] $\alpha_{p,q}$ is a norm on $X \otimes Y$ and we denote by $X \otimes_{\alpha_{p,q}} Y$ its completion.

3.3. (p, q)-FACTORABLE OPERATORS. If $T \in \mathcal{B}(X, Y^*)$ and $\xi = \sum_i x_i \otimes y_i \in X \otimes Y$, then in accordance with (1.2) we set

$$\langle T, \xi \rangle = \sum_i \langle T(x_i), y_i \rangle.$$

DEFINITION 3.1. Let $1 \leqslant p, q \leqslant \infty$ such that $1/p + 1/q \geqslant 1$. Let $T \in \mathcal{B}(X, Y^*)$. We say that $T \in \mathcal{L}_{p,q}(X, Y^*)$ if there exists a constant $C \geqslant 0$ such that

$$\forall \xi \in X \otimes Y, \quad |\langle T, \xi \rangle| \leqslant C \alpha'_{p,q}(\xi). \tag{3.1}$$

In this case, we write $L_{p,q}(T) = \inf\{C : C \text{ satisfying } (3.1)\}$.

Then $(\mathcal{L}_{p,q}(X, Y^*), L_{p,q})$ is a Banach space, called the *space of (p, q)-factorable operators*.

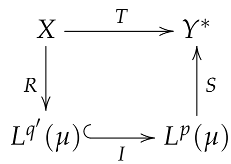
For a general definition of the spaces $\mathcal{L}_{p,q}(X, Y)$ (including the case when the range is not a dual space), see Chapter 17 in [4].

Since Y^* is 1-complemented in its bidual, Theorem 18.11 in [4] gives the following result.

THEOREM 3.2. *Let $1 \leqslant p, q \leqslant \infty$ such that $1/p + 1/q \geqslant 1$. Let $T \in \mathcal{B}(X, Y^*)$. The following two statements are equivalent:*

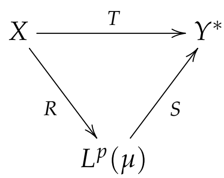
(i) $T \in \mathcal{L}_{p,q}(X, Y^*)$;

(ii) *there are a measure space (Ω, μ) (a probability space when $1/p + 1/q > 1$),* *operators* $R \in \mathcal{B}(X, L^{q'}(\mu))$ *and* $S \in \mathcal{B}(L^p(\mu), Y^*))$ *such that* $T = S \circ I \circ R$



where $I : L^{q'}(\mu) \to L^p(\mu)$ *is the inclusion mapping (well defined because* $q' \geqslant p$*).*
In this case, $L_{p,q}(T) = \inf \|S\|\|R\|$ *over all such factorizations.*

REMARK 3.3. Here we consider the case when $1/p + 1/q = 1$. Denote by p' the conjugate exponent of p. We have $T \in \mathcal{L}_{p,p'}(X, Y^*)$ if and only if there are a measure space (Ω, μ), operators $R \in \mathcal{B}(X, L^p(\mu))$ and $S \in \mathcal{B}(L^p(\mu), Y^*)$ such that $T = SR$



We usually write $\Gamma_p(X, Y^*)$ instead of $\mathcal{L}_{p,p'}(X, Y^*)$. Such operators are called *p-factorable*.

REMARK 3.4. Suppose that $X = L^1(\lambda)$ and $Y = L^1(\nu)$ for some localizable measure spaces (Ω_1, λ) and (Ω_2, ν). Consider $T \in \mathcal{B}(L^1(\lambda), L^\infty(\nu))$. By (1.6), there exists $\psi \in L^\infty(\lambda \times \nu)$ such that

$$T = u_\psi.$$

(See Subsection 1.1 for the notation.)

If $1 < q < +\infty$, $L^{q'}(\mu)$ has RNP so by (1.5),

$$\mathcal{B}(L^1(\lambda), L^{q'}(\mu)) = L^\infty(\lambda, L^{q'}(\mu)).$$

It means that if $R \in \mathcal{B}(X, L^{q'}(\mu))$, there exists $a \in L^\infty(\lambda, L^{q'}(\mu))$ such that

$$\forall f \in L^1(\lambda), \quad R(f) = \int_{\Omega_1} f(s)a(s)\mathrm{d}\lambda(s).$$

If $1 < p < +\infty$, then using (1.2), (1.3) and (1.4) we obtain

$$B(L^p(\mu), L^\infty(\nu)) = (L^p(\mu) \overset{\wedge}{\otimes} L^1(\nu))^* = L^\infty(\nu, L^{p'}(\mu)).$$

Hence, if $S \in \mathcal{B}(L^p(\mu), L^\infty(\nu))$, there exists $b \in L^\infty(\nu, L^{p'}(\mu))$ such that

$$\forall g \in L^p(\lambda), \quad S(g)(\cdot) = \langle g, b(\cdot) \rangle.$$

Thus, if $1 < p, q < +\infty$, there exist $a \in L^\infty(\lambda, L^{q'}(\mu))$ and $b \in L^\infty(\nu, L^{p'}(\mu))$ such that for almost every $(s,t) \in \Omega_1 \times \Omega_2$,

$$\psi(s,t) = \langle a(s), b(t)\rangle.$$

If T satisfies Theorem 3.2, the latter implies that for all $f \in L^1(\lambda)$,

$$T(f) = \int_{\Omega_1} \langle a(s), b(\cdot)\rangle f(s)\mathrm{d}s.$$

Using the same identifications we have the following cases:

(i) If $q = 1$ and $1 < p < +\infty$, then there exist $a \in L^\infty(\lambda \times \mu)$ and $b \in L^\infty(\nu, L^{p'}(\mu))$ such that for almost every $(s,t) \in \Omega_1 \times \Omega_2$,

$$\psi(s,t) = \langle a(s,\cdot), b(t)\rangle.$$

(ii) If $1 < q < +\infty$ and $p = +\infty$, then there exist $a \in L^\infty(\lambda, L^{q'}(\mu))$ and $b \in L^\infty(\nu \times \mu)$ such that for almost every $(s,t) \in \Omega_1 \times \Omega_2$,

$$\psi(s,t) = \langle a(s), b(t,\cdot)\rangle.$$

(iii) If $q = 1$ and $p = +\infty$, then there exist $a \in L^\infty(\lambda \times \mu)$ and $b \in L^\infty(\nu \times \mu)$ such that for almost every $(s,t) \in \Omega_1 \times \Omega_2$,

$$\psi(s,t) = \langle a(s,\cdot), b(t,\cdot)\rangle.$$

3.4. FINITE DIMENSIONAL CASE. If X and Y are finite dimensional, it follows from the very definition of the dual norm that

$$X \otimes_{\alpha'_{p,q}} Y = (X^* \otimes_{\alpha_{p,q}} Y^*)^*.$$

The next theorem describes the elements of this space.

THEOREM 3.5 ([4], Theorem 19.2). *Let E and F be Banach spaces. Let $p, q \in [1,\infty]$ with $1/p + 1/q \geqslant 1$ and $K \subset B_{E^*}$ and $L \subset B_{F^*}$ weak* compact norming sets for E and F, respectively. For $\phi : E \otimes F \to \mathbb{C}$ the following two statements are equivalent:*

(i) *$\phi \in (E \otimes_{\alpha_{p,q}} F)^*$;*

(ii) *there are a constant $A \geqslant 0$ and normalized Borel–Radon measures μ on K and ν on L such that for all $x \in E$ and $y \in F$,*

$$|\langle \phi, x \otimes y\rangle| \leqslant A\Big(\int_K |\langle x^*, x\rangle|^{q'} \mathrm{d}\mu(x^*)\Big)^{1/q'} \Big(\int_L |\langle y^*, y\rangle|^{p'} \mathrm{d}\mu(y^*)\Big)^{1/p'} \tag{3.2}$$

(if the exponent is ∞, we replace the integral by the norm).

In this case,

$$\|\phi\|_{(E \otimes_{\alpha_{p,q}} F)^*} = \inf\{A : A \text{ as in (ii)}\}.$$

This theorem will allow us to describe the predual of $\mathcal{L}_{p,q}(\ell_1^n, \ell_\infty^m)$, $n, m \in \mathbb{N}$. Let us apply the previous theorem with $E = \ell_\infty^n$ and $F = \ell_\infty^m$. Take $T \in \ell_1^n \otimes_{\alpha'_{p,q}} \ell_1^m = (\ell_\infty^n \otimes_{\alpha_{p,q}} \ell_\infty^m)^*$ and let

$$T = \sum_{i=1}^{n} \sum_{j=1}^{m} T(i,j) e_i \otimes e_j$$

be a representation of T. In the previous theorem, we can take $K = \{1, 2, \dots, n\}$ and $L = \{1, 2, \dots, m\}$. In this case, a normalized Borel–Radon measure μ on K is nothing but a sequence $\mu = (\mu_1, \dots, \mu_n)$ where, for all i, $\mu_i := \mu(\{i\}) \geqslant 0$ and $\sum_i \mu_i = 1$. Similarly, $\nu = (\nu_1, \dots, \nu_m)$ where, for all i, $\nu_i \geqslant 0$ and $\sum_i \nu_i = 1$. In this case, the inequality (3.2) means that for all sequences of complex numbers $x = (x_i)_{i=1}^n, y = (y_j)_{i=j}^m$,

$$\Big| \sum_{i=1}^{n} \sum_{j=1}^{m} T(i,j) x_i y_j \Big| \leqslant A \Big(\sum_{k=1}^{n} |x_k|^{q'} \mu_k \Big)^{1/q'} \Big(\sum_{k=1}^{m} |y_k|^{p'} \nu_k \Big)^{1/p'}.$$

Set $\alpha_k = x_k \mu_k^{1/q'}$, $\beta_k = y_k \nu_k^{1/p'}$ and define, for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$, $c(i,j)$ such that $T(i,j) = c(i,j) \mu_i^{1/q'} \nu_j^{1/p'}$ (we can assume $\mu_i > 0$ and $\nu_j > 0$). Then, the previous inequality becomes

$$\Big| \sum_{i=1}^{n} \sum_{j=1}^{m} c(i,j) \beta_j \alpha_i \Big| \leqslant A \|\alpha\|_{\ell_{q'}^n} \|\beta\|_{\ell_{p'}^m}.$$

This means that the operator $c : \ell_{q'}^n \to \ell_p^m$ whose matrix is $[c(i,j)]_{1 \leqslant j \leqslant m, 1 \leqslant i \leqslant n}$ has a norm smaller than A. Moreover, if we see T as a mapping from ℓ_∞^n into ℓ_1^m the relation between T and c means that T admits the following factorization

$$\begin{array}{ccc} \ell_\infty^n & \xrightarrow{\ T\ } & \ell_1^m \\ {\scriptstyle d_\mu}\big\downarrow & & \big\uparrow{\scriptstyle d_\nu} \\ \ell_{q'}^n & \xrightarrow[\ c\ ]{} & \ell_p^m \end{array}$$

where d_μ and d_ν are the operators of multiplication by $\mu = (\mu_1^{1/q'}, \dots, \mu_n^{1/q'})$ and $\nu = (\nu_1^{1/p'}, \dots, \nu_m^{1/p'})$. Those operators have norm 1.

Therefore, it is easy to check that

$$\|T\|_{(\ell_\infty^n \otimes_{\alpha_{p,q}} \ell_\infty^m)^*} = \inf\{\|c\| : T = d_\nu \circ c \circ d_\mu\}. \tag{3.3}$$

The elements of $(\ell_\infty^n \otimes_{\alpha_{p,q}} \ell_\infty^m)^*$ are called (q', p')-*dominated operators*. For more informations about this space in the infinite dimensional case (it is the predual of $\mathcal{L}_{p,q}$), see for instance Chapter 19 in [4].

By (3.3) and the fact that $\mathcal{L}_{p,q}(\ell_1^n, \ell_\infty^n) = (\ell_1^n \otimes_{\alpha'_{p,q}} \ell_1^m)^*$, we get the following result.

PROPOSITION 3.6. *Let $v = [v_{ij}] : \ell_1^n \to \ell_\infty^m$. Then*

$$L_{p,q}(v) = \sup |\mathrm{Tr}(vu)|$$

where the supremum runs over all $u : \ell_\infty^m \to \ell_1^n$ admitting the factorization

$$\begin{array}{ccc} \ell_\infty^m & \xrightarrow{\ u\ } & \ell_1^n \\ {\scriptstyle d_\mu}\downarrow & & \uparrow{\scriptstyle d_\nu} \\ \ell_{p'}^m & \xrightarrow[\ c\ ]{} & \ell_q^n \end{array}$$

with $\|d_\mu\| \leqslant 1, \|d_\nu\| \leqslant 1$ and $\|c\| \leqslant 1$.

Equivalently,

$$L_{p,q}(v) = \sup \Big\{ \Big| \sum_{i=1}^{m} \sum_{j=1}^{n} v_{ij} c_{ji} \mu_i \nu_j \Big| : \|c : \ell_{p'}^m \to \ell_q^n\| \leqslant 1, \|\mu\|_{\ell_{p'}^m} \leqslant 1, \|\nu\|_{\ell_{q'}^n} \leqslant 1 \Big\}.$$

4. THE MAIN RESULT

4.1. SCHUR MULTIPLIERS AND FACTORIZATION. Let p, q be two positive numbers such that $1 \leqslant q \leqslant p \leqslant \infty$. This condition is equivalent to $p, q \in [1, \infty]$ with $1/q + 1/p' \geqslant 1$, so that we can consider the space $\mathcal{L}_{q,p'}$.

The following results will allow us to give a description of the functions ϕ which are Schur multipliers.

LEMMA 4.1. *Let X, Y be Banach spaces and let $E \subset X, F \subset Y$ be 1-complemented subspaces of X and Y. For any $v \in E \otimes F$, denote by $\widetilde{\alpha}'_{q,p'}(v)$ the $\alpha'_{q,p'}$-norm of v as an element of $E \otimes F$ and by $\alpha'_{q,p'}(v)$ the $\alpha'_{q,p'}$-norm of v as an element of $X \otimes Y$. Then*

$$\widetilde{\alpha}'_{q,p'}(v) = \alpha'_{q,p'}(v).$$

Proof. The inequality $\widetilde{\alpha}'_{q,p'}(v) \geqslant \alpha'_{q,p'}(v)$ is easy to prove. For the converse inequality, take $v = \sum_k e_k \otimes f_k \in E \otimes F$ such that $\alpha'_{q,p'}(v) < 1$ and show that $\widetilde{\alpha}'_{q,p'}(v) < 1$. By assumption, there exists $M \subset X$ and $N \subset Y$ finite dimensional subspaces such that $v \in M \otimes N$ and

$$\alpha'(v, M, N) < 1.$$

By assumption, there exist two norm one projections P and Q, respectively from X onto E and from Y onto F. Set $M_1 = P(M) \subset E$ and $N_1 = Q(N) \subset F$. M_1 and N_1 are finite dimensional. Moreover, since $v \in E \otimes F$, it is easy to check that $(P \otimes Q)(v) = v$, where, for all $c = \sum_l a_l \otimes b_l \in X \otimes Y$,

$$(P \otimes Q)(c) = \sum_l P(a_l) \otimes Q(b_l).$$

Thus, $v \in M_1 \otimes N_1$. We will show that $\alpha'_{q,p'}(v, M_1, N_1) < 1$.

Let $z = \sum_{j=1}^{m} x_j^* \otimes y_j^* \in M_1^* \otimes N_1^*$ be such that $\alpha_{q,p'}(z) < 1$ and show that $|\langle v, z\rangle| \leqslant \alpha'_{q,p'}(v)$, so that $\alpha'_{q,p'}(v, M_1, N_1) \leqslant 1$.

Let $1 \leqslant r \leqslant \infty$ such that

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{p'} - 1.$$

The condition $\alpha_{q,p'}(z) < 1$ in $M_1^* \otimes N_1^*$ implies that z admits a representation $z = \sum_{j=1}^{m} \lambda_j m_j^* \otimes n_j^*$ where $m_j^* \in M_1^*, n_j^* \in N_1^*$ and

$$\|(\lambda_j)_j\|_{\ell_r} w_p(m_j^*, M_1^*) w_{q'}(n_j^*, N_1^*) < 1.$$

Set $\widetilde{z} := \sum_{j=1}^{m} \lambda_j P^*(m_j^*) \otimes Q^*(n_j^*)$ in $M^* \otimes N^*$. It is easy to check that

$$w_p(P^*(m_j^*), M^*) \leqslant w_p(m_j^*, M_1^*) \quad \text{and} \quad w_{q'}(Q^*(n_j^*), N^*) \leqslant w_{q'}(n_j^*, N_1^*).$$

Therefore, $\alpha_{q,p'}(\widetilde{z}, M^*, N^*) < 1$. Then, the condition $\alpha'_{q,p'}(v, M, N) < 1$ implies that

$$|\langle v, \widetilde{z}\rangle| \leqslant \alpha'_{q,p'}(v).$$

Finally, we have

$$\begin{aligned}\langle v, \widetilde{z}\rangle &= \sum_{j,k} \lambda_j \langle P^*(m_j^*), e_k\rangle \langle Q^*(n_j^*), f_k\rangle \\ &= \sum_{j,k} \lambda_j \langle m_j^*, P(e_k)\rangle \langle n_j^*, Q(f_k)\rangle = \sum_{j,k} \lambda_j \langle m_j^*, e_k\rangle \langle n_j^*, f_k\rangle = \langle v, z\rangle,\end{aligned}$$

and therefore

$$|\langle v, z\rangle| \leqslant \alpha'_{q,p'}(v).$$

This proves that $\widetilde{\alpha}'_{q,p'}(v) < 1$. ∎

We recall that if $\phi \in L^\infty(\Omega_1 \times \Omega_2)$, we denote by u_ϕ the mapping

$$\begin{aligned} u_\phi : L^1(\Omega_1) &\to L^\infty(\Omega_2) \\ f &\mapsto \int_{\Omega_1} \phi(s, \cdot) f(s) \mathrm{d}\mu_1(s).\end{aligned}$$

THEOREM 4.2. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be two localizable measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. Let $1 \leqslant q \leqslant p \leqslant \infty$. Then ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if and only if the operator u_ϕ belongs to $\mathcal{L}_{q,p'}(L^1(\Omega_1), L^\infty(\Omega_2))$. Moreover,*

$$\|T_\phi\| = L_{q,p'}(u_\phi).$$

Proof. Assume first that T_ϕ extends to a bounded operator

$$T_\phi : L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) \to L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$$

with norm $\leqslant 1$. To prove that $u_\phi \in \mathcal{L}_{q,p'}(L^1(\Omega_1), L^\infty(\Omega_2))$ with $L_{q,p'}(u_\phi) \leqslant 1$, we have to show that for any $v = \sum\limits_k f_k \otimes g_k \in L^1(\Omega_1) \otimes L^1(\Omega_2)$ with $\alpha'_{q,p'}(v) < 1$ we have

$$|u_\phi(v)| = \Big| \sum_k \langle u_\phi(f_k), g_k \rangle \Big| \leqslant 1.$$

By density, we can assume that f_k, g_k are simple functions. Hence, with the notations introduced in Section 2 there exist $n, m \in \mathbb{N}^*, A = (A_1, \dots, A_n) \in \mathcal{A}_{n,\Omega_1}$ and $B = (B_1, \dots, B_m) \in \mathcal{A}_{m,\Omega_2}$ such that, for all k, $f_k \in S_{A,1}$ and $g_k \in S_{B,1}$.

By Lemma 4.1, the $\alpha'_{q,p'}$-norm of v as an element of $S_{A,1} \otimes S_{B,1}$ is less than 1.

Let $\varphi_{A,1} : S_{A,1} \to \ell_1^n$ and $\psi_{B,1} : S_{B,1} \to \ell_1^m$ the isomorphisms defined in (2.2). Set $v' = \sum\limits_k \varphi_{A,1}(f_k) \otimes \psi_{B,1}(g_k) \in \ell_1^n \otimes \ell_1^m$. Since $\varphi_{A,1}$ and $\psi_{B,1}$ are isometries, we have $\alpha'_{q,p'}(v') < 1$. Using the identification (1.7), we obtain by (3.3) that v' admits a factorization

$$\begin{array}{ccc} \ell_\infty^n & \xrightarrow{\;v'\;} & \ell_1^m \\ {\scriptstyle d_\delta}\big\downarrow & & \big\uparrow{\scriptstyle d_\gamma} \\ \ell_p^n & \xrightarrow[\;c\;]{} & \ell_q^m \end{array}$$

where $\delta = (\delta_1, \dots, \delta_n)$, $\gamma = (\gamma_1, \dots, \gamma_m)$, d_δ and d_γ are the operators of multiplication and

$$\|d_\delta\| = \|\delta\|_{\ell_p} = 1, \quad \|d_\gamma\| = \|\gamma\|_{\ell_{q'}} = 1 \quad \text{and} \quad \|c\| < 1.$$

This factorization means that

$$v' = \sum_{i=1}^m \sum_{j=1}^n \gamma_i c(i,j) \delta_j e_j \otimes e_i.$$

Therefore, we have

$$v = \sum_{i=1}^m \sum_{j=1}^n \gamma_i c(i,j) \delta_j \; \varphi_{A,1}^{-1}(e_j) \otimes \psi_{B,1}^{-1}(e_i) = \sum_{i=1}^m \sum_{j=1}^n \gamma_i \frac{c(i,j)}{\mu_1(A_j)\mu_2(B_i)} \delta_j \, \chi_{A_j} \otimes \chi_{B_i}.$$

We compute

$$\begin{aligned} u_\phi(v) &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i \frac{c(i,j)}{\mu_1(A_j)\mu_2(B_i)} \delta_j \langle u_\phi(\chi_{A_j}), \chi_{B_i} \rangle \\ &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i \frac{c(i,j)}{\mu_1(A_j)\mu_2(B_i)} \delta_j \langle T_\phi(\chi_{A_j} \otimes \chi_{B_i})(\chi_{A_j}), \chi_{B_i} \rangle. \end{aligned}$$

Define

$$\widetilde{c} = \sum_{i=1}^{m}\sum_{j=1}^{n} \widetilde{c}(i,j)\chi_{A_j} \otimes \chi_{B_i} \in L^{p'}(\Omega_1) \otimes L^q(\Omega_2),$$

where $\widetilde{c}(i,j) = c_{i,j}\mu_1(A_j)^{-1/p'}\mu_2(B_i)^{-1/q}$.

Using the identification (1.7), it is easy to check that we have

$$\widetilde{c} = \psi_{B,q}^{-1} \circ c \circ \varphi_{A,p} : S_{A,p} \mapsto L^q(\Omega_2).$$

Therefore,

$$\|\widetilde{c}\|_{\vee} = \|c\|.$$

We have

$$\begin{aligned} u_\phi(v) &= \sum_{i=1}^{m}\sum_{j=1}^{n} \gamma_i \frac{\widetilde{c}(i,j)\mu_1(A_j)^{1/p'}\mu_2(B_i)^{1/q}}{\mu_1(A_j)\mu_2(B_i)} \delta_j \langle T_\phi(\chi_{A_j} \otimes \chi_{B_i})(\chi_{A_j}), \chi_{B_i}\rangle \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \gamma_i \widetilde{c}(i,j)\mu_1(A_i)^{-j1/p}\mu_2(B_i)^{-1/q'}\delta_j \langle T_\phi(\chi_{A_j} \otimes \chi_{B_i})(\chi_{A_j}), \chi_{B_i}\rangle \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \Big\langle T_\phi(\widetilde{c}(i,j)\chi_{A_j} \otimes \chi_{B_i})\Big(\frac{\delta_j}{\mu_1(A_j)^{1/p}}\chi_{A_j}\Big), \frac{\gamma_i}{\mu_2(B_i)^{1/q'}}\chi_{B_i}\Big\rangle \\ &= \langle T_\phi(\widetilde{c})(f), g\rangle_{L^q(\Omega_2), L^{q'}(\Omega_2)}, \end{aligned}$$

where

$$f = \sum_j \frac{\delta_j}{\mu_1(A_j)^{1/p}}\chi_{A_j} \quad \text{and} \quad g = \sum_i \frac{\gamma_i}{\mu_2(B_i)^{1/q'}}\chi_{B_i}.$$

Since $\|T_\phi\| \leqslant 1$, we deduce that

$$|u_\phi(v)| \leqslant \|T_\phi(\widetilde{c})\|\|f\|_p\|g\|_{q'} \leqslant \|\widetilde{c}\|\|\delta\|_{\ell_p}\|\gamma\|_{\ell_{q'}} = \|c\| \leqslant 1.$$

Conversely, assume that $u_\phi \in \mathcal{L}_{q,p'}(L^1(\Omega_1), L^\infty(\Omega_2))$ with $L_{q,p'}(u_\phi) \leqslant 1$. To prove that ϕ is a Schur multiplier, we will use Proposition 2.3. Let $n, m \in \mathbb{N}^*$, $A = (A_1, \dots, A_n) \in \mathcal{A}_{n,\Omega_1}$ and $B = (B_1, \dots, B_m) \in \mathcal{A}_{m,\Omega_2}$. Set

$$\phi_{ij} = \frac{1}{\mu_1(A_j)\mu_2(B_i)} \int_{A_j \times B_i} \phi \, \mathrm{d}\mu_1 \mathrm{d}\mu_2.$$

We want to show that the Schur multiplier on $\mathcal{B}(\ell_p^n, \ell_q^m)$ associated to the family $m = (\phi_{ij})_{i,j}$ has a norm less than 1. To prove that, let $c = \sum_{i,j} c(i,j) e_j \otimes e_i \in \mathcal{B}(\ell_p^n, \ell_q^m)$, $x = (x_j)_{j=1}^n$, $y = (y_i)_{i=1}^m$ in $\mathbb{C}$ be such that $\|c\| \leqslant 1$, $\|x\|_{\ell_p^n} = 1$, $\|y\|_{\ell_{q'}} = 1$. We have to show that

$$|\langle [T_m(c)](x), y\rangle_{\ell_q^m, \ell_{q'}^m}| \leqslant 1.$$

This inequality can be rewritten as

$$\Big|\sum_{i,j} c(i,j)\frac{x_j y_i}{\mu_1(A_j)\mu_2(B_i)}\Big(\int_{A_j\times B_i}\phi\Big)\Big| \leqslant 1. \tag{4.1}$$

Let $v = \sum\limits_{i,j} x_j c(i,j) y_i e_j \otimes e_i$. According to (3.3), $\alpha'_{q,p'}(v) \leqslant 1$. Now, let $\widetilde{v} = \sum\limits_{i,j} x_j c(i,j) y_i \varphi_{A,1}^{-1}(e_j) \otimes \psi_{B,1}^{-1}(e_i)$. We have

$$\alpha'_{q,p'}(\widetilde{v}) = \alpha'_{q,p'}(v) \leqslant 1 \quad \text{and} \quad \widetilde{v} = \sum_{i,j}\frac{x_j c(i,j) y_i}{\mu_1(A_j)\mu_2(B_i)}\chi_{A_j}\otimes\chi_{B_i}.$$

By assumption, $L_{q,p'}(u_\phi) \leqslant 1$, which implies that

$$|\langle u_\phi, \widetilde{v}\rangle| = \Big|\sum_{i,j} c(i,j)\frac{x_j y_i}{\mu_1(A_j)\mu_2(B_i)}\Big(\int_{A_j\times B_i}\phi\Big)\Big| \leqslant \alpha'_{q,p'}(\widetilde{v}) \leqslant 1,$$

and this is precisely the inequality (4.1). ∎

Theorem 3.2 and Remark 3.4 allow us to reformulate the previous theorem. The following two corollaries are generalizations of Theorem 1.1.

COROLLARY 4.3. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be two localizable measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. Let $1 \leqslant q \leqslant p \leqslant \infty$. The following statements are equivalent:*

(i) *ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$;*

(ii) *there are a measure space (a probability space when $p \neq q$) (Ω, μ), operators $R \in \mathcal{B}(L^1(\Omega_1), L^p(\mu))$ and $S \in \mathcal{B}(L^q(\mu), L^\infty(\Omega_2))$ such that $u_\phi = S \circ I \circ R$*

$$\begin{array}{ccc} L^1(\Omega_1) & \xrightarrow{\ u_\phi\ } & L^\infty(\Omega_2) \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle S} \\ L^p(\mu) & \underset{I}{\hookrightarrow} & L^q(\mu) \end{array}$$

where I is the inclusion mapping.

In the following cases, (i) *and* (ii) *are equivalent to:*

If $1 < q \leqslant p < +\infty$:

(iii) *there are a measure space (a probability space when $p \neq q$) (Ω, μ), elements $a \in L^\infty(\mu_1, L^p(\mu))$ and $b \in L^\infty(\mu_2, L^{q'}(\mu))$ such that, for almost every $(s,t) \in \Omega_1 \times \Omega_2$,*

$$\phi(s,t) = \langle a(s), b(t)\rangle.$$

If $1 = q < p < +\infty$:

(iii) *there are a probability space (Ω, μ), $a \in L^\infty(\mu_1 \times \mu)$ and $b \in L^\infty(\mu_2, L^{q'}(\mu))$ such that for almost every $(s,t) \in \Omega_1 \times \Omega_2$,*

$$\phi(s,t) = \langle a(s,\cdot), b(t)\rangle.$$

If $1 < q < +\infty$ *and* $p = +\infty$*:*

(iii) *there are a probability space* (Ω, μ)*,* $a \in L^\infty(\mu_1, L^p(\mu))$ *and* $b \in L^\infty(\mu_2 \times \mu)$ *such that for almost every* $(s,t) \in \Omega_1 \times \Omega_2$*,*

$$\phi(s,t) = \langle a(s), b(t,\cdot)\rangle.$$

If $q = 1$ *and* $p = +\infty$*:*

(iii) *there are a probability space* (Ω, μ)*,* $a \in L^\infty(\mu_1 \times \mu)$ *and* $b \in L^\infty(\mu_2 \times \mu)$ *such that for almost every* $(s,t) \in \Omega_1 \times \Omega_2$*,*

$$\phi(s,t) = \langle a(s,\cdot), b(t,\cdot)\rangle.$$

In this case,

$$\|T_\phi\| = \inf \|R\|\|I\|\|S\| = \inf \|a\|\|b\|.$$

REMARK 4.4. In the previous corollary, the condition (ii) implies that every $\phi \in L^\infty(\Omega_1 \times \Omega_2)$ is a Schur multiplier on $\mathcal{B}(L^1(\Omega_1), L^1(\Omega_2))$ and a Schur multiplier on $\mathcal{B}(L^\infty(\Omega_1), L^\infty(\Omega_2))$.

In the discrete case, the previous corollary can be reformulated as follow.

COROLLARY 4.5. *Let* $\phi = (c_{ij})_{i,j\in\mathbb{N}} \subset \mathbb{C}$*,* $C \geqslant 0$ *be a constant and let* $1 \leqslant q \leqslant p \leqslant +\infty$*. The following are equivalent:*

(i) ϕ *is a Schur multiplier on* $\mathcal{B}(\ell_p, \ell_q)$ *with norm* $\leqslant C$*;*

(ii) *there exist a measure space (a probability space when* $p \neq q$*)* (Ω, μ) *and two bounded sequences* $(x_j)_j$ *in* $L^p(\mu)$ *and* $(y_i)_i$ *in* $L^{q'}(\mu)$ *such that*

$$\forall i,j \in \mathbb{N}, \quad c_{ij} = \langle x_j, y_i\rangle \quad \text{and} \quad \sup_i \|y_i\|_{q'} \sup_j \|x_j\|_p \leqslant C.$$

4.2. AN APPLICATION: THE MAIN TRIANGLE PROJECTION. Let $m_{ij} = 1$ if $i \leqslant j$ and $m_{ij} = 0$ otherwise. Let T_m be the Schur multiplier associated with the family $m = (m_{ij})$. For any infinite matrix $A = [a_{ij}]$, $T_m(A)$ is the matrix $[b_{ij}]$ with $b_{ij} = a_{ij}$ if $i \leqslant j$ and $b_{ij} = 0$ otherwise. For that reason, T_m is called the *main triangle projection*. Similary, we define the *n-th main triangle projection* as the Schur multiplier on $\mathcal{M}_n(\mathbb{C})$ associated with the family $m_n = (m^n_{ij})_{1\leqslant i,j\leqslant n}$ where $m^n_{ij} = 1$ if $i \leqslant j$ and $m^n_{ij} = 0$ otherwise. In [8], Kwapień and Pelczyński proved that if $1 \leqslant q \leqslant p \leqslant +\infty, p \neq 1, q \neq +\infty$, there exists a constant $K > 0$ such that for all n,

$$\|T_{m_n} : \mathcal{B}(\ell^n_p, \ell^n_q) \to \mathcal{B}(\ell^n_p, \ell^n_q)\| \geqslant K \ln(n),$$

and this order of growth is obtained for the Hilbert matrices. Those estimates imply that T_m is not bounded on $\mathcal{B}(\ell_p, \ell_q)$. Bennett proved in [2] that when $1 < p < q < \infty$, T_m is bounded from $\mathcal{B}(\ell_p, \ell_q)$ into itself.

The results obtained in Subsection 4.1 allow us to give a very short proof of the unbounded case.

PROPOSITION 4.6. *Let* $1 \leqslant q \leqslant p \leqslant +\infty, p \neq 1, q \neq +\infty$*. Then* T_m *is not bounded on* $\mathcal{B}(\ell_p, \ell_q)$*.*

Proof. Assume that T_m is bounded on $\mathcal{B}(\ell_p, \ell_q)$. By Corollary 4.3, there exist a measure space (Ω, μ), $(a_n)_n \in L^p(\mu)$ and $(b_n)_n \in L^{q'}(\mu)$ two bounded sequences such that, for all $i, j \in \mathbb{N}$,

$$m_{ij} = \langle a_j, b_i \rangle. \tag{4.2}$$

By boundedness, $(a_n)_n$ and $(b_n)_n$ admit an accumulation point $a \in L^p(\mu)$ and $b \in L^{q'}(\mu)$, respectively, for the weak* topology. Fix $i \in \mathbb{N}$. For all $j \geqslant i$, we have

$$\langle a_i, b_j \rangle = 1$$

so that we get

$$\langle a_i, b \rangle = 1.$$

This equality holds for any i, hence

$$\langle a, b \rangle = 1.$$

Now fix $j \in \mathbb{N}$. For all $i > j$ we have

$$\langle a_i, b_j \rangle = 0.$$

From this, we deduce as above that

$$\langle a, b \rangle = 0.$$

We obtained a contradiction so T_m cannot be bounded. ∎

As a consequence, we have, by Proposition 2.3, the following corollary.

COROLLARY 4.7. *Let* $1 \leqslant q \leqslant p \leqslant +\infty, p \neq 1, q \neq +\infty$. *Let* $\Omega_1 = \Omega_2 = \mathbb{R}$ *with the Lebesgue measure. Then* $\phi \in L^\infty(\mathbb{R}^2)$ *defined by*

$$\phi(s,t) := \begin{cases} 1 & \text{if } s+t \geqslant 0, \\ 0 & \text{if } s+t < 0, \end{cases} \qquad s, t \in \mathbb{R}$$

is not a Schur multiplier on $\mathcal{B}(L^p(\mathbb{R}), L^q(\mathbb{R}))$.

REMARK 4.8. One could wonder whether the results of Subsection 4.1 can be extended to the case $1 \leqslant p < q \leqslant +\infty$, that is, if the boundedness of T_ϕ on $\mathcal{B}(L^p, L^q)$ implies that u_ϕ has a certain factorization. The fact that if $p < q$ the main triangle projection is bounded tells us that m is a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$. Nevertheless, the argument used in the previous proof shows that m cannot have a factorization like in (4.2). Therefore, the case $p < q$ is more tricky. For the discrete case, Theorem 4.3 in [3] gives a necessary and sufficient condition for a family $(m_{i,j}) \subset \mathbb{C}$ to be a Schur multiplier, for all values of p and q, using the theory of q-absolutely summing operators.

5. INCLUSION THEOREMS

In this section, we denote by $\mathcal{M}(p,q)$ the space of Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$.

First, we recall the inclusion relationships between the spaces $\mathcal{M}(p,q)$. Then we will establish new results as applications of those obtained in Subsection 4.1.

THEOREM 5.1 ([3], Theorem 6.1). *Let $p_1 \geqslant p_2$ and $q_1 \leqslant q_2$ be given. Then $\mathcal{M}(p_1,q_1) \subset \mathcal{M}(p_2,q_2)$ with equality in the following cases:*

(i) $p_1 = p_2 = 1$;

(ii) $q_1 = q_2 = \infty$;

(iii) $q_2 \leqslant 2 \leqslant p_2$;

(iv) $q_2 < p_1 = p_2 < 2$;

(v) $2 < q_1 = q_2 < p_2$.

Let (Ω_1, μ_1) and (Ω_2, μ_2) be two measure spaces. If $\mathcal{M}(p_1,q_1) \subset \mathcal{M}(p_2,q_2)$, then by Proposition 2.3, any Schur multiplier on $\mathcal{B}(L^{p_1}(\Omega_1), L^{q_1}(\Omega_2))$ is a Schur multiplier on $\mathcal{B}(L^{p_2}(\Omega_1), L^{q_2}(\Omega_2))$. Hence, the results in the previous theorem hold true for all the Schur multipliers on $\mathcal{B}(L^p, L^q)$.

In the sequel, we will need the notion of type for a Banach space X, for which we refer e.g. to [1]. Let $(\mathcal{E}_i)_{i\in\mathbb{N}}$ be a sequence of independent Rademacher random variables. We have the following definition.

DEFINITION 5.2. A Banach space X is said to *have Rademacher type p* (in short, type p) for some $1 \leqslant p \leqslant 2$ if there is a constant C such that for every finite set of vectors $(x_i)_{i=n}^n$ in X,

$$\Big(\mathbb{E}\Big\| \sum_{i=1}^n \mathcal{E}_i x_i \Big\|^p\Big)^{1/p} \leqslant C\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p}. \tag{5.1}$$

The smallest constant C for which (5.1) holds is called the type p constant of X.

We will use the fact that for $1 \leqslant p \leqslant 2$, L^p-spaces have type p and if $2 < p < +\infty$, L^p-spaces have type 2 and that those are the best types for infinite dimensional L^p-spaces (see for instance Theorem 6.2.14 in [1]). We will also use the fact that the type is stable by passing to quotients. Namely, if X has type p and $E \subset X$ is a closed subspace, then X/E has type p.

PROPOSITION 5.3. (i) *If* $1 \leqslant q < p \leqslant 2$, *then*

$$\mathcal{M}(q,1) \not\subseteq \mathcal{M}(p,p).$$

Consequently, for any $1 \leqslant r \leqslant q$,

$$\mathcal{M}(q,r) \not\subseteq \mathcal{M}(p,p).$$

(ii) *If* $2 \leqslant p < q \leqslant r$, *then*

$$\mathcal{M}(r,q) \not\subseteq \mathcal{M}(p,p).$$

(iii) *If* $1 < q < 2 < p < +\infty$ *or* $1 < p < 2 < q < +\infty$, *then*

$$\mathcal{M}(q,q) \nsubseteq \mathcal{M}(p,p).$$

To prove this proposition, we will need the following definitions and lemma.

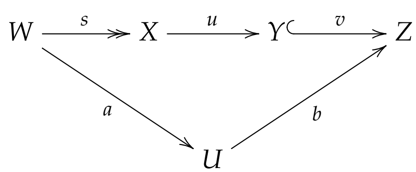
DEFINITION 5.4. Let X and Y be Banach spaces. A map $s : X \to Y$ is a *quotient map* if s is surjective and for all $y \in Y$ with $\|y\| < 1$, there exists $x \in X$ such that $\|x\| < 1$ and $s(x) = y$. This is equivalent to the fact that the injective map $\widehat{s} : X/\ker(s) \to Y$ induced by s is a surjective isometry.

DEFINITION 5.5. Let X and Y be Banach spaces, $u \in \mathcal{B}(X,Y)$ and $1 \leqslant p \leqslant \infty$. We say that $u \in SQ_p(X,Y)$ if there exists a closed subspace Z of a quotient of a L^p-space and two operators $A \in \mathcal{B}(X,Z)$ and $B \in \mathcal{B}(Z,Y)$ such that $u = BA$.

Then $\|u\|_{SQ_p} = \inf \|A\|\|B\|$ defines a norm on $SQ_p(X,Y)$ and $(SQ_p(X,Y), \|\cdot\|_{SQ_p})$ is a Banach space.

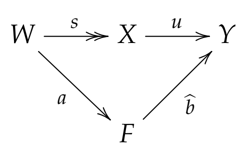
LEMMA 5.6. *Let* W, X, Y, Z *be Banach spaces and let* $u \in \mathcal{B}(X,Y), s \in \mathcal{B}(W,X)$, $v \in \mathcal{B}(Y,Z)$ *such that* s *is a quotient map,* v *is a linear isometry and* $vus \in \Gamma_p(W,Z)$. *Then* $u \in SQ_p(X,Y)$.

Proof. By assumption, there exist an L^p-space U and operators $a \in \mathcal{B}(W,U)$ and $b \in \mathcal{B}(U,Z)$ such that the following diagram commutes



Since v is an isometry, $V := v(Y) \subset Z$ is isometrically isomorphic to Y. Let $\psi : Y \to V$ be the isometric isomorphism induced by v.

Set $F := \{x \in U \text{ such that } b(x) \in V\}$. Since $vus = ba$, we have, for all $w \in W, v(us(w)) = b(a(w))$, so that $a(w) \in F$. This implies that $a(W) \subset F$. We still denote by a the mapping $a : W \to F$ and by b the restriction of b to F. Denote by $\widehat{b}$ the mapping $\widehat{b} = \psi^{-1} \circ b : F \to Y$. Then we have the following commutative diagram



Now, set $E := \overline{a(\ker(s))}$ and let $Q : F \to F/E$ be the canonical mapping. Clearly, $Q \circ a : W \to F/E$ vanishes on $\ker(s)$, so that we have a mapping

$$\widehat{Q \circ a} : W/\ker(s) \to F/E$$

induced by $Q \circ a$.

Since s is a quotient map, we denote by $\widehat{s}$ the isometric isomorphism

$$\widehat{s} : W/\ker(s) \to X.$$

Define

$$A = \widehat{Q \circ a} \circ \widehat{s}^{-1} : X \to F/E.$$

$\widehat{b}$ vanishes on E so that we have a mapping

$$B : F/E \to Y.$$

Finally, it is easy to check that $u = BA$, that is, we have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & Y \\ & {\scriptstyle A}\searrow \quad \nearrow{\scriptstyle B} & \\ & F/E & \end{array}$$

which concludes the proof. ∎

REMARK 5.7. To prove Lemma 5.6, one can use a result of Kwapień characterizing elements of SQ_p, as follows: a Banach space X is isomorphic to an SQ_q-space if and only if there exists a constant $K \geqslant 1$ such that for any $n \geqslant 1$, for any $n \times n$ matrix $[a_{ij}]$ and for any $x_1, \ldots, x_n$ in X,

$$\Big(\sum_i \Big\| \sum_j a_{ij} x_j \Big\|^q \Big)^{1/q} \leqslant K \| [a_{ij}] : \ell_q^n \to \ell_q^n \| \Big(\sum_j \|x_j\|^q \Big)^{1/q}.$$

However, the proof presented in this paper also works if we replace in the statement of the lemma Γ_p (respectively SQ_p) by the space of operators that can be factorized by some Banach space L (respectively by a subspace of a quotient of L).

Proof of Proposition 5.3. (i) Let $\Omega := [0,1]$ and λ be the Lebesgue measure on Ω. Let $I_q : L^q(\lambda) \to L^1(\lambda)$ be the inclusion mapping. By the classical Banach space theory (see Theorem 2.3.1 in [1] and Theorem 2.5.7 in [1]) there exist a quotient map $\sigma : \ell_1 \twoheadrightarrow L^q(\lambda)$ and an isometry $J : L^1(\lambda) \hookrightarrow \ell_\infty$. Let $\phi \in \ell_\infty(\mathbb{N}^2)$ be such that

$$u_\phi = J I_q \sigma$$

(by (1.6) any continuous linear map $\ell_1 \to \ell_\infty$ is a certain u_ϕ for $\phi \in L^\infty(\mathbb{N} \times \mathbb{N})$). We have the following factorization

$$\begin{array}{ccc} \ell_1 & \xrightarrow{\quad u_\phi \quad} & \ell_\infty \\ {\scriptstyle \sigma}\downarrow & & \uparrow{\scriptstyle J} \\ L^q(\lambda) & \xhookrightarrow[\quad I_q \quad]{} & L^1(\lambda) \end{array}$$

According to Theorem 4.3, $\phi \in \mathcal{M}(q,1)$.

Assume that $\phi \in \mathcal{M}(p,p)$. Then, again by Theorem 4.3, we have $u_\phi \in \Gamma_p(\ell_1,\ell_\infty)$ and therefore, by Lemma 5.6, there exist an SQ_p-space X and two operators $\alpha \in \mathcal{B}(L^q(\lambda),X)$ and $\beta \in \mathcal{B}(X,L^1(\lambda))$ such that $I_q = \beta\alpha$.

Let $(\mathcal{E}_i)_{i\in\mathbb{N}}$ be a sequence of independant Rademacher random variables. Let $n \in \mathbb{N}^*$ and $f_1,\dots,f_n \in L^q(\lambda)$.

$$\mathbb{E}\Big\| \sum_{j=1}^n \mathcal{E}_j f_j \Big\|_{L^1(\lambda)} = \mathbb{E}\Big\| \sum_{j=1}^n \mathcal{E}_j \beta\alpha(f_j) \Big\|_{L^1(\lambda)} \leqslant \|\beta\| \mathbb{E}\Big\| \sum_{j=1}^n \mathcal{E}_j \alpha(f_j) \Big\|_X.$$

But X has type p so there exists a constant $C_1 > 0$ such that

$$\mathbb{E}\Big\| \sum_{j=1}^n \mathcal{E}_j f_j \Big\|_{L^1(\lambda)} \leqslant C_1\|\beta\| \Big(\sum_{j=1}^n \|\alpha(f_j)\|_X^p \Big)^{1/p} \leqslant C_1\|\beta\|\|\alpha\| \Big(\sum_{j=1}^n \|f_j\|_{L^q(\lambda)}^p \Big)^{1/p}.$$

By Khintchine inequality, there exists $C_2 > 0$ such that

$$\Big\| \Big(\sum_{j=1} |f_j|^2 \Big)^{1/2} \Big\|_{L^1(\lambda)} \leqslant C_2 \mathbb{E}\Big\| \sum_{j=1}^n \mathcal{E}_j f_j \Big\|_{L^1(\lambda)}.$$

Thus, setting $K := C_1C_2\|\alpha\|\|\beta\|$, we obtained the inequality

$$\Big\| \Big(\sum_{j=1} |f_j|^2 \Big)^{1/2} \Big\|_{L^1(\lambda)} \leqslant K \Big(\sum_{j=1}^n \|f_j\|_{L^q(\lambda)}^p \Big)^{1/p}.$$

Let $E_1,\dots,E_n$ be disjoint measurable subsets of $[0,1]$ such that for all $1 \leqslant j \leqslant n, \lambda(E_j) = 1/n$. Set $f_j := \chi_{E_j}$. Then

$$\sum_j |f_j|^2 = 1 \quad \text{and} \quad \|f\|_{L^q(\lambda)} = n^{-1/q}.$$

Hence, applying the previous inequality to the f_j's, we obtain

$$1 \leqslant K n^{1/p-1/q}.$$

Since $q < p$, this inequality cannot hold for all n, so we obtained a contradiction.

Finally, notice that if $1 \leqslant r \leqslant q$, then by Theorem 5.1, $\mathcal{M}(q,1) \subset \mathcal{M}(q,r)$. Thus, $\mathcal{M}(q,r) \not\subseteq \mathcal{M}(p,p)$.

(ii) By Proposition 2.3 and using duality, it is easy to prove that for all $s,t \in [1,\infty]$, ϕ is a Schur multiplier on $\mathcal{B}(\ell_s,\ell_t)$ if and only if $\widetilde{\phi}$ is a Schur multiplier on $\mathcal{B}(\ell_{t'},\ell_{s'})$, where $\widetilde{\phi}$ is defined for all $i,j \in \mathbb{N}$ by $\widetilde{\phi}(i,j) = \phi(j,i)$.

Let $2 \leqslant p < q \leqslant r$. Then $1 \leqslant r' \leqslant q' < p' \leqslant 2$. If we assume that $\mathcal{M}(r,q) \subset \mathcal{M}(p,p)$ then the latter implies $\mathcal{M}(q',r') \subset \mathcal{M}(p',p')$, which is, by (i), a contradiction. This proves (ii).

(iii) By duality, it is enough to consider the case $1 < q < 2 < p < +\infty$. Assume that $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$. Using the notations introduced in the proof of (i), let $\sigma : \ell_1 \to \ell_q$ be a quotient map and $J : \ell_q \to \ell_\infty$ be an isometry. Let $\phi \in L^\infty(\mathbb{N}\times\mathbb{N})$ be such that

$$u_\phi = J I_{\ell_q} \sigma,$$

where $I_{\ell_q} : \ell_q \to \ell_q$ is the identity map. Then $\phi \in \mathcal{M}(q,q)$. By assumption, $\phi \in \mathcal{M}(p,p)$. By Lemma 5.6, this implies that $I_{\ell_q} \in SQ_p(\ell_q, \ell_q)$. Clearly, this implies that ℓ_q is isomorphic to an SQ_p-space. But ℓ_q does not have type 2 and any SQ_p has type 2. This is a contradiction, so $\mathcal{M}(q,q) \not\subseteq \mathcal{M}(p,p)$. ∎

THEOREM 5.8. *We have* $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$ *if and only if* $1 \leqslant p \leqslant q \leqslant 2$ *or* $2 \leqslant q \leqslant p \leqslant +\infty$.

Proof. By Proposition 5.3 and duality, we only have to show that when $1 \leqslant p \leqslant q \leqslant 2$, $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$.

We saw in the proof of Proposition 5.3(iii) that if $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$ then ℓ_q is isomorphic to an SQ_p-space. The converse holds true. Indeed, assume that ℓ_q is isomorphic to an SQ_p-space. Then by approximation, any L^q-space is isomorphic to an SQ_p-space. Hence any element of $\Gamma_q(\ell_1, \ell_\infty)$ factors through an SQ_p-space. By the lifting property of ℓ_1 and the extension property of ℓ_∞, this implies that any element of $\Gamma_q(\ell_1, \ell_\infty)$ factors through an L^p-space, that is $\Gamma_q(\ell_1, \ell_\infty) \subset \Gamma_p(\ell_1, \ell_\infty)$. By Corollary 4.5, this implies that $\mathcal{M}(q,q) \subset \mathcal{M}(p,p)$.

Assume that $1 \leqslant p \leqslant q \leqslant 2$. By Theorem 6.4.19 in [1], there exists an isometry from ℓ_q into an L^p-space, obtained by using q-stable processes. Hence, ℓ_q is an SQ_p-space. This concludes the proof. ∎

PROBLEM 5.9. Compare the other spaces of Schur multipliers. For example, if $1 < p \leqslant 2$, do we have

$$\mathcal{M}(p,1) = \mathcal{M}(p,p)?$$

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