

TOWARDS A PERTURBATION THEORY FOR EVENTUALLY POSITIVE SEMIGROUPS

DANIEL DANERS and JOCHEN GLÜCK

Communicated by Albrecht Böttcher

ABSTRACT. We consider eventually positive operator semigroups and study the question whether their eventual positivity is preserved by bounded perturbations of the generator or not. We demonstrate that eventual positivity is not stable with respect to large positive perturbations and that certain versions of eventual positivity react quite sensitively to small positive perturbations. In particular we show that if eventual positivity is preserved under arbitrary positive perturbations of the generator, then the semigroup is positive. We then provide sufficient conditions for a positive perturbation to preserve the eventual positivity. Some of these theorems are qualitative in nature while others are quantitative with explicit bounds.

KEYWORDS: *One-parameter semigroups of linear operators, semigroups on Banach lattices, eventually positive semigroup, Perron–Frobenius theory, perturbation theory.*

MSC (2010): 47D06, 47B65, 34G10.

1. INTRODUCTION

For positive C_0 -semigroups, it is easy to derive basic perturbation results. If, for instance, A generates a positive C_0 -semigroup on a Banach lattice E , B is a positive operator and M is a multiplication operator on E (see Section C-I-9 of [2]), then it is not difficult to show that the semigroup generated by $A + B + M$ is also positive. In the present paper we study the problem whether or not such a perturbation result is still true for *eventually* positive semigroups.

An *eventually positive semigroup* is a C_0 -semigroup (e^{tA}) on, say, a complex Banach lattice E such that, for every initial value $0 \leq f \in E$, the trajectory $e^{tA}f$ becomes positive for large enough t . Motivated by applications to partial differential equations (see e.g. [4], [13], [14]; see also [28] for an overview over related elliptic problems) and by the rapid development of a corresponding theory in finite dimensions (see for instance [9], [12], [22], [23]), a study of eventually positive semigroups on Banach lattices was initiated in a series of recent papers [5], [6], [7]. In particular, these papers clarified that there are several distinct notions of

eventual positivity such as an *individual* and a *uniform* one which are worthwhile studying. For the convenience of the reader we recall the exact definitions of these notions at the end of the introduction as we are going to need them throughout the paper.

We shall see that perturbation theory is much more subtle for eventually positive semigroups than it is for positive semigroups. We first demonstrate by a number of counterexamples in Section 2 what is *not* true. In particular we will see that, in sharp contrast to the case of positive semigroups, eventual positivity of a semigroup is in general lost if we perturb its generator by a positive operator of large norm; this is related to a recent result in Proposition 3.6 of [26] for perturbations of eventually positive matrices. Moreover, one of our examples shows that *individual* eventual positivity is not even stable with respect to small positive perturbations. This is the reason why we focus on *uniform* eventual positivity throughout the rest of the paper. In Section 3 we prove qualitative as well as quantitative perturbation results for eventually positive resolvents of operators, and in Section 4 we prove qualitative and quantitative perturbation results for C_0 -semigroups. In the appendix we consider rank-1-perturbations of linear operators and prove explicit formulas for their resolvents and for the semigroups generated by those operators; these formulas are needed in the main text.

We note that our results are far from constituting a complete perturbation theory for eventually positive semigroups. In fact, we leave more questions open than we solve. It is our hope though that, by exposing some surprising phenomena, the present article can serve a starting point for further research on the topic.

PRELIMINARIES. Throughout, we use the notation and the terminology from [5], [6], [7]. For the convenience of the reader we recall what we need throughout the paper. We assume familiarity with the theory of real and complex Banach lattices (see for instance [19], [25] for standard references on this topic) and with the basic theory of C_0 -semigroups (see for instance [10], [11], [24]).

For every $\lambda \in \mathbb{C}$ and every real number $r > 0$ we denote by $B(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| < r\}$ the open ball in \mathbb{C} of radius r .

If E, F are real or complex Banach spaces, then we denote the space of all bounded linear operators from E to F by $\mathcal{L}(E; F)$ and we abbreviate $\mathcal{L}(E) := \mathcal{L}(E; E)$. The *identity operator* on E is denoted by $I := I_E \in \mathcal{L}(E)$. For every $T \in \mathcal{L}(E)$ the *spectral radius* of T is denoted by $r(T)$. For every densely defined linear operator $A : E \supseteq D(A) \rightarrow F$ we denote by $A' : F' \supseteq D(A') \rightarrow E'$ the *dual operator* of A , where E' and F' are the *dual spaces* of E and F . For all $y \in F$ and all $\varphi \in E'$ we define $y \otimes \varphi \in \mathcal{L}(E; F)$ by $(y \otimes \varphi)z := \langle \varphi, z \rangle y$ for all $z \in E$. It is easy to see that the operator norm of $y \otimes \varphi$ is given by $\|y \otimes \varphi\| = \|y\| \|\varphi\|$. Recall that every rank-1-operator in $\mathcal{L}(E; F)$ is of the form $y \otimes \varphi$ for appropriate vectors $y \in F \setminus \{0\}$ and $\varphi \in E' \setminus \{0\}$.

Given a linear operator $A : E \supseteq D(A) \rightarrow E$ on a complex Banach space E we denote its *spectrum* and *resolvent set* by $\sigma(A)$ and $\rho(A) := \mathbb{C} \setminus \sigma(A)$, respectively.

Note that if $\rho(A) \neq \emptyset$, then A is necessarily closed. The *spectral bound* of A is given by

$$s(A) := \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\} \in [-\infty, \infty],$$

We say that the spectral bound $s(A)$ is a *dominant spectral value* of A if $s(A) \in \sigma(A)$ and if $\operatorname{Re} \lambda < s(A)$ for all other spectral values λ of A . We define the *resolvent* of A at $\lambda \in \rho(A)$ by $\mathcal{R}(\lambda, A) := (\lambda I - A)^{-1}$. The resolvent $\mathcal{R}(\cdot, A) : \rho(A) \rightarrow \mathcal{L}(E)$ is an analytic map. Any pole of this analytic map is an isolated point of $\sigma(A)$ and in fact an eigenvalue of A ; see Theorem 2 in Section VIII.8 of [29]. Let λ_0 be a pole of the resolvent of A . We call λ_0 a *geometrically simple eigenvalue* of A if the eigenspace $\ker(\lambda_0 I - A)$ is one-dimensional; we call λ_0 an *algebraically simple eigenvalue* of A if the spectral projection P associated with λ_0 has one-dimensional range. The eigenvalue λ_0 is algebraically simple if and only if it is geometrically simple and $\operatorname{Im}(P) = \ker(\lambda_0 I - A)$. Also, if λ_0 is algebraically simple, then λ_0 is a simple pole of $\mathcal{R}(\cdot, A)$. Let $A : E \supseteq D(A) \rightarrow E$ be a linear operator with non-empty resolvent set on a complex Banach space E . An operator $K \in \mathcal{L}(E)$ is called *A-compact* if there is a $\lambda_0 \in \rho(A)$ such that $K\mathcal{R}(\lambda_0, A)$ is compact. By the resolvent equation this is equivalent to $K\mathcal{R}(\lambda, A)$ being compact for every $\lambda \in \rho(A)$. Note that every compact operator $K \in \mathcal{L}(E)$ is naturally *A-compact*. Moreover, if A has compact resolvent, then every operator $K \in \mathcal{L}(E)$ is *A-compact*.

A complex Banach lattice E is by definition the complexification of a real Banach lattice $E_{\mathbb{R}}$ which we call the *real part* of E . The *positive cone* of a real or complex Banach lattice E is denoted by E_+ . A vector $f \in E$ is called *positive*, which we denote by $f \geq 0$, if $f \in E_+$. For two elements $f, g \in E$ in case of a real Banach lattice or $f, g \in E_{\mathbb{R}}$ in case of a complex Banach lattice we write, as usual, $f \leq g$ if $g - f \geq 0$. We write $f < g$ if $f \leq g$ but $f \neq g$. The dual space E' of a real or complex Banach lattice E is again a real or complex Banach lattice, where a functional $\varphi \in E'$ fulfils $\varphi \geq 0$ if and only if $\langle \varphi, x \rangle \geq 0$ for all $x \in E_+$; we denote the positive cone in E' by $E'_+ := (E')_+$.

Let E be a real or complex Banach lattice and let $u \in E_+$. The vector subspace

$$E_u := \{x \in E : \text{there exists } c \geq 0 \text{ with } |x| \leq cu\}$$

of E is called the *principal ideal* generated by u . We endow E_u with the *gauge norm* $\|\cdot\|_u$ with respect to u . The gauge norm is given by

$$\|x\|_u := \inf\{c \geq 0 : |x| \leq cu\}$$

for all $x \in E_u$ and is at least as strong as the norm induced by E , usually even stronger. Moreover it renders E_u a (real or complex) Banach lattice. A vector $u \in E$ is called a *quasi-interior point* of E_+ if $u \geq 0$ and if E_u is dense in E . If, for instance, E is an L^p -space over a σ -finite measure space (Ω, μ) (where $1 \leq p < \infty$) then $0 \leq u \in E$ is a quasi-interior point of E_+ if and only if $u(\omega) > 0$ for almost all $\omega \in \Omega$.

Let $u \in E_+$. We call a vector $f \in E$ *strongly positive with respect to u* , which we denote by $f \gg_u 0$, if there exists a number $\varepsilon > 0$ such that $f \geq \varepsilon u$. This condition is equivalent to the condition $f \geq 0$ and $u \in E_f$. An operator $T \in \mathcal{L}(E)$ is called *strongly positive with respect to u* , which we denote by $T \gg_u 0$, if $Tf \gg_u 0$ for all $0 < f \in E$.

Let E be a complex Banach lattice. A linear operator $A : E \supseteq D(A) \rightarrow E$ is called *real* if

$$D(A) = E_{\mathbb{R}} \cap D(A) + iE_{\mathbb{R}} \cap D(A)$$

and if A maps $E_{\mathbb{R}} \cap D(A)$ to $E_{\mathbb{R}}$. Clearly, an operator $T \in \mathcal{L}(E)$ is real if and only if $TE_{\mathbb{R}} \subseteq E_{\mathbb{R}}$. It is easy to see that a C_0 -semigroup $(e^{tA})_{t \geq 0}$ on E is real, meaning that e^{tA} is a real operator for every $t \geq 0$, if and only if A is real. A linear operator $T \in \mathcal{L}(E)$ on a real or complex Banach lattice E is called *positive* if $TE_+ \subseteq E_+$; we denote this by $T \geq 0$. In particular such an operator is real. A C_0 -semigroup $(e^{tA})_{t \geq 0}$ on E generated by A is called *positive* if $e^{tA} \geq 0$ for all $t \geq 0$. Furthermore, given $S, T \in \mathcal{L}(E)$ we write $S \leq T$ if S and T are both real operators and $T - S \geq 0$.

A real operator $T \in \mathcal{L}(E)$ is called a *multiplication operator* if there exists a number $c \geq 0$ such that $-cI_E \leq T \leq cI_E$; it is also possible to define non-real multiplication operators, but we have no need for this in the present article. All multiplication operators on a Banach lattice constitute a vector space which is usually called the *center* of the Banach lattice; see for instance Section 3.1 of [19] for more information. We recall how real multiplication operators can be characterised on two important classes of complex Banach lattices, also explaining the name “multiplication operator”. Let (Ω, μ) be a σ -finite measure space and K a compact Hausdorff space. Then the real operator T is a multiplication operator on $E = L^p(\Omega)$ with $1 \leq p < \infty$ or $E = C(K; \mathbb{C})$ if and only if there exists a function $h \in L^\infty(\Omega, \mu; \mathbb{R})$ or $h \in C(K; \mathbb{R})$, respectively, such that $Tf = hf$ for all $f \in E$.

NOTIONS OF EVENTUAL POSITIVITY. As in [5], [6], [7] we consider eventual positivity for resolvents of linear operators as well as for C_0 -semigroups. For the convenience of the reader we recall the most important definitions now. First we recall several notions of eventual positivity for resolvents.

DEFINITION 1.1. Let $A : E \supseteq D(A) \rightarrow E$ be a linear operator on a complex Banach lattice E and let $\lambda_0 \in [-\infty, \infty)$ be either a spectral value of A or $-\infty$.

(i) The resolvent of A is called *individually eventually positive at λ_0* if, for every $0 \leq f \in E$, there exists a real number $\lambda_1 > \lambda_0$ such that $(\lambda_0, \lambda_1] \subseteq \rho(A)$ and such that $\mathcal{R}(\lambda, A)f \geq 0$ for all $\lambda \in (\lambda_0, \lambda_1]$.

(ii) The resolvent of A is called *uniformly eventually positive at λ_0* if it is individually eventually positive at λ_0 and if the number λ_1 in (i) can be chosen to be independent of f .

Now assume in addition that u is a quasi-interior point of E_+ .

(iii) The resolvent of A is called *individually eventually strongly positive with respect to u at λ_0* if, for every $0 < f \in E$, there exists a real number $\lambda_1 > \lambda_0$ such that $(\lambda_0, \lambda_1] \subseteq \rho(A)$ and such that $\mathcal{R}(\lambda, A)f \gg_u 0$ for all $\lambda \in (\lambda_0, \lambda_1]$.

(iv) The resolvent of A is called *uniformly eventually strongly positive with respect to u at λ_0* if it is individually eventually strongly positive with respect to u at λ_0 and if the number λ_1 in (iii) can be chosen to be independent of f .

Note that one can also define various versions of *eventual negativity* of a resolvent as was for instance done in Definition 4.2 of [6]. We will, however, not discuss this notion in detail here.

The most interesting case in the above definition is the case $\lambda_0 = s(A)$. In fact, eventual positivity of the resolvent of A at the spectral bound is closely related to eventual positivity of the semigroup (see for instance Theorem 1.1 of [6]). Various version of eventual positivity of a semigroup can be found in the subsequent definition.

DEFINITION 1.2. Let $(e^{tA})_{t \geq 0}$ be a C_0 -semigroup on a complex Banach lattice E .

(i) The semigroup is called *individually eventually positive* if, for every $0 \leq f \in E$, there exists a time $t_0 \geq 0$ such that $e^{tA}f \geq 0$ for all $t \geq t_0$.

(ii) The semigroup is called *uniformly eventually positive* if it is individually eventually positive and if the time t_0 in (i) can be chosen to be independent of f .

Now assume in addition that u is a quasi-interior point of E_+ .

(iii) The semigroup is called *individually eventually strongly positive with respect to u* if, for every $0 < f \in E$, there exists a time $t_0 \geq 0$ such that $e^{tA}f \gg_u 0$ for all $t \geq t_0$.

(iv) The semigroup is called *uniformly eventually strongly positive with respect to u* if it is individually eventually strongly positive with respect to u and if the time t_0 from (iii) can be chosen to be independent of f .

It was demonstrated in Examples 5.7 and 5.8 of [7] that individual eventual strong positivity does not in general imply uniform eventual positivity, neither for resolvents nor for semigroups. In finite dimensions however, each of the above individual notions coincides with its uniform counterpart and we shall thus only speak of *eventual positivity* and *eventual strong positivity* if we work on finite dimensional Banach lattices. In the latter case the quasi-interior point u is not mentioned explicitly since the question whether a resolvent or a semigroup is eventually strongly positive with respect to u does not depend on the choice of u in finite dimensions.

In the present paper we mainly deal with eventual strong positivity with respect to a given quasi-interior point u (which is much easier to characterise than mere eventual positivity, as observed in Examples 7.1 of [6]). Mere eventual positivity will, however, occur in several counterexamples in this article.

2. LOSING EVENTUAL POSITIVITY UNDER POSITIVE PERTURBATIONS

If $(e^{tA})_{t \geq 0}$ is a positive C_0 -semigroup on a complex Banach lattice E (meaning that $e^{tA} \geq 0$ for all $t \geq 0$) and $B \in \mathcal{L}(E)$ is a positive operator, then it follows easily from the Dyson–Phillips series (see e.g. Theorem III.1.10 of [10]) that the perturbed semigroup $(e^{t(A+B)})_{t \geq 0}$ is positive, too. If, on the other hand, $B \in \mathcal{L}(E)$ is not necessarily positive, but real and a multiplication operator, then we can also conclude that $(e^{t(A+B)})_{t \geq 0}$ is positive. Indeed, we have $B + cI \geq 0$ for a sufficiently large number $c \geq 0$ and hence,

$$e^{t(A+B)} = e^{-c} e^{t(A+B+cI)} \geq 0$$

for all $t \geq 0$. It is the purpose of the current section to demonstrate that matters are much more complicated for eventually positive semigroups. In the first subsection we show how eventual positivity of the semigroup can get lost if we perturb A by a sufficiently large positive operator. In the second subsection we demonstrate that *individual* eventual positivity can be destroyed by positive perturbations of arbitrarily small norm.

2.1. LARGE PERTURBATIONS. It was demonstrated in Proposition 3.6 of [26] that eventual strong positivity of a matrix can always be destroyed by a suitable positive perturbation, unless the original matrix was positive itself. A similar phenomenon occurs for C_0 -semigroups. We first illustrate this by a concrete three dimensional example (Example 2.1). Afterwards we prove a general theorem which shows that the situation is similar in infinite dimensions (Theorem 2.3).

Let us now begin by studying a simple three dimensional matrix A that generates an eventually strongly positive semigroup on \mathbb{C}^3 . We will show that the eventual positivity is destroyed if we perturb A by a certain positive multiplication operator (i.e. by a diagonal matrix whose entries are all ≥ 0). Our example is a manifestation of the fact that certain sign patterns may or may not lead to eventual positivity as extensively discussed in [3], [12] and references therein.

EXAMPLE 2.1. We consider the symmetric matrix

$$A = \begin{bmatrix} -2 & -1 & 3 \\ -1 & -2 & 3 \\ 3 & 3 & -6 \end{bmatrix}$$

whose spectrum is $\sigma(A) = \{0, -1, -9\}$ and whose corresponding eigenvectors

$$u_1 = \frac{1}{\sqrt{3}} [1 \ 1 \ 1]^T, \quad u_2 = \frac{1}{\sqrt{2}} [1 \ -1 \ 0]^T \quad \text{and} \quad u_3 = \frac{1}{\sqrt{6}} [1 \ 1 \ -2]^T,$$

form an orthonormal basis in \mathbb{C}^3 . Hence 0 is the dominant eigenvalue of A ; the corresponding eigenspace $\ker A$ is one-dimensional and contains an eigenvector whose entries are all strictly positive. It thus follows from Theorem 6.7 of [6] that the semigroup $(e^{tA})_{t \geq 0}$ is eventually strongly positive. Yet, the semigroup is not

positive because A has negative entries outside the diagonal. We now show that a self-adjoint rank-1 perturbation of the form sB with $s > 0$ and $B := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ destroys the eventual positivity if $s > 4$. Indeed, it is easily verified that

$$v = [0 \quad 3 \quad 1]^T$$

is an eigenvector of $(A + 4B)$ corresponding to the eigenvalue 3. Computing the other eigenvalues we obtain

$$\sigma(A + 4B) = \left\{ 3, -\frac{1}{2}(9 \pm \sqrt{65}) \right\},$$

so 3 is the dominant eigenvalue. For $s = 4$ the eigenfunction is not strongly positive any more, and we will show that by choosing $s > 4$ the positivity is lost entirely.

Since all eigenvalues are simple, it follows from standard perturbation theory that there exists a curve $\lambda(s)$ and vectors $u(s) \neq 0$ depending analytically on s in an open interval J containing $s = 4$, such that $\lambda(s)u(s) = (A + sB)u(s)$ for all $s \in J$ with initial conditions $\lambda(4) = 3$ and $u(4) = v$; see Section II.1.7 of [18]. Differentiating the above equation with respect to s yields

$$(2.1) \quad \lambda'(s)u(s) + \lambda(s)u'(s) = Bu(s) + (A + sB)u'(s).$$

Taking the inner product of (2.1) with $u(s)$ and using the symmetry of $A + sB$ we see that

$$\begin{aligned} \lambda'(s)\|u(s)\|^2 + \lambda(s)\langle u'(s), u(s) \rangle &= \langle Bu(s), u(s) \rangle + \langle (A + sB)u'(s), u(s) \rangle \\ &= \langle Bu(s), u(s) \rangle + \langle u'(s), (A + sB)u(s) \rangle \\ &= \langle Bu(s), u(s) \rangle + \lambda(s)\langle u'(s), u(s) \rangle \end{aligned}$$

and so

$$\lambda'(s) = \frac{\langle Bu(s), u(s) \rangle}{\|u(s)\|^2}$$

for all $s \in J$. If we apply this to $s = 4$ we obtain

$$(2.2) \quad \lambda'(4) = \frac{\langle Bv, v \rangle}{\|v\|^2} = \frac{9}{10}.$$

To compute $w := u'(4)$ we rearrange (2.1) to get

$$(A + sB - \lambda(s)I)u'(s) = (\lambda'(s)I - B)u(s).$$

Setting $s = 4$ and making use of (2.2), we need to solve

$$(A + 4B - 3I)w = \left(\frac{9}{10}I - B\right)v.$$

Substituting the matrices A and B we seek $w = (w_1, w_2, w_3) \in \mathbb{R}^3$ so that

$$\begin{bmatrix} -5 & -1 & 3 \\ -1 & -1 & 3 \\ 3 & 3 & -9 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 9 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}.$$

Solving this equation we see that

$$w = u'(4) = \frac{1}{40} \begin{bmatrix} -3 \\ 15 \\ 0 \end{bmatrix} + \tau \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix},$$

for some $\tau \in \mathbb{R}$. Regardless of the value of τ , the first component of $u(s)$ has a negative derivative at $s = 4$, which means that first component changes sign from positive to negative at $s = 4$. Hence the eigenvector $u(s)$ of the dominant eigenvalue $\lambda(s)$ is not positive (or negative) for s in some interval $(4, 4 + \varepsilon)$, where $\varepsilon > 0$. Hence, the semigroup $(e^{t(A+sB)})_{t \geq 0}$ is *not* eventually positive for $s \in (4, 4 + \varepsilon)$. This follows for instance from Theorem 7.7(i) of [7].

Next we look at the above example in a different way.

EXAMPLE 2.2. Clearly the matrix

$$C_{a,s} = \begin{bmatrix} a & a & a \\ a & s & a \\ a & a & a \end{bmatrix}$$

generates a strongly positive semigroup $(e^{tC_{a,s}})_{t \geq 0}$ on \mathbb{C}^3 for every $a, s > 0$. Let A be given by (2.1). By Example 2.1 $(e^{tA})_{t \geq 0}$ is eventually strongly positive but not positive. We have also seen in Example 2.1 that for $a = 0$ the semigroup $(e^{t(C_{a,s}+A)})_{t \geq 0}$ is not eventually positive for suitable choice of $s > 4$. The reason is that the eigenvector corresponding to the dominant eigenvalue has strictly positive and strictly negative components. Having chosen such $s > 4$, the continuous dependence of the eigenvalues and eigenvectors on the coefficients of a matrix shows that we can choose $a > 0$ such that $(e^{t(C_{a,s}+A)})_{t \geq 0}$ is not eventually positive.

Hence we have the generator $C_{a,s}$ of a strongly positive semigroup and a bounded operator A generating an eventually strongly positive semigroup, but the semigroup generated by $C_{a,s} + A$ does not exhibit any positivity properties.

The above example demonstrates that strong positivity of a semigroup is possibly destroyed if the generator is perturbed by the generator of an eventually strongly positive semigroup; compare also Theorem 3.5 of [26].

We close this subsection with a general result asserting that, under certain technical assumptions, eventual strong positivity of a semigroup with respect to a quasi-interior point u is *always* unstable under suitable large positive perturbation unless the semigroup is positive. Recall from Theorem 7.6 of [7] that, if $(e^{tA})_{t \geq 0}$ is an eventually positive C_0 -semigroup and the spectrum $\sigma(A)$ is non-empty, then the spectrum contains the spectral bound $s(A)$. A finite dimensional analogue of the following theorem, which deals with powers of matrices rather than with time continuous semigroups, can be found in Proposition 3.6 of [26].

THEOREM 2.3. *Let E be a complex Banach lattice and let $(e^{tA})_{t \geq 0}$ be a real C_0 -semigroup on E which is individually eventually strongly positive with respect to a quasi-interior point u of E_+ . Suppose that $s(A)$ is not equal to $-\infty$ and that it is a pole of $\mathcal{R}(\cdot, A)$. Then the following assertions are equivalent:*

- (i) *For every positive operator $B \in \mathcal{L}(E)$ the perturbed semigroup $(e^{t(A+B)})_{t \geq 0}$ is individually eventually positive.*
- (ii) *For every positive rank-1 operator $B \in \mathcal{L}(E)$ the perturbed semigroup $(e^{t(A+B)})_{t \geq 0}$ is individually eventually positive.*
- (iii) *The semigroup $(e^{tA})_{t \geq 0}$ is positive.*

Proof. We may assume that $s(A) = 0$. Obviously, (iii) implies (i) and (i) implies (ii). To show “(ii) \Rightarrow (iii)”, assume that $(e^{t(A+B)})_{t \geq 0}$ is individually eventually positive for every positive rank-1 operator $B \in \mathcal{L}(E)$.

It suffices to prove that $\mathcal{R}(\mu, A) \geq 0$ for all $\mu > 0$. To this end, fix an arbitrary real number $\mu > 0$ and an arbitrary functional $0 < \varphi \in E'$. We show that $\mathcal{R}(\mu, A)' \varphi \geq 0$.

Since the spectral value $s(A) = 0$ is a pole of $\mathcal{R}(\cdot, A)$, it is an eigenvalue of A ([29], Theorem 2 in Section VIII.8), and it follows from Theorem 5.1 of [5] and Corollary 3.3 of [6] that A admits an eigenvector $v \gg_u 0$ for the eigenvalue 0. Since u is a quasi-interior point of E_+ , so is v and hence we have $\langle \varphi, v \rangle > 0$. We can thus find a scalar $\alpha > 0$ such that $\alpha \langle \varphi, v \rangle = \mu$.

Define $B := \alpha \varphi \otimes v \in \mathcal{L}(E)$. As B is a positive rank-1 operator, the semigroup $(e^{t(A+B)})_{t \geq 0}$ is by assumption individually eventually positive. It follows from Proposition A.2(i) in the Appendix that $s(A+B) = \alpha \langle \varphi, v \rangle = \mu$, that this number is a first order pole of the resolvent $\mathcal{R}(\cdot, A+B)$ and that the corresponding spectral projection Q is given by $Q = (\varphi \otimes v) \mathcal{R}(\mu, A) = (\mathcal{R}(\mu, A)' \varphi) \otimes v$.

Since $(\lambda - \mu) \mathcal{R}(\lambda, A+B) \rightarrow Q$ with respect to the operator norm as $\lambda \downarrow \mu$ and since the semigroup generated by $A+B$ is individually eventually positive, it follows from Corollary 7.3 of [7] that $Q \geq 0$. Thus, we conclude that $(\mathcal{R}(\mu, A)' \varphi) \otimes v \geq 0$ and hence, $\mathcal{R}(\mu, A)' \varphi \geq 0$, as claimed. ■

2.2. SMALL PERTURBATIONS. In this subsection we demonstrate that *individual* eventual positivity is very unstable with respect to small perturbations. The following example shows that it can be destroyed by positive perturbations of arbitrarily small norm. To do all necessary computations in our example we need a few formulas for rank-1-perturbations which can be found in the appendix of the paper. On any given set S we denote the constant function $S \rightarrow \mathbb{R}$ with value 1 by $\mathbf{1}$.

EXAMPLE 2.4. On the Banach lattice $E = C([-1, 1])$ there exist a bounded linear operator A and a positive rank-1-projection K with the following properties:

- (i) The spectral bound $s(A)$ equals 0, is a dominant spectral value of A and a first order pole of the resolvent $\mathcal{R}(\cdot, A)$.

For every $\alpha > 0$ the spectral bound $s(A + \alpha K)$ equals α , is a dominant spectral value of $A + \alpha K$ and a first order pole of the resolvent $\mathcal{R}(\cdot, A + \alpha K)$.

(ii) The resolvent $\mathcal{R}(\cdot, A)$ is individually but not uniformly eventually strongly positive with respect to $\mathbf{1}$ at 0.

Moreover, the semigroup $(e^{tA})_{t \geq 0}$ is individually but not uniformly eventually strongly positive with respect to $\mathbf{1}$.

(iii) The resolvent $\mathcal{R}(\cdot, A + \alpha K)$ is not individually eventually positive at $s(A + \alpha K)$ for any $\alpha > 0$.

Moreover, the semigroup $(e^{t(A+\alpha K)})_{t \geq 0}$ is not individually eventually positive for any $\alpha > 0$.

To prove this, we choose A to be the same operator which was constructed in Example 5.7 of [7]. For the convenience of the reader we briefly recall this construction:

Let $\varphi \in E'$ be the functional given by $\langle \varphi, f \rangle = \int_{-1}^1 f dx$ for every $f \in E$ and let $F = \ker \varphi$. Then we have $E = \langle \mathbf{1} \rangle \oplus F$, where $\mathbf{1}$ denotes the constant function with value 1 and $\langle \mathbf{1} \rangle$ is its span. Let $S \in \mathcal{L}(F)$ be the reflection operator given by $(Rf)(\omega) = f(-\omega)$ for every $f \in F$ and every $\omega \in [-1, 1]$ and let $A \in \mathcal{L}(E)$ be given by

$$A = 0_{\langle \mathbf{1} \rangle} \oplus (-2I_F - S).$$

We define $K := \mathbf{1} \otimes \delta_{-1}$, where δ_{-1} is the Dirac functional $\delta_{-1} : f \mapsto f(-1)$ on E . Hence, we have $Kf = f(-1)\mathbf{1}$ for every $f \in E$. Obviously, K is a positive rank-1-projection. Let us now show that the properties (i)–(iii) are fulfilled.

(i) Since $\sigma(S) = \{-1, 1\}$, we conclude that $\sigma(A) = \{-3, -1, 0\}$. Hence, the spectral bound $s(A)$ equals 0 and is a dominant spectral value of A ; clearly, it is also a first order pole of the resolvent $\mathcal{R}(\cdot, A)$. Note that $\mathbf{1}$ is an eigenvector of A for the eigenvalue 0.

Now, let $\alpha > 0$. We have $\alpha K = \alpha \delta_{-1} \otimes \mathbf{1}$ and it follows from Proposition A.2(i) that any complex number λ with $\operatorname{Re} \lambda > 0$ is a spectral value of $A + \alpha K$ if and only if $\lambda = \langle \alpha \delta_{-1}, \mathbf{1} \rangle = \alpha$. Hence, the spectral bound of $A + \alpha K$ equals α and is a dominant spectral value of $A + \alpha K$. The formula for $\mathcal{R}(\cdot, A + \alpha K)$ in Proposition A.2(i) immediately shows that the spectral value α is a first order pole of the resolvent.

(ii) This was shown in Example 5.7 of [7].

(iii) Fix $\alpha > 0$. We argue similarly as in Example 5.7 of [7]: for every $\varepsilon \in (0, 1)$ we can find a function $0 \leq f_\varepsilon \in E$ such that $f_\varepsilon(1) = \|f_\varepsilon\|_\infty = 1$, $\langle \varphi, f_\varepsilon \rangle = \varepsilon$ and $f_\varepsilon(-1) = 0$. A short computation (or compare with formula (5.3) in Example 5.7 of [7]) shows that for every $\lambda \in \rho(A)$ the resolvent of A is given by

$$\mathcal{R}(\lambda, A) = \frac{1}{\lambda} I_{\langle \mathbf{1} \rangle} \oplus \frac{1}{(\lambda + 2)^2 - 1} ((\lambda + 2) I_F - S).$$

Using this, an elementary calculation yields

$$(\mathcal{R}(\lambda, A)f_\varepsilon)(-1) = \frac{\varepsilon}{2} \left(\frac{1}{\lambda} - \frac{1}{\lambda+3} \right) - \frac{1}{(\lambda+2)^2 - 1}$$

for every $\lambda \in \rho(A)$. Hence, for every $\lambda > 0$ we can find an $\varepsilon \in (0, 1)$ such that $(\mathcal{R}(\lambda, A)f_\varepsilon)(-1) < 0$. Now we can show that $\mathcal{R}(\cdot, A + \alpha K)$ is not individually eventually positive at $s(A + \alpha K) = \alpha$: according to formula (A.2) we have

$$\mathcal{R}(\lambda, A + \alpha K)f_\varepsilon = \mathcal{R}(\lambda, A)f_\varepsilon + \frac{\alpha}{\lambda - \alpha} (\mathcal{R}(\lambda, A)f_\varepsilon)(-1) \mathbf{1}$$

and thus

$$(\mathcal{R}(\lambda, A + \alpha K)f_\varepsilon)(-1) = \left(1 + \frac{\alpha}{\lambda - \alpha} \right) (\mathcal{R}(\lambda, A)f_\varepsilon)(-1)$$

for all $\lambda \in \rho(A + \alpha K)$. Hence, if $\lambda > \alpha$ is given, then we only have to choose $\varepsilon > 0$ such that $(\mathcal{R}(\lambda, A)f_\varepsilon)(-1) < 0$ to obtain $\mathcal{R}(\lambda, A + \alpha K)f_\varepsilon \not\geq 0$.

It only remains to show that the semigroup $(e^{t(A+\alpha K)})_{t \geq 0}$ is not individually eventually positive. To this end, we choose $\varepsilon > 0$ such that $(\mathcal{R}(\alpha, A)f_\varepsilon)(-1) < 0$. It follows from formula (A.3) that we have

$$e^{-t\alpha} e^{t(A+\alpha K)} f_\varepsilon = e^{-t\alpha} e^{tA} f_\varepsilon + \alpha [(\mathcal{R}(\alpha, A)f_\varepsilon)(-1) - (e^{-t\alpha} e^{tA} \mathcal{R}(\alpha, A)f_\varepsilon)(-1)] \mathbf{1}$$

for every $t \geq 0$. Since the spectral bound of $A - \alpha I$ equals $-\alpha$ and the operator $A - \alpha I$ is bounded, we have $e^{-t\alpha} e^{tA} \rightarrow 0$ as $t \rightarrow \infty$ with respect to the operator norm. Hence, $e^{-t\alpha} e^{-t\alpha} e^{t(A+\alpha K)} f_\varepsilon$ converges to $\alpha (\mathcal{R}(\alpha, A)f_\varepsilon)(-1) \mathbf{1} < 0$ with respect to the $\|\cdot\|_\infty$ -norm as $t \rightarrow \infty$. In particular, $e^{t(A+\alpha K)} f_\varepsilon$ is not positive (in fact, it even fulfils $-e^{t(A+\alpha K)} f_\varepsilon \gg_{\mathbf{1}} 0$) for all sufficiently large t . ■

The above example indicates that if we want to prove any perturbation results for eventually positive resolvents or semigroups, then we should assume a version of *uniform* eventual positivity. This is our leitmotif for the rest of the paper.

3. PERTURBATION THEOREMS FOR RESOLVENTS

In this section we consider resolvents which are, at a spectral value λ_0 , uniformly eventually strongly positive with respect to a quasi-interior point u . In the first subsection we show that this property is stable with respect to sufficiently small perturbations which are either positive or real multiplication operators. In the second subsection we consider uniform eventual strong positivity at the spectral bound and prove a *quantitative* perturbation result for this property.

A concrete class of operators for which eventual positivity of resolvents has been studied for quite some time — though usually not under this name — is constituted by fourth order differential operators; see [28] for an overview; compare also Proposition 6.5 of [6]. For such operators, various perturbations results have been proved by quite concrete methods and estimates; see for instance [17].

Here, we rather focus on abstract functional analytical tools and prove results for abstract operators.

3.1. A QUALITATIVE RESULT. The main result of this subsection is the following qualitative perturbation result on eventually strongly positive resolvents at arbitrary real eigenvalues which are poles of the resolvent. Note that we do not make any kind of compactness assumption in this theorem.

THEOREM 3.1. *Let E be a complex Banach lattice and let A be a closed, densely defined real operator on E . Assume that $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ is a pole of $\mathcal{R}(\cdot, A)$ and suppose that $\mathcal{R}(\cdot, A)$ is uniformly eventually strongly positive with respect to u at λ_0 , where u is a quasi-interior point of E_+ .*

For all sufficiently small $r > 0$ there exists $\varepsilon > 0$ such that the following properties hold for every positive operator $B \in \mathcal{L}(E)$ of norm $\|B\| < \varepsilon$:

- (i) *The operator $A + B$ has a unique spectral value $\lambda_B \in B(\lambda_0, r)$.*
- (ii) *The spectral value λ_B is a real number, a pole of the resolvent $\mathcal{R}(\cdot, A + B)$ and an algebraically simple eigenvalue of $A + B$.*
- (iii) *The resolvent $\mathcal{R}(\cdot, A + B)$ is uniformly eventually strongly positive with respect to u at λ_B .*

One can prove a similar result for perturbations B which are not positive, but real multiplication operators; see Corollary 3.5 below.

In order to prove Theorem 3.1 we need two auxiliary results. The first one is a version of Proposition 4.2 of [7] on arbitrary Banach lattices. The fact that such a result holds was already remarked in the discussion after Definition 4.2 of [6]; however, the result was not stated explicitly there.

PROPOSITION 3.2. *Let $A : E \supseteq D(A) \rightarrow E$ be a real operator on a complex Banach lattice E and let λ_0 be either $-\infty$ or a spectral value of A in \mathbb{R} . Consider a real number $\lambda_1 > \lambda_0$ such that $(\lambda_0, \lambda_1] \subseteq \rho(A)$ and assume that $\mathcal{R}(\lambda_1, A) \geq 0$. Then the following assertions hold:*

- (i) *We have $\mathcal{R}(\lambda, A) \geq 0$ for all $\lambda \in (\lambda_0, \lambda_1]$.*
- (ii) *If u is a quasi-interior point of E_+ and if $\mathcal{R}(\lambda_1, A)^n \gg_u 0$ for some $n \in \mathbb{N}$, then $\mathcal{R}(\lambda, A) \gg_u 0$ for all $\lambda \in (\lambda_0, \lambda_1]$.*

The proof is exactly the same as the proof of Proposition 4.2 in [7].

The second ingredient for the proof of Theorem 3.1 is Lemma 3.3 below that guarantees that a pole of the resolvent which is, in addition, an algebraically simple real eigenvalue preserves these properties through a small perturbation by a real operator. This lemma is a typical result from standard perturbation theory; compare for instance Section IV.3 of [18]. Nevertheless, in order to have it available in exactly the version we need, we include a proof. In the preliminaries we introduced the concept of a real operator only on complex Banach lattices and to avoid the necessity of even more terminology, we shall state the lemma only on those spaces; compare however Remark 3.4 below.

LEMMA 3.3. *Let E be a complex Banach lattice and let A be a closed operator on E . Assume that $\lambda_0 \in \sigma(A)$ is a pole of the resolvent $\mathcal{R}(\cdot, A)$ and an algebraically simple eigenvalue of A with spectral projection P_0 . Let $r > 0$ be such that $\overline{B}(\lambda_0, r) \cap \sigma(A) = \{\lambda_0\}$ and set $\varepsilon = \min_{|\lambda-\lambda_0|=r} \|\mathcal{R}(\lambda, A)\|^{-1}$. For every $B \in \mathcal{L}(E)$ with $\|B\| < \varepsilon$ the following assertions are fulfilled:*

- (i) *$A + B$ has a unique spectral value $\lambda_B \in B(\lambda_0, r)$ and λ_B is a pole of the resolvent $\mathcal{R}(\cdot, A + B)$ and an algebraically simple eigenvalue of $A + B$.*
- (ii) *Denote by P_B the spectral projections associated with λ_B . Then $\lambda_B \rightarrow \lambda_0$ and $P_B \rightarrow P_0$ with respect to the operator norm as $\|B\| \rightarrow 0$.*
- (iii) *If $\lambda_0 \in \mathbb{R}$ and the operators A and B are real, then $\lambda_B \in \mathbb{R}$.*

Proof. Let C_r be the positively oriented circle of radius $r > 0$ centred at λ_0 as given in the statement of the lemma and let $B \in \mathcal{L}(E)$ with $\|B\| < \varepsilon$. For all $\lambda \in C_r$ we have $\|\mathcal{R}(\lambda, A)B\| \leq \|\mathcal{R}(\lambda, A)\|\|B\| \leq \|B\|/\varepsilon < 1$. Since $\lambda I - (A + B) = (I - B\mathcal{R}(\lambda, A))(\lambda I - A)$, a Neumann series expansion yields that $\lambda I - (A + B)$ is invertible and that

$$(3.1) \quad \mathcal{R}(\lambda, A + B) = \mathcal{R}(\lambda, A)[I - B\mathcal{R}(\lambda, A)]^{-1} = \mathcal{R}(\lambda, A) \sum_{k=0}^{\infty} [B\mathcal{R}(\lambda, A)]^k$$

for all $\lambda \in C_r$. In particular we can define the projection

$$P_B := \frac{1}{2\pi i} \int_{C_r} \mathcal{R}(\lambda, A + B) d\lambda.$$

We first show that P_B depends continuously on B . Indeed, let

$$\alpha := \min_{|\lambda-\lambda_0|=r} \|\mathcal{R}(\lambda, A + B)\|^{-1},$$

that is, we have $\alpha \cdot \|\mathcal{R}(\lambda, A + B)\| \leq 1$ for all $\lambda \in C_r$. Let $\delta \in (0, 1)$. Another Neumann series argument shows that whenever an operator $\tilde{B} \in \mathcal{L}(E)$, say of norm $\|\tilde{B}\| < \varepsilon$, is closer to B than $\alpha\delta$, then

$$\|\mathcal{R}(\lambda, A + \tilde{B}) - \mathcal{R}(\lambda, A + B)\| \leq \frac{\delta}{\alpha(1 - \delta)}$$

for all $\lambda \in C_r$, and thus $\|P_{\tilde{B}} - P_B\| \leq \delta r / (\alpha(1 - \delta))$. This proves that $P_{\tilde{B}} \rightarrow P_B$ in $\mathcal{L}(E)$ as $\tilde{B} \rightarrow B$ in $\mathcal{L}(E)$.

Now it follows from Lemma I.4.10 of [18] (the proof there does not rely on E being finite dimensional) and our assumption that $\dim(\text{Im } P_B) = \dim(\text{Im } P_0) = 1$ whenever $\|B\| < \varepsilon$. In particular, $A + B$ has only one spectral value λ_B in the disk $B(\lambda_0, r)$; since the corresponding spectral projection P_B has rank one, it follows that λ_B is a pole of the resolvent $\mathcal{R}(\cdot, A + B)$ ([18], Section III.6.5) and an algebraically simple eigenvalue. We thus proved (i) and the second part of (ii). Because $r > 0$ can be chosen arbitrarily small, we conclude that $\lambda_B \rightarrow \lambda_0$ as $\|B\| \rightarrow 0$, which proves the first part of (ii).

To prove (iii), suppose that A, B are real and that $\lambda_0 \in \mathbb{R}$. If $\lambda_B \notin \mathbb{R}$, then $\bar{\lambda}_B$ is a second spectral value of $A + B$ in the disk $B(\lambda_0, r)$, which contradicts (i). Thus, $\lambda_B \in \mathbb{R}$. ■

REMARK 3.4. The proof of Lemma 3.3 actually shows a bit more. Assertions (i) and (ii) of the lemma remain true if E is only assumed to be a complex Banach space. Assertion (iii) does not make sense if E is only a complex Banach space since the notion of a *real operator* is not defined on such spaces. If, however, E is a so-called *complexification* of a real Banach space $E_{\mathbb{R}}$, then the notion of a real operator makes sense; in this situation, assertion (iii) of Lemma 3.3 remains true.

For a detailed treatment of complexifications we refer the reader for example to [21]. Here we only point out that every complex Banach lattice is a certain complexification of a real Banach lattice and thus of a real Banach space (see Section II.11 of [25] or Section 2.2 of [19]).

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. It follows from Theorem 4.2 and Proposition 4.1 of [5] that λ_0 is an algebraically simple eigenvalue of A . By assumption we can choose $r > 0$ such that $B(\lambda_0, r) \cap \sigma(A) = \{\lambda_0\}$ and that $\mathcal{R}(\lambda_0 + r, A) \gg_u 0$. Choose $\varepsilon > 0$ as in Lemma 3.3 and let $B \in \mathcal{L}(E)$ be positive with norm $\|B\| < \varepsilon$. Then by that lemma there exists a unique $\lambda_B \in B(\lambda_0, r) \cap \sigma(A + B)$. Moreover λ_B is a pole of the resolvent $\mathcal{R}(\cdot, A + B)$ and an algebraically simple eigenvalue of $A + B$ and we have $\lambda_B \in (\lambda_0 - r, \lambda_0 + r)$. This proves (i) and (ii).

Since $B \mathcal{R}(\lambda_0 + r, A) \geq 0$, identity (3.1) with λ replaced with $r + \lambda_0$ implies that $\mathcal{R}(\lambda_0 + r, A + B) \geq \mathcal{R}(\lambda_0 + r, A) \gg_u 0$. Proposition 3.2(ii) now shows that $\mathcal{R}(\lambda, A + B) \gg_u 0$ for all $\lambda \in (\lambda_B, \lambda_0 + r]$. ■

Let us now consider the case where the perturbation B is not positive, but a real multiplication operator.

COROLLARY 3.5. *Let E be a complex Banach lattice and let A be a closed, densely defined real operator on E . Assume that $\lambda_0 \in \sigma(A) \cap \mathbb{R}$ is a pole of $\mathcal{R}(\cdot, A)$ and suppose that $\mathcal{R}(\cdot, A)$ is uniformly eventually strongly positive with respect to u at λ_0 , where u is a quasi-interior point of E_+ .*

Then, for all sufficiently small $r > 0$ there exists $\varepsilon > 0$ such that the assertions (i)–(iii) from Theorem 3.1 hold for every real multiplication operator $B \in \mathcal{L}(E)$ of norm $\|B\| < \varepsilon$.

The proof of this corollary relies on the following observation concerning multiplication operators.

LEMMA 3.6. *Let E be a complex Banach lattice and let $T \in \mathcal{L}(E)$ be a real multiplication operator. Then we have*

$$\|T\| \geq \min\{c \geq 0 : -cI \leq T \leq cI\}.$$

Proof. By $T_{\mathbb{R}}$ we denote the restriction $T|_{E_{\mathbb{R}}}$ of T to the real part of $E_{\mathbb{R}}$ of E . We have

$$\min\{c \geq 0 : -cI_E \leq T \leq cI_E\} = \min\{c \geq 0 : -cI_{E_{\mathbb{R}}} \leq T_{\mathbb{R}} \leq cI_{E_{\mathbb{R}}}\} = \|T_{\mathbb{R}}\|,$$

where the latter equality can be found in Theorem 3.1.11 of [19]. This proves the assertion since we clearly have $\|T_{\mathbb{R}}\| \leq \|T\|$. ■

REMARK 3.7. We suspect that there is equality in Lemma 3.6 as is true on real Banach lattices ([19], Theorem 3.1.11). This is, however, not important for our purposes.

We are now ready to prove Corollary 3.5.

Proof of Corollary 3.5. According to Theorem 3.1 we can, for each sufficiently small $r > 0$, find $\tilde{\varepsilon} > 0$ such that for each operator $0 \leq \tilde{B} \in \mathcal{L}(E)$ of norm $\|\tilde{B}\| < \tilde{\varepsilon}$ the following holds: there exists exactly one spectral value of $A + \tilde{B}$ in the disk $B(\lambda_0, r/3)$ and no other spectral value in the disk $B(\lambda_0, r)$ and the assertions (ii) and (iii) of the theorem are fulfilled for this spectral value and for the operator $A + \tilde{B}$.

Now, define $\varepsilon := \min\{r/3, \tilde{\varepsilon}/2\}$ and let $B \in \mathcal{L}(E)$ be a real multiplication operator of norm $\|B\| < \varepsilon$. According to Lemma 3.6 we have $\tilde{B} := B + \|B\|I \geq 0$; moreover, the positive operator \tilde{B} has norm $\|\tilde{B}\| < \tilde{\varepsilon}$.

Hence there exists exactly one spectral value of $A + \tilde{B}$ in the disk $B(\lambda_0, r/3)$ and no other spectral value in the disk $B(\lambda_0, r)$; furthermore, assertions (ii) and (iii) of Theorem 3.1 are fulfilled for this spectral value and for the operator $A + \tilde{B}$. This implies that the operator $A + \tilde{B} - \|B\|I = A + B$ has exactly one spectral value in the disk $B(\lambda_0, 2r/3)$ and that assertions (ii) and (iii) of Theorem 3.1 are fulfilled for this spectral value and for the operator $A + B$. ■

For matrices we can prove a stronger result than Theorem 3.1. Relying on the fact that the set of strongly positive matrices is open in the space of all real matrices we obtain stability of eventual strong positivity even with respect to negative perturbations.

PROPOSITION 3.8. *The set of matrices in $\mathbb{R}^{d \times d}$ having an eigenvalue at which its resolvent is eventually strongly positive is open in $\mathbb{R}^{d \times d}$.*

Proof. Let $A \in \mathbb{R}^{d \times d}$ be a matrix having an eigenvalue λ_0 at which $\mathcal{R}(\cdot, A)$ is eventually strongly positive. By Theorem 4.4 of [7] the corresponding spectral projection P_0 fulfils $P_0 \gg 0$, by which we mean that every entry of P_0 is strictly positive. Moreover, according to Proposition 3.1 of [7] λ_0 is the only eigenvalue of A having a positive eigenvector, and λ_0 is algebraically simple. Lemma 3.3 implies the existence of $\varepsilon_0 > 0$ such that $A + B$ has an algebraically simple eigenvalue $\lambda_B \in \mathbb{R}$ near λ_0 if $\|B\| < \varepsilon_0$. Moreover, the corresponding spectral projection P_B converges to P_0 as $\|B\| \rightarrow 0$. Since $P_0 \gg 0$ there exists $\varepsilon \in (0, \varepsilon_0]$ such that $P_B \gg 0$ whenever $\|B\| < \varepsilon$. Now, Theorem 4.4 of [7] implies that $\mathcal{R}(\cdot, A + B)$ is

eventually strongly positive at λ_B whenever $\|B\| < \varepsilon$. Hence all matrices in the ε -neighbourhood of A have an eigenvalue at which their resolvent is eventually strongly positive. ■

3.2. A QUANTITATIVE RESULT. In this subsection we consider uniform eventual strong positivity of resolvents at the spectral bound of an operator A and prove a quantitative perturbation result, meaning that we give an estimate of how large a positive perturbation may be in norm in order not to destroy the eventual strong positivity. Eventual positivity at $s(A)$ is of particular importance since it is related to eventual positivity of $(e^{tA})_{t \geq 0}$ (in case that A is a generator); compare the perturbation result in Theorem 4.9 which we obtain as a consequence of the perturbation result in the present subsection.

To formulate the next theorem we need the following notation: for every operator $A : E \supseteq D(A) \rightarrow E$ on a complex Banach space E we define the *real spectral bound* $s_{\mathbb{R}}(A)$ of A to be the supremum of all real spectral values of A , i.e. $s_{\mathbb{R}}(A) := \sup(\sigma(A) \cap \mathbb{R})$; we clearly have $-\infty \leq s_{\mathbb{R}}(A) \leq s(A) \leq \infty$.

THEOREM 3.9. *Let $E \neq \{0\}$ be a complex Banach lattice, let $u \in E$ be a quasi-interior point of E_+ and let A be a densely defined and real linear operator on E such that $s(A)$ is a spectral value of A and a pole of $\mathcal{R}(\cdot, A)$. Suppose there exists $\lambda_1 > s(A)$ such that $\mathcal{R}(\lambda, A) \gg_u 0$ for all $\lambda \in (s(A), \lambda_1)$ and assume furthermore that*

$$M := \sup_{\operatorname{Re} \lambda \geq \lambda_1} \|\mathcal{R}(\lambda, A)\| < \infty.$$

Then, for every operator $0 \leq K \in \mathcal{L}(E)$ with norm $\|K\| < 1/M$ the real spectral bound $s_{\mathbb{R}}(A + K)$ fulfils the following properties:

- (i) $s_{\mathbb{R}}(A + K) \leq s(A + K) < \lambda_1$.
- (ii) $\mathcal{R}(\lambda, A + K) \gg_u 0$ for all $\lambda \in (s_{\mathbb{R}}(A + K), \lambda_1)$.
- (iii) If K is A -compact, then $s_{\mathbb{R}}(A + K) \geq s(A)$ and $s_{\mathbb{R}}(A + K)$ is a pole of $\mathcal{R}(\cdot, A + K)$.
- (iv) If K is A -compact and non-zero, then $s_{\mathbb{R}}(A + K) > s(A)$.

Note that the condition $M < \infty$ in the above theorem is equivalent to the condition that λ_1 be larger than the so-called *pseudo-spectral bound* $s_0(A)$ of A (see p. 356 of [1] for a definition); in particular, we have $M < \infty$ in case that $s(A) = s_0(A)$. The latter equality is for instance fulfilled if the operator A generates a C_0 -semigroup on E which is eventually norm continuous (see Corollary IV.3.11 of [10] and Theorem 5.1.9 of [1]) or which is individually eventually positive (see Corollary 7.5 of [7]).

Proof of Theorem 3.9. Let $0 \leq K \in \mathcal{L}(H)$ and note that

$$\lambda I - (A + K) = [I - K \mathcal{R}(\lambda, A)](\lambda I - A),$$

for each $\lambda \in \rho(A)$. Hence, if $\lambda \in \rho(A)$ and $r(K \mathcal{R}(\lambda, A)) < 1$, then $\lambda \in \rho(A + K)$ and

$$(3.2) \quad \mathcal{R}(\lambda, A + K) = \mathcal{R}(\lambda, A)[I - K \mathcal{R}(\lambda, A)]^{-1}.$$

(i) Obviously, $s_{\mathbb{R}}(A + K) \leq s(A + K)$. If $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \lambda_1$ then we have $r(K\mathcal{R}(\lambda, A)) \leq \|K\mathcal{R}(\lambda, A)\| < (1/M)\|\mathcal{R}(\lambda, A)\| \leq 1$ and thus $\lambda \in \rho(A + K)$. This proves that $s(A + K) < \lambda_1$.

(ii) As noted in the proof of (i) we have $\|K\mathcal{R}(\lambda_1, A)\| < 1$. Since the mapping $\lambda \mapsto \|K\mathcal{R}(\lambda, A)\|$ is continuous we also obtain the estimate

$$r(K\mathcal{R}(\lambda, A)) \leq \|K\mathcal{R}(\lambda, A)\| < 1$$

for all $\lambda \in (\lambda_1 - \varepsilon, \lambda_1)$ if $\varepsilon > 0$ is chosen small enough. Each such λ is contained in $\rho(A + K)$ and (3.2) holds. Since $K\mathcal{R}(\lambda, A)$ is positive and has spectral radius < 1 , the inverse $[I - K\mathcal{R}(\lambda, A)]^{-1}$ is also positive. We thus have $[I - K\mathcal{R}(\lambda, A)]^{-1}f > 0$ for each $f > 0$. As $\mathcal{R}(\lambda, A) \gg_u 0$, formula (3.2) now yields $\mathcal{R}(\lambda, A + K) \gg_u 0$. According to Proposition 3.2(ii) this implies that $\mathcal{R}(\lambda, A + K) \gg_u 0$ holds in fact for all $\lambda \in (s_{\mathbb{R}}(A), \lambda_1)$.

(iv) Assume now in addition that K is A -compact and non-zero. To prove (iv) it suffices to show that $A + K$ has a spectral value $\lambda \in (s(A), \lambda_1)$. To this end, let $P \in \mathcal{L}(E)$ be the spectral projection of A associated with $s(A)$. By Theorem 4.1 of [5] and Corollary 3.3 of [6], P is a rank-1 operator which fulfils $P \gg_u 0$ and we have $P = \lim_{\lambda \downarrow s(A)} (\lambda - s(A))\mathcal{R}(\lambda, A)$ with respect to the operator norm. Let us now define a mapping $\gamma : [s(A), \lambda_1) \rightarrow \mathcal{L}(E)$ which is given by

$$\gamma(\lambda) = \begin{cases} (\lambda - s(A))K\mathcal{R}(\lambda, A) & \text{if } \lambda > s(A), \\ KP & \text{if } \lambda = s(A). \end{cases}$$

Note that γ is continuous with respect to the operator norm and that $\gamma(\lambda)$ is a compact, positive operator for every $\lambda \in [s(A), \lambda_1)$. Moreover, we recall that the restriction of the mapping $\mathcal{L}(E) \rightarrow [0, \infty)$, $T \mapsto r(T)$ to the set of compact operators is continuous with respect to the operator norm; this follows e.g. from Remark IV.3.3 and the discussion in Section IV.3.5 of [18] or from Theorem 2.1(a) of [8]. Hence, $r(\gamma(\cdot)) : [s(A), \lambda_1) \rightarrow [0, \infty)$ is continuous.

Let us show that $r(\gamma(s(A))) = r(KP) > 0$. Since P has rank 1 and since $P \gg_u 0$, we can find a strictly positive functional $\varphi \in E'$ and a vector $0 \ll_u v \in E$ such that $P = \varphi \otimes v$ and hence, $KP = \varphi \otimes Kv$. Since v is a quasi-interior point of E_+ and K is non-zero, it follows that $Kv \neq 0$. Using that φ is strictly positive, we deduce that $\langle \varphi, Kv \rangle > 0$ and hence $\sigma(KP) = \sigma(\varphi \otimes Kv) \ni \langle \varphi, Kv \rangle > 0$. Thus, $r(KP) > 0$.

We conclude that for all sufficiently small $\lambda > s(A)$ we have that

$$r((\lambda - s(A))K\mathcal{R}(\lambda, A)) \geq \frac{r(KP)}{2} > 0.$$

We can thus find $\lambda \in (s(A), \lambda_1)$ such that $r(K\mathcal{R}(\lambda, A)) > 1$. On the other hand we have $\|K\mathcal{R}(\lambda_1, A)\| < 1$. Hence we have $r(K\mathcal{R}(\lambda, A)) \leq \|K\mathcal{R}(\lambda, A)\| < 1$ for all $\lambda \in (s(A), \lambda_1)$ which are sufficiently close to λ_1 . Using again that the spectral radius is continuous on the compact operators with respect to the norm topology ([8], Theorem 2.1(a)) we conclude from the intermediate value theorem that

$r(K \mathcal{R}(\lambda, A)) = 1$ for some $\lambda \in (s(A), \lambda_1)$. For this λ the operator $\lambda I - A$ is invertible, but the operator $I - K \mathcal{R}(\lambda, A)$ is not since the spectral radius of $K \mathcal{R}(\lambda, A)$ is contained in its spectrum (this is a general fact for positive operators, see Proposition V.4.1 of [25]). Hence, it follows from (3.1) that $\lambda \in \sigma(A + K)$.

(iii) Assume that K is A -compact. If $K = 0$, then assertion (iii) is obvious. If K is non-zero, then it follows from (iv) that $s_{\mathbb{R}}(A + K) > s(A)$. We use formula (3.2) to prove that $s_{\mathbb{R}}(A + K)$ is a pole of $\mathcal{R}(\cdot, A + K)$. Let $\Omega := \{z \in \mathbb{C} : \operatorname{Re} z > s(A)\}$. On this set, the mappings $\lambda \mapsto \mathcal{R}(\lambda, A)$ and $\lambda \mapsto K \mathcal{R}(\lambda, A)$ are analytic and the latter one takes only compact operators as its values. Since $I - K \mathcal{R}(\lambda, A)$ is invertible for at least one $\lambda \in \Omega$, it follows from the so-called *analytic Fredholm theorem* (see e.g. Theorem 1 of [27]) that $[I - K \mathcal{R}(\lambda, A)]^{-1}$ is meromorphic on Ω . Hence, $\mathcal{R}(\cdot, A + K)$ is either analytic at $s_{\mathbb{R}}(A + K)$ or it has a pole there; yet, since $s_{\mathbb{R}}(A + K)$ is, of course, a spectral value of $A + K$, the latter alternative must be true. ■

4. PERTURBATION THEOREMS FOR SEMIGROUPS

In this final section we consider perturbations of semigroup generators. We do however *not* prove theorems of the type “If $(e^{tA})_{t \geq 0}$ is eventually strongly positive, then so is $(e^{t(A+B)})_{t \geq 0}$ for appropriate B ”. Those results would, of course, be desirable, but it seems to be a difficult task to prove them. Instead we assume that the resolvent of the semigroup generator A is uniformly eventually strongly positive at the spectral bound $s(A)$. Using the results of Section 3 we then show that the resolvent of the perturbed operator $A + B$ is also uniformly eventually strongly positive at $s(A + B)$ and, by means of the characterisation results in Sections 4 and 5 of [6], this yields at least *individual* eventual strong positivity of the semigroup generated by $A + B$. In case that the underlying space is an L^2 -space, one even obtains uniform eventual strong positivity of this semigroup, see Theorem 4.9 below and Theorem 10.2.1 of [15].

4.1. A QUALITATIVE RESULT. We start again with a subsection containing qualitative perturbation results. To prove our main theorems we need the following auxiliary results. As we did with Lemma 3.3, we only formulate the following result on a complex Banach lattice, although the proof shows that it is actually true on arbitrary complexifications of real Banach spaces.

LEMMA 4.1. *Let E be a complex Banach lattice and let $(e^{tA})_{t \geq 0}$ be a real eventually norm continuous C_0 -semigroup on E . Suppose furthermore that $s(A)$ is a dominant spectral value of A , a pole of the resolvent and an algebraically simple eigenvalue; denote the spectral projection associated with $s(A)$ by P_0 . Then there exists an $\varepsilon > 0$ such that the following properties are fulfilled for every real operator $B \in \mathcal{L}(E)$ with $\|B\| < \varepsilon$:*

- (i) *The spectral bound $s(A + B)$ of $A + B$ is a dominant spectral value of $A + B$, a pole of the resolvent $\mathcal{R}(\cdot, A + B)$ and an algebraically simple eigenvalue.*

(ii) We have $s(A + B) \rightarrow s(A)$ and $P_B \rightarrow P_0$ with respect to the operator norm as $\|B\| \rightarrow 0$; here, P_B denotes the spectral projection of $A + B$ associated with the isolated spectral value $s(A + B)$.

Proof. Since $s(A)$ is a dominant spectral value and $(e^{tA})_{t \geq 0}$ is eventually norm continuous, we can find a number $r > 0$ such that $\operatorname{Re} \lambda \leq s(A) - 2r$ for all $\lambda \in \sigma(A) \setminus \{s(A)\}$. The spectral bound of the restriction of A to the kernel of P_0 fulfils $s(A|_{\ker P_0}) \leq s(A) - 2r$ and since $(e^{tA}|_{\ker P_0})_{t \geq 0}$ is eventually norm continuous, it follows that the growth bound of this restricted semigroup is also no larger than $s(A) - 2r$ ([10], Corollary IV.3.11). In particular, we obtain from the Laplace transform representation of the resolvent that

$$\sup_{\operatorname{Re} \lambda \geq s(A) - r} \|\mathcal{R}(\lambda, A|_{\ker P_0})\| < \infty.$$

On the other hand,

$$\sup_{|\lambda - s(A)| \geq r} \|\mathcal{R}(\lambda, A|_{\operatorname{Im} P_0})\| < \infty.$$

Hence, $\|\mathcal{R}(\cdot, A)\|$ is bounded by a constant $C \in (0, \infty)$ on the set

$$\overline{\Omega} := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq s(A) - r \text{ and } |\lambda - s(A)| \geq r\}.$$

Define $\varepsilon = 1/C$ and let $B \in \mathcal{L}(E)$ with $\|B\| < \varepsilon$. According to Lemma 3.3, $A + B$ has a uniquely determined spectral value $\lambda_B \in B(r, s(A))$, and this spectral value λ_B is real, a pole of the resolvent $A + B$ and an algebraically simple eigenvalue of $A + B$. Moreover, $\lambda_B \rightarrow s(A)$ and $P_B \rightarrow P_0$ with respect to the operator norm as $\|B\| \rightarrow 0$.

It only remains to show that $A + B$ has no spectral value within the set $\overline{\Omega}$ since this implies that $s(A + B) = \lambda_B$ has the claimed properties. So, let $\lambda \in \overline{\Omega}$. Then we have

$$\lambda - (A + B) = [I - B \mathcal{R}(\lambda, A)](\lambda I - A).$$

Since $\|B \mathcal{R}(\lambda, A)\| < \varepsilon C = 1$ the above operator is invertible and hence, $\lambda \in \rho(A + B)$. ■

Now we formulate and prove the first main result of this subsection.

THEOREM 4.2. *Let E be a complex Banach lattice and let $(e^{tA})_{t \geq 0}$ be a real C_0 -semigroup on E . Suppose that $s(A)$ is a dominant spectral value of A and a pole of the resolvent. Suppose that $\mathcal{R}(\cdot, A)$ is uniformly eventually strongly positive at $s(A)$ with respect to a quasi-interior point u of E_+ . Assume moreover that at least one of the following assumptions is fulfilled:*

- (i) $(e^{tA})_{t \geq 0}$ is analytic and $D(A) \subseteq E_u$.
- (ii) $(e^{tA})_{t \geq 0}$ is immediately norm-continuous and $E_u = E$.

Then there exists an $\varepsilon > 0$ such that for every operator $0 \leq B \in \mathcal{L}(E)$ with $\|B\| < \varepsilon$ the semigroup generated by $A + B$ is individually eventually strongly positive with respect to u .

Proof. According to Theorem 3.1 and Lemma 4.1 we can find an $\varepsilon > 0$ with the following property: for all $0 \leq B \in \mathcal{L}(E)$ with $\|B\| < \varepsilon$ the spectral bound of $A + B$ is a dominant and isolated spectral value of $A + B$, an algebraically simple eigenvalue and a first order pole of the resolvent. Moreover, the resolvent $\mathcal{R}(\cdot, A + B)$ is uniformly eventually strongly positive at $s(A + B)$ with respect to u . We now see from Theorem 4.2 of [5] that the spectral projection P associated with $s(A + B)$ is strongly positive with respect to u .

Next we observe that assumptions (i) and (ii) imply that $(e^{t(A+B)})_{t \geq 0}$ is eventually (in fact: immediately) norm continuous and that $e^{tA}E \subseteq E_u$ for all $t > 0$. Indeed, if (i) is fulfilled, then it follows from Proposition III.1.12(i) of [10] that the perturbed semigroup $(e^{t(A+B)})_{t \geq 0}$ is analytic, too. From $D(A + B) = D(A) \subseteq E_u$ we can thus conclude that $e^{t(A+B)}E \subseteq D(A + B) \subseteq E_u$ for every $t > 0$. If, on the other hand, (ii) is fulfilled, then $(e^{tA})_{t \geq 0}$ is immediately norm continuous according to Theorem III.1.16(i) of [10]. Moreover, we obviously have $e^{tA}E \subseteq E = E_u$.

Let us finally show that the two properties proved above imply that the semigroup $(e^{t(A+B)})_{t \geq 0}$ is individually eventually strongly positive with respect to u . Since the perturbed semigroup is eventually norm continuous and the spectral bound $s(A + B)$ is a dominant spectral value of $A + B$ and a first order pole of its resolvent, it follows that the rescaled semigroup $(e^{t(A+B-s(A+B)I)})_{t \geq 0}$ is bounded. Since the spectral projection P associated with $s(A + B)$ is strongly positive with respect to u , we conclude from the characterisation theorem given in Theorem 5.2 of [6] that $(e^{t(A+B)})_{t \geq 0}$ is individually eventually strongly positive with respect to u . ■

A typical space where the condition $E_u = E$ in assumption (ii) of the above theorem is fulfilled is the space $\mathcal{C}(K; \mathbb{C})$ of all complex-valued continuous functions on a compact Hausdorff space K ; this holds independently of the choice of the quasi-interior point u .

EXAMPLES 4.3. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain of class \mathcal{C}^2 . Consider one of the following situations:

(i) $E = \mathcal{C}_0(\Omega; \mathbb{C})$ (the space of all complex-valued continuous functions on Ω which vanish at the boundary) and $A = -\Delta_D^2$, where Δ_D denotes the Dirichlet Laplace operator on E .

(ii) $E = \mathcal{C}(\overline{\Omega}; \mathbb{C})$ and $A = -(\Delta_R^c)^2$, where Δ_R^c denotes the Laplace operator on E with Robin boundary conditions (see Section 6.4 of [7] for details).

Then E and A fulfil the assumptions of Theorem 4.2. For (i), this is shown in the proof of Theorem 6.1 of [6] and for (ii), this follows from Sections 6.3 and 6.4 of [7].

In case that the perturbation B is compact, we can replace assumption (ii) in Theorem 4.2 with a weaker condition. This is the subject of the next theorem, our second main result in this section.

THEOREM 4.4. *Suppose that the assumptions of Theorem 4.2 are fulfilled, but instead of (i) or (ii) assume the following condition:*

(iii) $(e^{tA})_{t \geq 0}$ is eventually norm continuous and $E_u = E$.

Then there is an $\varepsilon > 0$ such that for every compact operator $0 \leq B \in \mathcal{L}(E)$ with $\|B\| < \varepsilon$ the semigroup generated by $A + B$ is individually eventually strongly positive with respect to u .

Proof. The proof is the same as for Theorem 4.2. The only difference is that we need the compactness of B to conclude that the perturbed semigroup $(e^{t(A+B)})_{t \geq 0}$ is eventually norm continuous since the original semigroup $(e^{tA})_{t \geq 0}$ is only assumed to be eventually but not necessarily immediately norm continuous; see Proposition III.1.14 of [10] ■

Let us also comment on perturbation by (non-positive) multiplication operators.

COROLLARY 4.5. *The Theorems 4.2 and 4.4 remain true if we replace the assumption of B being positive with the assumption that B be a real multiplication operator (where, however, ε has to be chosen half as large as in the theorems).*

Proof. The operator $\tilde{B} := B + \|B\|I$ has norm at most $2\|B\|$ and is positive according to Lemma 3.6. Hence, the corollary follows from Theorems 4.2 and 4.4 and from the formula

$$e^{t(A+B)} = e^{-t\|B\|} e^{t(A+\tilde{B})}$$

which is true for all $t \geq 0$. ■

We can prove a much stronger result than in the above theorems in case that E is finite dimensional.

PROPOSITION 4.6. *Let $d \in \mathbb{N}$, $d \geq 1$. The set of all generators of eventually strongly positive C_0 -semigroups on \mathbb{C}^d is an open subset of $\mathbb{R}^{d \times d}$.*

Proof. Let $A \in \mathbb{C}^{d \times d}$ be the generator of an eventually strongly positive semigroup. Then obviously, $A \in \mathbb{R}^{d \times d}$. By the characterisation result in Corollary 5.6 of [7] this implies that $s(A)$ is a dominant spectral value of A and that the corresponding spectral projection P_0 has only strictly positive entries. Hence, it follows from Proposition 3.1 of [7] that $s(A)$ is an algebraically simple eigenvalue of A . We now conclude from Lemma 4.1 that for all $B \in \mathbb{R}^{d \times d}$ which are sufficiently small in norm, the spectral bound $s(A+B)$ is a dominant spectral value of $A+B$. Moreover, the spectral projection P_B corresponding to $s(A+B)$ fulfils $P_B \rightarrow P_0$ as $\|B\| \rightarrow 0$. Since P_B is real, it thus contains only strictly positive entries whenever $\|B\|$ is sufficiently small and thus, we can again employ the characterisation result in Corollary 5.6 of [7] to conclude that the semigroup $(e^{t(A+B)})_{t \geq 0}$ is eventually strongly positive for all such B . ■

It is a natural question whether the set of all generators of strongly positive matrix semigroups $(e^{tA})_{t \geq 0}$ (meaning that each matrix e^{tA} has only strictly positive entries whenever $t > 0$) is also open in $\mathbb{R}^{d \times d}$. Surprisingly, the answer depends on the dimension d : it is positive if $d = 2$ (and, obviously, also if $d = 1$), but negative if $d \geq 3$. The details can be found in the next corollary and the subsequent example.

COROLLARY 4.7. *The set all generators of strongly positive C_0 -semigroups on \mathbb{C}^2 is an open subset of $\mathbb{R}^{2 \times 2}$.*

Proof. The generator of a strongly positive C_0 -semigroup on \mathbb{C}^2 is obviously a real matrix and it was shown in Proposition 6.2 of [7] that a matrix $A \in \mathbb{R}^{2 \times 2}$ generates a strongly positive C_0 -semigroup if and only if it generates an eventually strongly positive C_0 -semigroup. Hence, the corollary follows from Proposition 4.6. ■

EXAMPLE 4.8. We showed in Proposition 4.6 that eventual strong positivity of matrix semigroups is robust with respect to small, not necessarily positive perturbations. We now give an example that this is not the case for strong positivity of the semigroup. Consider the generator

$$A := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

on \mathbb{C}^3 . Then it is easily checked that

$$A^{2k} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad A^{2k+1} = A$$

for all $k \geq 1$ and thus $e^{tA} = \sum_{k=0}^{\infty} (t^k/k!)A^k \gg 0$ for all $t > 0$. Moreover, $\sigma(A) = \{0, \pm 1\}$, where the eigenspace associated with 1 is spanned by the positive eigenvector $(1, 1, \sqrt{2})$. If we set

$$B := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$A + \varepsilon B = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -\varepsilon & 1 \\ -\varepsilon & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

cannot generate a positive semigroup for any $\varepsilon > 0$. However, if ε is small enough, then Theorem 4.2 implies that $(e^{t(A+\varepsilon B)})_{t \geq 0}$ is eventually strongly positive.

Clearly, the above example can be generalised to any finite dimension $d \geq 3$ by defining

$$A := \frac{1}{\sqrt{d-1}} \begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 1 \\ 1 & \dots & 1 & 0 \end{bmatrix} \in \mathbb{C}^{d,d}.$$

Hence, Proposition 4.6 and the above example show that, in dimension $d \geq 3$, eventual strong positivity of semigroups is a much more stable concept than strong positivity.

4.2. A QUANTITATIVE RESULT. In this final section we prove a quantitative perturbation theorem for semigroups. It is based on the quantitative result about resolvents in Theorem 3.9. It seems, in general, unclear how to ensure that the real spectral bound $s_{\mathbb{R}}(A + B)$ (see the discussion before Theorem 3.9 for a definition) is a dominant spectral value of $A + B$ (and thus coincides with $s(A + B)$). For this reason we restrict ourselves to self-adjoint semigroups and perturbations on Hilbert spaces in the following theorem; since the spectrum of self-adjoint operators is always real we clearly have $s_{\mathbb{R}}(A + B) = s(A + B)$ in case that A and B are self-adjoint.

THEOREM 4.9. *Let $\{0\} \neq H$ be a complex-valued L^2 -space over an arbitrary measure space, let $u \in H_+$ be a quasi-interior point and let $(e^{tA})_{t \geq 0}$ be a self-adjoint and real C_0 -semigroup on H with $D(A) \subseteq H_u$. Suppose that there is a $\lambda_1 > s(A)$ such that $\mathcal{R}(\lambda, A) \gg_u 0$ for all $\lambda \in (s(A), \lambda_1)$.*

If $B \in \mathcal{L}(H)$ is positive and self-adjoint with $\|B\| < \lambda_1 - s(A)$, then the semigroup $(e^{t(A+B)})_{t \geq 0}$ is uniformly eventually strongly positive with respect to u .

Note that if the underlying measure space of H is σ -finite, then a vector $u \in H_+$ is a quasi-interior point if and only if $u(\omega) > 0$ for almost all ω in the measure space. The fact that we obtain *uniform* eventual strong positivity for the perturbed semigroup in the above theorem is due to a recent result of the authors in the Hilbert space case which appeared in the second author’s Ph.D. Dissertation ([15], Theorem 10.2.1).

Proof of Theorem 4.9. Since $e^{t_0 A} H \subseteq H_u$, it follows from Theorem 2.3(ii) of [5] that $e^{t_0 A}$ is compact. Therefore, the semigroup $(e^{tA})_{t \geq 0}$ is eventually compact, and since it is analytic, it must in fact be immediately compact; see Exercise II.4.30(6) of [10]. Hence, its generator A has compact resolvent ([10], Theorem II.4.29). In particular, $s(A)$ is a pole of $\mathcal{R}(\cdot, A)$ and B is A -compact.

We have $M := \sup_{\operatorname{Re} \lambda \geq \lambda_1} \|\mathcal{R}(\lambda, A)\| = 1/(\lambda_1 - s(A))$ since A is self-adjoint.

Moreover, $s_{\mathbb{R}}(A + B)$ equals $s(A + B)$ since $A + B$ is self-adjoint. It therefore follows from Theorem 3.9 that $s(A + B)$ is a pole of $\mathcal{R}(\cdot, A + B)$ and that $\mathcal{R}(\cdot, A + B)$ is uniformly eventually strongly positive at $s(A + B)$ with respect to u . Hence,

the spectral projection P associated with the spectral value $s(A + B)$ of $A + B$ fulfils $P \gg_u 0$ according to Theorem 4.1 of [5]. Since $D(A + B) = D(A) \subseteq H_u$ it thus follows from the characterisation of eventual positivity in Theorem 10.2.1 of [15] that $(e^{t(A+B)})_{t \geq 0}$ is uniformly eventually strongly positive with respect to u . ■

Let us conclude the paper with the following example of a Laplace operator on $(0, 1)$ with non-local boundary conditions. Eventual positivity properties of the unperturbed operator were discussed in Section 6 of [6] and in Section 11.7 of [15].

EXAMPLE 4.10. Let H denote the Hilbert space $L^2(0, 1)$ and consider the sesqui-linear form

$$a : H^1(0, 1) \times H^1(0, 1) \rightarrow \mathbb{C}$$

which is given by

$$a(u, v) = \int_0^1 u' \bar{v}' dx + [u(0) \quad u(1)] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v(0) \\ v(1) \end{bmatrix}.$$

The operator A on H associated with a is self-adjoint; it is a (negative) Laplace operator with non-local boundary conditions, given by

$$D(A) = \{u \in H^2(0, 1) : u'(0) = -u'(1) = u(0) + u(1)\},$$

$$Au = -u''.$$

It was shown in Theorem 11.7.3 of [15] that $s(-A) < 0$ and that the semigroup $(e^{-tA})_{t \geq 0}$ is not positive, but uniformly eventually strongly positive with respect to $\mathbf{1}$ (where $\mathbf{1}$ denotes the constant function on $(0, 1)$ with value 1). Let us now prove the following assertion:

If $0 \leq B \in \mathcal{L}(H)$ is self-adjoint and $\|B\| < 1$, then the semigroup generated by $-A + B$ is uniformly eventually strongly positive with respect to $\mathbf{1}$.

Proof. Obviously, $u = \mathbf{1}$ is a quasi-interior point of H_+ . Moreover, we have $D(A) \subseteq H^1(0, 1) \subseteq L^\infty(0, 1) = H_1$. It was shown in Theorem 6.11(i) of [6] that $\mathcal{R}(0, -A) \gg_u 0$ and in the proof of that theorem the following formula for $\mathcal{R}(0, -A)$ was given:

$$(\mathcal{R}(0, -A)f)(x) = \frac{1}{2} \int_0^x \int_y^1 f(z) dz dy + \frac{1}{2} \int_x^1 \int_0^y f(z) dz dy$$

for all $f \in H$ and all $x \in [0, 1]$. We want to apply Theorem 4.9 and thus we have to estimate the number $-s(-A)$. This number coincides with the distance of 0 to $\sigma(-A)$ which is in turn equal to $1/\|\mathcal{R}(0, -A)\|$ since $-A$ is self-adjoint. Hence, we have to estimate the norm of $\mathcal{R}(0, -A)$. Since this is a positive operator, we

only have to consider positive functions $f \in H$. For each such f we have

$$\| \mathcal{R}(0, -A)f \|_2 \leq \| \mathcal{R}(0, -A)f \|_\infty \leq \int_0^1 \int_0^1 f(z) \, dz \, dy = \|f\|_1 \leq \|f\|_2.$$

Hence, we have $\| \mathcal{R}(0, -A) \| \leq 1$ and thus $1/\| \mathcal{R}(0, -A) \| \geq 1$. The assertion now follows from Theorem 4.9. ■

Appendix A. FORMULAS FOR RANK-1-PERTURBATIONS

Let $A : E \supseteq D(A) \rightarrow E$ be a linear operator on a complex Banach space E . In this appendix we study what happens to the spectrum and the resolvent of A if we perturb A by a rank-1-operator. If A generates a C_0 -semigroup and if the perturbation is, in a sense, well-adapted to A , we also derive a formula for the perturbed C_0 -semigroup.

PROPOSITION A.1. *Let $A : E \supseteq D(A) \rightarrow E$ be a linear operator on a complex Banach space E and let $\lambda \in \rho(A)$. Moreover, assume that $\varphi \in E'$ and $w \in E$.*

Then $\lambda \in \rho(A + \varphi \otimes w)$ if and only if $1 \neq \langle \varphi, \mathcal{R}(\lambda, A)w \rangle$. In this case we have

$$(A.1) \quad \mathcal{R}(\lambda, A + \varphi \otimes w) = \mathcal{R}(\lambda, A) + \frac{1}{1 - \langle \varphi, \mathcal{R}(\lambda, A)w \rangle} \mathcal{R}(\lambda, A)(\varphi \otimes w) \mathcal{R}(\lambda, A).$$

Proof. If $\langle \varphi, \mathcal{R}(\lambda, A)w \rangle \neq 1$, then the right hand side of (A.1) is well defined and, using that $(\varphi \otimes w) \mathcal{R}(\lambda, A) (\varphi \otimes w) = \langle \varphi, \mathcal{R}(\lambda, A)w \rangle (\varphi \otimes w)$, one can check by a simple computation that it is the inverse of $\lambda I - (A + \varphi \otimes w)$; this implies that $\lambda \in \rho(A + \varphi \otimes w)$ and that the formula holds.

Now, assume that $\lambda \in \rho(A + \varphi \otimes w)$. If $w = 0$, then clearly $\langle \varphi, \mathcal{R}(\lambda, A)w \rangle = 0 \neq 1$, so let $w \neq 0$. Since both operators $\lambda I - A$ and

$$\lambda I - (A + \varphi \otimes w) = (I - (\varphi \otimes w) \mathcal{R}(\lambda, A))(\lambda I - A)$$

are bijective from $D(A)$ to E , it follows that $I - (\varphi \otimes w) \mathcal{R}(\lambda, A)$ is a bijection on E ; in particular, the latter operator is injective, so

$$0 \neq (I - (\varphi \otimes w) \mathcal{R}(\lambda, A))w = (1 - \langle \varphi, \mathcal{R}(\lambda, A)w \rangle)w.$$

This proves that $\langle \varphi, \mathcal{R}(\lambda, A)w \rangle \neq 1$. ■

If A is a square matrix and $\lambda = 0$, then (A.1) is a special case of the *Sherman–Morrison–Woodbury formula* from numerical analysis; see Section 2.1.3 of [16] or Lemma on p. 68 of [20].

If A generates a C_0 -semigroup $(e^{tA})_{t \geq 0}$, then we do not obtain such a nice formula for the perturbed semigroup $(e^{t(A + \varphi \otimes w)})_{t \geq 0}$, in general. However, if w is an eigenvector of A , then an explicit formula for the perturbed semigroup can be given, and the perturbation formula for the resolvent from the previous proposition can be considerably simplified.

PROPOSITION A.2. Let $A : E \supseteq D(A) \rightarrow E$ be a linear operator on a complex Banach space E . Let $\varphi \in E'$ and let $v \in D(A)$ be an eigenvector of A for an eigenvalue $\lambda_0 \in \mathbb{C}$.

(i) If $\lambda \in \rho(A)$, then $\lambda \in \rho(A + \varphi \otimes v)$ if and only if $\lambda - \lambda_0 \neq \langle \varphi, v \rangle$. In this case

$$(A.2) \quad \mathcal{R}(\lambda, A + \varphi \otimes v) = \mathcal{R}(\lambda, A) + \frac{1}{(\lambda - \lambda_0) - \langle \varphi, v \rangle} (\varphi \otimes v) \mathcal{R}(\lambda, A).$$

(ii) If A generates a C_0 -semigroup on E and if $\langle \varphi, v \rangle + \lambda_0 \notin \sigma(A)$, then

$$(A.3) \quad e^{t(A+\varphi \otimes v)} = e^{tA} + (\varphi \otimes v)(e^{t(\langle \varphi, v \rangle + \lambda_0)} I - e^{tA}) \mathcal{R}(\langle \varphi, v \rangle + \lambda_0, A)$$

for all $t \geq 0$.

Proof. (i) This follows immediately from Proposition A.1.

(ii) The right hand side of (A.3) is clearly strongly continuous with respect to $t \in [0, \infty)$ and a direct computation verifies that it is a semigroup. We denote by B the generator of this semigroup. Then one immediately checks that $D(B) \supseteq D(A) = D(A + \varphi \otimes v)$ and that $Bf = (A + \varphi \otimes v)f$ for all $f \in D(A) = D(A + \varphi \otimes v)$. Hence, B is an extension of $A + \varphi \otimes v$. Since $A + \varphi \otimes v$ and B are both semigroup generators, their resolvent sets have non-empty intersection and thus, we must have $B = A + \varphi \otimes v$. ■

Acknowledgements. Jochen Glück was partially supported by a scholarship within the scope of the LGFG Baden-Württemberg, Germany.

REFERENCES

- [1] W. ARENDT, C.J.K. BATTY, M. HIEBER, F. NEUBRANDER, *Vector-valued Laplace Transforms and Cauchy Problems*, second ed., Monogr. Math., vol. 96, Birkhäuser/Springer Basel AG, Basel 2011.
- [2] W. ARENDT, A. GRABOSCH, G. GREINER, U. GROH, H.P. LOTZ, U. MOUSTAKAS, R. NAGEL, F. NEUBRANDER, U. SCHLOTTERBECK, *One-parameter Semigroups of Positive Operators*, Lecture Notes in Math., vol. 1184, Springer-Verlag, Berlin 1986.
- [3] A. BERMAN, M. CATRAL, L.M. DEALBA, A. ELHASHASH, F.J. HALL, L. HOGBEN, I.-J. KIM, D.D. OLESKY, P. TARAZAGA, M.J. TSATSOMEROS, P. VAN DEN DRIESSCHE, Sign patterns that allow eventual positivity, *Electron. J. Linear Algebra* **19**(2009), 108–120.
- [4] D. DANERS, Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator, *Positivity* **18**(2014), 235–256.
- [5] D. DANERS, J. GLÜCK, The role of domination and smoothing conditions in the theory of eventually positive semigroups, *Bull. Austral. Math. Soc.* **96**(2017), 286–298.
- [6] D. DANERS, J. GLÜCK, J.B. KENNEDY, Eventually and asymptotically positive semigroups on Banach lattices, *J. Differential Equations* **261**(2016), 2607–2649.

- [7] D. DANERS, J. GLÜCK, J.B. KENNEDY, Eventually positive semigroups of linear operators, *J. Math. Anal. Appl.* **433**(2016), 1561–1593.
- [8] G. DEGLA, An overview of semi-continuity results on the spectral radius and positivity, *J. Math. Anal. Appl.* **338**(2008), 101–110.
- [9] E.M. ELLISON, L. HOGBEN, M.J. TSATSOMEROS, Sign patterns that require eventual positivity or require eventual nonnegativity, *Electron. J. Linear Algebra* **19**(2009), 98–107.
- [10] K.-J. ENGEL, R. NAGEL, *One-parameter Semigroups for Linear Evolution Equations*, Grad. Texts Math., vol. 194, Springer-Verlag, New York 2000.
- [11] K.-J. ENGEL, R. NAGEL, *A Short Course on Operator Semigroups*, Universitext, Springer, New York 2006.
- [12] C. ERICKSON, Sign patterns that require eventual exponential nonnegativity, *Electron. J. Linear Algebra* **30**(2015), 171–195.
- [13] A. FERRERO, F. GAZZOLA, H.-C. GRUNAU, Decay and eventual local positivity for biharmonic parabolic equations, *Discrete Contin. Dyn. Syst.* **21**(2008), 1129–1157.
- [14] F. GAZZOLA, H.-C. GRUNAU, Eventual local positivity for a biharmonic heat equation in \mathbb{R}^n , *Discrete Contin. Dyn. Syst. Ser. S* **1**(2008), 83–87.
- [15] J. GLÜCK, Invariant sets and long time behaviour of operator semigroups, Ph.D. Dissertation, Universität Ulm, Ulm 2016.
- [16] G.H. GOLUB, C.F. VAN LOAN, *Matrix Computations*, second ed., Johns Hopkins Ser. Math. Sci., vol. 3, Johns Hopkins Univ. Press, Baltimore, MD 1989.
- [17] H.-C. GRUNAU, G. SWEERS, Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions, *Math. Nachr.* **179**(1996), 89–102.
- [18] T. KATO, *Perturbation Theory for Linear Operators*, second ed., Grund. Math. Wiss., vol. 132, Springer-Verlag, Berlin 1976.
- [19] P. MEYER-NIEBERG, *Banach Lattices*, Universitext, Springer-Verlag, Berlin 1991.
- [20] K.S. MILLER, On the inverse of the sum of matrices, *Math. Mag.* **54**(1981), 67–72.
- [21] G.A. MUÑOZ, Y. SARANTOPOULOS, A. TONGE, Complexifications of real Banach spaces, polynomials and multilinear maps, *Studia Math.* **134**(1999), 1–33.
- [22] D. NOUTSOS, M.J. TSATSOMEROS, Reachability and holdability of nonnegative states, *SIAM J. Matrix Anal. Appl.* **30**(2008), 700–712.
- [23] D.D. OLESKY, M.J. TSATSOMEROS, P. VAN DEN DRIESSCHE, M_V -matrices: a generalization of M -matrices based on eventually nonnegative matrices, *Electron. J. Linear Algebra* **18**(2009), 339–351.
- [24] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci., vol. 44, Springer-Verlag, New York 1983.
- [25] H.H. SCHAEFER, *Banach Lattices and Positive Operators*, Grund. Math. Wiss., vol. 215, Springer-Verlag, New York 1974.
- [26] F. SHAKERI, R. ALIZADEH, Nonnegative and eventually positive matrices, *Linear Algebra Appl.* **519**(2017), 19–26.

- [27] S. STEINBERG, Meromorphic families of compact operators, *Arch. Rational Mech. Anal.* **31**(1968/1969), 372–379.
- [28] G. SWEERS, On sign preservation for clotheslines, curtain rods, elastic membranes and thin plates, *Jahresber. Dtsch. Math.-Ver.* **118**(2016), 275–320.
- [29] K. YOSIDA, *Functional Analysis*, Classics in Math., Springer-Verlag, Berlin 1995.

DANIEL DANERS, SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, NSW 2006, AUSTRALIA

E-mail address: daniel.daners@sydney.edu.au

JOCHEN GLÜCK, INSTITUT FÜR ANGEWANDTE ANALYSIS, UNIVERSITÄT ULM, D-89069 ULM, GERMANY

E-mail address: jochen.glueck@uni-ulm.de

Received March 29, 2017.