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THE CESÀRO OPERATOR ON DUALS OF POWER SERIES SPACES OF INFINITE TYPE

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ABSTRACT. A detailed investigation is made of the continuity, spectrum and mean ergodic properties of the Cesàro operator C when acting on the strong duals of power series spaces of infinite type. There is a dramatic difference in the nature of the spectrum of C depending on whether or not the strong dual space (which is always Schwartz) is nuclear.

KEYWORDS: *Cesàro operator, duals of power series spaces, spectrum, (LB)-space, mean ergodic operator, nuclear space.*

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1. INTRODUCTION AND NOTATION

The discrete Cesàro operator C is defined on the linear space $\mathbb{C}^{\mathbb{N}}$ (consisting of all scalar sequences) by

$$\mathsf{C}x := \Big(x_1, \frac{x_1 + x_2}{2}, \dots, \frac{x_1 + \cdots + x_n}{n}, \dots\Big), \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \tag{1.1}$$

The linear operator C is said to *act* in a vector subspace $X \subseteq \mathbb{C}^{\mathbb{N}}$ if it maps X into itself. Of particular interest is the situation when X is a Fréchet space or an (LF)-space. Two fundamental questions in this case are: is $\mathsf{C} : X \to X$ continuous and, if so, what is its spectrum? For a large collection of classical Banach spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ where precise answers are known we refer to the Introductions in [4], [6], for example. The discrete Cesàro operator C acting on the Fréchet sequence space $\mathbb{C}^{\mathbb{N}}$, on $\ell^{p+} := \bigcap\limits_{q>p} \ell_q$, and on the power series spaces $\Lambda_0(\alpha) := \Lambda_0^1(\alpha)$ of finite type was investigated in [2], [5], [6], respectively. The aim of this paper is to investigate the behaviour of C when it acts on the *strong duals* $(\Lambda_\infty^1(\alpha))'$ of power series spaces $\Lambda_\infty^1(\alpha)$ of *infinite type*. Power series spaces of infinite type play an important role in the isomorphic classification of Fréchet spaces, [17], [21], [22]. The reason for concentrating on the infinite type dual spaces $(\Lambda_\infty^1(\alpha))'$ is that the

Cesàro operator C *fails* to be continuous on "most" of the finite type dual spaces $(\Lambda_0^1(\alpha))'$. This is explained more precisely in an Appendix (Section 5) at the end of the paper.

In order to describe the main results we require some notation and definitions.

Let X be a locally convex Hausdorff space (briefly, lcHs) and Γ_X a system of continuous seminorms determining the topology of X. Let X' denote the space of all continuous linear functionals on X. The family of all bounded subsets of X is denoted by $\mathcal{B}(X)$. Denote the identity operator on X by I. Let $\mathcal{L}(X)$ denote the space of all continuous linear operators from X into itself. For $T \in \mathcal{L}(X)$, the *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{\mathrm{pt}}(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. If we need to stress the space X, then we also write $\sigma(T; X)$, $\sigma_{\mathrm{pt}}(T; X)$ and $\rho(T; X)$. Given $\lambda, \mu \in \rho(T)$ the *resolvent identity* $R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T)$ holds. Unlike for Banach spaces, it may happen that $\rho(T) = \emptyset$ (cf. Remark 2.6(ii)) or that $\rho(T)$ is not open in $\mathbb{C}$; see Proposition 2.9(i) for example. That is why some authors prefer the subset $\rho^*(T)$ of $\rho(T)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that the open disc $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\} \subseteq \rho(T)$ and $\{R(\mu, T) : \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(X)$. If X is a Fréchet space or even an (LF)-space, then it suffices that such sets are bounded in $\mathcal{L}_s(X)$, where $\mathcal{L}_s(X)$ denotes $\mathcal{L}(X)$ endowed with the strong operator topology τ_s which is determined by the seminorms $T \mapsto q_x(T) := q(Tx)$, for all $x \in X$ and $q \in \Gamma_X$. The advantage of $\rho^*(T)$, whenever it is non-empty, is that it is open and the resolvent map $R : \lambda \mapsto R(\lambda, T)$ is holomorphic from $\rho^*(T)$ into $\mathcal{L}_b(X)$, ([3], Proposition 3.4). Here $\mathcal{L}_b(X)$ denotes $\mathcal{L}(X)$ endowed with the lcH-topology τ_b of uniform convergence on members of $\mathcal{B}(X)$; it is determined by the seminorms $T \mapsto q_B(T) := \sup_{x \in B} q(Tx)$, for $T \in \mathcal{L}(X)$, for all $B \in \mathcal{B}(X)$ and $q \in \Gamma_X$. Define $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$, which is a closed set containing $\sigma(T)$. If $T \in \mathcal{L}(X)$ with X a Banach space, then $\sigma(T) = \sigma^*(T)$. In Remark 3.5(vi), p. 265 of [3] an example of a continuous linear operator T on a Fréchet space X is presented such that $\overline{\sigma(T)} \subset \sigma^*(T)$ properly. For undefined concepts concerning lcHs' see [12], [17].

Each positive, strictly increasing sequence $\alpha = (\alpha_n)$ which tends to infinity generates a power series space $\Lambda_\infty^1(\alpha)$ of infinite type; see Section 2. The strong dual $E_\alpha \subseteq \mathbb{C}^{\mathbb{N}}$ of $\Lambda_\infty^1(\alpha)$ is then a co-echelon space, i.e., a particular kind of inductive limit of Banach spaces (of sequences), which is necessarily a Schwartz space in our setting. It turns out (cf. Proposition 2.1) that always $\mathsf{C} \in \mathcal{L}(E_\alpha)$. Furthermore, it is known that the nuclearity of the space E_α is characterized by the condition $\sup_{n \in \mathbb{N}} \frac{\log(n)}{\alpha_n} < \infty$. Remarkably, this is equivalent to the operator $\mathsf{C} \in \mathcal{L}(E_\alpha)$ being invertible, i.e., $0 \in \rho(\mathsf{C}; E_\alpha)$; see Proposition 2.4. Actually, the main results of this section (namely, Proposition 2.9 and Corollary 2.10) establish the equivalence of the following assertions:

(i) E_α *is nuclear;*
(ii) $\sigma(\mathsf{C}; E_\alpha) = \sigma_{\mathrm{pt}}(\mathsf{C}; E_\alpha)$;
(iii) $\sigma(\mathsf{C}; E_\alpha) = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Moreover, in this case we have $\sigma^*(\mathsf{C}; E_\alpha) = \{0\} \cup \sigma(\mathsf{C}; E_\alpha)$. So, whenever E_α is nuclear, the spectra $\sigma_{\mathrm{pt}}(\mathsf{C}; E_\alpha)$, $\sigma(\mathsf{C}; E_\alpha)$ and $\sigma^*(\mathsf{C}; E_\alpha)$ are completely identified. In particular, these spectra of C are independent of α.

The operator $D \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$ of differentiation (defined in the obvious way) is closely connected to the Cesàro operator $\mathsf{C} \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$ via the identity (valid in $\mathcal{L}(\mathbb{C}^{\mathbb{N}})$)

$$\mathsf{C}^{-1} = (I - S_{\mathrm{r}}) D S_{\mathrm{r}},$$

where $S_{\mathrm{r}} \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$ is the right-shift operator. It is always the case that $S_{\mathrm{r}} \in \mathcal{L}(E_\alpha)$ whenever $\alpha_n \uparrow \infty$. Moreover, it follows from (i)–(iii) above that $\mathsf{C}^{-1} \in \mathcal{L}(E_\alpha)$ precisely when E_α is nuclear. So, the above identity for C^{-1} suggests that there should be a connection between the continuity of D on E_α and the nuclearity of E_α. This is clarified by Proposition 2.5. Namely, D is continuous on E_α if and only if E_α is both nuclear and $\sup_{n\in\mathbb{N}} \frac{\alpha_{n+1}}{\alpha_n} < \infty$. Remark 2.6(i) shows that these two conditions are independent of one another.

Section 3 identifies the spectra of $\mathsf{C} \in \mathcal{L}(E_\alpha)$ in the case when E_α is not nuclear. We have seen if E_α is nuclear, then $\sigma(\mathsf{C}; E_\alpha)$ is a bounded, infinite and countable set with no accumulation points. For E_α *non-nuclear* the spectrum of C is very different. Indeed, in this case

$$\sigma(\mathsf{C}; E_\alpha) = \{0,1\} \cup \Big\{\lambda \in \mathbb{C} : \Big|\lambda - \frac{1}{2}\Big| < \frac{1}{2}\Big\} \quad \text{and} \quad \sigma^*(\mathsf{C}; E_\alpha) = \Big\{\lambda \in \mathbb{C} : \Big|\lambda - \frac{1}{2}\Big| \leqslant \frac{1}{2}\Big\}$$

whenever $\sup_{n\in\mathbb{N}} \frac{\log(\log(n))}{\alpha_n} < \infty$, whereas

$$\sigma(\mathsf{C}; E_\alpha) = \sigma^*(\mathsf{C}; E_\alpha) = \Big\{\lambda \in \mathbb{C} : \Big|\lambda - \frac{1}{2}\Big| \leqslant \frac{1}{2}\Big\}$$

otherwise; see Proposition 3.4. Again the spectra of C are independent of α.

J. von Neumann (1931) proved that unitary operators T in Hilbert space are mean ergodic, i.e., the sequence of its averages $\frac{1}{n} \sum_{m=1}^{n} T^m$, for $n \in \mathbb{N}$, converges for the strong operator topology (to a projection). Ever since, intensive research has been undertaken to identify the mean ergodicity of individual (and classes) of operators both in Banach spaces and non-normable lcHs'; see [1], [15] for example, and the references therein. In Section 4 it is shown, for every sequence α with $\alpha_n \uparrow \infty$, that the Cesàro operator $\mathsf{C} \in \mathcal{L}(E_\alpha)$ is always power bounded, (uniformly) mean ergodic and $E_\alpha = \operatorname{Ker}(I - \mathsf{C}) \oplus \overline{(I - \mathsf{C})(E_\alpha)}$; see Proposition 4.1. Actually, even the sequence $\{\mathsf{C}^m\}_{m=1}^{\infty}$ of the iterates of C (not just its averages) turns out to be convergent, not only in $\mathcal{L}_s(E_\alpha)$ but also in $\mathcal{L}_b(E_\alpha)$; see Proposition 4.2. Furthermore, if E_α is nuclear, then the range $(I - \mathsf{C})^m(E_\alpha)$ of the operator $(I - \mathsf{C})^m$ is a closed subspace of E_α for each $m \in \mathbb{N}$ (cf. Proposition 4.3). For $m = 1$ this is an analogue, for the operator $\mathsf{C} \in \mathcal{L}(E_\alpha)$, of a result of M. Lin for arbitrary uniformly mean ergodic Banach space operators T which satisfy $\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0$, [16].

2. THE SPECTRUM OF C IN THE NUCLEAR CASE

Let $\alpha := (\alpha_n)$ be a positive, strictly increasing sequence tending to infinity, briefly, $\alpha_n \uparrow \infty$. Let $(s_k) \subseteq (1, \infty)$ be another strictly increasing sequence satisfying $s_k \uparrow \infty$. For each $k \in \mathbb{N}$, define $v_k : \mathbb{N} \to (0, \infty)$ by $v_k(n) := s_k^{-\alpha_n}$ for $n \in \mathbb{N}$. Then $v_k(n) \geqslant v_k(n+1)$, for $n \in \mathbb{N}$, i.e., v_k is a decreasing sequence, and $v_k \geqslant v_{k+1}$ pointwise on $\mathbb{N}$ for all $k \in \mathbb{N}$. Set $\mathcal{V} := (v_k)$ and note that $v_k \in c_0$ for all $k \in \mathbb{N}$.

Define the co-echelon spaces $E_\alpha := \operatorname{ind}_k c_0(v_k)$, that is, E_α is the (increasing) union of the weighted Banach spaces $c_0(v_k)$, $k \in \mathbb{N}$, endowed with the finest lcH-topology such that each natural inclusion map $c_0(v_k) \hookrightarrow E_\alpha$ is continuous. Since $\lim_{n\to\infty} \frac{v_{k+1}(n)}{v_k(n)} = 0$, for $k \in \mathbb{N}$, implies that $\ell_\infty(v_k) \subseteq c_0(v_{k+1})$ continuously, for $k \in \mathbb{N}$, it follows that also $E_\alpha := \operatorname{ind}_k \ell_\infty(v_k)$. Observing that the power series space $\Lambda^1_\infty(\alpha) := \operatorname{proj}_k \ell_1(v_k^{-1})$ of *infinite type* is Fréchet–Schwartz (hence, distinguished), ([17], p. 357), it follows that $E_\alpha := \operatorname{ind}_k c_0(v_k) = \operatorname{ind}_k \ell_\infty(v_k) = (\Lambda^1_\infty(\alpha))'$ is the *strong dual* of $\Lambda^1_\infty(\alpha)$, ([17], Remark 25.13). The condition $\frac{v_{k+1}}{v_k} \in c_0$ for $k \in \mathbb{N}$ implies that E_α is always a (DFS)-space, ([17], p. 304), and in particular, a Montel space, ([17], Remark 24.24). Note that power series spaces in Chapter 24 of [17] are defined using ℓ_2-norms. It follows from Proposition 29.6 of [17] that $\Lambda^1_\infty(\alpha)$ is a nuclear Fréchet space (equivalently, E_α is a (DFN)-space) if and only if $\sup_{n\in\mathbb{N}} \frac{\log n}{\alpha_n} < \infty$. This criterion plays a relevant role throughout this section. As the space E_α does not change if (s_k) is replaced by any other strictly increasing sequence in $(1, \infty)$ tending to infinity, we sometimes choose $s_k := e^k$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the norm

$$q_k(x) := \sup_{n\in\mathbb{N}} v_k(n)|x_n|, \quad x = (x_n) \in \ell_\infty(v_k),$$

whose restriction to $c_0(v_k)$ is the norm of $c_0(v_k)$. Observe, for each $k \in \mathbb{N}$, that $c_0(v_k) \subseteq c_0(v_l)$ for every $l \in \mathbb{N}$ with $l \geqslant k$, and

$$q_l(x) \leqslant q_k(x), \quad x \in c_0(v_k). \tag{2.1}$$

As general references for co-echelon spaces we refer to [8], [9], [14], [17], for example.

PROPOSITION 2.1. *For each $\alpha_n \uparrow \infty$ the Cesàro operator satisfies* $\mathsf{C} \in \mathcal{L}(E_\alpha)$.

Proof. Since each sequence v_k, for $k \in \mathbb{N}$, is decreasing, Corollary 2.3(i) of [4] implies that the Cesàro operator at each step, namely $\mathsf{C} : c_0(v_k) \to c_0(v_k)$, for $k \in \mathbb{N}$, is continuous. The result then follows from the general theory of (LB)-spaces as $E_\alpha = \operatorname{ind}_k c_0(v_k)$. ∎

LEMMA 2.2. *Let $\alpha_n \uparrow \infty$. The following conditions are equivalent:*

(i) $\sup_{n\in\mathbb{N}} \frac{\log n}{\alpha_n} < \infty$;

(ii) *for each $\gamma > 0$ there exists $M(\gamma) \in \mathbb{N}$ such that* $\sup_{n\in\mathbb{N}} n^\gamma e^{-M(\gamma)\alpha_n} < \infty$;

(iii) *for some $\gamma > 0$ and $M(\gamma) \in \mathbb{N}$ we have* $\sup_{n\in\mathbb{N}} n^\gamma e^{-M(\gamma)\alpha_n} < \infty$.

Proof. (i) ⇒ (ii) Fix any $\gamma > 0$. By assumption there exists $D > 0$ such that $\log n \leqslant D\alpha_n$ for all $n \in \mathbb{N}$. Let $M(\gamma) \in \mathbb{N}$ satisfy $M(\gamma) \geqslant \gamma D$. Then $\gamma \log n \leqslant \gamma D\alpha_n \leqslant M(\gamma)\alpha_n$ for all $n \in \mathbb{N}$ and hence, $n^\gamma \leqslant e^{M(\gamma)\alpha_n}$ for all $n \in \mathbb{N}$.

(ii) ⇒ (iii) is clear.

(iii) ⇒ (i) By assumption $\sup_{n\in\mathbb{N}} n^\gamma e^{-M(\gamma)\alpha_n} < \infty$. So, there exists $D > 1$ such that $n^\gamma \leqslant De^{M(\gamma)\alpha_n}$ for all $n \in \mathbb{N}$. It follows for each $n \in \mathbb{N}$ that $\frac{\log n}{\alpha_n} \leqslant \frac{\log D}{\gamma\alpha_n} + \frac{M}{\gamma}$. Since $\alpha_n \to \infty$, we can conclude that $\sup_{n\in\mathbb{N}} \frac{\log n}{\alpha_n} < \infty$. ∎

We now turn our attention to the spectrum of $\mathsf{C} \in \mathcal{L}(E_\alpha)$, for which we introduce the notation $\Sigma := \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\Sigma_0 := \{0\} \cup \Sigma$. The Cesàro matrix C, when acting in $\mathbb{C}^\mathbb{N}$, is similar to the diagonal matrix $\operatorname{diag}((\frac{1}{n}))$. Indeed, $\mathsf{C} = \Delta \operatorname{diag}((\frac{1}{n}))\Delta$ with $\Delta = \Delta^{-1} = (\Delta_{nk})_{n,k\in\mathbb{N}} \in \mathcal{L}(\mathbb{C}^\mathbb{N})$ the lower triangular matrix where, for each $n \in \mathbb{N}$, $\Delta_{nk} = (-1)^{k-1}\binom{n-1}{k-1}$, for $1 \leqslant k < n$ and $\Delta_{nk} = 0$ if $k > n$, ([13], pp. 247–249). Thus $\sigma_{pt}(\mathsf{C}; \mathbb{C}^\mathbb{N}) = \Sigma$ and each eigenvalue $\frac{1}{n}$ has multiplicity 1 with eigenvector Δe_n, where $e_n := (\delta_{nk})_{k\in\mathbb{N}}$, for $n \in \mathbb{N}$, are the canonical basis vectors in $\mathbb{C}^\mathbb{N}$. Moreover, $\lambda I - \mathsf{C}$ is invertible for each $\lambda \in \mathbb{C} \setminus \Sigma$. If X is a lcHs continuously contained in $\mathbb{C}^\mathbb{N}$ and $\mathsf{C}(X) \subseteq X$, then

$$\sigma_{pt}(\mathsf{C}; X) = \left\{\frac{1}{n} : n \in \mathbb{N},\ \Delta e_n \in X\right\} \subseteq \Sigma. \tag{2.2}$$

In case the space φ (of all finitely supported vectors in $\mathbb{C}^\mathbb{N}$) is densely contained in X, then $\varphi \subseteq X'$ and $\Sigma \subseteq \sigma_{pt}(\mathsf{C}'; X') \subseteq \sigma(\mathsf{C}; X)$, where C' is the dual operator of C. Observe that always $\Delta e_1 = \mathbf{1} := (1)_{n\in\mathbb{N}} \in c_0(v_1) \subseteq E_\alpha$ whenever $\alpha_n \uparrow \infty$. Since φ is dense in E_α for *every* α with $\alpha_n \uparrow \infty$, we conclude that *always*

$$1 \in \sigma_{pt}(\mathsf{C}; E_\alpha) \subseteq \Sigma \subseteq \sigma(\mathsf{C}; E_\alpha). \tag{2.3}$$

We point out that C does *not* act in the vector space $\varphi := \operatorname{ind}_k \mathbb{C}^k \subseteq \mathbb{C}^\mathbb{N}$ because $e_1 \in \varphi$ but $\mathsf{C}e_1 = (\frac{1}{n}) \notin \varphi$.

PROPOSITION 2.3. *For α with $\alpha_n \uparrow \infty$ the following assertions are equivalent:*

(i) *E_α is nuclear;*

(ii) $\sup_{n\in\mathbb{N}} \frac{\log n}{\alpha_n} < \infty$;

(iii) $\sigma_{\rm pt}(\mathsf{C}; E_\alpha) = \Sigma$;

(iv) $\sigma_{\rm pt}(\mathsf{C}; E_\alpha) \setminus \{1\} \neq \emptyset$.

Proof. (i) $\Leftrightarrow$ (ii) See the introduction to this section.

(ii) $\Rightarrow$ (iii) Observe that Δe_m, for fixed $m \in \mathbb{N}$, behaves asymptotically like $(n^{m-1})_{n\in\mathbb{N}}$, i.e., $|(\Delta e_m)| \simeq n^{m-1}$ for $n \to \infty$. By Lemma 2.2 each $\Delta e_m \in E_\alpha$ for $m \in \mathbb{N}$. Hence, (2.2) yields that $\sigma_{\rm pt}(\mathsf{C}; E_\alpha) = \Sigma$.

(iii) $\Rightarrow$ (iv) Obvious.

(iv) $\Rightarrow$ (ii) For this proof select $v_k(n) := \mathrm{e}^{-k\alpha_n}$, $n \in \mathbb{N}$, for each $k \in \mathbb{N}$. By (2.3) and the assumption (iv) there exists $m \in \mathbb{N}$ with $m > 1$ such that $\frac{1}{m} \in \sigma_{\rm pt}(\mathsf{C}; E_\alpha)$, i.e., $\Delta e_m \in E_\alpha$. As seen in the proof of (ii) $\Rightarrow$ (iii) we then have $(n^{m-1})_{n\in\mathbb{N}} \in E_\alpha$. Hence, for some $k \in \mathbb{N}$, $(n^{m-1})_{n\in\mathbb{N}} \in c_0(v_k)$ and so there exists $M > 1$ such that $n^{m-1}v_k(n) = n^{m-1}\mathrm{e}^{-k\alpha_n} \leqslant M$ for all $n \in \mathbb{N}$. It follows from Lemma 2.2 that (ii) holds. ∎

PROPOSITION 2.4. *Let $\alpha_n \uparrow \infty$. The following conditions are equivalent:*

(i) $\sup_{n\in\mathbb{N}} \frac{\log n}{\alpha_n} < \infty$, *i.e., E_α is nuclear;*

(ii) $\mathsf{C} \in \mathcal{L}(E_\alpha)$ *is invertible, i.e.,* $0 \in \rho(\mathsf{C}; E_\alpha)$.

Proof. Note that $\mathsf{C} : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ is bijective with inverse $\mathsf{C}^{-1} : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ given by

$$\mathsf{C}^{-1}y = (ny_n - (n-1)y_{n-1}), \quad y = (y_n) \in \mathbb{C}^\mathbb{N}, \tag{2.4}$$

with $y_0 := 0$. Accordingly, $0 \notin \sigma(\mathsf{C}; E_\alpha)$ if and only if $\mathsf{C}^{-1} : E_\alpha \to E_\alpha$ is continuous if and only if for each $k \in \mathbb{N}$ there exists $l \geqslant k$ such that $\mathsf{C}^{-1} : c_0(v_k) \to c_0(v_l)$ is continuous.

For the rest of the proof we select $v_k(n) := \mathrm{e}^{-k\alpha_n}$ for $k,\ n \in \mathbb{N}$, i.e., $s_k := \mathrm{e}^k$.

(i) $\Rightarrow$ (ii) By Lemma 2.2 there exists $m \in \mathbb{N}$ with $D := \sup_{n\in\mathbb{N}} n\mathrm{e}^{-m\alpha_n} < \infty$. Fix $k \in \mathbb{N}$ and set $l := m + k$. Let $y = (y_n) \in c_0(v_k)$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} v_l(n)(\mathsf{C}^{-1}y) &= \mathrm{e}^{-l\alpha_n}|ny_n - (n-1)y_{n-1}| \leqslant \mathrm{e}^{-l\alpha_n}n|y_n| + \mathrm{e}^{-l\alpha_{n-1}}(n-1)|y_{n-1}| \\ &\leqslant D(\mathrm{e}^{-k\alpha_n}|y_n| + \mathrm{e}^{-k\alpha_{n-1}}|y_{n-1}|) \leqslant 2Dq_k(y). \end{aligned}$$

Forming the supremum relative to $n \in \mathbb{N}$ yields $q_l(\mathsf{C}^{-1}y) \leqslant 2Dq_k(y)$ for all $y \in c_0(v_k)$. Accordingly, $\mathsf{C}^{-1} : c_0(v_k) \to c_0(v_l)$ is continuous. Since $k \in \mathbb{N}$ is arbitrary, it follows that $\mathsf{C}^{-1} : E_\alpha \to E_\alpha$ is continuous and so $0 \in \rho(\mathsf{C}; E_\alpha)$.

(ii) $\Rightarrow$ (i) By assumption $\mathsf{C}^{-1} : E_\alpha \to E_\alpha$ is continuous. So, there exists $l \in \mathbb{N}$ such that $\mathsf{C}^{-1} : c_0(v_1) \to c_0(v_l)$ is continuous, that is, there exists $D > 1$ such that $q_l(\mathsf{C}^{-1}y) \leqslant Dq_1(y)$ for all $y \in c_0(v_1)$. Since $\mathsf{C}^{-1}e_n = ne_n - ne_{n+1}$ and $q_l(\mathsf{C}^{-1}e_n) = \max\{nv_l(n), nv_l(n+1)\} = nv_l(n) = n\mathrm{e}^{-l\alpha_n}$, with $q_1(e_n) = v_1(n) = \mathrm{e}^{-\alpha_n}$, for all

$n \in \mathbb{N}$, it follows that $n e^{-l\alpha_n} \leqslant D e^{-\alpha_n}$, for $n \in \mathbb{N}$. Hence, $n e^{(1-l)\alpha_n} \leqslant D$, for $n \in \mathbb{N}$, which implies that $\sup_{n\in\mathbb{N}} \frac{\log n}{\alpha_n} < \infty$. ∎

The operator of differentiation D acts on $\mathbb{C}^{\mathbb{N}}$ via

$$D(x_1, x_2, x_3, \ldots) := (x_2, 2x_3, 3x_4, \ldots), \quad x = (x_n) \in \mathbb{C}^{\mathbb{N}}.$$

Clearly $D \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$. According to (2.4) and a routine calculation the inverse operator $\mathsf{C}^{-1} \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$ is given by

$$\mathsf{C}^{-1} = (I - S_{\mathrm{r}}) D S_{\mathrm{r}}, \tag{2.5}$$

where $S_{\mathrm{r}} \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$ is the right-shift operator, i.e., $S_{\mathrm{r}} x := (0, x_1, x_2, \ldots)$ for $x \in \mathbb{C}^{\mathbb{N}}$. Fix $k \in \mathbb{N}$. Since v_k is decreasing on $\mathbb{N}$, it follows that

$$q_k(S_{\mathrm{r}} x) := \sup_{n\in\mathbb{N}} v_k(n+1)|x_n| \leqslant \sup_{n\in\mathbb{N}} v_k(n)|x_n| = q_k(x), \quad x \in c_0(v_k).$$

Hence, $S_{\mathrm{r}} : c_0(v_k) \to c_0(v_k)$ is continuous for each $k \in \mathbb{N}$ which implies (for *every* $\alpha_n \uparrow \infty$) that $S_{\mathrm{r}} \in \mathcal{L}(E_\alpha)$. Moreover, Proposition 2.4 shows that $\mathsf{C}^{-1} \in \mathcal{L}(E_\alpha)$ if and only if E_α is nuclear. The identity (2.5) suggests there should be a connection between the nuclearity of E_α and the continuity of D on E_α. The following result addresses this point. Recall that E_α is *shift stable* if $\limsup_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} < \infty$, [23].

PROPOSITION 2.5. *For α with $\alpha_n \uparrow \infty$ the following assertions are equivalent:*

(i) *$D(E_\alpha) \subseteq E_\alpha$, i.e., D acts in E_α;*

(ii) *the differentiation operator $D \in \mathcal{L}(E_\alpha)$;*

(iii) *for every $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that $D : c_0(v_k) \to c_0(v_l)$ is continuous;*

(iv) *for every $k \in \mathbb{N}$ there exist $l \in \mathbb{N}$ with $l > k$ and $M > 0$ such that*

$$n v_l(n) \leqslant M v_k(n+1), \quad n \in \mathbb{N};$$

(v) *the space E_α is both nuclear and shift stable.*

Proof. (i) ⇔ (ii) is immediate from the closed graph theorem for (LB)-spaces, ([17], Theorem 24.31 and Remark 24.36).

(ii) ⇔ (iii) is a general fact about continuous linear operators between (LB)-spaces.

(iii) ⇒ (iv) Fix $k \in \mathbb{N}$. By (iii) there exists $l \in \mathbb{N}$ with $l > k$ such that $D : c_0(v_k) \to c_0(v_l)$ is continuous. Hence, there is $M > 0$ satisfying

$$q_l(Dx) = \sup_{n\in\mathbb{N}} v_l(n)|(Dx)| \leqslant M q_k(x) = M \sup_{n\in\mathbb{N}} v_k(n)|x_n|, \quad x \in c_0(v_k).$$

For each $j \in \mathbb{N}$ with $j \geqslant 2$ substitute $x := e_j$ in the previous inequality (noting that $Dx = De_j = (j-1)e_{j-1}$) yields $(j-1)v_l(j-1) \leqslant M v_k(j)$. Since $j \geqslant 2$ is arbitrary, this is precisely (iv).

(iv) $\Rightarrow$ (iii) Given any $k \in \mathbb{N}$ select $l > k$ and $M > 0$ which satisfy (iv). Fix $x \in c_0(v_k)$. Then, for each $n \in \mathbb{N}$, we have via (iv) that

$$v_l(n)|(Dx)| = nv_l(n)|x_{n+1}| \leqslant Mv_k(n+1).$$

Forming the supremum relative to $n \in \mathbb{N}$ of both sides of this inequality yields

$$q_l(Dx) \leqslant Mq_k(x), \quad x \in c_0(v_k),$$

which is precisely (iii).

(iv) $\Rightarrow$ (v) For $k = 1$, condition (iv) ensures the existence of $l > 1$ and $M > 1$ such that

$$nv_l(n) \leqslant Mv_1(n+1) \leqslant Mv_1(n), \quad n \in \mathbb{N}. \tag{2.6}$$

For the remainder of the proof of this proposition, choose $s_k := \mathrm{e}^k$ for $k \in \mathbb{N}$. It follows from (2.6) that $n\mathrm{e}^{-l\alpha_n} \leqslant M\mathrm{e}^{-\alpha_n}$ for all $n \in \mathbb{N}$. By Lemma 2.2 one can conclude that E_α is *nuclear*.

To prove that E_α is shift stable observe that the left-inequality in (2.6) is $n\mathrm{e}^{-l\alpha_n} \leqslant M\mathrm{e}^{-\alpha_{n+1}}$ for $n \in \mathbb{N}$. Taking logarithms and rearranging yields

$$\frac{\alpha_{n+1}}{\alpha_n} \leqslant l + \frac{\log(M)}{\alpha_n} - \frac{\log(n)}{\alpha_n}, \quad n \in \mathbb{N}.$$

Since $\sup_{n\in\mathbb{N}} \frac{\log(n)}{\alpha_n} < \infty$ (as E_α is nuclear) and $\sup_{n\in\mathbb{N}} \frac{\log(M)}{\alpha_n} < \infty$ it follows that $\sup_{n\in\mathbb{N}} \frac{\alpha_{n+1}}{\alpha_n} < \infty$, i.e., E_α is *shift-stable*.

(v) $\Rightarrow$ (iv) Fix $k \in \mathbb{N}$. Since E_α is shift stable, there exists $h \in \mathbb{N}$ such that $\alpha_{n+1} \leqslant h\alpha_n$ for $n \in \mathbb{N}$. Because of the nuclearity of E_α, Lemma 2.2 implies the existence of $M \in \mathbb{N}$ which satisfies $L := \sup_{n\in\mathbb{N}} n\mathrm{e}^{-M\alpha_n} < \infty$. Set $l := M + hk$. Then $l \in \mathbb{N}$ and, for each $n \in \mathbb{N}$, it follows that

$$nv_l(n) = n\mathrm{e}^{-l\alpha_n} = n\mathrm{e}^{-M\alpha_n}\mathrm{e}^{-hk\alpha_n} \leqslant L\mathrm{e}^{-k(h\alpha_n)} \leqslant L\mathrm{e}^{-k\alpha_{n+1}} = Lv_k(n+1).$$

This is precisely condition (iv). ∎

REMARK 2.6. (i) There exist nuclear spaces E_α for which D is *not* continuous on E_α. Let $\alpha_n := n^n$ for $n \in \mathbb{N}$. Then E_α is nuclear but, not shift stable. Proposition 2.5 implies that $D \notin \mathcal{L}(E_\alpha)$. On the other hand, for $\alpha_n := \log(\log(n))$ for $n \geqslant 3$, the space E_α is shift stable but, not nuclear; again $D \notin \mathcal{L}(E_\alpha)$.

(ii) Because $v_1 \downarrow 0$, it is clear that $\ell_\infty \subseteq \ell_\infty(v_1) \subseteq E_\alpha := \operatorname{ind}_k \ell_\infty(v_k)$ for every α with $\alpha_n \uparrow \infty$. Accordingly, if $x_\lambda := \left(\frac{\lambda^{n-1}}{(n-1)!}\right)_{n\in\mathbb{N}}$ for $\lambda \in \mathbb{C}$, then clearly $\{x_\lambda : \lambda \in \mathbb{C}\} \subseteq \ell_\infty$ and so $\{x_\lambda : \lambda \in \mathbb{C}\} \subseteq E_\alpha$. Since $Dx_\lambda = \lambda x_\lambda$ for each $\lambda \in \mathbb{C}$, we have established (via Proposition 2.5) the following fact.

Let α with $\alpha_n \uparrow \infty$ be a sequence such that E_α is both nuclear and shift stable. Then $D \in \mathcal{L}(E_\alpha)$ and

$$\sigma_{\mathrm{pt}}(D; E_\alpha) = \sigma(D; E_\alpha) = \sigma^*(D; E_\alpha) = \mathbb{C}.$$

In order to determine $\sigma(\mathsf{C};E_\alpha)$ we require some further preliminaries. Define the continuous function $a:\mathbb{C}\setminus\{0\}\to\mathbb{R}$ by $a(z):=\mathrm{Re}(\frac{1}{z})$ for $z\in\mathbb{C}\setminus\{0\}$. The following result is a refinement of Lemma 7 in [19].

LEMMA 2.7. *Let $\lambda\in\mathbb{C}\setminus\Sigma_0$. Then there exists $\delta=\delta_\lambda>0$ and positive constants d_δ, D_δ such that $\overline{B(\lambda,\delta)}\cap\Sigma_0=\emptyset$ and*

$$\frac{d_\delta}{N^{a(\mu)}}\leqslant\prod_{n=1}^{N}\Big|1-\frac{1}{n\mu}\Big|\leqslant\frac{D_\delta}{N^{a(\mu)}},\quad \forall N\in\mathbb{N},\ \mu\in B(\lambda,\delta). \tag{2.7}$$

Proof. Fix $\lambda\in\mathbb{C}\setminus\Sigma_0$ and write $\frac{1}{\lambda}=\alpha+\mathrm{i}\beta$ with $\alpha,\beta\in\mathbb{R}$, i.e., $\alpha=a(\lambda)$. Observe that

$$1-\frac{2\alpha}{n}+\frac{(\alpha^2+\beta^2)}{n^2}=\Big(1-\frac{\alpha}{n}\Big)^2+\frac{\beta^2}{n^2}>0,\quad n\in\mathbb{N}.$$

Using the inequality $(1+x)\leqslant e^x$ for $x\in\mathbb{R}$ we conclude that $(1+x)^{1/2}\leqslant e^{x/2}$ for all $x\geqslant-1$. In particular, for $x:=-\frac{2\alpha}{n}+\frac{(\alpha^2+\beta^2)}{n^2}$ it follows that

$$\Big(1-\frac{2\alpha}{n}+\frac{(\alpha^2+\beta^2)}{n^2}\Big)^{1/2}\leqslant\exp\Big(-\frac{\alpha}{n}+\frac{(\alpha^2+\beta^2)}{2n^2}\Big),\quad n\in\mathbb{N}.$$

Fix $N\in\mathbb{N}$. Since $\sum_{n=1}^{N}\frac{1}{n^2}<2$, we conclude that

$$\begin{aligned}\prod_{n=1}^{N}\Big|1-\frac{1}{n\lambda}\Big|&=\prod_{n=1}^{N}\Big(1-\frac{2\alpha}{n}+\frac{(\alpha^2+\beta^2)}{n^2}\Big)^{1/2}\\&\leqslant\exp\Big(\sum_{n=1}^{N}-\frac{\alpha}{n}+\frac{(\alpha^2+\beta^2)}{2n^2}\Big)\leqslant\exp(\alpha^2+\beta^2)\exp\Big(-\alpha\sum_{n=1}^{N}\frac{1}{n}\Big)\\&=\exp\Big(\frac{1}{|\lambda|^2}\Big)\exp\Big(-\alpha\sum_{n=1}^{N}\frac{1}{n}\Big).\end{aligned}$$

By considering separately the cases when $\alpha\leqslant0$ and $\alpha>0$ and employing the inequalities

$$\log(k+1)\leqslant\sum_{n=1}^{k}\frac{1}{n}\leqslant1+\log(k),\quad k\in\mathbb{N}, \tag{2.8}$$

it turns out that

$$\exp\Big(-\alpha\sum_{n=1}^{N}\frac{1}{n}\Big)\leqslant\frac{e^{|a(\lambda)|}}{N^{a(\lambda)}}\leqslant\frac{e^{1/|\lambda|}}{N^{a(\lambda)}}.$$

Accordingly, we have that

$$\prod_{n=1}^{N}\Big|1-\frac{1}{n\lambda}\Big|\leqslant\frac{\exp\Big(\frac{1}{|\lambda|}+\frac{1}{|\lambda|^2}\Big)}{N^{a(\lambda)}},\quad N\in\mathbb{N}. \tag{2.9}$$

From above, for each $n \in \mathbb{N}$, we have $|1-\frac{1}{n\lambda}|^{-1} = (1+x_n)^{-1/2}$, where $x_n := -\frac{2\alpha}{n} + \frac{(\alpha^2+\beta^2)}{n^2}$ satisfies $x_n > -1$. Applying Taylor's formula to the function $f(x) = (1+x)^{-1/2}$ for $x > -1$ yields, for each $n \in \mathbb{N}$, that

$$(1+x_n)^{-1/2} = f(0) + f'(0)x_n + \frac{f''(\theta_n x_n)}{2!}x_n^2 = 1 - \frac{1}{2}x_n + \frac{3}{4}(1+\theta_n x_n)^{-5/2}x_n^2$$

for some $\theta_n \in (0,1)$. Substituting for x_n its definition and rearranging we get

$$(1+x_n)^{-1/2} = 1+\frac{\alpha}{n} - \frac{(\alpha^2+\beta^2)}{2n^2} + \frac{3}{4}\Big(1-\theta_n+\theta_n\Big|1-\frac{1}{\lambda n}\Big|\Big)^{-5/2}\Big(-\frac{2\alpha}{n}+\frac{(\alpha^2+\beta^2)}{n^2}\Big)^2,$$

for each $n \in \mathbb{N}$. Defining $d(\lambda) := \operatorname{dist}(\lambda, \Sigma_0) \leqslant |\lambda|$ we have

$$\Big|1-\frac{1}{\lambda n}\Big| = \frac{1}{|\lambda|}\cdot\Big|\lambda - \frac{1}{n}\Big| \geqslant \frac{d(\lambda)}{|\lambda|}, \quad n \in \mathbb{N}.$$

Hence, for each $n \in \mathbb{N}$, it follows that

$$1-\theta_n+\theta_n\Big|1-\frac{1}{\lambda n}\Big| \geqslant 1-\theta_n+\theta_n\frac{d(\lambda)}{|\lambda|} \geqslant \min\Big\{1, \frac{d(\lambda)}{|\lambda|}\Big\} = \frac{d(\lambda)}{|\lambda|},$$

where we have used the inequality

$$1 - x + \gamma x \geqslant \min\{1,\gamma\}, \quad \forall \gamma \in \mathbb{R},\ x \in [0,1].$$

Accordingly, $\big(1-\theta_n+\theta_n|1-\frac{1}{\lambda n}|\big)^{-5/2} \leqslant \big(\frac{|\lambda|}{d(\lambda)}\big)^{5/2}$, for $n \in \mathbb{N}$, which implies (see above), for each $n \in \mathbb{N}$, that

$$\begin{aligned}\Big|1-\frac{1}{n\lambda}\Big|^{-1} &\leqslant 1+\frac{\alpha}{n}+\frac{1}{n^2}\Big(-\frac{(\alpha^2+\beta^2)}{2}+\frac{3}{4}\Big(\frac{|\lambda|}{d(\lambda)}\Big)^{5/2}\Big(-2\alpha+\frac{(\alpha^2+\beta^2)}{n}\Big)^2\Big)\\ &\leqslant 1+\frac{\alpha}{n}+\frac{3}{4n^2}\Big(\frac{|\lambda|}{d(\lambda)}\Big)^{5/2}(2|\alpha|+\alpha^2+\beta^2)^2.\end{aligned}$$

But, $(2|\alpha|+\alpha^2+\beta^2)^2 \leqslant \big(\frac{2}{|\lambda|}+\frac{1}{|\lambda|^2}\big)^2 \leqslant 4\big(\frac{1}{|\lambda|}+\frac{1}{|\lambda|^2}\big)^2$ and so

$$\Big|1-\frac{1}{n\lambda}\Big|^{-1} \leqslant 1+\frac{\alpha}{n}+\frac{D(\lambda)}{n^2}, \quad n \in \mathbb{N},$$

with $D(\lambda) := \frac{3(1+|\lambda|)^2}{|\lambda|^{3/2}(d(\lambda))^{5/2}}$. Accordingly, for fixed $N \in \mathbb{N}$, we have

$$\begin{aligned}\prod_{n=1}^{N}\Big|1-\frac{1}{\lambda n}\Big|^{-1} &\leqslant \prod_{n=1}^{N}\Big(1+\frac{\alpha}{n}+\frac{D(\lambda)}{n^2}\Big) \leqslant \exp\Big(\alpha\sum_{n=1}^{N}\frac{1}{n}\Big)\exp\Big(D(\lambda)\sum_{n=1}^{N}\frac{1}{n^2}\Big)\\ &\leqslant \mathrm{e}^{2D(\lambda)}\exp\Big(\alpha\sum_{n=1}^{N}\frac{1}{n}\Big).\end{aligned}$$

By considering separately the cases when $\alpha < 0$ and $\alpha \geqslant 0$ and applying (2.8) yields

$$\exp\Big(\alpha\sum_{n=1}^{N}\frac{1}{n}\Big) \leqslant \mathrm{e}^{|\alpha|}N^{\alpha} \leqslant \mathrm{e}^{1/|\lambda|}N^{a(\lambda)}.$$

Accordingly, $\prod_{n=1}^{N} \left|1-\frac{1}{\lambda n}\right|^{-1} \leqslant N^{a(\lambda)} \exp\left(2D(\lambda)+\frac{1}{|\lambda|}\right)$ and hence,

$$\frac{\exp\Big(-\frac{1}{|\lambda|}-2D(\lambda)\Big)}{N^{a(\lambda)}} \leqslant \prod_{n=1}^{N}\left|1-\frac{1}{n\lambda}\right|, \quad N\in\mathbb{N}. \tag{2.10}$$

It follows from (2.9) and (2.10), for any given $\lambda \in \mathbb{C}\setminus\Sigma_0$, that

$$\frac{u(\lambda)}{N^{a(\lambda)}} \leqslant \prod_{n=1}^{N}\left|1-\frac{1}{\lambda n}\right| \leqslant \frac{v(\lambda)}{N^{a(\lambda)}}, \quad N\in\mathbb{N}, \tag{2.11}$$

where $v(\lambda) := \exp\left(\frac{1}{|\lambda|}+\frac{1}{|\lambda|^2}\right)$ and $u(\lambda) := \exp\left(-\frac{1}{|\lambda|}-\frac{6(1+|\lambda|^2)}{|\lambda|^{3/2}(d(\lambda))^{5/2}}\right)$.

Fix now a point $\lambda\in\mathbb{C}\setminus\Sigma_0$ and choose any $\delta>0$ satisfying $\overline{B(\lambda,\delta)}\cap\Sigma_0 = \emptyset$. According to (2.11) we have

$$\frac{u(\mu)}{N^{a(\mu)}} \leqslant \prod_{n=1}^{N}\left|1-\frac{1}{n\mu}\right| \leqslant \frac{v(\mu)}{N^{a(\mu)}}, \quad \forall N\in\mathbb{N},\ \mu\in\overline{B(\lambda,\delta)}. \tag{2.12}$$

By the continuity (and form) of the functions u and v on $\mathbb{C}\setminus\Sigma_0$ and the compactness of the set $\overline{B(\lambda,\delta)}\subseteq(\mathbb{C}\setminus\Sigma_0)$ it follows that $D_\delta := \sup\{v(\mu) : \mu\in\overline{B(\lambda,\delta)}\} < \infty$ and $d_\delta := \inf\{u(\mu) : \mu\in\overline{B(\lambda,\delta)}\} > 0$. It is then clear that (2.4) follows from (2.12). ∎

LEMMA 2.8. *Let $w=(w_n)$ be any strictly positive, decreasing sequence. Then*

$$\sigma(\mathsf{C};c_0(w)) \subseteq \Big\{\lambda\in\mathbb{C} : \Big|\lambda-\frac{1}{2}\Big| \leqslant \frac{1}{2}\Big\}. \tag{2.13}$$

Moreover, for each $\lambda\in\mathbb{C}$ satisfying $\left|\lambda-\frac{1}{2}\right| > \frac{1}{2}$ there exist constants $\delta_\lambda>0$ and $M_\lambda>0$ such that

$$\|(\mu I-\mathsf{C})^{-1}\|_{\mathrm{op}} \leqslant \frac{M_\lambda}{1-a(\mu)}, \quad \mu\in B(\lambda,\delta_\lambda),$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm in $\mathcal{L}(c_0(w))$.

Proof. According to Corollary 2.3(i) of [4] the Cesàro operator $\mathsf{C} : c_0(w)\to c_0(w)$ is continuous. Then Corollary 3.6 of [4] implies that (2.13) is satisfied.

Set $A := \left\{\lambda\in\mathbb{C} : \left|\lambda-\frac{1}{2}\right| \leqslant \frac{1}{2}\right\}$ and fix $\lambda\in\mathbb{C}\setminus A$. Define $\delta_\lambda := \frac{1}{2}\mathrm{dist}(\lambda,A)>0$ and $C_\lambda := \overline{B(\lambda,\delta)}$, in which case (2.13) implies that

$$\mathrm{dist}(C_\lambda,\sigma(\mathsf{C};c_0(w))) \geqslant \mathrm{dist}(C_\lambda,A)=\delta_\lambda.$$

According to Lemma 6.11, p. 590 of [10] there is a constant $K>0$ such that (setting $\varepsilon := \delta_\lambda$ in that lemma)

$$\|(\mu I-\mathsf{C})^{-1}\|_{\mathrm{op}} < \frac{K}{\delta_\lambda}, \quad \mu\in C_\lambda. \tag{2.14}$$

Now, each $\mu \in B(\lambda, \delta_\lambda)$ satisfies $a(\mu) < 1$, ([4], Remark 3.5), and so

$$\frac{K}{\delta_\lambda} = \frac{K\delta_\lambda^{-1}(1 - a(\mu))}{1 - a(\mu)} \leqslant \frac{K\delta_\lambda^{-1}\left(1 + \frac{1}{|\mu|}\right)}{1 - a(\mu)} \leqslant \frac{M_\lambda}{1 - a(\mu)}, \tag{2.15}$$

where $M_\lambda := \sup\left\{\frac{K}{\delta_\lambda}\left(1 + \frac{1}{|z|}\right) : z \in C_\lambda\right\} < \infty$ as the set $C_\lambda \subseteq (\mathbb{C} \setminus \{0\})$ is compact and the function $z \mapsto \frac{K}{\delta_\lambda}\left(1 + \frac{1}{|z|}\right)$ is continuous on $\mathbb{C} \setminus \{0\}$. The desired inequality follows from (2.14) and (2.15). ∎

Recall that a Hausdorff inductive limit $E = \operatorname{ind}_k E_k$ of Banach spaces is called *regular* if every $B \in \mathcal{B}(E)$ is contained and bounded in some step E_k. In particular, for every α with $\alpha_n \uparrow \infty$ the space $E_\alpha = \operatorname{ind}_k c_0(v_k)$ is regular, ([17], Proposition 25.19).

PROPOSITION 2.9. *Let α satisfy $\alpha_n \uparrow \infty$ with E_α nuclear. Then*
(i) $\sigma(\mathsf{C}; E_\alpha) = \sigma_{\text{pt}}(\mathsf{C}; E_\alpha) = \Sigma$, *and*
(ii) $\sigma^*(\mathsf{C}; E_\alpha) = \sigma(\mathsf{C}; E_\alpha) \cup \{0\} = \Sigma_0$.

Proof. By Proposition 2.3 we have $\Sigma = \sigma_{\text{pt}}(\mathsf{C}; E_\alpha) \subseteq \sigma(\mathsf{C}; E_\alpha)$ and hence,

$$\Sigma_0 = \overline{\Sigma} \subseteq \overline{\sigma(\mathsf{C}; E_\alpha)} \subseteq \sigma^*(\mathsf{C}; E_\alpha).$$

Moreover, Proposition 2.4 yields $0 \notin \sigma(\mathsf{C}; E_\alpha)$. So, it remains to show that $(\mathbb{C} \setminus \Sigma_0) \subseteq \rho^*(\mathsf{C}; E_\alpha)$. To this end, we need to show, for each $\lambda \in \mathbb{C} \setminus \Sigma_0$, that there exists $\delta > 0$ with the property that $(\mathsf{C} - \mu I)^{-1} : E_\alpha \to E_\alpha$ is continuous for each $\mu \in B(\lambda, \delta)$ and the set $\{(\mathsf{C} - \mu I)^{-1} : \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(E_\alpha)$. We recall that $(\mathsf{C} - \mu I)^{-1} : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ exists in $\mathcal{L}(\mathbb{C}^{\mathbb{N}})$ for each $\mu \in \mathbb{C} \setminus \Sigma$.

For this proof we select the weights $v_k(n) = \mathrm{e}^{-k\alpha_n}$, $n \in \mathbb{N}$, for each $k \in \mathbb{N}$. Fix $\lambda \in \mathbb{C} \setminus \Sigma_0$. First, choose $\delta_1 > 0$ such that $\overline{B(\lambda, \delta_1)} \cap \Sigma_0 = \emptyset$. Later $\delta > 0$ will be selected in such a way that $0 < \delta < \delta_1$.

According to Lemma 5.4 in the Appendix it suffices to find a $\delta > 0$ satisfying the following condition: for each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l \geqslant k$ and $D_k > 0$ such that

$$q_l((\mathsf{C} - \mu I)^{-1}x) \leqslant D_k q_k(x), \quad \forall \mu \in B(\lambda, \delta),\ x \in c_0(v_k). \tag{2.16}$$

(i) Suppose that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$ (equivalently, $a(\lambda) < 1$, ([4], Remark 3.5)). To establish the condition (2.16) we proceed as follows. Fix $k \in \mathbb{N}$. Since $a(\lambda) < 1$, we can select $\varepsilon > 0$ such that $a(\lambda) < 1 - \varepsilon$. By continuity of the function $a : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ there exists $\delta_2 \in (0, \delta_1)$ such that $a(\mu) < 1 - \varepsilon$ for all $\mu \in \overline{B(\lambda, \delta_2)}$. Applying Lemma 2.8 (with v_k in place of w), it follows that there exist $\delta \in (0, \delta_2)$ and $M_{k,\lambda} > 0$ satisfying

$$q_k((\mathsf{C} - \mu I)^{-1}x) \leqslant \frac{M_{k,\lambda}}{1 - a(\mu)} q_k(x) \leqslant \frac{M_{k,\lambda}}{\varepsilon} q_k(x)$$

for all $\mu \in \overline{B(\lambda, \delta)}$ and $x \in c_0(v_k)$. So, inequality (2.16) is then satisfied with $l := k$ and $D_k := \frac{M_{k,\lambda}}{\varepsilon}$. Since $k \in \mathbb{N}$ is arbitrary, condition (2.16) holds.

(ii) Suppose now that $\left|\lambda - \frac{1}{2}\right| \leqslant \frac{1}{2}$ (equivalently, $a(\lambda) \geqslant 1$, ([4], Remark 3.5)). We recall the formula for the inverse operator $(\mathsf{C} - \mu I)^{-1} : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ whenever $\mu \notin \Sigma_0$, ([19], p. 266). For $n \in \mathbb{N}$ the n-th row of the matrix for $(\mathsf{C} - \mu I)^{-1}$ has the entries

$$\frac{-1}{n\mu^2 \prod_{k=m}^{n}\left(1 - \frac{1}{\mu k}\right)}, \quad 1 \leqslant m < n; \quad \frac{n}{1 - n\mu} = \frac{1}{\frac{1}{n} - \mu}, \quad m = n,$$

and all the other entries in row n are equal to 0. So, we can write

$$(\mathsf{C} - \mu I)^{-1} = D_\mu - \frac{1}{\mu^2} E_\mu, \quad \mu \in \mathbb{C} \setminus \Sigma_0, \tag{2.17}$$

where the diagonal operator $D_\mu = (d_{nm}(\mu))_{n,m\in\mathbb{N}}$ is given by $d_{nn}(\mu) := \frac{1}{(1/n) - \mu}$ and $d_{nm}(\mu) := 0$ if $n \neq m$. The operator $E_\mu = (e_{nm}(\mu))_{n,m\in\mathbb{N}}$ is then the lower triangular matrix with $e_{1m}(\mu) = 0$ for all $m \in \mathbb{N}$, and for every $n \geqslant 2$ with $e_{nm}(\mu) := \frac{1}{n\prod_{k=m}^{n}\left(1 - \frac{1}{\mu k}\right)}$ if $1 \leqslant m < n$ and $e_{nm}(\mu) := 0$ if $m \geqslant n$.

Since $d_0(\lambda) := \operatorname{dist}(\overline{B(\lambda, \delta_1)}, \Sigma_0) > 0$, we have $|d_{nn}(\mu)| \leqslant \frac{1}{d_0(\lambda)}$ for all $\mu \in \overline{B(\lambda, \delta_1)}$ and $n \in \mathbb{N}$. Fix $k \in \mathbb{N}$. Then, for every $x \in c_0(v_k)$ and $\mu \in \overline{B(\lambda, \delta_1)}$, we have

$$q_k(D_\mu(x)) = \sup_{n\in\mathbb{N}} |d_{nn}(\mu) x_n| v_k(n) \leqslant \frac{1}{d_0(\lambda)} \sup_{n\in\mathbb{N}} |x_n| v_k(n) = \frac{1}{d_0(\lambda)} q_k(x).$$

So, $\{D_\mu : \mu \in \overline{B(\lambda, \delta_1)}\} \subseteq \mathcal{L}(c_0(v_k))$. Moreover, for every $l \in \mathbb{N}$ with $l \geqslant k$ it follows that

$$q_l(D_\mu(x)) \leqslant q_k(D_\mu(x)) \leqslant \frac{1}{d_0(\lambda)} q_k(x), \quad \forall x \in c_0(v_k), \mu \in \overline{B(\lambda, \delta_1)}. \tag{2.18}$$

Via (2.17) it remains to investigate the operator $E_\mu : E_\alpha \to E_\alpha$ in order to show the validity of condition (2.16) for $(\mathsf{C} - \mu I)^{-1}$. To this end we first observe, for each $k \in \mathbb{N}$, that $c_0(v_k)$ is isometrically isomorphic to c_0 via the linear multiplication operator $\Phi_k : c_0(v_k) \to c_0$ given by $\Phi_k(x) := (v_k(n) x_n)$, for $x = (x_n) \in c_0(v_k)$. Of course, each Φ_k is also a bicontinuous isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto $\mathbb{C}^{\mathbb{N}}$. So, it suffices to show, for every $k \in \mathbb{N}$, that there exist $l \in \mathbb{N}$ with $l \geqslant k$ and $D_k > 0$ such that $\|\Phi_l E_\mu \Phi_k^{-1} x\|_0 \leqslant D_k \|x\|_0$ for all $x \in c_0$ and $\mu \in \overline{B(\lambda, \delta_1)}$; here $\|\cdot\|_0$ denotes the usual norm of c_0. For each k, $l \in \mathbb{N}$ with $l \geqslant k$, define $\widetilde{E}_{\mu,k,l} := \Phi_l E_\mu \Phi_k^{-1} \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$, for $\mu \in \mathbb{C} \setminus \Sigma_0$.

Fix $k \in \mathbb{N}$. For each $l \geqslant k$ the operator $\widetilde{E}_{\mu,k,l}$, for $\mu \in B(\lambda, \delta_1)$, is the restriction to c_0 of

$$\widetilde{E}_{\mu,k,l}(x) = ((\widetilde{E}_{\mu,k,l}(x))) = \Big(v_l(n) \sum_{m=1}^{n-1} \frac{e_{nm}(\mu)}{v_k(m)} x_m\Big), \quad x = (x_n) \in \mathbb{C}^{\mathbb{N}},$$

with $(\widetilde{E}_{\mu,k,l}(x))_1 := 0$. Moreover, observe that $\widetilde{E}_{\mu,k,l} = (\widetilde{e}^{k,l}_{nm}(\mu))_{n,m\in\mathbb{N}}$ is the lower triangular matrix given by $\widetilde{e}^{k,l}_{1m}(\mu) = 0$ for $m \in \mathbb{N}$ and $\widetilde{e}^{k,l}_{nm}(\mu) = \frac{v_l(n)}{v_k(m)} e_{nm}(\mu)$ for $n \geqslant 2$ and $1 \leqslant m < n$.

So, it suffices to verify, for some $l \geqslant k$ and $\delta > 0$, that $\widetilde{E}_{\mu,k,l} \in \mathcal{L}(c_0)$ for $\mu \in B(\lambda,\delta)$ and $\{\widetilde{E}_{\mu,k,l} : \mu \in B(\lambda,\delta)\}$ is equicontinuous in $\mathcal{L}(c_0)$. To prove this first observe from the definition of $e_{nm}(\mu)$ that Lemma 2.7 implies, for every $l \geqslant k$, every $m,\ n \in \mathbb{N}$ and all $\mu \in \overline{B(\lambda,\delta_2)}$ that

$$|\widetilde{e}^{k,l}_{nm}(\mu)| = \frac{v_l(n)}{v_k(m)} |e_{nm}(\mu)| \leqslant D'_\lambda \frac{n^{a(\mu)-1} v_l(n)}{m^{a(\mu)} v_k(m)}, \tag{2.19}$$

for some constant $D'_\lambda > 0$ and $\delta_2 \in (0,\delta_1)$. Because the function $a : \mathbb{C}\setminus\{0\} \to \mathbb{R}$ is continuous, there exists $\delta \in (0,\delta_2)$ such that $a(\lambda) - \frac{1}{2} < a(\mu) < a(\lambda) + \frac{1}{2}$, for all $\mu \in \overline{B(\lambda,\delta)}$. This implies, for each $\mu \in \overline{B(\lambda,\delta)}$ that $a(\mu) > a(\lambda) - \frac{1}{2} \geqslant \frac{1}{2}$; recall that $a(\lambda) \geqslant 1$. Let $c := \max\left\{2, a(\lambda) + \frac{1}{2}\right\}$. According to Lemma 2.2 there exists $t \in \mathbb{N}$ such that $S_\lambda := \sup_{n\in\mathbb{N}} n^c e^{-t\alpha_n} < \infty$. Set $l := k + t$. By (2.19) and the fact that $\widetilde{e}^{k,l}_{nm}(\mu) = 0$ for $1 \leqslant m < n$, it follows for every $n \in \mathbb{N}$ and $\mu \in \overline{B(\lambda,\delta)}$ that

$$\begin{aligned}
\sum_{m=1}^{\infty} |\widetilde{e}^{k,l}_{nm}(\mu)| &= \sum_{m=1}^{n-1} |\widetilde{e}^{k,l}_{nm}(\mu)| \leqslant D'_\lambda n^{a(\mu)-1} v_l(n) \sum_{m=1}^{n-1} \frac{1}{m^{a(\mu)} v_k(m)} \\
&= D'_\lambda n^{a(\mu)-1} e^{-l\alpha_n} \sum_{m=1}^{n-1} \frac{e^{k\alpha_m}}{m^{a(\mu)}} \leqslant D'_\lambda n^{a(\mu)-1} e^{-l\alpha_n} \sum_{m=1}^{n-1} e^{k\alpha_m} \\
&\leqslant D'_\lambda n^{a(\mu)-1} e^{-l\alpha_n} (n-1) e^{k\alpha_n} \leqslant D'_\lambda n^{a(\mu)} e^{(k-l)\alpha_n} \\
&= D'_\lambda n^{a(\mu)} e^{-t\alpha_n} \leqslant D'_\lambda n^c e^{-t\alpha_n} \leqslant D'_\lambda S_\lambda.
\end{aligned}$$

Hence, for every $\mu \in \overline{B(\lambda,\delta)}$, we have the inequality

$$\sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |\widetilde{e}^{k,l}_{nm}(\mu)| \leqslant D'_\lambda S_\lambda,$$

that is, condition (ii) of Lemma 2.1 in [4] is satisfied for all $\mu \in \overline{B(\lambda,\delta)}$. Moreover, since $n^{a(\mu)-1} v_l(n) = n^{a(\mu)-1} e^{-l\alpha_n} = n^{a(\mu)-1-c} n^c e^{-t\alpha_n} e^{-k\alpha_n} \to 0$ for $n \to \infty$ (because $S_\lambda = \sup_{n\in\mathbb{N}} n^c e^{-t\alpha_n} < \infty$, $e^{-k\alpha_n} \leqslant 1$, and $a(\mu) < a(\lambda) + \frac{1}{2} \leqslant c + 1$), the inequality (2.19) implies for each fixed $\mu \in \overline{B(\lambda,\delta)}$ and $m \in \mathbb{N}$ that

$$\lim_{n\to\infty} \widetilde{e}^{k,l}_{nm}(\mu) = 0.$$

Also the condition (i) of Lemma 2.1 in [4] is satisfied, for all $\mu \in \overline{B(\lambda,\delta)}$. Accordingly, Lemma 2.1 of [4] implies, for every $\mu \in \overline{B(\lambda,\delta)}$, that $\widetilde{E}_{\mu,k,l} \in \mathcal{L}(c_0)$ with $\|\widetilde{E}_{\mu,k,l}\|_{\mathrm{op}} \leqslant D'_\lambda S_\lambda$, that is, $\{\widetilde{E}_{\mu,k,l} : \mu \in \overline{B(\lambda,\delta)}\}$ is equicontinuous in $\mathcal{L}(c_0)$. Finally, in view of (2.18), we have shown that condition (2.16) is indeed satisfied. ∎

COROLLARY 2.10. *For α with $\alpha_n \uparrow \infty$ the following assertions are equivalent:*
(i) *E_α is nuclear;*
(ii) *$\sigma(\mathsf{C}; E_\alpha) = \sigma_{\mathrm{pt}}(\mathsf{C}; E_\alpha)$;*
(iii) *$\sigma(\mathsf{C}; E_\alpha) = \Sigma$.*

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are clear from Proposition 2.9(i).

(ii) $\Rightarrow$ (i) The equality in (ii) together with the fact that $\sigma_{\mathrm{pt}}(\mathsf{C}; E_\alpha) \subseteq \Sigma$ (see the discussion prior to Proposition 2.3) implies $0 \in \rho(\mathsf{C}; E_\alpha)$. Hence, E_α is nuclear; see Proposition 2.4.

(iii) $\Rightarrow$ (i) The equality in (iii) implies $0 \in \rho(\mathsf{C}; E_\alpha)$ and so E_α is nuclear (cf. Proposition 2.4). ∎

Recall that an operator $T \in \mathcal{L}(X)$, with X a lcHs, is *compact* (respectively *weakly compact*) if there exists a neighbourhood U of 0 such that $T(U)$ is a relatively compact (respectively relatively weakly compact) subset of X.

COROLLARY 2.11. *Let α satisfy $\alpha_n \uparrow \infty$ with E_α nuclear. Then the Cesàro operator $\mathsf{C} \in \mathcal{L}(E_\alpha)$ is neither compact nor weakly compact.*

Proof. Since E_α is Montel, there is no distinction between C being compact or weakly compact. So, suppose that C is compact. Then $\sigma(\mathsf{C}; E_\alpha)$ is necessarily a compact set in $\mathbb{C}$, ([11], Theorem 9.10.2), which contradicts Proposition 2.9(i). ∎

The identity $\mathsf{C} = \Delta \mathrm{diag}((\frac{1}{n}))\Delta$ holds in $\mathcal{L}(\mathbb{C}^{\mathbb{N}})$ and all the three operators C, Δ and $\mathrm{diag}((\frac{1}{n}))$ are continuous; see the discussion prior to Proposition 2.3. For *every* positive sequence $\alpha_n \uparrow \infty$ we also have that $\mathsf{C} \in \mathcal{L}(E_\alpha)$ (cf. Proposition 2.1) and $\mathrm{diag}((\frac{1}{n})) \in \mathcal{L}(E_\alpha)$ (because $\mathrm{diag}((\frac{1}{n})) \in \mathcal{L}(c_0(v_k))$ for every $k \in \mathbb{N}$). If Δ acts in E_α, then $\Delta e_n \in E_\alpha$ for all $n \in \mathbb{N}$ and so $\sigma_{\mathrm{pt}}(\mathsf{C}; E_\alpha) = \Sigma$; see (2.2). Accordingly, E_α is necessarily nuclear via Proposition 2.3. However, this condition alone is not sufficient for the continuity of Δ.

PROPOSITION 2.12. *For α with $\alpha_n \uparrow \infty$ the following assertions are equivalent:*
(i) *the operator $\Delta \in \mathcal{L}(E_\alpha)$;*
(ii) $\sup_{n \in \mathbb{N}} \frac{n}{\alpha_n} < \infty$.

Proof. For each $k \in \mathbb{N}$, the surjective isometric isomorphism $\Phi_k : c_0(v_k) \to c_0$ was defined in the proof of Proposition 2.9. Because $E_\alpha = \mathrm{ind}_k\, c_0(v_k)$ it follows that $\Delta \in \mathcal{L}(E_\alpha)$ if and only if for each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that $\Delta : c_0(v_k) \to c_0(v_l)$ is continuous. Moreover, the continuity of $\Delta : c_0(v_k) \to c_0(v_l)$ is equivalent to continuity of the operator $D^{k,l} : c_0 \to c_0$, where $D^{k,l} := \Phi_l \Delta \Phi_k^{-1}$. Note that $\Phi_l = \mathrm{diag}((v_l(n)))$ and $\Phi_k^{-1} = \mathrm{diag}((\frac{1}{v_k(n)}))$ are diagonal matrices and $\Delta = (\Delta_{nm})_{n,m \in \mathbb{N}}$ is a lower triangular matrix. A direct calculation shows that $D^{k,l} = (d^{k,l}_{nm})_{n,m \in \mathbb{N}}$ is the lower triangular matrix where, for each $n \in \mathbb{N}$, $d^{k,l}_{nm} = (-1)^{m-1} \frac{v_l(n)}{v_k(m)} \binom{n-1}{m-1}$, for $1 \leqslant m < n$ and $d^{k,l}_{nm} = 0$ if $m > n$. It follows from

Theorem 4.51-C of [20] that a matrix $A = (a_{nm})_{n,m\in\mathbb{N}}$ acts continuously on c_0 if and only if the matrix $(|a_{nm}|)_{n,m\in\mathbb{N}}$ does so and hence, by the same result in [20], that $\Delta \in \mathcal{L}(E_\alpha)$ if and only if for each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that the lower triangular matrix $(|d_{nm}^{k,l}|)_{n,m\in\mathbb{N}}$ satisfies both

$$\lim_{n\to\infty} |d_{nm}^{k,l}| = \lim_{n\to\infty} \frac{v_l(n)}{v_k(m)}\binom{n-1}{m-1} = 0, \quad \forall m \in \mathbb{N}, \quad \text{and} \tag{2.20}$$

$$\sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |d_{nm}^{k,l}| = \sup_{n\in\mathbb{N}} \sum_{m=1}^{n} \frac{v_l(n)}{v_k(m)}\binom{n-1}{m-1} < \infty. \tag{2.21}$$

Actually, (2.21) implies (2.20). Indeed, if (2.21) holds, then there exists $L > 0$ satisfying $v_l(n) \sum\limits_{m=1}^{n} \frac{1}{v_k(m)}\binom{n-1}{m-1} \leqslant L$ for all $n \in \mathbb{N}$ and hence, as $\frac{1}{v_k(m)} = e^{k\alpha_m} > 1$ for all $m \in \mathbb{N}$, also $2^{n-1}v_l(n) = v_l(n)\sum\limits_{m=1}^{n}\binom{n-1}{m-1} \leqslant L$ for all $n \in \mathbb{N}$. Then, for fixed $m \in \mathbb{N}$, it follows that

$$n^{m-1}v_l(n) = \frac{n^{m-1}}{2^{n-1}} \cdot 2^{n-1}v_l(n) \leqslant \frac{L \cdot n^{m-1}}{2^{n-1}}, \quad n \in \mathbb{N}.$$

Since $(\frac{n^{m-1}}{2^{n-1}})_{n\in\mathbb{N}}$ is a null sequence and $\binom{n-1}{m-1} \simeq n^{m-1}$ for $n \to \infty$ the condition (2.20) follows. So, we have established that the continuity of $\Delta : E_\alpha \to E_\alpha$ is *equivalent* to the following:

Condition (δ): For every $k \in \mathbb{N}$ there exists $l > k$ such that (2.21) is satisfied.

(i) $\Rightarrow$ (ii) Since Condition (δ) holds, for the choice $k = 1$ there exist $l \in \mathbb{N}$ with $l > 1$ and $M > 1$ such that

$$2^{n-1}v_l(n) = v_l(n)\sum_{m=1}^{n}\binom{n-1}{m-1} \leqslant \sum_{m=1}^{n}\frac{v_l(n)}{v_1(m)}\binom{n-1}{m-1} \leqslant M, \quad n \in \mathbb{N}.$$

Hence, $2^n v_l(n) \leqslant 2M$ from which it follows that

$$\exp(n\log(2) - la_n) \leqslant 2M = \exp(\log(2M)), \quad n \in \mathbb{N}.$$

Rearranging this inequality yields

$$\frac{n}{\alpha_n} \leqslant \frac{l}{\log(2)} + \frac{\log(2M)}{\alpha_n \log(2)}, \quad n \in \mathbb{N}.$$

Since $\alpha_n \uparrow \infty$, it follows that $\sup\limits_{n\in\mathbb{N}} \frac{n}{\alpha_n} < \infty$.

(ii) $\Rightarrow$ (i) Choose $M \in \mathbb{N}$ such that $n \leqslant M\alpha_n$ for $n \in \mathbb{N}$. In order to verify Condition (δ) fix $k \in \mathbb{N}$. Then $l := (k+M) \in \mathbb{N}$ and $l > k$. Since v_k is decreasing on $\mathbb{N}$ we have

$$\sum_{m=1}^{n}\frac{v_l(n)}{v_k(m)}\binom{n-1}{m-1} \leqslant \frac{v_l(n)}{v_k(n)}\sum_{m=1}^{n}\binom{n-1}{m-1} \leqslant 2^n\frac{v_l(n)}{v_k(n)}, \quad n \in \mathbb{N}.$$

Furthermore, for each $n \in \mathbb{N}$, it is also the case that

$$2^n \frac{v_l(n)}{v_k(n)} = 2^n e^{-\alpha_n(l-k)} = e^{n\log(2)} e^{-M\alpha_n} \leqslant e^n e^{-M\alpha_n} \leqslant 1.$$

The previous two sets of inequalities imply (2.21) and hence, Condition (δ) is satisfied, i.e., $\Delta \in \mathcal{L}(E_\alpha)$. ∎

REMARK 2.13. (i) Clearly $\sup_{n\in\mathbb{N}} \frac{n}{\alpha_n} < \infty$ implies E_α is a nuclear space (cf. Proposition 2.4). On the other hand, the sequence $\alpha_n := \log(n)$, $n \in \mathbb{N}$, has the property that E_α is nuclear, but $\Delta \notin \mathcal{L}(E_\alpha)$ by Proposition 2.12.

(ii) The continuity of the operators Δ and D on E_α is unrelated. Indeed, consider $\alpha_n := \sqrt{n}$, for $n \in \mathbb{N}$. Then D is continuous because E_α is both nuclear and shift stable (cf. Proposition 2.5) whereas Δ is not continuous (cf. Proposition 2.12). On the other hand, Δ is continuous on E_α for $\alpha_n := n^n$, $n \in \mathbb{N}$ (via Proposition 2.12), but D fails to be continuous on this space; see Remark 2.6.

We end this section with an application. Consider the space of germs of holomorphic functions at 0, namely the regular (LB)-space defined by $H_0 := \operatorname{ind}_k A\big(\overline{B(0,\frac{1}{k})}\big)$. Here, for each $k \in \mathbb{N}$, $A\big(\overline{B(0,\frac{1}{k})}\big)$ is the disc algebra consisting of all holomorphic functions on the open disc $B(0,\frac{1}{k}) \subseteq \mathbb{C}$ which have a continuous extension to its closure $\overline{B(0,\frac{1}{k})}$: it is a Banach algebra for the norm

$$\|f\|_k := \sup_{|z|\leqslant 1/k} |f(z)| = \sup_{|z|=1/k} |f(z)|, \quad f \in A\Big(\overline{B\Big(0,\frac{1}{k}\Big)}\Big).$$

It is known that the linking maps $A\big(\overline{B(0,\frac{1}{k})}\big) \to A\big(\overline{B(0,\frac{1}{k+1})}\big)$ for $k \in \mathbb{N}$, which are given by restriction, are injective and absolutely summing. By Köthe duality theory, H_0 is isomorphic to the strong dual of the nuclear Fréchet space $H(\mathbb{C})$. In particular, H_0 is a (DFN)-space. We refer to Section 2, Example 5 of [8] and Chapter 5.27, Sections 3,4 of [14] for further information concerning spaces of holomorphic germs and their strong duals. Define $\alpha = (\alpha_n)$ by $\alpha_n := n$ for $n \in \mathbb{N}$ in which case $\lim_{n\to\infty} \frac{\log(n)}{\alpha_n} = 0$. Then $H(\mathbb{C})$ is isomorphic to the power series space $\Lambda^1_\infty(\alpha)$ of infinite type, ([17], Example 29.4(2)), and its strong dual E_α is isomorphic to H_0. Indeed, a topological isomorphism of H_0 onto E_α is given by the linear map which sends $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (an element of $A\big(\overline{B(0,\frac{1}{k})}\big)$ for some $k \in \mathbb{N}$) to $(a_{n-1})_{n\in\mathbb{N}} \in E_\alpha$. The proof of this (known) fact relies on the following estimates.

(i) If $f \in A(\overline{B(0,\varepsilon)})$ for some $0 < \varepsilon < 1$ (with $f(z) = \sum_{n=0}^{\infty} a_n z^n$), then the Cauchy estimates for f imply $|a_n| \leqslant \frac{1}{\varepsilon^n} \max_{|z|=\varepsilon} |f(z)|$ for $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Hence, if $f \in A\big(\overline{B(0,\frac{1}{k})}\big)$ for some $k \in \mathbb{N}$, then

$$|a_n| \leqslant k^n \max_{|z|=1/k} |f(z)| = k^n \|f\|_k, \quad n \in \mathbb{N}_0.$$

(ii) Let $a := (a_n)_{n\in\mathbb{N}_0} \in \ell_\infty(v_k)$ for some $k \in \mathbb{N}$, where $v_k(n) := \frac{1}{(1+k)^n}$ for $n \in \mathbb{N}_0$, $k \in \mathbb{N}$; we have taken here $s_k := \log(k+1)$. Then $|a_n| \leqslant q_k(a)k^n$ for $n \in \mathbb{N}_0$ and each fixed $k \in \mathbb{N}$. Hence, if $|z| \leqslant \frac{1}{2k}$, then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies

$$|f(z)| \leqslant \sum_{n=0}^{\infty} |a_n| \cdot |z|^n \leqslant q_k(a) \sum_{n=0}^{\infty} k^n \frac{1}{(2k)^n} = 2q_k(a).$$

Accordingly, $f \in A\big(\overline{B(0,\frac{1}{2k})}\big)$.

The above facts, combined with Proposition 2.9 and Corollary 2.11, yield the following result.

PROPOSITION 2.14. *The Cèsaro operator* $\mathsf{C} : H_0 \to H_0$ *is continuous with spectra*

$$\sigma(\mathsf{C}; H_0) = \sigma_{\mathrm{pt}}(\mathsf{C}; H_0) = \Sigma \quad \text{and} \quad \sigma^*(\mathsf{C}; H_0) = \Sigma_0.$$

In particular, C *is not (weakly) compact.*

3. THE SPECTRUM OF C IN THE NON-NUCLEAR CASE

The aim of this section is to give a complete description of the spectrum of $\mathsf{C} \in \mathcal{L}(E_\alpha)$ for the case when E_α is *not* nuclear. It turns out that $\sigma(\mathsf{C}; E_\alpha)$ and $\sigma^*(\mathsf{C}; E_\alpha)$ are dramatically different to that when E_α is nuclear. The following fact, which we record for the sake of explicit reference, is immediate from (2.3) and Propositions 2.3 and 2.4.

PROPOSITION 3.1. *For* α *with* $\alpha_n \uparrow \infty$ *the following assertions are equivalent:*

(i) E_α *is not nuclear;*

(ii) $\sigma_{\mathrm{pt}}(\mathsf{C}; E_\alpha) = \{1\}$;

(iii) $0 \in \sigma(\mathsf{C}; E_\alpha)$.

The following general result will be useful in the sequel. For each $r > 0$ we adopt the notation $D(r) := \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2r}| < \frac{1}{2r}\}$.

PROPOSITION 3.2. *Let* α *satisfy* $\alpha_n \uparrow \infty$. *Then*

$$\Sigma \subseteq \sigma(\mathsf{C}; E_\alpha) \subseteq \overline{D(1)}.$$

Proof. Since $\mathsf{C} \in \mathcal{L}(E_\alpha)$, its dual operator C' is defined, continuous on the strong dual $E'_\alpha = \bigcap_{k\in\mathbb{N}} \ell_1\big(\frac{1}{v_k}\big) = \operatorname{proj}_k \ell_1\big(\frac{1}{v_k}\big)$ of $E_\alpha = \operatorname{ind}_k c_0(v_k)$ and is given by the formula

$$\mathsf{C}'y := \Big(\sum_{j=n}^{\infty} \frac{y_j}{j}\Big)_{n\in\mathbb{N}}, \quad y = (y_n) \in E'_\alpha;$$

see (3.7) in p. 774 of [4], for example, after noting that $E'_\alpha \subseteq \ell_1(\frac{1}{v_1})$. Given $\lambda \in \Sigma$ there is $m \in \mathbb{N}$ with $\lambda = \frac{1}{m}$. Define $u^{(m)}$ by $u_n^{(m)} := \prod_{k=1}^{n-1}(1 - \frac{1}{\lambda k})$ for $1 < n \leqslant m$ (with $u_1^{(m)} := 1$) and $u_n^{(m)} := 0$ for $n > m$. It is routine to verify that $u^{(m)} \in E'_\alpha$ (as $u^{(m)} \in \varphi$) and $\mathsf{C}'u^{(m)} = \frac{1}{m}u^{(m)}$, i.e., $\lambda \in \sigma_{\mathrm{pt}}(\mathsf{C}'; E'_\alpha)$. It follows that $\lambda \in \sigma(\mathsf{C}; E_\alpha)$. Indeed, if not, then $\lambda \in \rho(\mathsf{C}; E_\alpha)$ and so $(\mathsf{C} - \lambda I)(E_\alpha) = E_\alpha$. This implies, for each $z \in E_\alpha$ that there exists $x \in E_\alpha$ satisfying $(\mathsf{C} - \lambda I)x = z$. Hence,

$$\langle z, u^{(m)} \rangle = \langle (\mathsf{C} - \lambda I)x, u^{(m)} \rangle = \langle x, (\mathsf{C}' - \lambda I)u^{(m)} \rangle = 0,$$

that is, $\langle z, u^{(m)} \rangle = 0$ for all $z \in E_\alpha$. Since $u^{(m)} \neq 0$, this is a contradiction. So, $\lambda \in \sigma(\mathsf{C}; E_\alpha)$. This establishes that $\Sigma \subseteq \sigma(\mathsf{C}; E_\alpha)$.

According to Lemma 2.8 we see that $\sigma(\mathsf{C}_k; c_0(v_k)) \subseteq \overline{D(1)}$ for all $k \in \mathbb{N}$, where $\mathsf{C}_k : c_0(v_k) \to c_0(v_k)$ is the restriction of $\mathsf{C} \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$. Hence,

$$\bigcap_{m \in \mathbb{N}} \Big(\bigcup_{k=m}^{\infty} \sigma(\mathsf{C}_k; c_0(v_k)) \Big) \subseteq \overline{D(1)}$$

and so $\sigma(\mathsf{C}; E_\alpha) \subseteq \overline{D(1)}$; see Lemma 5.5 in the Appendix. ∎

The following result identifies a large part of $\sigma(\mathsf{C}; E_\alpha)$.

PROPOSITION 3.3. *Let α satisfy $\alpha_n \uparrow \infty$ and such that E_α is not nuclear. Then*

$$\{0, 1\} \cup D(1) \subseteq \sigma(\mathsf{C}; E_\alpha) \subseteq \overline{D(1)}.$$

Proof. It follows from Propositions 3.1 and 3.2 that $\Sigma_0 \subseteq \sigma(\mathsf{C}; E_\alpha) \subseteq \overline{D(1)}$. So, it remains to verify that $(D(1) \setminus \Sigma) \subseteq \sigma(\mathsf{C}; E_\alpha)$. This is achieved via a contradiction argument.

Let $\lambda \in D(1) \setminus \Sigma$ and suppose that $\lambda \in \rho(\mathsf{C}; E_\alpha)$. Note that $\beta := \mathrm{Re}(\frac{1}{\lambda}) > 1$. Since $(\mathsf{C} - \lambda I)^{-1} : E_\alpha \to E_\alpha$ is continuous, for $k = 1$ there exists $l \in \mathbb{N}$ with $l > 1$ such that $(\mathsf{C} - \lambda I)^{-1} : c_0(v_1) \to c_0(v_l)$ is continuous. In the notation of the proof of Proposition 2.9 it follows that the linear map $\widetilde{E}_{\lambda,1,l} : c_0 \to c_0$ is continuous, where $\widetilde{E}_{\lambda,1,l} = (\widetilde{e}^{1,l}_{nm}(\lambda))_{n,m \in \mathbb{N}}$ is the lower triangular matrix given by

$$\widetilde{e}^{1,l}_{nm}(\lambda) = \frac{v_l(n)}{v_1(m)} e_{nm}(\lambda), \quad \forall n \geqslant 2, 1 \leqslant m < n, \tag{3.1}$$

and $\widetilde{e}^{1,l}_{nm}(\lambda) = 0$ otherwise. Here $e_{n,m}(\lambda) = \frac{1}{n \prod_{k=m}^{n}\left(1 - \frac{1}{\lambda k}\right)}$ if $1 \leqslant m < n$ and $e_{nm}(\lambda) = 0$ if $m \geqslant n$. According to the inequality (3.10) in p. 776 of [4], there exist positive constants c, d such that

$$\frac{c}{n^{1-\beta}} \leqslant |e_{n1}(\lambda)| \leqslant \frac{d}{n^{1-\beta}}, \quad n \geqslant 2. \tag{3.2}$$

Since $\widetilde{E}_{\lambda,1,l} \in \mathcal{L}(c_0)$, a well known criterion, ([4], Lemma 2.1; [20], Theorem 4.51-C) implies that necessarily

$$\lim_{n\to\infty} \widetilde{e}^{1,l}_{nm}(\lambda) = 0, \quad m \in \mathbb{N}. \tag{3.3}$$

It now follows from (3.1), the left-inequality in (3.2), and (3.3) with $m = 1$, that

$$\lim_{n\to\infty} n^{\beta-1} \mathrm{e}^{-l\alpha_n} = \lim_{n\to\infty} n^{\beta-1} v_l(n) = 0.$$

Since $\beta > 1$, it follows from Lemma 2.2 that $\sup_{n\in\mathbb{N}} \frac{\log(n)}{\alpha_n} < \infty$ which contradicts the non-nuclearity of E_α (cf. Proposition 2.3). Hence, *no* $\lambda \in D(1) \setminus \Sigma$ exists with $\lambda \in \rho(\mathsf{C}; E_\alpha)$. ∎

We now come to the main result of this section.

PROPOSITION 3.4. *Let α satisfy $\alpha_n \uparrow \infty$ and such that E_α is not nuclear.*

(i) *if* $\sup_{n\in\mathbb{N}} \frac{\log(\log(n))}{\alpha_n} < \infty$, *then*

$$\sigma(\mathsf{C}; E_\alpha) = \{0, 1\} \cup D(1) \quad \textit{and} \quad \sigma^*(\mathsf{C}; E_\alpha) = \overline{D(1)};$$

(ii) *if* $\sup_{n\in\mathbb{N}} \frac{\log(\log(n))}{\alpha_n} = \infty$, *then*

$$\sigma(\mathsf{C}; E_\alpha) = \overline{D(1)} = \sigma^*(\mathsf{C}; E_\alpha).$$

Proof. In the notation of the proof of Proposition 2.9, for each $\lambda \in \mathbb{C} \setminus \Sigma_0$ the inverse operator $(\mathsf{C} - \lambda I)^{-1} \in \mathcal{L}(\mathbb{C}^N)$ satisfies

$$(\mathsf{C} - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda;$$

see (2.17). It is also argued there (as a consequence of the fact that the diagonal in D_λ is a bounded sequence) that $(\mathsf{C} - \lambda I)^{-1} : E_\alpha \to E_\alpha$ is continuous if and only if $E_\lambda \in \mathcal{L}(E_\alpha)$; the nuclearity of E_α is not used for this part of the argument. Moreover, since E_α is an inductive limit, general theory yields that $E_\lambda \in \mathcal{L}(E_\alpha)$ if and only if for each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that $E_\lambda : c_0(v_k) \to c_0(v_l)$ is continuous. With $\widetilde{E}_{\lambda,k,l} = (\widetilde{e}^{k,l}_{nm}(\lambda))_{n,m\in\mathbb{N}}$, where $\widetilde{e}^{k,l}_{nm}(\lambda) := \frac{v_l(n)}{v_k(m)} e_{nm}(\lambda)$ for $n, m \in \mathbb{N}$, it follows via the argument used in case (ii) of the proof of Proposition 2.9 (see also the proof of Proposition 3.3, where $k = 1$ can be replaced by an arbitrary $k \in \mathbb{N}$) that $E_\lambda : c_0(v_k) \to c_0(v_l)$ is continuous if and only if $\widetilde{E}_{\lambda,k,l} : c_0 \to c_0$ is continuous. Via Theorem 4.51-C of [20] this is equivalent to both of the following conditions being satisfied:

$$\lim_{n\to\infty} |\widetilde{e}^{k,l}_{nm}(\lambda)| = \lim_{n\to\infty} \frac{v_l(n)}{v_k(m)} |e_{nm}(\lambda)| = 0, \quad \forall m \in \mathbb{N}, \quad \text{and} \tag{3.4}$$

$$\sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} \frac{v_l(n)}{v_k(m)} |e_{nm}(\lambda)| = \sup_{n\in\mathbb{N}} \sum_{m=1}^{n-1} \frac{v_l(n)}{v_k(m)} |e_{nm}(\lambda)| < \infty. \tag{3.5}$$

Next, if $\lambda \notin \{0,1\}$ belongs to the boundary $\partial D(1)$ of $D(1)$, then $\beta := \mathrm{Re}(\frac{1}{\lambda}) = 1$ and $\lambda \notin \Sigma_0$. Accordingly, Lemma 3.3 of [4] ensures the existence of positive constants c, d such that $c \leqslant |e_{n1}(\lambda)| \leqslant d$ for all $n \in \mathbb{N}$ and

$$\frac{c}{m} \leqslant |e_{nm}(\lambda)| \leqslant \frac{d}{m}, \quad \forall n \in \mathbb{N}, 2 \leqslant m < n. \tag{3.6}$$

In order to deduce (3.6) from Lemma 3.3 of [4] we have used the formula

$$|e_{nm}(\lambda)| = \frac{1}{(m-1)} \cdot \frac{(m-1)\prod_{k=1}^{m-1} \left|1 - \frac{1}{\lambda k}\right|}{n \prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right|}, \quad \forall n \in \mathbb{N}, 2 \leqslant m < n.$$

Henceforth we use $v_r(n) := \mathrm{e}^{-r\alpha_n}$ for all $r, n \in \mathbb{N}$. Note that (3.4) is satisfied for every $\lambda \in \partial D(1) \setminus \{0,1\}$. Indeed, for fixed $m \in \mathbb{N}$, we have via (3.6) that

$$\frac{v_l(n)}{v_k(m)} |e_{nm}(\lambda)| \leqslant \frac{d\mathrm{e}^{k\alpha_m}}{m\mathrm{e}^{l\alpha_n}} \leqslant \frac{d'}{\mathrm{e}^{l\alpha_n}}, \quad n \in \mathbb{N},$$

from which (3.4) is clear.

(i) Since $\sup_{n\in\mathbb{N}} \frac{\log(\log(n))}{\alpha_n} < \infty$, there exists $M \in \mathbb{N}$ such that $\log(\log(n)) \leqslant M\alpha_n$, equivalently $\log(n) \leqslant \mathrm{e}^{M\alpha_n}$ for $n \in \mathbb{N}$. Fix $\lambda \in \partial D(1) \setminus \{0,1\}$; in particular, $\lambda \notin \Sigma_0$. Given $k \in \mathbb{N}$ define $l := k + M$. Then, for every $n \geqslant 2$, it follows from (2.8), (3.6) and $(l-k) = M$ that

$$\begin{aligned} \sum_{m=1}^{n-1} \frac{v_l(n)}{v_k(m)} |e_{nm}(\lambda)| &\leqslant \frac{d}{\mathrm{e}^{l\alpha_n}} \sum_{m=1}^{n-1} \frac{\mathrm{e}^{k\alpha_m}}{m} \leqslant \frac{d\mathrm{e}^{k\alpha_n}}{\mathrm{e}^{l\alpha_n}} \sum_{m=1}^{n-1} \frac{1}{m} \\ &\leqslant \frac{1+\log(n)}{\mathrm{e}^{M\alpha_n}} = \mathrm{e}^{-M\alpha_n} + \frac{\log(n)}{\mathrm{e}^{M\alpha_n}} \leqslant 2. \end{aligned}$$

Accordingly, (3.5) is satisfied. Since (3.4) holds, we conclude that $\widetilde{E}_{\lambda,k,l} : c_0 \to c_0$ is continuous, equivalently that $(\mathsf{C} - \lambda I)^{-1} \in \mathcal{L}(E_\alpha)$. It follows that $\partial D(1) \setminus \{0,1\} \subseteq \rho(\mathsf{C}; E_\alpha)$ and so $\sigma(\mathsf{C}; E_\alpha) = \{0,1\} \cup D(1)$; see Proposition 3.3.

It was shown in the proof of Proposition 3.2 that $\bigcup_{k=1}^{\infty} \sigma(\mathsf{C}_k; c_0(v_k)) \subseteq \overline{D(1)}$. Since $\sigma(\mathsf{C}; E_\alpha) = \{0,1\} \cup D(1)$, we have $\overline{\sigma(\mathsf{C}; E_\alpha)} = \overline{D(1)}$ and so

$$\bigcup_{k=1}^{\infty} \sigma(\mathsf{C}_k; c_0(v_k)) \subseteq \overline{\sigma(\mathsf{C}; E_\alpha)}.$$

It follows from Lemma 5.5(iii) in the Appendix that $\sigma^*(\mathsf{C}; E_\alpha) = \overline{D(1)}$.

(ii) Fix $\lambda \in \partial D(1) \setminus \{0,1\}$. Observe first, for $k = 1$ and $l \in \mathbb{N}$ *arbitrary*, that it follows from (2.8) and (3.6) that

$$\sum_{m=1}^{n-1} \frac{v_l(n)}{v_k(m)} |e_{nm}(\lambda)| \geqslant \frac{c}{\mathrm{e}^{l\alpha_n}} \sum_{m=1}^{n-1} \frac{\mathrm{e}^{\alpha_m}}{m} \geqslant \frac{c\mathrm{e}^{\alpha_1}}{\mathrm{e}^{l\alpha_n}} \sum_{m=1}^{n-1} \frac{1}{m} \geqslant \frac{c\log(n)}{\mathrm{e}^{l\alpha_n}}, \tag{3.7}$$

for all $n \geqslant 2$. Suppose now that $\lambda \in \rho(\mathsf{C}; E_\alpha)$. Then for $k = 1$ there exists $l \in \mathbb{N}$ with $l > 1$ such that (3.5) is satisfied. It then follows from (3.7) that $\sup_{n\in\mathbb{N}} \frac{\log(n)}{\mathrm{e}^{l\alpha_n}} < \infty$.

So, there exists $K > 1$ such that $\log(n) \leqslant K e^{l\alpha_n}$, equivalently that

$$\log(\log(n)) \leqslant l\alpha_n + \log(K), \quad n \geqslant 3.$$

A rearrangement yields

$$\frac{\log(\log(n))}{\alpha_n} \leqslant l + \frac{\log(K)}{\alpha_n}, \quad n \geqslant 3,$$

and so $\sup_{n\in\mathbb{N}} \frac{\log(\log(n))}{\alpha_n} < \infty$; contradiction! So, *no* $\lambda \in \partial D(1) \setminus \{0,1\}$ exists which satisfies $\lambda \in \rho(\mathsf{C}; E_\alpha)$, i.e., $\partial D(1) \setminus \{0,1\} \subseteq \sigma(\mathsf{C}; E_\alpha)$. It now follows from Proposition 3.3 that $\sigma(\mathsf{C}; E_\alpha) = \overline{D(1)}$.

It was observed in the proof of part (i) that $\bigcup_{k=1}^{\infty} \sigma(\mathsf{C}_k; c_0(v_k)) \subseteq \overline{D(1)}$. Since $\overline{D(1)} = \sigma(\mathsf{C}; E_\alpha) = \overline{\sigma(\mathsf{C}; E_\alpha)}$, it again follows from Lemma 5.5(iii) in the Appendix that $\sigma^*(\mathsf{C}; E_\alpha) = \sigma(\mathsf{C}; E_\alpha)$. ∎

REMARK 3.5. (i) Let α satisfy $\alpha_n \uparrow \infty$. Then $\sigma(\mathsf{C}; E_\alpha)$ is a compact subset of $\mathbb{C}$ if and only if $\sup_{n\in\mathbb{N}} \frac{\log(\log(n))}{\alpha_n} = \infty$. This follows from Corollary 2.10, Proposition 3.4 and the fact that the condition $\sup_{n\in\mathbb{N}} \frac{\log(\log(n))}{\alpha_n} = \infty$ implies $\sup_{n\in\mathbb{N}} \frac{\log(n)}{\alpha_n} = \infty$, i.e., E_α is automatically non-nuclear.

(ii) The sequence $\alpha_n := \log(\log(n))$ for $n \geqslant 3^3 > e^e$ (with $1 < \alpha_1 < \cdots < \alpha_{26} < \log(\log(3^3))$ arbitrary) satisfies $1 < \alpha_n \uparrow \infty$ with E_α not nuclear and $\sup_{n\in\mathbb{N}} \frac{\log(\log(n))}{\alpha_n} < \infty$. Proposition 3.4(i) shows that $\sigma(\mathsf{C}; E_\alpha) = \{0,1\} \cup D(1)$. On the other hand, the sequence $\alpha_n := \log(\log(\log(n)))$ for $n \geqslant 3^{27} > e^{e^e}$ (with $1 < \alpha_1 < \cdots < \alpha_{3^{27}-1} < \log(\log(\log(3^{27})))$ arbitrary) satisfies $1 < \alpha_n \uparrow \infty$ with E_α not nuclear and $\sup_{n\in\mathbb{N}} \frac{\log(\log(n))}{\alpha_n} = \infty$. In this case Proposition 3.4(ii) reveals that $\sigma(\mathsf{C}; E_\alpha) = \overline{D(1)}$.

4. MEAN ERGODICITY OF THE CESÀRO OPERATOR

An operator $T \in \mathcal{L}(X)$, with X a lcHs, is *power bounded* if $\{T^n\}_{n=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$. Given $T \in \mathcal{L}(X)$, the averages

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N},$$

are called the Cesàro means of T. The operator T is said to be *mean ergodic* (respectively *uniformly mean ergodic*) if $\{T_{[n]}\}_{n=1}^{\infty}$ is a convergent sequence in $\mathcal{L}_s(X)$ (respectively, in $\mathcal{L}_b(X)$). A relevant text for mean ergodic operators is [15].

PROPOSITION 4.1. *Let $\alpha_n \uparrow \infty$. The Cesàro operator $\mathsf{C} \in \mathcal{L}(E_\alpha)$ is power bounded and uniformly mean ergodic. In particular,*

$$E_\alpha = \operatorname{Ker}(I - \mathsf{C}) \oplus \overline{(I - \mathsf{C})(E_\alpha)} \tag{4.1}$$

with

$$\operatorname{Ker}(I - \mathsf{C}) = \{\mathbf{1}\} \quad \textit{and} \quad \overline{(I - \mathsf{C})(E_\alpha)} = \{x \in E_\alpha : x_1 = 0\} = \overline{\operatorname{span}\{e_n\}_{n \geqslant 2}}. \tag{4.2}$$

Proof. Since each weight v_k for $k \in \mathbb{N}$ is decreasing, it is known that $\mathsf{C} \in \mathcal{L}(c_0(v_k))$ and $q_k(\mathsf{C}x) \leqslant q_k(x)$ for all $x \in c_0(v_k)$, ([4], Corollary 2.3(i)). It follows, via (2.1), for every $k \in \mathbb{N}$ that

$$q_k(\mathsf{C}^m x) \leqslant q_k(x), \quad \forall x \in c_0(v_k),\ m \in \mathbb{N}.$$

Accordingly, for each $k \in \mathbb{N}$, (5.5) is satisfied with $l := k$ and $D = 1$. Then Lemma 5.4 in the Appendix implies that $\mathcal{H} := \{\mathsf{C}^m : m \in \mathbb{N}\} \subseteq \mathcal{L}(E_\alpha)$ is equicontinuous, i.e., the Cesàro operator C is power bounded in E_α. Since E_α is Montel, it follows via Proposition 2.8 of [1] that the Cesàro operator C is uniformly mean ergodic in E_α and hence, (4.1) is also satisfied, ([1], Theorem 2.4). The facts that each $x \in E_\alpha$ belongs to $c_0(v_k)$ for some $k \in \mathbb{N}$, that the inclusion $c_0(v_k) \subseteq E_\alpha$ is continuous and that the canonical vectors $e_n := (\delta_{nk})_{k \in \mathbb{N}}$, for $n \in \mathbb{N}$, form a Schauder basis in $c_0(v_k)$ implies $\{e_n : n \in \mathbb{N}\}$ is a Schauder basis for E_α. The proof of the identities in (4.2) now follow by applying the same (algebraic) arguments as used in the proof of Proposition 4.1 in [2]. ∎

PROPOSITION 4.2. *Let $\alpha_n \uparrow \infty$. The sequence $\{\mathsf{C}^m\}_{m \in \mathbb{N}}$ converges in $\mathcal{L}_b(E_\alpha)$ to the projection onto* $\operatorname{span}\{\mathbf{1}\}$ *along* $\overline{(I - \mathsf{C})(E_\alpha)}$.

Proof. Using Proposition 4.1 we proceed as in the proof of the analogous result when C acts in the Frèchet space $\Lambda_0(\alpha)$, ([6], Proposition 3.2). Indeed, for each $x \in E_\alpha$, we have that $x = y + z$ with $y \in \operatorname{Ker}(I - \mathsf{C}) = \operatorname{span}\{\mathbf{1}\}$ and $z \in \overline{(I - \mathsf{C})(E_\alpha)} = \overline{\operatorname{span}\{e_n\}_{n \geqslant 2}}$. So, for each $m \in \mathbb{N}$ we have $\mathsf{C}^m x = \mathsf{C}^m y + \mathsf{C}^m z$, with $\mathsf{C}^m y = y \to y$ in E_α as $m \to \infty$. The claim is that the sequence $\{\mathsf{C}^m z\}_{m \in \mathbb{N}}$ is also convergent in E_α. Indeed, proceeding as in the proof of Proposition 3.2 of [6] one shows, for each $r \geqslant 2$ and $m,\ n \in \mathbb{N}$, that $|(\mathsf{C}^m e_r)(n)| \leqslant \frac{1}{r-1} a_m$, where $(a_m)_{m \in \mathbb{N}}$ is a sequence of positive numbers satisfying $\lim_{m \to \infty} a_m = 0$. Since $v_1(n)|(\mathsf{C}^m e_r)(n)| \leqslant \frac{v_1(n)}{r-1} a_m$, for each $r \geqslant 2$ and $n, m \in \mathbb{N}$, with $1 \geqslant v_1(1) \geqslant v_1(n)$ for all $n \in \mathbb{N}$ it follows that $q_1(\mathsf{C}^m e_r) \leqslant \frac{1}{r-1} a_m$. We deduce, for each $r \geqslant 2$, that $\mathsf{C}^m e_r \to 0$ in $c_0(v_1)$ and hence, also in E_α as $m \to \infty$. Since $\{\mathsf{C}^m\}_{m \in \mathbb{N}} \subseteq \mathcal{L}(E_\alpha)$ is equicontinuous and (by (4.2)) the linear span of $\{e_n\}_{n \geqslant 2}$ is dense in $\overline{(I - \mathsf{C})(E_\alpha)}$, it follows that $\mathsf{C}^m z \to 0$ in E_α as $m \to \infty$ for each $z \in \overline{(I - \mathsf{C})(E_\alpha)}$. So, it has been shown that $\mathsf{C}^m x = \mathsf{C}^m y + \mathsf{C}^m z \to y$ in E_α as $m \to \infty$, for each $x \in E_\alpha$, i.e., $\{\mathsf{C}^m\}_{m \in \mathbb{N}}$ converges in $\mathcal{L}_s(E_\alpha)$. Since E_α is a Montel space, $\{\mathsf{C}^m\}_{m \in \mathbb{N}}$ also converges in $\mathcal{L}_b(E_\alpha)$. ∎

PROPOSITION 4.3. *Let $\alpha_n \uparrow \infty$ with E_α nuclear. Then the range $(I - \mathsf{C})^m(E_\alpha)$ is a closed subspace of E_α for each $m \in \mathbb{N}$.*

Proof. Consider first $m = 1$. Set $X(\alpha) := \{x \in E_\alpha : x_1 = 0\}$. The claim is that

$$(I - \mathsf{C})(E_\alpha) = (I - \mathsf{C})(X(\alpha)). \tag{4.3}$$

First recall that each sequence v_k, for $k \in \mathbb{N}$, is strictly positive and decreasing with $v_k \in c_0$ and so $\overline{(I - \mathsf{C})(c_0(v_k))} = \{x \in c_0(v_k) : x_1 = 0\} =: X_k$ and $(I - \mathsf{C})(X_k) = (I - \mathsf{C})(c_0(v_k))$, ([4], Lemmas 4.1 and 4.5). Now, if $x \in X(\alpha)$, then $x \in X_k$ for some $k \in \mathbb{N}$ and hence,

$$(I - \mathsf{C})x \in (I - \mathsf{C})(X_k) = (I - \mathsf{C})(c_0(v_k)) \subseteq (I - \mathsf{C})(E_\alpha).$$

This establishes one inclusion in (4.3). For the reverse inclusion let $x \in E_\alpha$. Then $x \in c_0(v_k)$ for some $k \in \mathbb{N}$ and hence, $(I - \mathsf{C})x \in (I - \mathsf{C})(c_0(v_k)) = (I - \mathsf{C})(X_k) \subseteq (I - \mathsf{C})(X(\alpha))$. Thus, the reverse inclusion in (4.3) is also valid.

Because of (4.3) and the containment $(I - \mathsf{C})(E_\alpha) \subseteq \overline{(I - \mathsf{C})(E_\alpha)} = X(\alpha)$, which is immediate from Proposition 4.1, to show that $(I - \mathsf{C})(E_\alpha)$ is closed in E_α it suffices to show that the continuous linear restriction operator

$$(I - \mathsf{C})|_{X(\alpha)} : X_\alpha \to X_\alpha$$

is bijective, actually surjective. Indeed, if we show that $(I - \mathsf{C})(X(\alpha)) = X(\alpha)$, then $(I - \mathsf{C})(E_\alpha) = X(\alpha)$ by (4.3) and hence, $(I - \mathsf{C})(E_\alpha)$ is a closed subspace of E_α.

To establish that $(I - \mathsf{C})|_{X_\alpha}$ is bijective we require the identity $(X(\alpha), \tau) = \operatorname{ind}_k X_k$, where τ is the relative topology in $X(\alpha)$ induced from E_α. This identity follows from the general fact that if $(E, \widetilde{\tau}) = \operatorname{ind}_n E_n$ is a (LB)-space and $F \subseteq E$ is a closed subspace with finite codimension, then $(F, \widetilde{\tau}|_F) = \operatorname{ind}_n (F \cap E_n)$ is also a (LB)-space, ([18], Lemma 6.3.1). Actually, setting $\widetilde{v}_k(n) := v_k(n+1)$ for all $k,\ n \in \mathbb{N}$, we have that $X(\alpha)$ is topologically isomorphic to $E(\widetilde{\alpha}) := \operatorname{ind}_k c_0(\widetilde{v}_k)$. Indeed, the left-shift operator $S : X(\alpha) \to E(\widetilde{\alpha})$ given by $S(x) := (x_2, x_3, \ldots)$ for $x = (x_n)_{n \in \mathbb{N}} \in X(\alpha)$ is such an isomorphism (because, for each $k \in \mathbb{N}$, the left shift operator $S : X_k \to c_0(v_k)$ is a surjective isometry). Consider now the operator $A := S \circ (I - \mathsf{C})|_{X(\alpha)} \circ S^{-1} \in \mathcal{L}(E(\widetilde{\alpha}))$. The claim is that A is bijective with $A^{-1} \in \mathcal{L}(E(\widetilde{\alpha}))$.

To establish the above claim observe, when interpreted to be acting in the space $\mathbb{C}^{\mathbb{N}}$, that the operator $A : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is bijective (which is a routine verification) and its inverse $B := A^{-1} : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is determined by the lower triangular matrix $B = (b_{nm})_{n,m \in \mathbb{N}}$ with entries given as follows: for each $n \in \mathbb{N}$ we have $b_{nm} = 0$ if $m > n$, $b_{nm} = \frac{n+1}{n}$ if $m = n$ and $b_{nm} = \frac{1}{m}$ if $1 \leqslant m < n$. To show that B is also the inverse of A acting on $E(\widetilde{\alpha})$, we only need to verify that $B \in \mathcal{L}(E(\widetilde{\alpha}))$. To establish this it suffices to show, for each $k \in \mathbb{N}$, that there exists $l \geqslant k$ such that

$\Phi_{\widetilde{v}_l} \circ B \circ \Phi_{\widetilde{v}_k}^{-1} \in \mathcal{L}(c_0)$, where for each $h \in \mathbb{N}$ the operator $\Phi_{\widetilde{v}_h} : c_0(\widetilde{v}_h) \to c_0$ given by $\Phi_{\widetilde{v}_h}(x) = (\widetilde{v}_h(n+1)x_n)$ for $x \in c_0(\widetilde{v}_h)$ is a surjective isometry. To this end, given $k \in \mathbb{N}$ set $l := k+1$, say. Then the lower triangular matrix corresponding to $\Phi_{\widetilde{v}_l} \circ B \circ \Phi_{\widetilde{v}_k}^{-1}$ is given by $D := \left(\frac{v_l(n+1)}{v_k(m+1)} b_{nm}\right)_{n,m\in\mathbb{N}}$. Moreover, for each fixed $m \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{v_l(n+1)}{v_k(m+1)} b_{nm} = \frac{1}{m v_k(m+1)} \lim_{n\to\infty} v_l(n+1) = 0$$

and, for each $n \in \mathbb{N}$, that

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{v_l(n+1)}{v_k(m+1)} b_{nm} &= \frac{(n+1)}{n} \frac{v_l(n+1)}{v_k(n+1)} + v_l(n+1) \sum_{m=1}^{n-1} \frac{1}{m v_k(m+1)} \\ &\leqslant 2 + (s_l)^{-\alpha_{n+1}} \sum_{m=1}^{n-1} \frac{s_k^{\alpha_{m+1}}}{m} \leqslant 2 + \left(\frac{s_k}{s_l}\right)^{\alpha_{n+1}} \sum_{m=1}^{n-1} \frac{1}{m} \\ &\leqslant 2 + \left(\frac{s_k}{s_l}\right)^{\alpha_{n+1}} (1 + \log(n)) \leqslant 2 + 2a^{\alpha_{n+1}} \log(n+1),\end{aligned}$$

where $a := \frac{s_k}{s_l} \in (0,1)$. Since E_α is nuclear, there exists $M \geqslant 1$ such that $\log(n) \leqslant M\alpha_n$ for all $n \in \mathbb{N}$ and hence, $a^{\alpha_n} \log(n) \leqslant M\alpha_n a^{\alpha_n}$ for $n \in \mathbb{N}$. Since $f(x) := xa^x$, for $x \in (0,\infty)$, satisfies $f'(x) < 0$ for $x > \frac{1}{\log\left(\frac{1}{a}\right)}$, the function f is decreasing on $\left(\frac{1}{\log\left(\frac{1}{a}\right)}, \infty\right)$ which implies $\sup_{n\in\mathbb{N}} a^{\alpha_n} \log(n) < \infty$, i.e., $\sum_{m=1}^{\infty} \frac{v_l(n+1)}{v_k(m+1)} < \infty$ for each $n \in \mathbb{N}$. Thus, both the conditions (i), (ii) of Lemma 2.1 in [4] are satisfied. Accordingly, $\Phi_{\widetilde{v}_l} \circ B \circ \Phi_{\widetilde{v}_k}^{-1} \in \mathcal{L}(c_0)$. The proof that $(I - \mathsf{C})(E_\alpha)$ is closed is thereby complete.

Since $(I - \mathsf{C})(E_\alpha)$ is closed, (4.1) implies $E_\alpha = \mathrm{Ker}(I - \mathsf{C}) \oplus (I - \mathsf{C})(E_\alpha)$. The proof of (2) ⇒ (5) in Remark 3.6 of [2] then shows that $(I - \mathsf{C})^m(E_\alpha)$ is closed in E_α for all $m \in \mathbb{N}$. ∎

An operator $T \in \mathcal{L}(X)$, with X a separable lcHs, is called *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in X. If, for some $z \in X$ the projective orbit $\{\lambda T^n z : n \in \mathbb{N}_0,\ \lambda \in \mathbb{C}\}$ is dense in X, then T is called *supercyclic*. Clearly, hypercyclicity implies supercyclicity.

PROPOSITION 4.4. *Let α satisfy $\alpha_n \uparrow \infty$. Then $\mathsf{C} \in \mathcal{L}(E_\alpha)$ is not supercyclic and hence, also not hypercyclic.*

Proof. It is known that C is *not* supercyclic in $\mathbb{C}^{\mathbb{N}}$, ([5], Proposition 4.3). Since E_α is dense (as it contains φ) and continuously included in $\mathbb{C}^{\mathbb{N}}$, the supercyclicity of C in any one of the spaces E_α would imply that $\mathsf{C} \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$ is supercyclic. ∎

5. APPENDIX

In this section we elaborate on the point raised in Section 1 that the behaviour of the Cesàro operator on the strong dual $(\Lambda_0^1(\alpha))'$ of power series spaces $\Lambda_0^1(\alpha)$ of *finite type*, is not so relevant in relation to continuity. It turns out that C *fails* to act in $(\Lambda_0^1(\alpha))'$ for *every* α with $\alpha_n \uparrow \infty$ such that $(\Lambda_0^1(\alpha))'$ is nuclear. Moreover, there exist $\alpha_n \uparrow \infty$ such that $(\Lambda_0^1(\alpha))'$ is not nuclear and $\mathsf{C} \in \mathcal{L}((\Lambda_0^1(\alpha))')$ (cf. Example 5.2) as well as other $\alpha_n \uparrow \infty$ such that $(\Lambda_0^1(\alpha))'$ is not nuclear and $\mathsf{C} \notin \mathcal{L}((\Lambda_0^1(\alpha))')$; see Example 5.3.

In order to be able to formulate the above claims more precisely, let $(v_k)_{k\in\mathbb{N}}$ be a sequence of functions $v_k : \mathbb{N} \to (0,\infty)$ satisfying $v_k(n) \uparrow_n \infty$, for each $k \in \mathbb{N}$, with $v_k \geqslant v_{k+1}$ pointwise on $\mathbb{N}$ and $\lim_{n\to\infty} \frac{v_{k+1}(n)}{v_k(n)} = 0$ for all $k \in \mathbb{N}$. Then $\ell_\infty(v_k) \subseteq c_0(v_{k+1})$ continuously for each $k \in \mathbb{N}$ and so

$$k_0(V) := \operatorname*{ind}_k c_0(v_k) = \operatorname*{ind}_k \ell_\infty(v_k).$$

In the notation of Köthe echelon spaces

$$\lambda_1\Big(\frac{1}{v}\Big) := \operatorname*{proj}_k \ell_1\Big(\frac{1}{v_k}\Big)$$

is a Fréchet–Schwartz space whose strong dual space, i.e., the co-echelon space

$$\Big(\lambda_1\Big(\frac{1}{v}\Big)\Big)'_\beta = \operatorname*{ind}_k \ell_\infty(v_k) = k_0(V),$$

is a (DFS)-space. It is known that the regular (LB)-space $k_0(V)$ is nuclear if and only if the Fréchet–Schwartz space $\lambda_1(\frac{1}{v})$ is nuclear if and only if the Grothendieck–Pietsch criterion is satisfied: for every $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that the sequence $\big(\frac{v_l(n)}{v_k(n)}\big)_{n\in\mathbb{N}} \in \ell_1$, ([12], Section 21.6). In case $v_k(n) := e^{\alpha_n/k}$, for $k, n \in \mathbb{N}$, with $\alpha_n \uparrow \infty$, then $k_0(V)$ is the strong dual of the finite type power series space (of order 1) $\Lambda_0^1(\alpha) := \operatorname*{proj}_k \ell_1(\frac{1}{v_k})$. This Fréchet space is nuclear if and only if $\lim_{n\to\infty} \frac{\log(n)}{\alpha_n} = 0$, ([17], Proposition 29.6). Whenever this nuclearity condition is satisfied we have $\Lambda_0^1(\alpha) = \operatorname*{proj}_j c_0(\frac{1}{v_k})$ which is precisely the power series space $\Lambda_0(\alpha)$ in which the operator C was investigated in [6].

For the rest of this section, whenever $\alpha_n \uparrow \infty$ we only consider the weights $v_k(n) := e^{\alpha_n/k}$ for $k, n \in \mathbb{N}$.

PROPOSITION 5.1. *Let the sequence α_n satisfy $\alpha_n \uparrow \infty$ and $\lim_{n\to\infty} \frac{\log(n)}{\alpha_n} = 0$. Then the Cesàro operator C does not act in $k_0(V) = \operatorname*{ind}_k c_0(v_k)$.*

Proof. Since $\lim_{n\to\infty} \frac{\log(n)}{\alpha_n} = 0$, it follows from Lemma 2.2 of [6] that

$$\lim_{n\to\infty} n^t e^{-\alpha_n} = 0$$

for each $t \in \mathbb{N}$, which implies $\lim_{n\to\infty} n e^{-\alpha_n/l} = 0$ for each $l \in \mathbb{N}$. In particular,

$$\sup_{n\in\mathbb{N}} \frac{e^{\alpha_n/l}}{n} = \infty, \quad \forall l \in \mathbb{N}. \tag{5.1}$$

Suppose that $\mathsf{C} \in \mathcal{L}(k_0(V))$, i.e., for every $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that $\mathsf{C} : c_0(v_k) \to c_0(v_l)$ is continuous. Then, for $k := 1$ there exists $l_1 > 1$ such that $\mathsf{C} : c_0(v_1) \to c_0(v_{l_1})$ is continuous, equivalently

$$M := \sup_{n\in\mathbb{N}} \frac{v_{l_1}(n)}{n} \sum_{m=1}^{n} \frac{1}{v_1(m)} < \infty, \tag{5.2}$$

([4], Proposition 2.2(i)). But, via (5.2), we then have for each $n \in \mathbb{N}$ that

$$\frac{e^{\alpha_n/l_1}}{n} = v_1(1) \cdot \frac{v_{l_1}(n)}{n v_1(n)} \leqslant v_1(1) \cdot \frac{v_{l_1}(n)}{n} \sum_{m=1}^{n} \frac{1}{v_1(m)} \leqslant M v_1(1).$$

This contradicts (5.1) for $l := l_1$. Hence, C does *not* act in $k_0(V)$. ∎

EXAMPLE 5.2. Define $\alpha_n := \log(n+1)$ for $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\log(n)}{\alpha_n} = 1 \neq 0$, the space $k_0(V)$ is *not* nuclear. To see that $\mathsf{C} \in \mathcal{L}(k_0(V))$ fix any $k \in \mathbb{N}$ and set $l := k+1$. Noting that $v_r(n) = (n+1)^{1/r}$ for $r, n \in \mathbb{N}$, it follows that

$$\frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_k(m)} = \frac{(n+1)^{1/l}}{n} \sum_{m=1}^{n} \frac{1}{(m+1)^{1/k}} \leqslant \frac{2(n+1)^{1/l}}{(n+1)} \sum_{m=1}^{n+1} \frac{1}{m^{1/k}}, \tag{5.3}$$

for each $n \in \mathbb{N}$. If $k = 1$, then $l = 2$ and it follows from (5.3) and the inequality $\sum_{m=1}^{n+1} \frac{1}{m} \leqslant 1 + \log(n+1)$ that the left-side of (5.3) is at most $\frac{2(1+\log(n+1))}{(n+1)^{1/2}}$, for $n \in \mathbb{N}$. For $k > 1$, using the inequality $\sum_{m=1}^{n+1} \frac{1}{m^{\delta}} \leqslant 1 + \frac{(n+1)^{1-\delta}}{1-\delta}$, $n \in \mathbb{N}$ (valid for each $\delta \in (0,1)$), with $\delta := \frac{1}{k}$ it follows from (5.3) (with $l = k+1$) that

$$\frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_k(m)} \leqslant (n+1)^{(1/(k+1))-1} + \frac{k(n+1)^{(1/(k+1))-(1/k)}}{(k-1)}, \quad n \in \mathbb{N}.$$

In both the cases (i.e., $k = 1$ and $k > 1$) we see that $\sup_{n\in\mathbb{N}} \frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_k(m)} < \infty$ and so $\mathsf{C} : c_0(v_k) \to c_0(v_l)$ is continuous, ([4], Proposition 2.2(i)). Since this is valid for *every* $k \in \mathbb{N}$ and with $l := k+1$, it follows that $\mathsf{C} \in \mathcal{L}(k_0(V))$.

EXAMPLE 5.3. Let $(j(k))_{k\in\mathbb{N}} \subseteq \mathbb{N}$ be the sequence given by $j(1) := 1$ and $j(k+1) := 2(k+1)(j(k))^k$, for $k \geqslant 1$. Observe that $j(k+1) > k(j(k))^k + 1 > j(k)$ for all $k \in \mathbb{N}$. Define $\beta = (\beta_n)_{n\in\mathbb{N}}$ via $\beta_n := k(j(k))^k$ for $n = j(k), \ldots, j(k+1)-1$.

Then β is non-decreasing with $\lim_{n\to\infty} \beta_n = \infty$. Let $\gamma = (\gamma_n)_{n\in\mathbb{N}}$ be any strictly increasing sequence satisfying $2 < \gamma_n \uparrow 3$. Then the sequence $\alpha_n := \log(\beta_n + \gamma_n)$, for $n \in \mathbb{N}$, satisfies $1 < \alpha_n \uparrow \infty$ and $\lim_{n\to\infty} \frac{\log(n)}{n} \neq 0$, ([6], Remark 2.17). In particular, $k_0(V)$ it *not* nuclear. To establish that C does *not* act in $k_0(V)$ is suffices to show, for $k := 1$, that

$$\sup_{n\in\mathbb{N}} \frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_1(m)} = \infty, \quad \forall l \in \mathbb{N}. \tag{5.4}$$

So, fix any $l \in \mathbb{N}$. Select $n = j(k)$, for any $k \in \mathbb{N}$, and observe (for *this* n) that

$$\begin{aligned}\frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_1(m)} &= \frac{(\beta_{j(k)} + \gamma_{j(k)})^{1/l}}{j(k)} \sum_{m=1}^{j(k)} \frac{1}{\beta_m + \gamma_m} \geqslant \frac{(\beta_{j(k)} + \gamma_{j(k)})^{1/l}}{j(k)} \cdot \frac{1}{(\beta_1 + \gamma_1)} \\ &\geqslant \frac{(k(j(k))^k + \gamma_{j(k)})^{1/l}}{4j(k)} \geqslant \frac{k^{1/l}(j(k))^{(k/l)-1}}{4} \geqslant \frac{k^{1/l}k^{(k/l)-1}}{4},\end{aligned}$$

where we have used $\frac{1}{\beta_1+\gamma_1} > \frac{1}{4}$ and $j(k) \geqslant k$. Accordingly,

$$\sup_{n\in\mathbb{N}} \frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_1(m)} \geqslant \sup_{k\in\mathbb{N}} \frac{v_l(j(k))}{j(k)} \sum_{m=1}^{j(k)} \frac{1}{v_1(m)} \geqslant \sup_{k\in\mathbb{N}} \frac{k^{1/l}k^{(k/l)-1}}{4} = \infty.$$

So, (5.4) is satisfied and hence, C does not act in $k_0(V)$.

The final two (abstract) results are recorded here in order not to disturb the flow of the text in earlier sections (where these results are needed). We begin with a fact which is surely known; a proof is included for the sake of self containment.

LEMMA 5.4. *Let $E = \operatorname{ind}_k (E_k, \|\cdot\|_k)$ be a regular inductive limit of Banach spaces. Then a subset $\mathcal{H} \subseteq \mathcal{L}(E)$ is equicontinuous if and only if the following condition is satisfied: for every $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l \geqslant k$ and $D > 0$ such that*

$$\|Tx\|_l \leqslant D\|x\|_k, \quad \forall T \in \mathcal{H},\ x \in E_k. \tag{5.5}$$

Proof. First, assume that $\mathcal{H}$ is equicontinuous. Fix $k \in \mathbb{N}$, in which case the closed unit ball B_k of E_k is bounded in E. The claim is that $C := \bigcup_{T\in\mathcal{H}} T(B_k)$ is bounded in E. Indeed, by equicontinuity of $\mathcal{H}$, given any 0-neighbourhood V in E there exists a 0-neighbourhood U in E such that $T(U) \subseteq V$ for all $T \in \mathcal{H}$. Since B_k is bounded in E, there exists $\lambda > 0$ such that $B_k \subseteq \lambda U$ and hence, $T(B_k) \subseteq \lambda T(U) \subseteq \lambda V$ for all $T \in \mathcal{H}$. It follows that $C \subseteq \lambda V$. Since V is arbitrary, it follows that C is bounded in E. But, E is regular and so there exists $l \geqslant k$ such that C is contained and bounded in E_l. Thus, there exists $D > 0$ such that $\|Tx\|_l \leqslant D$ for all $x \in B_k$ and $T \in \mathcal{H}$. Accordingly, the stated condition (5.5) is satisfied.

Assume that the stated condition (5.5) is satisfied. Since E is barrelled, the Banach–Steinhaus principle is available and so it suffices to show that the set

$\{Ty : T \in \mathcal{H}\}$ is bounded in E for each $y \in E$. So, fix $y \in E$ in which case $y \in E_k$ for some $k \in \mathbb{N}$. Selecting $l \geqslant k$ and $D > 0$ according to condition (5.5), we have $\|Ty\|_l \leqslant D\|y\|_k$ for all $T \in \mathcal{H}$. Hence, the set $\{Ty : T \in \mathcal{H}\}$ is bounded in E_l and so, also in E. ∎

The following result occurs in Lemma 5.2 of [7].

LEMMA 5.5. *Let* $E = \operatorname{ind}_n (E_n, \|\cdot\|_n)$ *be a Hausdorff inductive limit of Banach spaces. Let* $T \in \mathcal{L}(E)$ *satisfy the following condition:*

(A) *for each* $n \in \mathbb{N}$ *the restriction* T_n *of* T *to* E_n *maps* E_n *into itself and belongs to* $\mathcal{L}(E_n)$.

Then the following properties are satisfied:

(i) $\sigma_{\text{pt}}(T;E) = \bigcup_{n\in\mathbb{N}} \sigma_{\text{pt}}(T_n;E_n)$;

(ii) $\sigma(T;E) \subseteq \bigcap_{m\in\mathbb{N}} \Big(\overline{\bigcup_{n=m}^{\infty} \sigma(T_n;E_n)} \Big)$. *Moreover, if* $\lambda \in \bigcap_{n=m}^{\infty} \rho(T_n;E_n)$ *for some* $m \in \mathbb{N}$, *then* $R(\lambda, T_n)$ *coincides with the restriction of* $R(\lambda, T)$ *to* E_n *for each* $n \geqslant m$;

(iii) *if* $\bigcup_{n=m}^{\infty} \sigma(T_n;E_n) \subseteq \overline{\sigma(T;E)}$ *for some* $m \in \mathbb{N}$, *then* $\sigma^*(T;E) = \overline{\sigma(T;E)}$.

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