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A STONE–ČECH THEOREM FOR $C_0(X)$-ALGEBRAS

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ABSTRACT. For a $C_0(X)$-algebra A, we study $C(K)$-algebras B that we regard as compactifications of A, generalising the notion of (the algebra of continuous functions on) a compactification of a completely regular space. We show that A admits a Stone–Čech-type compactification A^β, a $C(\beta X)$-algebra with the property that every bounded continuous section of the C^*-bundle associated with A has a unique extension to a continuous section of the bundle associated with A^β. Moreover, A^β satisfies a maximality property amongst compactifications of A (with respect to appropriately chosen morphisms) analogous to that of βX. We investigate the structure of the space of points of βX for which the fibre algebras of A^β are non-zero, and partially characterise those $C_0(X)$-algebras A for which this space is precisely βX.

KEYWORDS: *C^*-algebra, $C_0(X)$-algebra, C^*-bundle, Stone–Čech compactification.*

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INTRODUCTION

Bundles, or (semi-)continuous fields arise as a natural way to study the structure of non-simple C^*-algebras. Indeed, the Gelfand–Naimark representation of a commutative C^*-algebra may be viewed in this way, and thus suggests the approach of representing a general (i.e. non-commutative) C^*-algebra as the algebra of continuous sections of a bundle of C^*-algebras, over a suitably constructed base space. To this end, Kasparov [22] introduced the notion of a $C_0(X)$-algebra, which represents a C^*-algebra A as the section algebra of a bundle over a locally compact Hausdorff space X. This generalised earlier work of Fell [18], Dixmier and Douady [16], Dauns and Hofmann [14], Lee [30], and others. Working in the framework of $C_0(X)$-algebras, it is often possible to adapt and generalise the techniques and results from the commutative setting to study more general C^*-algebras. Recently $C_0(X)$-algebras have proved to be extremely useful in advancing the study of the structure and classification of non-simple C^*-algebras, for example in [8], [9], [11], [12], [13] and [23].

Typically when working with bundles arising from non-unital C^*-algebras, the natural choice of base space X is non-compact and the original algebra is identified with the algebra of continuous sections of the bundle that vanish at infinity over X. Recent results of Archbold and Somerset [5], [7] motivate the study of the larger algebra of all bounded continuous sections of the bundle, with a view to understanding the structure of multiplier and corona algebras of non-simple C^*-algebras. Indeed, examples of this algebra have also arisen in the study of extensions [36], [37] and tensor products [39]. The corresponding object in the commutative case, namely the algebra $C^b(X)$, is often studied by embedding X in a larger, compact space.

A compactification of a space X is a compact Hausdorff space K together with a homeomorphic embedding of X as a dense subspace of K. If X is in addition locally compact, then Gelfand duality gives an equivalent formulation of this notion in the language of commutative C^*-algebras: a compactification of X is equivalent to a unitisation of the C^*-algebra $C_0(X)$. Of particular interest are the minimal (one-point) and maximal (Stone–Čech) compactifications of X, corresponding to the minimal unitisation and the multiplier algebra of $C_0(X)$ respectively. Our goal here is to study compactifications in the framework of C^*-bundles. More precisely, given a bundle over a non-compact space X, when and how can it be extended to a bundle over a compactification of X (leaving the fibres over points of X unchanged)? A similar question for locally trivial bundles with finite-dimensional fibres (i.e. those arising from homogeneous C^*-algebras) was studied by Phillips in [35]. Here we consider this problem in the more general setting of bundles arising from $C_0(X)$-algebras.

Of particular interest is the question of whether or not such a bundle over X extends to a bundle over its Stone–Čech compactification βX in such a way that the natural C^*-bundle analogue of the Stone–Čech extension property holds: every bounded continuous section over X has a unique extension to a continuous section over βX. One motivation for studying this question is the desire to obtain a more detailed decomposition of the C^*-algebra of bounded continuous sections of our original bundle, in line with the classical identification of $C^b(X)$ with $C(\beta X)$.

For a locally compact Hausdorff space X, a $C_0(X)$-algebra is a C^*-algebra A which carries the structure of a non-degenerate Banach $C_0(X)$-module. The maximal ideals of $C_0(X)$ give rise to quotient C^*-algebras $\{A_x : x \in X\}$ of A, which we regard as the fibres of a bundle of C^*-algebras over X. Each $a \in A$ then gives rise to a cross-section $X \to \coprod A_x, x \mapsto a(x)$, and there is a natural topology on $\coprod A_x$ such that A is isomorphic to the C^*-algebra of all continuous sections of this bundle that vanish at infinity on X. The norm functions $x \mapsto \|a(x)\|$ ($a \in A$) are in general only upper-semicontinuous on X; when they are in addition continuous for all $a \in A$ we speak of a *continuous* $C_0(X)$*-algebra*. Continuous

$C_0(X)$-algebras are equivalent to the continuous fields of C^*-algebras studied by Fell [18], Dixmier and Douady [16], and many others.

When A is a $C_0(X)$-algebra with all fibres nonzero, we define a *compactification* of A to be a $C(K)$-algebra B where K is a compactification of X in the usual sense, B contains A as an essential ideal, and the fibre algebras B_x of B are naturally isomorphic to those of A over points of X. This is equivalent to requiring that the C^*-bundle over K defined by B, when restricted to the dense subspace X, coincides with the bundle over X defined by A.

In seeking compactifications of a $C_0(X)$-algebra A, it would seem natural at first to take $B = M(A)$ (the multiplier algebra of A) and $K = \beta X$. While $M(A)$ is indeed a $C(\beta X)$-algebra, it often fails to be a compactification of A, since the fibre algebras $M(A)_x$ can be much larger than those of A. For example, this occurs whenever the A_x are non-unital, since quotients of $M(A)$ are necessarily unital. We remark that it was shown by Archbold and Somerset that the identity $M(A)_x = M(A_x)$ can also fail in general [6], and that when A is a continuous $C_0(X)$-algebra it is often the case that $M(A)$ fails to be continuous [4].

We study the C^*-algebra A^β of bounded continuous sections associated with a $C_0(X)$-algebra A, which, in general, lies between A and its multiplier algebra $M(A)$. In Section 3 we show that for any compactification K of X, A^β carries the structure of a $C(K)$-algebra and moreover defines a compactification of A (Theorem 3.4). We show in the same theorem that if A is a continuous $C_0(X)$-algebra and $K = \beta X$, then A^β is a continuous $C(\beta X)$-algebra. Moreover, Corollary 6.5 shows that βX is essentially the unique compactification of X with this property. In general, the operation of constructing A^β from A is functorial with respect to the natural morphisms between $C_0(X)$-algebras A and $C_0(Y)$-algebras B (Corollary 4.6), and is a closure operation (Corollary 6.6). These properties generalise the Stone extension of a continuous map $X \to Y$ to a continuous map $\beta X \to \beta Y$, and the fact that $\beta(\beta X) = \beta X$ respectively.

In the special case of a trivial $C_0(X)$-algebra $A = C_0(X, B)$ for some C^*-algebra B, we have $A^\beta = C^b(X, B)$ (Corollary 4.7). In particular this shows that if $A = C_0(X)$ then $A^\beta = C^b(X)$. More generally, we show that the $C(\beta X)$-algebra A^β, has the property that every bounded continuous section on X of the bundle defined by A extends uniquely to a continuous section on βX of A^β (Theorem 4.2). This shows that the bundle arising from the $C(\beta X)$-algebra A^β has an extension property analogous to that of the classical Stone–Čech compactification.

The corona algebra of a $C_0(X)$-algebra is well-known to exhibit interesting and pathological behaviour [5], [7], [25]. With this in mind, we investigate the nature of the fibres of the $C(\beta X)$-algebra A^β at points of the "corona" set $\beta X \backslash X$, and in particular, the question of whether or not these fibres are all non-zero C^*-algebras. We show in Theorem 5.5 that A^β has no zero-algebra fibres whenever A belongs to a large class of $C_0(X)$-algebras, including all σ-unital, continuous $C_0(X)$-algebras. Interestingly, Example 5.10 shows that this can fail when A is not continuous. Nonetheless, we show in Theorem 5.9 that the set of nonzero fibres of A^β in $\beta X\backslash X$ is at least dense in $\beta X\backslash X$ for all σ-unital A. The existence of zero-algebra fibres of A^β is closely related to the existence of so-called *remote points* of Stone–Čech remainders, a topic of significant interest in the field of general topology since the work of Fine and Gilmann in [19].

It is well-known that βX is maximal amongst compactifications of X: given any compactification K of X, there is a unital, injective $*$-homomorphism $C(K) \to C(\beta X)$ (commuting with the restriction homomorphisms $f \mapsto f|_X$). We show that the compactification A^β of the $C_0(X)$-algebra A has an analogous maximality and uniqueness property (Theorems 6.1 and 6.4).

The structure of the paper is as follows: in Section 1 we introduce the basic definitions and properties of $C_0(X)$-algebras and illustrate how a $C_0(X)$-algebra A defines a C^*-bundle over X. Since a general $C_0(X)$-algebra A may have some fibres equal to the zero C^*-algebra, it is often necessary to restrict our attention to the subspace X_A corresponding to the nonzero fibres. Often, X may be essentially replaced with X_A as a base space, though, in general, care is needed since X_A may fail to be locally compact (it is always completely regular and so we may safely speak of βX_A). The notion of a morphism between a $C_0(X)$-algebra A and a $C_0(Y)$-algebra B is made clear in Section 3, where we introduce precise definition of a compactification of a $C_0(X)$-algebra A (allowing for $X_A \subsetneq X$) and its equivalent formulation in the language of C^*-bundles.

The main results concerning the Stone–Čech compactification of a $C_0(X)$-algebra appear in Sections 3 and 4. In order to account for the possibility of having some fibre algebras equal to zero, the statements of these results are first presented in the framework of more general compactifications K of the space X_A, before restricting to the case $K = \beta X$. Section 5 concerns the study of the fibre algebras of A^β of points of the "corona" sets $K\backslash X_A$. Finally, the maximality and uniqueness results appear in Section 6.

1. PRELIMINARIES ON $C_0(X)$-ALGEBRAS

DEFINITION 1.1. Let X be a locally compact Hausdorff space. A *$C_0(X)$-algebra* is a C^*-algebra A together with a $*$-homomorphism $\mu_A : C_0(X) \to ZM(A)$ with the property that $\mu_A(C_0(X))A = A$.

When the space X is compact, the non-degeneracy condition $\mu_A(C_0(X))A = A$ above is equivalent to the $*$-homomorphism μ_A being unital. In this case we will say that (A, X, μ_A) is a $C(X)$-algebra.

It follows from the Dauns–Hofmann theorem [14] that there is a $*$-isomorphism $\theta_A : C^b(\mathrm{Prim}(A)) \to ZM(A)$ with the property that

$$\theta_A(f)a + P = f(P)(a + P), \quad \text{for } a \in A, f \in C^b(\mathrm{Prim}(A)), P \in \mathrm{Prim}(A). \tag{1.1}$$

This gives an equivalent formulation of Definition 1.1: a $C_0(X)$-algebra is a C^*-algebra A together with a continuous map $\phi_A : \mathrm{Prim}(A) \to X$. The maps μ_A and ϕ_A are related via $\mu_A(f) = \theta_A(f \circ \phi_A)$ for all $f \in C_0(X)$ ([40], Proposition C.5). We call ϕ_A the *base map* and μ_A the *structure map*.

For clarity we will denote any $C_0(X)$-algebra A by the triple (A, X, μ_A). For $x \in X$ we define the ideal J_x via

$$J_x = \mu_A(\{f \in C_0(X) : f(x) = 0\})A = \bigcap\{P \in \mathrm{Prim}(A) : \phi_A(P) = x\}, \tag{1.2}$$

see Section 2 of [34] for example.

We do not require that the base map $\phi_A : \mathrm{Prim}(A) \to X$ be surjective, or even that $\phi_A(\mathrm{Prim}(A))$ be dense in X. It is shown in Corollary 1.3 of [4] that ϕ_A has dense range if and only if the structure map μ_A is injective.

If $x \in X\backslash\mathrm{Im}(\phi_A)$, then we may still define the ideal J_x of A via $J_x = \mu_A(\{f \in C_0(X) : f(x) = 0\})A$; it is shown in Section 1 of [4] that $J_x = A$ for all such x. This is consistent with our second definition of J_x in (1.2), when we regard the intersection of the empty set $\{P \in \mathrm{Prim}(A) : \phi_A(P) = x\}$ of ideals of A as A itself.

DEFINITION 1.2. Let (A, X, μ_A) be a $C_0(X)$-algebra. Define the subset X_A of X to be

$$X_A = \mathrm{Im}(\phi_A) = \{x \in X : J_x \neq A\}.$$

For each $x \in X$ let $A_x = A/J_x$ be the quotient C^*-algebra, and for $a \in A$ let $a(x) = a + J_x \in A_x$ be the image of a under the quotient $*$-homomorphism. Then the following properties of (A, X, μ_A) are well-known, see Proposition C.10 of [40] or [34] for example:

(i) $\|a\| = \sup_{x \in X} \|a(x)\|$ for all $a \in A$;

(ii) for all $a \in A, f \in C_0(X)$ and $x \in X$ we have

$$(\mu_A(f)a)(x) = f(x)a(x);$$

(iii) the function $x \mapsto \|a(x)\|$ is upper-semicontinuous and vanishes at infinity on X.

As a consequence, we may regard A as a C^*-algebra of cross-sections $X \to \coprod_{x \in X} A_x$, identifying $a \in A$ with $x \mapsto a(x)$. Note that under this identification, property (ii) shows that the $*$-homomorphism $\mu_A : C_0(X) \to ZM(A)$ is given by pointwise multiplication of sections by functions in $C_0(X)$. Note that for $x \in X\backslash X_A$, the fibre algebras A_x are zero, and we have

$$X_A = \{x \in X : A_x \neq 0\}. \tag{1.3}$$

In fact, there is a unique topology on $\coprod_{x \in X} A_x$ which defines an upper-semicontinuous C^*-bundle in such a way that A is canonically isomorphic to the C^*-algebra of *all* continuous cross-sections of this bundle that vanish at infinity on

X ([40], Theorem C.26). While we will not explicitly define the C^*-bundle topology on $\coprod_{x\in X} A_x$, it will be useful to identify those cross-sections $X \to \coprod_{x\in} A_x$ that are continuous with respect to this topology. Definition 1.3 of a continuous cross-section $X \to \coprod_{x\in X} A_x$ is easily seen to be equivalent to the one used in Theorem C.25 of [40] and Definition 5.1 of [2], and generalises the definitions used by Fell [18] and Dixmier ([15], Chapter 10) in the context of continuous fields.

DEFINITION 1.3. Let (A, X, μ_A) be a $C_0(X)$-algebra and let $Y \subseteq X$. We say that a cross-section $b : Y \to \coprod_{x\in Y} A_x$ is *continuous* (with respect to (A, X, μ_A)) if for all $x \in Y$ and $\varepsilon > 0$ there is a neighbourhood U of x in Y and an element $a \in A$ such that

$$\|b(y) - a(y)\| < \varepsilon \quad \text{for all } y \in U.$$

Further, we define $\Gamma((A, X, \mu_A))$ to be the collection of all sections $b : X_A \to \coprod_{x\in X_A} A_x$ that are continuous with respect to (A, X, μ_A). Then $\Gamma((A, X, \mu_A))$ is a $*$-algebra with respect to pointwise operations. The $*$-subalgebra $\Gamma^b((A, X, \mu_A))$ (respectively $\Gamma_0((A, X, \mu_A))$) consisting of those continuous sections b for which the norm-function $x \mapsto \|b(x)\|$ is bounded (respectively vanishes at infinity on X_A), equipped with the supremum norm is a C^*-algebra.

When there is no ambiguity regarding the $C_0(X)$-algebra structure on a given C^*-algebra A, we shall write $\Gamma(A)$, $\Gamma^b(A)$ and $\Gamma_0(A)$ respectively to denote the section algebras of Definition 1.3.

PROPOSITION 1.4. *Let (A, X, μ_A) be a $C_0(X)$-algebra.*

(i) *For any $b \in \Gamma(A)$, the function $x \mapsto \|b(x)\|$ is upper-semicontinuous on X_A.*

(ii) *Let $b : X \to \coprod_{x\in X} A_x$ be a cross-section with the property that $x \mapsto \|b(x)\|$ vanishes at infinity on X_A. Then b is continuous on X if and only if b is continuous on X_A (with respect to (A, X, μ_A)).*

(iii) *The section algebra $\Gamma_0(A)$ is an essential ideal of $\Gamma^b(A)$.*

(iv) *If X_A is compact then the section algebras $\Gamma(A)$, $\Gamma^b(A)$ and $\Gamma_0(A)$ coincide.*

Proof. (i) Let $\alpha > 0$ and let $x \in X_A$ be such that $\|b(x)\| < \alpha$. We will show that there is an open neighbourhood V of x in X_A such that $\|b(y)\| < \alpha$ for all $y \in V$.

With $\varepsilon = (1/2)(\alpha - \|b(x)\|)$, continuity of b ensures that there is an open neighbourhood U_1 of x in X_A and an element $a \in A$ such that

$$\|a(y) - b(y)\| < \varepsilon \quad \text{for all } y \in U_1.$$

Since $a \in A$, the function $y \mapsto \|a(y)\|$ is upper-semicontinuous on X, and so the set U_2 defined by

$$U_2 := \{y \in X_A : \|a(y)\| < \alpha - \varepsilon\}$$

is open in X_A, and this set contains x since

$$\|a(x)\| \leqslant \|a(x) - b(x)\| + \|b(x)\| < \varepsilon + \|b(x)\| = \alpha - \varepsilon.$$

Hence the set $V := U_1 \cap U_2$ is an open neighbourhood of x in X_A with

$$\|b(y)\| \leqslant \|b(y) - a(y)\| + \|a(y)\| < \varepsilon + (\alpha - \varepsilon) = \alpha$$

for all $y \in V$.

(ii) Let b be continuous on X_A and let $x \in X$ and $\varepsilon > 0$. If $x \in X_A$, then since b is continuous on X_A, there is an open neighbourhood U of x in X and an element $a \in A$ with the property that

$$\|a(y) - b(y)\| < \varepsilon \quad \text{for all } y \in U \cap X_A.$$

If $y \in U \backslash X_A$ then $A_y = \{0\}$ and hence $a(y) = b(y) = 0$. It follows that $\|a(y) - b(y)\| = 0$ for $y \in U \backslash X_A$, and so $\|a(y) - b(y)\| < \varepsilon$ for all $y \in U$.

Now assume that $x \in X \backslash X_A$. Since $A_x = \{0\}$, we have $\|b(x)\| = 0$, and as $y \mapsto \|b(y)\|$ vanishes at infinity on X_A, the set $\{y \in X_A : \|b(y)\| \geqslant \varepsilon\}$ is compact and hence closed as a subset of X. Thus if we take the element 0 of A and the open subset $U := \{y \in X : \|b(y)\| < \varepsilon\}$ of X, we have

$$\|0(y) - b(y)\| = \|b(y)\| < \varepsilon$$

for all $y \in U$.

Hence if b is continuous on X_A, it is continuous on X, and the converse is trivial.

(iii) Let $a \in \Gamma_0(A)$, b a nonzero element of $\Gamma^b(A)$ and $\alpha > 0$. Then since $ba \in \Gamma(A)$, the norm function $x \mapsto \|(ba)(x)\|$ is upper-semicontinuous on X_A by part (i), and so the set

$$\{x \in X_A : \|(ba)(x)\| \geqslant \alpha\}$$

is closed in X_A. For all $x \in X_A$, we have

$$\|(ba)(x)\| \leqslant \|b(x)\| \cdot \|a(x)\| \leqslant \|b\| \cdot \|a(x)\|,$$

and so if $\|(ba)(x)\| \geqslant \alpha$ then $\|a(x)\| \geqslant \alpha / \|b\|$. Since $a \in \Gamma_0(A)$, the set $\{x \in X_A : \|a(x)\| \geqslant \alpha / \|b\|\}$ is compact, and by the above argument, contains the closed subset $\{x \in X_A : \|(ba)(x)\| \geqslant \alpha\}$. Hence the latter is also compact, which shows that $ba \in \Gamma_0(A)$.

Since both $\Gamma_0(A)$ and $\Gamma^b(A)$ are equipped with the supremum norm, an element $a \in \Gamma_0(A)$ is zero if and only if $a(x) = 0$ for all $x \in X_A$. Moreover, for any $x \in X_A$ there is $a \in \Gamma_0(A)$ with $a(x) \neq 0$. Hence if $b \in \Gamma^b(A)$ with $ba = ab = 0$ for all $a \in \Gamma_0(A)$, then it must be the case that $b(x) = 0$ for all $x \in X_A$, which implies that $b = 0$. Thus $\Gamma_0(A)$ is an essential ideal of $\Gamma^b(A)$.

(iv) If $b \in \Gamma(A)$ then $x \mapsto \|b(x)\|$ is upper-semicontinuous on X_A by part (i). Since an upper-semicontinuous function on a compact space is bounded above ([21], Theorem 1), it follows that the norm function of b is bounded, i.e., that $b \in \Gamma^b(A)$. Moreover, given any $\varepsilon > 0$, upper-semicontinuity of $x \mapsto \|b(x)\|$ ensures that the set $\{x \in X_A : \|b(x)\| \geqslant \varepsilon\}$ is closed, hence compact. Thus we have $\Gamma(A) \subseteq \Gamma^b(A)$ and $\Gamma(A) \subseteq \Gamma_0(A)$, and since the reverse inclusions are evident, all three section algebras must coincide. ∎

REMARK 1.5. Usually when considering a $C_0(X)$-algebra (A, X, μ_A) as an algebra of cross-sections, elements of $\Gamma_0(A)$ are regarded as continuous cross-sections $X \to \coprod_{x\in X} A_x$ (rather than being defined on X_A as in Definition 1.3). By Proposition 1.4(ii), the definition of $\Gamma_0(A)$ is independent of whether we take X or X_A to be the domain of the cross-sections under consideration.

In particular, this shows that the definition of $\Gamma_0(A)$ made in Definition 1.3 is equivalent to the one used in Appendix C of [40].

The definition of $\Gamma^b(A)$, however, does depend on whether sections are defined on X or on X_A (see Example 1.10). Hence using X_A rather than X allows us flexibility in changing the ambient space of X_A without affecting $\Gamma^b(A)$.

The following theorem recalls some known results about $\Gamma_0(A)$.

THEOREM 1.6. *Let (A, X, μ_A) be a $C_0(X)$-algebra. Then the section algebra $\Gamma_0(A)$ has the following properties:*

(i) *the natural action of functions in $C_0(X)$ on sections in $\Gamma_0(A)$ by pointwise multiplication, equips $\Gamma_0(A)$ with the structure of a $C_0(X)$-algebra, with fibres given by $\Gamma_0(A)_x \cong A_x$ (*[40]*, Proposition* C.23*);*

(ii) *identifying each $a \in A$ with the cross section $x \mapsto a(x)$ defines a $C_0(X)$-linear $*$-isomorphism $A \to \Gamma_0(A)$ (*[34], [40]*, Theorem* C.26*);*

(iii) *let B be a C^*-subalgebra of $\Gamma_0(A)$ such that*

(a) *for all $f \in C_0(X)$ and $b \in B$ the section $f \cdot b$, where $(f \cdot b)(x) = f(x)b(x)$ for all $x \in X$, belongs to B, and*

(b) *for each $x \in X$ and $c \in A_x$ there is some $b \in B$ with $b(x) = c$,*

then $B = \Gamma_0(A)$ ([40]*, Proposition* C.24*).*

Let (A, X, μ_A) be a $C_0(X)$-algebra with base map $\phi_A : \mathrm{Prim}(A) \to X$, and let Y be any locally compact Hausdorff space containing (a homeomorphic image of) X_A. Then we may regard ϕ_A as a map from $\mathrm{Prim}(A)$ to Y. Hence A defines a $C_0(Y)$-algebra with this base map.

Since X_A is a subspace of the locally compact Hausdorff space X, X_A is completely regular, and so it admits a homeomorphic embedding into its Stone–Čech compactification βX_A. Thus the above remarks apply with $Y = \beta X_A$. It shall be convenient to fix some notation to describe this process.

DEFINITION 1.7. Let (A, X, μ_A) be a $C_0(X)$-algebra and let Y be a locally compact Hausdorff space containing a homeomorphic image of X_A. The $C_0(Y)$-algebra constructed by regarding ϕ_A as a map $\mathrm{Prim}(A) \to Y$ shall be denoted by (A, Y, μ_A^Y). In the particular case where $Y = \beta X_A$, we shall set

$$(A, \beta X_A, \nu_A) := (A, \beta X_A, \mu_A^{\beta X_A}).$$

Note that for $x \in X_A$, the ideals J_x of (1.2) are independent of the ambient space containing X_A. In particular, for each $a \in A$ the cross-section $a : X_A \to \coprod_{x \in X_A} A_x$, $x \mapsto a(x)$ is unchanged by the process of varying the base space. This shows that the definitions of $\Gamma_0(A)$ and $\Gamma^b(A)$ are unambiguous, in that they are also independent of the choice of ambient space containing X_A.

When X_A is locally compact, we may take the base space Y of Definition 1.7 to be X_A. Hence $(A, X_A, \mu_A^{X_A})$ is a $C_0(X_A)$-algebra, with (nonzero) fibres A_x for all $x \in X_A$. Note that in this case X_A is open in βX_A, and so $C_0(X_A)$ is the ideal $\{f \in C(\beta X_A) : f|_{\beta X_A \setminus X_A} \equiv 0\}$. It follows that $\mu_A^{X_A} = \nu_A|_{C_0(X_A)}$.

DEFINITION 1.8. A $C_0(X)$-algebra (A, X, μ_A) is said to be *continuous* if the functions $x \mapsto \|a(x)\|$ are continuous on X for all $a \in A$.

REMARK 1.9. If (A, X, μ_A) is a continuous $C_0(X)$-algebra and $b \in \Gamma(A)$, the norm function $x \mapsto \|b(x)\|$ is continuous on X_A. The proof of this fact is almost identical to the proof of Proposition 1.4(i) (or can alternatively be deduced from Theorem C.27 of [40] and Proposition 1.4(ii)).

By Lee's theorem ([30], Theorem 5), (A, X, μ_A) is a continuous $C_0(X)$-algebra if and only if the base map ϕ_A is an open map. In this case the local compactness of $\mathrm{Prim}(A)$ ([15], Corollary 3.3.8) passes to X_A. It is evident that (A, X, μ_A) is a continuous $C_0(X)$-algebra if and only if $(A, X_A, \mu_A^{X_A})$ is a continuous $C_0(X_A)$-algebra.

Since we wish to identify A with both $\Gamma_0((A, X_A, \mu_A^{X_A}))$ and $\Gamma_0((A, X, \mu_A))$, it is natural to ask that $\Gamma^b((A, X_A, \mu_A^{X_A}))$ and $\Gamma^b((A, X, \mu_A))$ be isomorphic also. The following example shows how this could fail if we were to allow elements of $\Gamma^b((A, X, \mu_A))$ to have domain X rather than X_A.

EXAMPLE 1.10. Let $A = C_0((0,1))$, $X = [0,1]$ and $\mu_A : C([0,1]) \to M(C_0(0,1))$ be given by pointwise multiplication, so that (A, X, μ_A) is a $C([0,1])$-algebra. Then $X_A = (0,1)$ and the fibres of (A, X, μ_A) are given by $A_t = \mathbb{C}$ for $t \in X_A$ and $A_0 = A_1 = \{0\}$.

Suppose that $b : X \to \coprod_{t \in X} A_t$ is continuous with respect to (A, X, μ_A). Then since $A_0 = A_1 = \{0\}$, b necessarily satisfies $b(0) = b(1) = 0$. By continuity, it follows that for any $\varepsilon > 0$ and $t \in [0,1]$ there is $a \in A$ and $\delta > 0$ with

$$\|b(s) - a(s)\| < \varepsilon \quad \text{for all } s \in (t - \delta, t + \delta).$$

Hence b is continuous and vanishes at infinity on $(0,1)$, and so in fact $\Gamma^b(A) \cong C_0(0,1)$.

On the other hand, if we replace X with X_A, we see that $\Gamma^b(A)$ consists of cross-sections $c : (0,1) \to \coprod_{t \in (0,1)} A_t$. It is easily seen that in this case $\Gamma^b(A)$ is naturally isomorphic to $C^b((0,1))$.

We now describe how a $C_0(X)$-algebra (A, X, μ_A) gives rise to a $C(\beta X)$-algebra $(M(A), \beta X, \mu_{M(A)})$.

PROPOSITION 1.11 ([4], Proposition 1.2). *Let (A, X, μ_A) be a $C_0(X)$-algebra. Then the structure map μ_A extends to a unital $*$-homomorphism $\mu_{M(A)} : C(\beta X) \to ZM(A)$, hence $(M(A), \beta X, \mu_{M(A)})$ is a $C(\beta X)$-algebra.*

REMARK 1.12. When X is compact, note that $\mu_{M(A)} = \mu_A$. In particular this applies to the $C(\beta X_A)$-algebra $(A, \beta X_A, \nu_A)$ of Definition 1.7.

It shall be convenient to fix some notation associated with (A, X, μ_A) and $(M(A), \beta X, \mu_{M(A)})$.

(•) By analogy with the ideals J_x of A defined in (1.2), we define for $y \in \beta X$ the ideals H_y of $M(A)$ via

$$H_y = \mu_{M(A)}(\{f \in C(\beta X : f(y) = 0\})M(A). \tag{1.4}$$

(•) For $x \in X_A$, let $\pi_x : A \to A_x$ be the quotient $*$-homomorphism. Then π_x extends to a strictly continuous $*$-homomorphism $\widetilde{\pi}_x : M(A) \to M(A_x)$ defined by

$$(\widetilde{\pi}_x(b))\pi_x(a) = \pi_x(ba) \quad \text{and} \quad \pi_x(a)(\widetilde{\pi}_x(b)) = \pi_x(ab) \tag{1.5}$$

for all $a \in A$ and $b \in M(A)$ [10].

Archbold and Somerset have studied the structure of the $C(\beta X)$-algebra $(M(A), \beta X, \mu_{M(A)})$ extensively [4], [5], [6], [7]. In particular, two important pathologies in the behaviour of $(M(A), \beta X, \mu_{M(A)})$ should be observed:

(i) It always holds that $\ker(\widetilde{\pi}_x) \supseteq H_x$, but equality does not hold in general [6].

(ii) If (A, X, μ_A) is a continuous $C_0(X)$-algebra, it does not necessarily follow that $(M(A), \beta X, \mu_{M(A)})$ is a continuous $C(\beta X)$-algebra. In Theorem 3.8 of [4] was obtained a complete characterisation of those $C_0(X)$-algebras (A, X, μ_A), (with A σ-unital) for which $(M(A), \beta X, \mu_{M(A)})$ is continuous.

LEMMA 1.13. *Let (A, X, μ_A) be a $C_0(X)$-algebra, let $x \in X$ and $m \in M(A)$ be such that $\widetilde{\pi}_x(m) = 0$. Then $m \in H_x$ if and only if $\|\widetilde{\pi}_y(m)\| \to 0$ as $y \to x$.*

Proof. By Lemma 1.5(ii) of [4], we have

$$\|m + H_x\| = \inf_W \sup_{y \in W} \|\widetilde{\pi}_y(m)\|,$$

as W ranges over all open neighbourhoods of x in X, from which the result follows. ∎

Note that the fibres $M(A)_y$ of the $C(\beta X)$-algebra $(M(A), \beta X, \mu_{M(A)})$ are the quotient C^*-algebras $M(A)/H_y$. Hence for $m \in M(A)$ and $y \in Y$, $m(y)$ denotes $m + H_y$ (not $\widetilde{\pi}_y(m)$ defined above).

EXAMPLE 1.14. Let $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$, and let $A = C(\widehat{\mathbb{N}}, c_0) \cong C_0(\widehat{\mathbb{N}} \times \mathbb{N})$ be the trivial continuous $C(\widehat{\mathbb{N}})$-algebra with fibre c_0. Then $\mathrm{Prim}(A)$ is canonically isomorphic to $\widehat{\mathbb{N}} \times \mathbb{N}$, and the base map $\phi_A : \mathrm{Prim}(A) \to \widehat{\mathbb{N}}$ is the projection onto the first coordinate. Since $M(c_0) = \ell^\infty = C(\beta\mathbb{N})$, it follows from Corollary 3.4 of [1] that $M(A) = C(\widehat{\mathbb{N}}, \ell^\infty_\beta)$, where ℓ^∞_β denotes ℓ^∞ with the strict topology induced by regarding ℓ^∞ as the multiplier algebra of c_0.

The algebra $M(A)$, together with the structure map $\mu_{M(A)} = \mu_A$, defines a $C(\widehat{\mathbb{N}})$-algebra. By Corollary 3.9 of [4], $(M(A), \widehat{\mathbb{N}}, \mu_{M(A)})$ fails to be a continuous $C(\widehat{\mathbb{N}})$-algebra.

For each $n \in \widehat{\mathbb{N}}$, let $\pi_n : A \to A_n \cong c_0$ be the quotient map, and $\widetilde{\pi}_n : M(A) \to M(A_n) \cong \ell^\infty$ its strictly continuous extension to $M(A)$. By Theorem 4.9 of [5], we have

$$M(A)_n := M(A)/H_n = M(A_n) \cong \ell^\infty \quad \text{for all } n \in \mathbb{N},$$

while $H_\infty \subsetneq \ker(\widetilde{\pi}_\infty)$ so that $M(A)_\infty \neq \ell^\infty$. Moreover, by the same reference, there are uncountably many distinct norm-closed ideals J of $M(A)$ satisfying $H_\infty \subsetneq J \subsetneq \ker(\widetilde{\pi}_\infty)$. Note that by Lemma 1.13 we have

$$\ker(\widetilde{\pi}_\infty) = \{b \in C(\widehat{\mathbb{N}}, \ell^\infty_\beta) : b(\infty) = 0\},$$

while

$$H_\infty = \{b \in C(\widehat{\mathbb{N}}, \ell^\infty_\beta) : \|b(n)\| \to 0 \text{ as } n \to \infty\}.$$

2. MORPHISMS AND COMPACTIFICATIONS OF $C_0(X)$-ALGEBRAS

For a locally compact Hausdorff space X and a pair of $C_0(X)$-algebras (A, X, μ_A) and (B, X, μ_B), the natural notion of a morphism between (A, X, μ_A) and (B, X, μ_B) is a $C_0(X)$-linear $*$-homomorphism. It is well-known that a $C_0(X)$-linear $*$-homomorphism $A \to B$ induces $*$-homomorphisms $A_x \to B_x$ of the corresponding fibre algebras. In this section we describe how this notion can be extended to morphisms from a $C_0(X)$-algebra (A, X, μ_A) to a $C_0(Y)$-algebra (B, Y, μ_B), and we clarify what it means for such a morphism to be injective.

We introduce the notion of a compactification of a $C_0(X)$-algebra (A, X, μ_A), generalising the classical definition for locally compact Hausdorff spaces. Intuitively, we define a compactification of a $C_0(X)$-algebra as a $C(K)$-algebra (B, K, μ_B), where K is a compactification of X_A, such that A is isomorphic to the ideal of continuous sections of B that vanish at infinity on X_A. At the level of C^*-bundles, this is equivalent to (B, K, μ_B) defining a bundle over K whose restriction to X_A coincides with that defined by (A, X, μ_A).

The following definition is due to Kwasniewski [26].

DEFINITION 2.1. Let (A, X, μ_A) be a $C_0(X)$-algebra and (B, Y, μ_B) a $C_0(Y)$-algebra. By a *morphism* $(A, X, \mu_A) \to (B, Y, \mu_B)$ we mean a pair (Ψ, ψ) where

(i) $\Psi : A \to B$ and $\psi : C_0(X) \to C_0(Y)$ are $*$-homomorphisms, and

(ii) for all $f \in C_0(X)$ and $a \in A$ we have

$$\Psi(\mu_A(f)a) = \mu_B(\psi(f))\Psi(a).$$

REMARK 2.2. Note that in the case that $Y = X$ and $\psi = \mathrm{Id}_{C_0(X)}$, a $*$-homomorphism $\Psi : A \to B$ gives rise to a morphism $(\Psi, \psi) : (A, X, \mu_A) \to (B, Y, \mu_B)$ if and only if Ψ is $C_0(X)$-linear.

Let $(\Psi, \psi) : (A, X, \mu_A) \to (B, Y, \mu_B)$ be a morphism. Note that dual to the $*$-homomorphism ψ there is an open subset U_ψ of Y and a continuous, proper map $\psi^* : U_\psi \to X$, such that ψ is given by the formula

$$\psi(f)(y) = \begin{cases} f(\psi^*(y)) & \text{if } y \in U_\psi, \\ 0 & \text{if } y \notin U_\psi. \end{cases} \tag{2.1}$$

It is shown in Proposition 3.5 of [26] that the pair (Ψ, ψ) induces $*$-homomorphisms $\Psi_y : A_{\psi^*(y)} \to B_y$ (for $y \in U_\psi$) of the fibre algebras, so that $\Psi : A \to B$ satisfies

$$\Psi(a)(y) = \begin{cases} \Psi_y(a(\psi^*(y))) & \text{if } y \in U_\psi, \\ 0 & \text{if } y \notin U_\psi. \end{cases} \tag{2.2}$$

Identifying A with $\Gamma_0(A)$ and B with $\Gamma_0(B)$, Ψ may be regarded as a $*$-homomorphism $\Gamma_0(A) \to \Gamma_0(B)$, and is completely determined by the formula (2.2) for all y in the subspace $Y_B = \{y \in Y : B_y \neq \{0\}\}$ of Y (defined analogously to X_A of (1.3)) and $a \in A$ ([26], Propositions 3.2 and 3.5).

Note that in general, neither of the relations $U_\psi \subseteq Y_B$ nor $Y_B \subseteq U_\psi$ hold. Indeed, let (B, Y, μ_B) be any $C_0(Y)$-algebra with $Y_B \subsetneq Y$ and consider the identity morphism $(\mathrm{Id}_B, \mathrm{Id}_{C_0(Y)})$. Then $U_\psi = Y$, which is not contained in Y_B. An example where Y_B is not contained in U_ψ is given in Remark 2.7.

REMARK 2.3. Let $\psi : C_0(X) \to C_0(Y)$ be a $*$-homomorphism with the property that the corresponding dual map ψ^* has domain Y. The ψ extends to a $*$-homomorphism $\psi^b : C^b(X) \to C^b(Y)$ defined via $\psi^b(f)(y) = f(\psi^*(y))$ for all $f \in C^b(X)$ and $y \in Y$.

Analogously, let $(\Psi, \psi) : (A, X, \mu_A) \to (B, Y, \mu_B)$ be a morphism with the property that the domain U_ψ of ψ^* contains Y_B. Then Ψ extends to a morphism $\Psi^b : \Gamma^b(A) \to \Gamma^b(B)$ via

$$(\Psi^b(c))(y) = \Psi_y(c(\psi^*(y)))$$

for all $c \in \Gamma^b(A)$ and $y \in Y_B$. Indeed, it is clear that $\Psi^b(c)$ is a cross section $Y_B \to \coprod_{y \in Y_B} B_y$ since $c(x) \in A_x$ for all $x \in X_A$. To see that $\Psi^b(c)$ is continuous, let

$y_0 \in Y_B$ and $\varepsilon > 0$. Then there is $a \in A$ and a neighbourhood U of $\psi^*(y_0)$ in X_A with

$$\|a(x) - c(x)\| < \varepsilon$$

for all $x \in U$, and hence

$$\|\Psi(a)(y) - \Psi^b(c)(y)\| = \|\Psi_y[a(\psi^*(y)) - c(\psi^*(y))]\| \leqslant \|a(\psi^*(y)) - c(\psi^*(y))\| < \varepsilon$$

for all y in the open neighbourhood $(\psi^*)^{-1}(U) \cap Y_B$ of y_0. Since $\Psi(a) \in B$, it follows that $\Psi^b(c)$ is indeed continuous.

DEFINITION 2.4. A morphism $(\Psi, \psi) : (A, X, \mu_A) \to (B, Y, \mu_B)$ is called

(i) *injective* if the $*$-homomorphisms Ψ and $\{\Psi_y : y \in Y_B \cap U_\psi\}$ are all injective;

(ii) an *isomorphism* if it is invertible, i.e. there exists a morphism $(\Phi, \phi) : (B, Y, \mu_B) \to (A, X, \mu_A)$ such that $\Phi = \Psi^{-1}$ and

$$\mu_A((\phi \circ \psi)(f)) = \mu_A(f) \quad \text{and} \quad \mu_B((\psi \circ \phi)(g)) = \mu_B(g)$$

for all $f \in C_0(X)$ and $g \in C_0(Y)$.

The motivation for this definition of an injective morphism is as follows: consider $A \cong \Gamma_0(A)$ and $B \cong \Gamma_0(B)$ as algebras of cross-sections in the usual way. Let $(\Psi, \psi) : (A, X, \mu_A) \to (B, Y, \mu_B)$ be an injective morphism, and identify $A \subseteq B$ and $A_{\psi(y)} \subseteq B_y$ as C^*-subalgebras for all $y \in U_\psi \cap Y_B$. Then each $a \in A$ may be naturally regarded as a cross-section $a : Y_B \to \coprod_{y \in Y_B} B_y$ such that $a(y) \in A_{\psi(y)} \subseteq B_y$ for all $y \in U_\psi \cap Y_B$ and $a(y) = 0$ otherwise. In particular, we may identify $\Gamma_0(A)$ canonically with a C^*-subalgebra of $\Gamma_0(B)$.

REMARK 2.5. While it is not immediately obvious from the definition, it is true that an isomorphism $(\Psi, \psi) : (A, X, \mu_A) \to (B, Y, \mu_B)$ (with inverse (Φ, ϕ)) is necessarily injective. To see this, we first note that for $g \in C_0(Y)$, the requirement that $\mu_B([\psi \circ \phi](g)) = \mu_B(g)$ implies that for all $b \in B$ and $y \in Y_B$ we have

$$g(y) \cdot b(y) = [(\psi \circ \phi)(g)](y) \cdot b(y),$$

and since for all such y there is $b \in B$ with $b(y) \neq 0$ we see that

$$[(\psi \circ \phi)(g)](y) = g(y) \quad \text{for all } y \in Y_B.$$

Now let $a \in A$ and suppose that $y \in Y_B$ with $\Psi(a)(y) = 0$. Then by the Cohen factorisation theorem ([17], Theorem 16.1), there is some $b \in B$ and $g \in C_0(Y)$ with $g(y) = 0$ such that $\Psi(a) = \mu_B(g) \cdot b$. Hence

$$a = (\Phi \circ \Psi)(a) = \Phi(\mu_B(g)b) = \mu_A(\phi(g)) \cdot \Phi(b),$$

and so

$$\begin{aligned} a(\psi^*(y)) &= [\phi(g)](\psi^*(y)) \cdot [\Phi(b)](\psi^*(y)) = [(\psi \circ \phi)(g)](y) \cdot [\Phi(b)](\psi^*(y)) \\ &= g(y) \cdot [\Phi(b)](\psi^*(y)) = 0. \end{aligned}$$

This shows that for $y \in Y_B$ and $a \in A$, $\Psi(a)(y) = 0$ if and only if $a(\psi^*(y)) = 0$, hence $\Psi_y : A_{\psi^*(y)} \to B_y$ is injective.

For our purposes, injective morphisms and isomorphisms shall mostly arise in the manner described in the following proposition.

PROPOSITION 2.6. *Let (A, X, μ_A) and (B, X, μ_B) be $C_0(X)$-algebras and $\psi : A \to B$ a $C_0(X)$-linear $*$-homomorphism. Then the morphism $(\Psi, \mathrm{Id}_{C_0(X)}) : (A, X, \mu_A) \to (B, X, \mu_B)$ is injective (respectively, an isomorphism) if and only if Ψ is injective (respectively, a $*$-isomorphism).*

Proof. Clearly a necessary condition for $(\Psi, \mathrm{Id}_{C_0(X)})$ to be injective (respectively, an isomorphism) is that Ψ be injective (respectively, an isomorphism).

Assume that Ψ is injective and $x \in X$ and let $I_x \subseteq A$ and $J_x \subseteq B$ denote the kernels of the evaluation maps $A \to A_x$ and $B \to B_x$ respectively, that is to say,

$$I_x = \mu_A(\{f \in C_0(X) : f(x) = 0\}) \cdot A \quad \text{and} \quad J_x = \mu_B(\{f \in C_0(X) : f(x) = 0\}) \cdot B$$

(see Section 1). Then by Lemma 2.11(ii) of [33] we have

$$\Psi(I_x) = J_x \cap \Psi(A).$$

Hence for $a \in A$ and $x \in X_B$, we have $\Psi(a)(x) = 0$ if and only if $a(x) = 0$, which shows that each $*$-homomorphism $\Psi_x : A_x \to B_x$ is injective, and so $(\Psi, \mathrm{Id}_{C_0(X)})$ is an injective morphism.

If Ψ is an isomorphism then Ψ^{-1} is also $C_0(X)$-linear, since for $f \in C_0(X)$ and $b \in B$ we have

$$\mu_B(f) \cdot b = \mu_B(f) \cdot \Psi(\Psi^{-1}(b)) = \Psi(\mu_A(f) \cdot \Psi^{-1}(b))$$

and hence

$$\Psi^{-1}(\mu_B(f) \cdot b) = \mu_A(f) \cdot \Psi^{-1}(b).$$

This shows that $(\Psi, \mathrm{Id}_{C_0(X)})$ is an isomorphism. ∎

REMARK 2.7. Note that if $(\Psi, \psi) : (A, X, \mu_A) \to (B, Y, \mu_B)$ is a morphism such that $\{\Psi_y : y \in Y\}$ are all injective then it does not necessarily follow that Ψ is injective. For example let $X = [0, 1) \cup (1, 2]$, $Y = [0, 1)$, $A = C_0(X)$ and $B = C_0(Y)$. Let $\Psi = \psi : A \to B$ be the restriction homomorphism, then clearly (Ψ, ψ) is a morphism.

We have $U_\psi = [0, 1)$ and $\psi^* : [0, 1) \to [0, 1) \cup (1, 2]$ is the inclusion map. Then for all $y \in U_\psi$, $\Psi_y : A_y \to B_y$ is the identity mapping, hence is injective, while clearly Ψ is not injective.

REMARK 2.8. Injectivity of Ψ and ψ does not imply injectivity of the Ψ_y. Let C be a C^*-algebra, Y a compact Hausdorff space which contains more than one point, $B = C(Y, C)$ regarded as a $C(Y)$ algebra in the usual way. If $X = \{x\}$ is a one-point space, equip $A = C(Y, C)$ with the structure of a $C(X)$-algebra, and consider the morphism $(\Psi, \psi) : (A, X, \mu_A) \to (B, Y, \mu_B)$ where Ψ is the identity morphism and ψ is the embedding of $\mathbb{C}$ as the constant functions on Y. Then Ψ

and ψ are injective. Moreover, ψ^* maps every point of Y to x, and $\Psi_y : A = B \to B_y$ is the evaluation map for each y, which is not injective.

DEFINITION 2.9. Let (A, X, μ_A) be a $C_0(X)$-algebra. A *compactification* of A is given by a $C(K)$-algebra (B, K, μ_B) together with an injective $C(K)$-linear $*$-homomorphism $\Psi : A \to B$ such that:

(i) K is a compactification of X_A,

(ii) $\Psi(A)$ is an essential ideal of B, and

(iii) the induced $*$-homomorphisms $\Psi_x : A_x \to B_x$ are isomorphisms for all $x \in X_A$.

When the $*$-homomorphism Ψ is clear, we shall often say that (B, K, μ_B) is a compactification of (A, X, μ_A) and consider $A \subseteq B$.

Note that if the $C_0(X)$-algebra (A, X, μ_A) in Definition 2.9 is simply $C_0(X)$, then $C(K)$ is a compactification of $C_0(X)$ for any compactification K of X. More generally, if $A = C_0(X, B)$, then for any such K, $C(K, B)$ defines a compactification of (A, X, μ_A), though in general there exist other compactifications in this case, see Corollary 4.7.

Based upon these elementary examples, one might conjecture that conditions (i) and (iii) of Definition 2.9 above imply condition (ii). However, the following example, from p. 684 of [24] shows that this is not the case.

EXAMPLE 2.10. Let C be a non-nuclear C^*-algebra. Then there is a C^*-algebra D and a C^*-subalgebra $D_0 \subseteq D$ for which the canonical map $C \otimes_{\max} D_0 \to C \otimes_{\max} D$ is not injective. Fixing $\alpha \in [0,1]$, let $X = [0,1] \backslash \{\alpha\}$ and let (A, X, μ_A) be the trivial $C_0(X)$-algebra $C_0(X, C \otimes_{\max} D)$.

Let $(D_\alpha, [0,1], \mu_{D_\alpha})$ be the continuous $C([0,1])$-algebra

$$D_\alpha = \{f \in C([0,1], D) : f(\alpha) \in D_0\},$$

and let $B = C \otimes_{\max} D_\alpha$. Then $(B, [0,1], 1_C \otimes_{\max} \mu_{D_\alpha})$ is a $C([0,1])$-algebra with fibres $B_t = C \otimes_{\max} D$ for $t \neq \alpha$ and $B_\alpha = C \otimes_{\max} D_0$ which is discontinuous at α [24]. Moreover, it is easily seen that the natural embedding of (A, X, μ_A) into $(B, [0,1], 1_C \otimes_{\max} \mu_{D_\alpha})$ satisfies properties (i) and (iii) of Definition 2.9.

To see that A is not an essential ideal of B, let $\pi : C \otimes_{\max} D_0 \to C \otimes_{\max} D$ be the $*$-homomorphism given by the universal property of the maximal tensor product. One can then construct an element $b \in B$ for which $b(x) = 0$ for all $x \in X$ and $\|b(\alpha)\| = 1$. It is clear then that ba and ab are zero for all $a \in A$.

REMARK 2.11. For a compactification (B, K, μ_B) of a $C_0(X)$-algebra (A, X, μ_A), we do not require that all of the fibre algebras B_y, where $y \in K \backslash X_A$, be nonzero. Indeed, for a given $C_0(X)$-algebra (A, X, μ_A), it is not clear whether or not there exists any compactification (B, K, μ_B) of (A, X, μ_A) with this property. We study this question in depth in Section 5.

We also remark that if (B, K, μ_B) is a compactification of (A, X, μ_A), it is not true in general that there exists an injective morphism (or indeed any morphism) $(\Psi, \psi) : (A, X, \mu_A) \to (B, K, \mu_B)$, unless K is also a compactification of X

(e.g. when $X_A = X$). Indeed, if $X_A \subsetneq X$, then there is no obvious $*$-homomorphism $\psi : C_0(X) \to C(K)$.

The following proposition establishes a number of useful facts about compactifications that shall be used in subsequent sections.

PROPOSITION 2.12. *Let (B, K, μ_B) be a compactification of (A, X, μ_A) and $\Psi : A \to B$ the associated $C(K)$-linear $*$-homomorphism.*

(i) *For all $b \in B$ we have*

$$\|b\| = \sup\{\|b(x)\| : x \in X_A\}.$$

(ii) *Identifying A_x with B_x for each $x \in X_A$, the map sending $b \in B$ to the cross-section $b|_{X_A} : X_A \to \coprod_{x \in X_A} A_x$, $x \mapsto b(x)$, defines an injective $*$-homomorphism $B \to \Gamma^b(A)$.*

(iii) *For all $b \in B$ and $y \in K \backslash X_A$,*

$$\|b(y)\| = \inf_W \sup_{x \in W \cap X_A} \|b(x)\|,$$

where W ranges over all neighbourhoods of y in K.

(iv) $\Psi(A) = \{b \in B : b(y) = 0$ *for all* $y \in K \backslash X_A\}$.

(v) *If in addition X_A is locally compact, then we may identify $C_0(X_A)$ with an ideal of $C(K)$. Under this identification, we have $\mu_B(C_0(X_A)) \cdot B = \Psi(A)$.*

Proof. (i) Since $\Psi(A)$ is an essential ideal of B, there is an injective $*$-homomorphism $B \to M(\Psi(A))$ which is the identity on $\Psi(A)$ ([10], Proposition 3.7(i) and (ii)). It then follows (using the construction of $M(\Psi(A))$ described in Section 2 of [10]) that for all $b \in B$ we have

$$\|b\| = \sup\{\|b \cdot \Psi(a)\| : a \in A, \|a\| \leqslant 1\}.$$

Suppose for a contradiction that there were some $b \in B$ with $\|b\| = 1$ but with $\sup\{\|b(x)\| : x \in X_A\} = \alpha$ for some $0 \leqslant \alpha < 1$. Then for all $a \in A$ with $\|a\| \leqslant 1$ and $x \in X_A$ we would have

$$\|(b \cdot \Psi(a))(x)\| = \|b(x) \cdot (\Psi(a))(x)\| \leqslant \alpha \cdot \|\Psi(a)(x)\| \leqslant \alpha\|a\| \leqslant \alpha.$$

Since $b \cdot \Psi(a) \in \Psi(A)$ for all $a \in A$, it would then follow that

$$\|b \cdot \Psi(a)\| = \sup_{x \in X_A} \|(b \cdot \Psi(a))(x)\| \leqslant \alpha$$

whenever $\|a\| \leqslant 1$. In particular, this would imply that

$$\|b\| = \sup\{\|b \cdot \Psi(a)\| : a \in A, \|a\| \leqslant 1\} \leqslant \alpha < 1,$$

which is a contradiction.

(ii) We first show that for each $b \in B$, the associated cross-section is continuous on X_A with respect to (A, X, μ_A). To this end, let $x \in X_A$ and $\varepsilon > 0$. Since $A_x = B_x$, there is some $a \in A$ with $a(x) = b(x)$. In particular, $\|(a - b)(x)\| = 0$.

Now, since (B, K, μ_B) is a $C(K)$-algebra and $\Psi(a) \in B$, the norm function $y \mapsto \|(b - \Psi(a))(y)\|$ is upper-semicontinuous on K by Proposition 1.4(i), and hence is upper-semicontinuous on X_A. In particular, there is a neighbourhood U of x in X_A such that $\|(b - \Psi(a))(y)\| < \varepsilon$ for all $y \in U$. Since Ψ_y is injective for all $y \in X_A$, it follows that, for all $y \in U$,

$$\|b(y) - a(y)\|_{A_y} = \|(b - \Psi(a))(y)\|_{B_y} < \varepsilon.$$

This shows that $b|_{X_A}$ is continuous on X_A with respect to (A, X, μ_A), and hence defines an element of $\Gamma^b(A)$. Moreover, it is clear from the definition of $x \mapsto b(x)$ that the map sending b to $b|_{X_A}$ is a $*$-homomorphism. Since $\Gamma^b(A)$ is equipped with the supremum norm, part (i) shows that this map is injective.

(iii) Let $y \in K\backslash X_A$ and $\varepsilon > 0$. Since $x \mapsto \|b(x)\|$ is upper-semicontinuous on K, there is an open neighbourhood U of y such that

$$\sup_{x \in U} \|b(x)\| \leqslant \|b(y)\| + \varepsilon.$$

It follows that

$$\inf_{W} \sup_{x \in W \cap X_A} \leqslant \|b(y)\|,$$

as W ranges over all neighbourhoods of y in K.

Suppose that there were some open neighbourhood U of y in K such that

$$\sup_{x \in U \cap X_A} \|b(x)\| < \|b(y)\|.$$

Let $f \in C(K)$, $0 \leqslant f \leqslant 1$ with $f(y) = 1$ and $f|_{K\backslash U} \equiv 0$, and let $x \in X_A$. If $x \notin U$, then $f(x) = 0$ and hence $\|(\mu_B(f)b)(x)\| = 0$. If $x \in U$, then

$$\|(\mu_B(f)b)(x)\| \leqslant \sup_{x \in U \cap X_A} \|b(x)\| < \|b(y)\|.$$

Together with part (i), this shows that

$$\|\mu_B(f)b\| = \sup_{x \in X_A} \|(\mu_B(f)b)(x)\| < \|b(y)\| = \|(\mu_B(f)b)(y)\|,$$

which is a contradiction.

(iv) Note that for $y \in K\backslash X_A$, the fibre algebra A_y is zero, and so (2.2) ensures that we have $\Psi(a)(y) = 0$ for all $a \in A$.

Now let $b \in B$ be such that $b(y) = 0$ for all y in $K\backslash X_A$. Note that $b|_{X_A}$ belongs to $\Gamma^b(A)$ by part (ii); we shall show that $b|_{X_A} \in \Gamma_0(A)$.

Since $y \mapsto \|b(y)\|$ is upper-semicontinuous on K, for each $\alpha > 0$ the set

$$\{y \in K : \|b(y)\| < \alpha\}$$

is open, and contains $K\backslash X_A$ by assumption. Hence

$$\{y \in K : \|b(y)\| \geqslant \alpha\}$$

is a closed and hence compact subset of K, and moreover, is contained in X_A. It follows that $(x \mapsto b(x))$ defines an element of $\Gamma_0(A)$, which by Theorem 1.6(i), can only be the case if $b \in \Psi(A)$.

(v) Since X_A is locally compact and K is a compactification of X_A, X_A is also an open subset of K ([20], 3.15(d)). Then we may identify $C_0(X_A)$ with the ideal of $C(K)$ consisting of those $f \in C(K)$ vanishing on $K \backslash X_A$. In particular,

$$\Psi(\mu_A^K(f)a) = \mu_B(f)\Psi(a)$$

for all $f \in C_0(X_A)$ and $a \in A$.

Since X_A is locally compact, we may regard A as a $C_0(X_A)$ algebra with the same base map as (A, X, μ_A) (restricting the range to X_A) as in Definition 1.7. The corresponding structure map is then $\mu_A^{X_A} = \mu_A^K|_{C_0(X_A)}$, so that $\mu_A^K(C_0(X_A)) \cdot A = A$. Hence

$$\Psi(A) = \mu_B(C_0(X_A)) \cdot \Psi(A) \subseteq \mu_B(C_0(X_A)) \cdot B,$$

so it remains to show the reverse inclusion.

Letting $B^0 = \mu_B(C_0(X_A)) \cdot B$, we see that $(B^0, X_A, \mu_B|_{C_0(X_A)})$ is a $C_0(X_A)$-algebra. Moreover, if $b \in B$ and $x \in X_A$, choosing $f \in C_0(X_A)$ with $f(x) = 1$ we have $\mu_B(f)b \in B^0$ and $(\mu_B(f)b)(x) = b(x)$. Hence for all $x \in X_A$ the fibre algebras B_x^0 are equal to B_x, which are in turn isomorphic to A_x (since (B, K, μ_B) is a compactification of (A, X, μ_A)).

In particular, for every $x \in X_A$ and $c \in B_x^0$ there is $a \in A$ with $a(x) = c$. Hence by Theorem 1.6(iii), $\Psi(A) = B^0$, which completes the proof. ∎

REMARK 2.13. Proposition 2.12(ii) in some sense characterises compactifications of a $C_0(X)$-algebra (A, X, μ_A) completely. Indeed, suppose that K is a compactification of X_A and that (B, K, μ_B) is a $C(K)$-algebra with the property that the map sending $b \in B$ to the cross section $b|_{X_A}$ is an injective $*$-homomorphism of B onto a subalgebra of $\Gamma^b(A)$ containing $\Gamma_0(A)$. Then (B, K, μ_B) gives rise to a compactification of (A, X, μ_A) in a natural way.

Indeed, identifying A with $\Gamma_0(A)$ and B with a subalgebra of $\Gamma^b(A)$ containing $\Gamma_0(A)$ defines an injective $*$-homomorphism from A to B, which is $C(K)$-linear since

$$(\mu_A^K(f)a)(x) = f(x)a(x) = (\mu_B(f)a)(x)$$

for all $f \in C(K)$, $a \in A$ and $x \in X_A$. Moreover, since the homomorphism $B \to \Gamma^b(A)$ is injective, $\|b\| = \sup_{x \in X_A} \|b(x)\|$, and so $b = 0$ if and only if $b(x) = 0$ for all $x \in X_A$, which occurs if and only if $ba = ab = 0$ for all $a \in A$. Property (iii) of Definition 2.9 is immediate from the fact that $b(x) \in A_x$ for all $a \in A$.

3. THE STONE–ČECH COMPACTIFICATION OF A $C_0(X)$-ALGEBRA

In this section we show that any $C_0(X)$-algebra (A, X, μ_A) gives rise to a C^*-algebra A^β, with $A \subseteq A^\beta \subseteq M(A)$, such that $(A^\beta, \beta X_A, \mu_{A^\beta})$ is a $C(\beta X_A)$-algebra, and defines a compactification of (A, X, μ_A) in a natural way.

DEFINITION 3.1. Let (A, X, μ_A) be a $C_0(X)$-algebra, and define the closed ideal A^β of $M(A)$ via

$$A^\beta = \bigcap\{A + H_x : x \in X_A\}, \tag{3.1}$$

where H_x, for $x \in X_A$, are the ideals in $M(A)$ given by (1.4).

Note that A^β is an essential ideal of $M(A)$ since it contains A. We shall show that A^β gives rise naturally to a compactification of (A, X, μ_A), which has many properties analogous to those of the Stone–Čech compactification of a locally compact Hausdorff space.

It is important to note that the definition of A^β is independent of the base space X containing X_A. More precisely, replacing (A, X, μ_A) with the $C_0(Y)$-algebra (A, Y, μ_A^Y) of Definition 1.7, the ideals H_x for $x \in X_A$ are unchanged ([5], Lemma 2.4), hence the same is true for A^β.

Part (ii) of Proposition 3.2 was established in Theorem 3.3 of [7].

PROPOSITION 3.2. *For $C_0(X)$-algebra (A, X, μ_A) we have*
(i) *$A^\beta = M(A)$ if and only if $\mu_A(C_0(X)) \cap A \not\subseteq J_x$ for any $x \in X_A$;*
(ii) *if X_A is compact, then $A^\beta = A$; if A is σ-unital, then $A^\beta = A$ if and only if X_A is compact.*

Proof. To see (i), suppose first that $A^\beta = M(A)$, so that in particular, $A + H_x = M(A)$ for all $x \in X_A$. If there were some $x \in X_A$ with $\mu_A(C_0(X)) \cap A \subseteq J_x$, then by Lemma 2.1 (i) $\Rightarrow$ (iii) of [4], there would be some $R \in \mathrm{Prim}(M(A))$ with $R \supseteq A$ and $\phi_{M(A)}(R) = x$. Hence (using (1.4)) $R \supseteq H_x$, so that $R \supseteq A + H_x$. This would imply that $A + H_x \subsetneq M(A)$, which is a contradiction.

Conversely, suppose that $\mu_A(C_0(X)) \cap A \not\subseteq J_x$ for any $x \in X_A$. Then we have that the canonical embedding of A_x into $M(A)_x$ is surjective for every $x \in X_A$ by Proposition 2.2(iii) of [4]. But then for any $m \in M(A)$, there is $a \in A$ with $m(x) = a(x)$, and hence $m - a \in H_x$. It follows that $m \in A + H_x$, so $A + H_x = M(A)$. Since this is true for all $x \in X_A$, $A^\beta = M(A)$.

(ii) is shown in Theorem 3.3 of [7] (note that the proof of (ii) $\Rightarrow$ (i) therein does not require A to be σ-unital). ∎

EXAMPLE 3.3. Let B be a C^*-algebra and let (A, X, μ_A) be the trivial $C_0(X)$-algebra $A = C_0(X, B)$. Then by Corollary 3.4 of [1], $M(A) = C^b(X, M(B)_\beta)$, where $M(B)_\beta$ denotes $M(B)$ equipped with the strict topology.

For each $x \in X$, the $*$-homomorphism $\widetilde{\pi}_x : M(A) \to M(B)$ of (1.5) coincides with the evaluation map at x. Then by Lemma 1.13, H_x is the ideal

$$\{c \in M(A) : \|\widetilde{\pi}_y(c)\| \to 0 \text{ as } y \to x\}$$

of $M(A)$. Hence for $m \in M(A)$, m belongs to $A + H_x$ if and only if there is some $a \in A$ such that $m - a \in H_x$, i.e., if and only if there is some $b \in B$ such that

$$\|m(y) - b\| \to 0 \quad \text{as } y \to x.$$

It follows that A^β consists of the subalgebra $C^b(X,B)$ of $M(A)$.

Note that in this example, we have $X_A = X$, and hence the results of Proposition 3.2 are easily observed. Indeed, it is clear that $A = A^\beta$ if and only if $C_0(X,B) = C^b(X,B)$, which occurs if and only if X is compact.

Moreover, the structure map $\mu_A : C_0(X) \to ZM(A)$ in this case is given by pointwise multiplication by elements of $C_0(X)$. Hence if B is unital,

$$\mu_A(C_0(X)) \cap A = \{f \cdot 1_B : f \in C_0(X)\} \not\subseteq J_x$$

for any $x \in X$, while if B is non-unital, $\mu_A(C_0(X)) \cap A = \emptyset$.

It is clear the $A^\beta = M(A)$ if and only if A^β is unital, which occurs if and only if B is, hence if and only if $\mu_A(C_0(X)) \cap A \not\subseteq J_x$ for any $x \in X_A = X$.

Let (A, X, μ_A) be a $C_0(X)$-algebra and K a compactification of X_A. Then for any $f \in C(K)$, $f|_{X_A}$ belongs to $C^b(X_A)$. It follows that $f|_{X_A}$ extends to a continuous function $\overline{f|_{X_A}} \in C(\beta X_A)$. In particular, we get a unital, injective $*$-homomorphism $C(K) \to C(\beta X_A)$, and so we may identify $C(K)$ with a unital C^*-subalgebra of $C(\beta X_A)$.

In the following theorem we shall make use of the $C(\beta X_A)$-algebra $(A, \beta X_A, \nu_A)$ of Definition 1.7.

THEOREM 3.4. *Let (A, X, μ_A) be a $C_0(X)$-algebra and $(A, \beta X_A, \nu_A)$ be the $C(\beta X_A)$-algebra canonically defined by (A, X, μ_A). Then*

(i) *There is an injective $*$-homomorphism $\pi : ZM(A) \to ZM(A^\beta)$, and hence letting $\mu_{A^\beta} : C(\beta X_A) \to ZM(A^\beta)$ be the composition $\mu_{A^\beta} = \pi \circ \nu_A$, the triple $(A^\beta, \beta X_A, \mu_{A^\beta})$ is a $C(\beta X_A)$-algebra. Moreover, if K is any compactification of X_A, then letting $\mu^K_{A^\beta} = \mu_{A^\beta}|_{C(K)}$, $(A^\beta, K, \mu^K_{A^\beta})$ is a $C(K)$-algebra.*

(ii) *For any compactification K of X_A, $(A^\beta, K, \mu^K_{A^\beta})$ is a compactification of (A, X, μ_A).*

(iii) *If (A, X, μ_A) is a continuous $C_0(X)$-algebra, then $(A^\beta, \beta X_A, \mu_{A^\beta})$ is a continuous $C(\beta X_A)$-algebra.*

Proof. (i) Since A^β is an essential ideal of $M(A)$, there is an injective $*$-homomorphism $\iota : M(A) \to M(A^\beta)$ which is the identity on A^β. Setting $\pi = \iota|_{ZM(A)}$, then since A is essential in A^β, π maps $ZM(A)$ into $ZM(A^\beta)$. It follows that there is a unital $*$-homomorphism $\mu_{A^\beta} := \pi \circ \nu_A : C(\beta X_A) \to ZM(A^\beta)$, so that $(A^\beta, \beta X_A, \mu_{A^\beta})$ is a $C(\beta X_A)$-algebra.

If K is a compactification of X_A, then since $C(K)$ is a unital C^*-subalgebra of $C(\beta X_A)$, the $*$-homomorphism $\mu^K_{A^\beta} : C(K) \to ZM(A^\beta)$ is unital, and in particular, non-degenerate.

(ii) It is clear from the definition of $\mu^K_{A^\beta}$ that we have $\mu^K_{A^\beta}(f)a = \mu^K_A(f)a$ for all $f \in C(K)$ and $a \in A$, hence $A \to A^\beta$ is $C(K)$-linear and we have a morphism $(A, K, \mu^K_A) \to (A^\beta, K, \mu^K_{A^\beta})$. It is an injective morphism (i.e. the maps $A_x \to A^\beta_x$ are injective) by Proposition 2.6. Hence conditions (i) and (ii) of Definition 2.9 are satisfied.

To see condition (iii) of Definition 2.9, we first claim that the fibre algebras A_x^β of $(A^\beta, K, \mu^K_{A^\beta})$, where $x \in X_A$, do not depend on our choice of compactification K of X_A. For clarity, let us denote by $\psi_K : C(K) \to C(\beta X_A)$ the usual injective $*$-homomorphism, and by $\psi_K^* : \beta X_A \to K$ the dual continuous surjection. Then $\psi_K^*|_{X_A}$ is the identity map, and moreover, $\psi_K^*(\beta X_A \backslash X_A) = K \backslash X_A$ by Theorem 6.12 of [20].

Denote by $\phi^K_{A^\beta} : \mathrm{Prim}(A^\beta) \to K$ the base map of the $C(K)$-algebra $(A^\beta, K, \mu^K_{A^\beta})$. Then it is clear from the definition of $\mu^K_{A^\beta} : C(K) \to ZM(A^\beta)$ above that

$$\phi^K_{A^\beta} = \psi_K^* \circ \phi^{\beta X_A}_{A^\beta}.$$

In particular, this implies that for $P \in \mathrm{Prim}(A^\beta)$ and $x \in X_A$

$$\phi^K_{A^\beta}(P) = x \quad \text{if and only if } \phi^{\beta X}_{A^\beta}(P) = x.$$

Hence for all such x, we have

$$\mu^K_{A^\beta}(C_0(K \backslash \{x\})) \cdot A^\beta = \mu^{\beta X}_{A^\beta}(C_0(\beta X \backslash \{x\})) \cdot A^\beta.$$

Since the fibres corresponding to points of X_A of $(A^\beta, K, \mu^K_{A^\beta})$ and $(A^\beta, \beta X_A, \mu_{A^\beta})$ are quotients of A^β by these ideals, it follows that A_x^β does not depend on the choice of compactification K of X_A.

Now let $x \in X_A$ and note that for $c \in A^\beta$, $c(x) = 0$ (as a section of $(A^\beta, K, \mu^K_{A^\beta})$) whenever $c \in H_x$ by the definition of $\mu^K_{A^\beta}$. Now if $b \in A^\beta$, write $b = a + c$ for some $a \in A$ and $c \in H_x$, then it is clear that $a(x) = (a+c)(x) = b(x)$. In particular, the canonical map $A_x \to A_x^\beta$ is surjective for all $x \in X_A$.

(iii) We first show that norm functions of elements of A^β are continuous on X_A. Note that X_A is locally compact since (A, X, μ_A) is continuous.

Take $b \in A^\beta$ and let $f_b : X_A \to \mathbb{R}$ be the function $f_b(x) = \|b(x)\|$. For $x_0 \in X$ let K_{x_0} be a compact neighbourhood of x_0 in X_A and $g \in C_0(X_A)$ with $g|_{K_{x_0}} \equiv 1$. Then by Proposition 2.12(v), $\mu_{A^\beta}(g)b \in A$, and $x \mapsto \|(\mu_{A^\beta}(g)b)(x)\|$ is continuous on X_A. Since $x \mapsto \|(\mu_{A^\beta}(g)b)(x)\|$ agrees with f_b on K_{x_0}, it follows that f_b is continuous at x_0 for any x_0, and so f_b is continuous on X_A.

Denote by $\overline{f}_b$ the unique extension of f_b to a continuous function on βX_A, and by g_b the function $g_b(y) = \|b(y)\|$ on βX_A. Following the proof of Proposition 2.8 of [4], we will show that $\overline{f}_b = g_b$.

Let $y \in \beta X_A$ and let (x_α) be a net in X_A converging to y. Note that for each α the function $b \mapsto \|b(x_\alpha)\|$, where $b \in A^\beta$, is a C^*-seminorm on A^β. It follows that $b \mapsto \overline{f}_b(y) = \lim_\alpha f_b(x_\alpha)$ is a C^*-seminorm on A^β. Since the C^*-norm on A_y^β is unique, it suffices to prove that $\overline{f}_b(y) = 0$ if and only if $b(y) = 0$.

Suppose first that $b(y) = 0$ and $\varepsilon > 0$. Then since $x \mapsto \|b(x)\|$ is upper-semicontinuous on βX_A, there is an open neighbourhood U of y in βX_A such that

$\|b(x)\| < \varepsilon$ for all $x \in U$. It follows that there is α_0 with $f_b(x_\alpha) = \|b(x_\alpha)\| < \varepsilon$ whenever $\alpha \geqslant \alpha_0$. Since $\overline{f}_b$ is continuous and $x_\alpha \to y$, this implies that $\overline{f}_b(y) = 0$.

Now suppose that $\overline{f}_b(y) = 0$, and let $\varepsilon > 0$. Let $U = \{x \in \beta X_A : \overline{f}_b(x) < \varepsilon\}$ and take $g \in C(\beta X_A)$ with $0 \leqslant g \leqslant 1$, $g(y) = 1$ and $g|_{\beta X_A \setminus U} \equiv 0$. Then

$$\begin{aligned}\|b - \mu_{A^\beta}(1-g)\cdot b\| &= \|\mu_{A^\beta}(g)\cdot b\| = \sup_{x\in\beta X_A} \|(\mu_{A^\beta}(g)\cdot b)(x)\| \\ &= \sup_{x\in X_A} \|(\mu_{A^\beta}(g)\cdot b)(x)\| = \sup_{x\in U\cap X_A} |g(x)|\|b(x)\| \leqslant \varepsilon.\end{aligned}$$

Since ε was arbitrary, it follows that $b \in H_y$. Hence $b(y) = 0$ by the definition of A^β_y. ∎

EXAMPLE 3.5. If (A, X, μ_A) is as in Example 3.3 and K is a compactification of X then $(A^\beta, K, \mu^K_{A^\beta})$ is a compactification of (A, X, μ_A). Here $\mu^K_{A^\beta} : C(K) \to ZM(A^\beta)$ is given by pointwise multiplication.

It is easy to see that $(A^\beta, K, \mu^K_{A^\beta})$ need not be a continuous $C(K)$-algebra in general. Indeed, consider $X = \mathbb{N}$, $A = c_0$ $(= C_0(\mathbb{N}, \mathbb{C}))$ and K the one-point compactification of $\mathbb{N}$. Then $A^\beta = \ell^\infty$ and for $b \in A^\beta$, the norm function $y \mapsto \|b(y)\|$ is continuous on K if and only if the sequence $(|b(n)|)_{n\in\mathbb{N}}$ converges. Hence $(A^\beta, K, \mu^K_{A^\beta})$ is not a continuous $C(K)$-algebra.

4. THE ALGEBRA OF BOUNDED CONTINUOUS SECTIONS

In this section, we examine the structure of the algebra of continuous sections of $(A^\beta, K, \mu^K_{A^\beta})$, and in particular, of $(A^\beta, \beta X_A, \mu_{A^\beta})$. We show that A^β has the following Stone–Čech-type property: every continuous bounded section in $\Gamma^b(A)$ has a unique extension to a continuous section in $\Gamma(A^\beta)$ (irrespective of our choice of compactification K of X_A over which A^β defines a $C(K)$-algebra). As a consequence, we show that every trivial C^*-bundle over a locally compact Hausdorff space X extends uniquely to a continuous C^*-bundle over βX (though this extension may fail to be trivial) with the Stone–Čech extension property above.

Let (A, X, μ_A) be a $C_0(X)$-algebra and $(A^\beta, K, \mu^K_{A^\beta})$ the compactification of (A, X, μ_A) obtained from Theorem 3.4. In order to clarify the compactification K of X_A under consideration, we shall return to the full notation of Definition 1.3, and denote the associated section algebras by $\Gamma_0((A^\beta, K, \mu^K_{A^\beta}))$, $\Gamma^b((A^\beta, K, \mu^K_{A^\beta}))$ and $\Gamma((A^\beta, K, \mu^K_{A^\beta}))$. We make the following observations about $(A^\beta, K, \mu^K_{A^\beta})$:

(i) Associated with $(A^\beta, K, \mu^K_{A^\beta})$ there is the subspace $K_{A^\beta} := \{y \in K : A^\beta_y \neq \{0\}\}$ of K defined analogously to X_A of (1.3). Condition (iii) of Definition 2.9 ensures that $X_A \subseteq K_{A^\beta}$, but we do not necessarily have $K_{A^\beta} = K$ (see Section 5).

(ii) There is a natural isomorphism $A^\beta \cong \Gamma_0(((A^\beta, K, \mu^K_{A^\beta})))$ given by Theorem 1.6. We do not yet know whether or not K_{A^β} is compact, and hence whether or not $\Gamma_0((A^\beta, K, \mu^K_{A^\beta})) = \Gamma^b((A^\beta, K, \mu^K_{A^\beta})) = \Gamma((A^\beta, K, \mu^K_{A^\beta}))$; the latter shall be shown in Theorem 4.2.

We shall continue to use the notation $\Gamma^b(A)$ for the algebra of bounded cross sections $X_A \to \coprod_{x \in X_A} A_x$ that are continuous with respect to (A, X, μ_A), since this is unambiguous (i.e. independent of the space containing X_A).

LEMMA 4.1. *Let (A, X, μ_A) be a $C_0(X)$-algebra, K a compactification of X_A, and consider the compactification $(A^\beta, K, \mu^K_{A^\beta})$ of (A, X, μ_A). Then for any $c \in \Gamma^b((A^\beta, K, \mu^K_{A^\beta}))$,*

(i) *the restriction $c|_{X_A}$ of c to X_A belongs to $\Gamma^b(A)$, and*

(ii) *the norm of c is given by $\|c\| = \sup_{x \in X_A} \|c(x)\|$.*

Proof. (i) Let $x \in X_A$ and $\varepsilon > 0$. Then since c is continuous with respect to $(A^\beta, K, \mu^K_{A^\beta})$, there is an element $b \in A^\beta$ and an open neighbourhood U_1 of x in K_{A^β} such that

$$\|c(y) - b(y)\| < \frac{\varepsilon}{2} \quad \text{for all } y \in U_1.$$

By Proposition 2.12(ii), $b|_{X_A}$ is continuous with respect to (A, X, μ_A), and so there is $a \in A$ and an open neighbourhood U_2 of x in X_A with

$$\|b(y) - a(y)\| < \frac{\varepsilon}{2} \quad \text{for all } y \in U_2.$$

Hence for all y in the open neighbourhood $U := (U_1 \cap X_A) \cap U_2$ of x in X_A we have

$$\|c(y) - a(y)\| \leqslant \|c(y) - b(y)\| + \|b(y) - a(y)\| < \varepsilon.$$

This shows that the section $c|_{X_A}$ is continuous on X_A with respect to (A, X, μ_A), i.e., $c|_{X_A} \in \Gamma^b(A)$.

(ii) Identify A^β with $\Gamma_0((A^\beta, K, \mu^K_{A^\beta}))$ and note that by Proposition 1.4(iii), $\Gamma_0((A^\beta, K, \mu^K_{A^\beta}))$ is an essential ideal of $\Gamma^b((A^\beta, K, \mu^K_{A^\beta}))$. Hence $c = 0$ if and only if $bc = cb = 0$ for all $b \in \Gamma_0((A^\beta, K, \mu^K_{A^\beta}))$. Since $(A^\beta, K, \mu^K_{A^\beta})$ is a compactification of A, we have $\|b\| = \sup_{x \in X_A} \|b(x)\|$ for any $b \in A^\beta$ by Proposition 2.12(i). Thus if $c(x) = 0$ for all $x \in X_A$, it follows that $(bc)(x) = (bc)(x) = 0$ for all $b \in \Gamma_0((A^\beta, K, \mu^K_{A^\beta}))$ and $x \in X_A$, and so $bc = cb = 0$ for all such b.

This shows that $c = 0$ if and only if $c(x) = 0$ for all $x \in X_A$, which is equivalent to the claim that $\|c\| = \sup_{x \in X_A} \|c(x)\|$ by 1.9.12(c) of [15]. ∎

THEOREM 4.2. *Let (A, X, μ_A) be a $C_0(X)$-algebra, K a compactification of X_A and let $(A^\beta, K, \mu^K_{A^\beta})$ be the compactification of (A, X, μ_A) obtained from Theorem* 3.4.

(i) *The map sending* $b \in A^\beta$ *to the cross-section* $X_A \to \coprod_{x \in X_A} A_x$, $x \mapsto b(x)$, *is a* $*$*-isomorphism of* A^β *onto* $\Gamma^b(A)$.

(ii) *The section algebras* $\Gamma_0((A^\beta, K, \mu^K_{A^\beta}))$, $\Gamma^b((A^\beta, K, \mu^K_{A^\beta}))$ *and* $\Gamma((A^\beta, K, \mu^K_{A^\beta}))$ *all agree.*

(iii) *Every bounded continuous section* $a \in \Gamma^b(A)$ *has a unique extension to a continuous section* $\overline{a} \in \Gamma((A^\beta, K, \mu^K_{A^\beta}))$.

Proof. (i) By Proposition 2.12(ii), this map is an injective $*$-homomorphism of A^β into $\Gamma^b(A)$, thus it remains to prove that it is surjective.

Suppose that $m \in \Gamma^b(A)$ and $x \in X_A$, then there is some $a \in A$ with $a(x) = m(x)$. Then since $a - m \in \Gamma^b(A)$ it follows that $y \mapsto \|(a-m)(y)\|$ is upper-semicontinuous on X_A by Proposition 1.4(i). As $(a-m)(x) = 0$, there is some open neighbourhood U of x in X_A with $\|(a-m)(y)\| < \varepsilon$ for all $y \in U$.

Denote by $\widetilde{\pi}_y : M(A) \to M(A_x)$ the $*$-homomorphism extending $\pi_y : A \to A_y$ of equation (1.5) for each $y \in X_A$. Then since $\|\widetilde{\pi}_y(b)\| \leqslant \|b(y)\|$ for all $y \in X_A$ and $b \in M(A)$, we have

$$\|\widetilde{\pi}_y(a-m)\| \leqslant \|(a-m)(y)\| < \varepsilon$$

for all $y \in U$. Hence $\|\widetilde{\pi}_y(a-m)\| \to 0$ as $y \to x$, so that $a - m \in H_x$ by Lemma 1.13.

This ensures that for any $m \in \Gamma^b(A)$ and $x \in X_A$, $m \in A + H_x$, so that $\Gamma^b(A) \subseteq A^\beta$.

(ii) It is clear from the definitions of these section algebras that the inclusions

$$\Gamma_0((A^\beta, K, \mu^K_{A^\beta})) \subseteq \Gamma^b((A^\beta, K, \mu^K_{A^\beta})) \subseteq \Gamma((A^\beta, K, \mu^K_{A^\beta}))$$

hold, we shall show the reverse inclusions. Let $c \in \Gamma^b((A^\beta, K, \mu^K_{A^\beta}))$, then by Lemma 4.1(i), $c|_{X_A}$ defines an element of $\Gamma^b(A)$. By part (i), it follows that there is some $b \in A^\beta$ (or equivalently, $b \in \Gamma_0((A^\beta, K, \mu^K_{A^\beta})))$ with $b(x) = c(x)$ for all $x \in X_A$. Hence $(b-c)(x) = 0$ on X_A, and so by Lemma 4.1(ii), $b = c$ when both are regarded as elements of $\Gamma^b((A^\beta, K, \mu^K_{A^\beta}))$. Since A^β is isomorphic to $\Gamma_0((A^\beta, K, \mu^K_{A^\beta}))$, this shows that $c \in \Gamma_0((A^\beta, K, \mu^K_{A^\beta}))$ and hence

$$\Gamma_0((A^\beta, K, \mu^K_{A^\beta})) = \Gamma^b((A^\beta, K, \mu^K_{A^\beta})).$$

To see that $\Gamma((A^\beta, K, \mu^K_{A^\beta})) \subseteq \Gamma^b((A^\beta, K, \mu^K_{A^\beta}))$, assume for a contradiction that there is some $c \in \Gamma((A^\beta, K, \mu^K_{A^\beta}))$ with $y \mapsto \|c(y)\|$ unbounded on K_{A^β}. Then since $y \mapsto \|c(y)\|$ is upper semicontinuous on K_{A^β} (Proposition 1.4(i)), K_{A^β} is not countably compact ([21], Theorem 1), and so K_{A^β} contains a countably infinite discrete subset $Y := \{y_n : n \in \mathbb{N}\}$ (i.e. without limit points in K_{A^β}). Let $\{U_n : n \in \mathbb{N}\}$ be a collection of pairwise disjoint open sets with $y_n \in U_n$ for each n, and let $b_n \in A^\beta$ with $\|b_n\| = \|b_n(y_n)\| = 1$. Since K_{A^β} is completely regular, there exist functions $f_n \in C^b(K_{A^\beta})$ with $0 < f_n \leqslant 1$, $f_n(y_n) = 1$ and $f_n|_{K_{A^\beta} \setminus U_n} \equiv 0$.

Define a cross section $b : K_{A^\beta} \to \coprod_{y \in K_{A^\beta}} A^\beta_y$ via

$$b(y) = \sum_{n=1}^{\infty} f_n(y) \cdot b_n(y),$$

which is well defined since each $y \in K_{A^\beta}$ belongs to the support of at most one of the functions f_n. Moreover, since $f_n \cdot b_n$ is continuous with respect to $(A^\beta, K, \mu^K_{A^\beta})$ for each n, it follows that b is continuous and hence $b \in \Gamma^b((A^\beta, K, \mu^K_{A^\beta}))$. However, the set

$$\{y \in K_{A^\beta} : \|b(y)\| \geqslant 1\}$$

contains the countably infinite discrete set Y, and thus cannot be compact, so that $y \mapsto \|b(y)\|$ does not vanish at infinity on K_{A^β}. This contradicts the fact that $\Gamma_0((A^\beta, K, \mu^K_{A^\beta})) = \Gamma^b((A^\beta, K, \mu^K_{A^\beta}))$.

(iii) By part (i), there is an element $\overline{a} \in A^\beta$ with $\overline{a}(x) = a(x)$ for all $x \in X_A$, and by part (ii), A^β is canonically isomorphic to $\Gamma((A^\beta, K, \mu^K_{A^\beta}))$. ∎

The conclusion of Theorem 4.2 might appear counter-intuitive at first, in that the extension property of $(A^\beta, K, \mu^K_{A^\beta})$ is true for *any* compactification K of X_A. Indeed, for a commutative C^*-algebra $C_0(X)$, we know that βX is the *unique* compactification of X having the property that every $f \in C^b(X)$ extends to a continuous function $\overline{f} \in C(\beta X)$. Nonetheless, given any compactification K of X, $C(\beta X)$ may be equipped with the structure of a $C(K)$-algebra. The unique extension $\overline{f} \in C(\beta X)$ of $f \in C^b(X)$ above then defines a continuous section $K \to \coprod_{y \in K} C(\beta X)_y$, as the following example describes.

EXAMPLE 4.3. Let (A, X, μ_A) be the $C_0(X)$-algebra defined by $C_0(X)$. Now, $A^\beta \cong C^b(X) \cong C(\beta X)$, and by Theorem 3.4(i), the $C(K)$-algebra $(A^\beta, K, \mu^K_{A^\beta})$ gives rise to a compactification of (A, X, μ_A) for any compactification K of X. Note that $\mu^K_{A^\beta} : C(K) \to ZM(A^\beta)$ is given by the natural (unital) embedding of $C(K)$ into $C(\beta X)$ in each case. Hence the corresponding base map $\phi^K_{A^\beta}$: $\mathrm{Prim}(A^\beta) \cong \beta X \to K$ is the Stone–Čech extension to βX of the homeomorphic embedding of X into K.

Since $(A^\beta, \beta X, \mu_{A^\beta})$ is a compactification of (A, X, μ_A), the fibre algebras are given by $A^\beta_x = A_x = \mathbb{C}$ for all $x \in X$. For $y \in K \backslash X$ we have $\mathrm{Prim}(A^\beta_y) \cong (\phi^K_{A^\beta})^{-1}(y)$, so that

$$A^\beta_y = C((\phi^K_{A^\beta})^{-1}(y))$$

for all such y.

Note that in the particular case where $K = \widehat{X} := X \cup \{\infty\}$ (the one-point compactification of X), we have

$$A^\beta_\infty = C(\beta X \backslash X).$$

REMARK 4.4. The fact that the fibre algebras of $(A^\beta, K, \mu^K_{A^\beta})$ at points of x are given by $\mathbb{C}$ may be deduced from the fact that $\phi^K_{A^\beta} : \beta X \to K$ maps $\beta X \backslash X$ to $K \backslash X$ ([20], Theorem 6.12). Indeed, it then follows that $(\phi^K_{A^\beta})^{-1}(\{x\}) = \{x\}$ for any $x \in X$, and so $\mathrm{Prim}(A^\beta_x) = \{x\}$ consists of a single point for all such x.

REMARK 4.5. The fact that βX is the unique compactification of X with the property that any $f \in C^b(X)$ extends to $\overline{f} \in C(\beta X)$ can still be recovered in the language of $C_0(X)$-algebra compactifications. Indeed, let the continuous $C_0(X)$-algebra (A, X, μ_A) be given by $A = C_0(X)$ in the usual way. Then since $A^\beta = C^b(X)$, every positive function $g \in C^b(X)_+$ occurs as the norm-function $x \mapsto \|g(x)\|$.

Suppose that K is a compactification of X such that $(A^\beta, K, \mu^K_{A^\beta})$ is a continuous $C(K)$-algebra, every such g extends uniquely to a continuous function $\overline{g} \in C(K)$, namely, the norm function of the continuous section $\overline{g} \in \Gamma((A^\beta, K, \mu^K_{A^\beta}))$ of Theorem 4.2. Since this is true for all $g \in C^b(X)_+$, it follows that $K = \beta X$.

In summary, $(A^\beta, \beta X, \mu_{A^\beta})$ is unique in the following sense: let $A = C_0(X)$ and (A, X, μ_A) the corresponding continuous $C_0(X)$-algebra. Let (B, K, μ_B) be a compactification of (A, X, μ_A) such that:

(i) (B, K, μ_B) is a continuous $C(K)$-algebra, and
(ii) every $b \in \Gamma^b((A, X, \mu_A))$ extends to $\overline{b} \in \Gamma((B, K, \mu_B))$.

Then there is an isomorphism $(B, K, \mu_B) \cong (A^\beta, \beta X, \mu_{A^\beta})$. We shall extend this to more general continuous $C_0(X)$-algebras in Section 6.

The Stone–Čech compactification of a completely regular space Y is functorial in the sense that if X is another completely regular space and $\phi : Y \to X$ a continuous map, ϕ has a unique extension to $\phi^\beta : \beta Y \to \beta X$. If in addition X and Y are locally compact and ϕ is a proper map, this is equivalent to the property that the dual $*$-homomorphism $C_0(X) \to C_0(Y)$ extends uniquely to a $*$-homomorphism $C(\beta X) \to C(\beta Y)$.

As a consequence of Theorem 4.2, we can now show that $(A^\beta, \beta X_A, \mu_{A^\beta})$ has the corresponding property with respect to morphisms of $C_0(X)$-algebras.

COROLLARY 4.6. *Let (A, X, μ_A) be a $C_0(X)$-algebra, (B, Y, μ_B) a $C_0(Y)$-algebra and $(\Psi, \psi) : (A, X, \mu_A) \to (B, Y, \mu_B)$ a morphism satisfying the following nondegeneracy condition:*

(4.1) *the domain of the dual map ψ^* of $\psi : C_0(X) \to C_0(Y)$ contains Y_B.*

Then there is a morphism

$$(\Psi^\beta, \psi^\beta) : (A^\beta, \beta X_A, \mu_{A^\beta}) \to (B^\beta, \beta Y_B, \mu_{B^\beta})$$

where Ψ^β extends Ψ and $\psi^\beta(f)\big|_{Y_B} = \psi(f)\big|_{Y_B}$ for all $f \in C_0(X)$.

Hence the assignment of $(A^\beta, \beta X_A, \mu_{A^\beta})$ to (A, X, μ_A) and (Ψ^β, ψ^β) to (Ψ, ψ) gives rise to a covariant functor from the category of C^*-bundles over locally compact *spaces, together with morphisms satisfying* (4.1)*, to the category of C^*-bundles over compact spaces with the same morphisms.*

Proof. As discussed in Remark 2.3, we have a $*$-homomorphism $\Psi^b : \Gamma^b(A) \to \Gamma^b(B)$ defined via

$$\Psi^b(c)(y) = \Psi_y(c(\psi^*(y)))$$

for all $c \in \Gamma^b(A)$ and $y \in Y_B$. Taking the composition with the isomorphisms $A^\beta \cong \Gamma^b(A)$ and $B^\beta \cong \Gamma^b(B)$ of Theorem 4.2 gives a $*$-homomorphism $\Psi^\beta : A^\beta \to B^\beta$ extending Ψ.

To construct ψ^β we first define a $*$-homomorphism $\psi^b : C^b(X_A) \to C^b(Y_B)$ via the composition

$$\psi^b(f)(y) = f(\psi^*(y)),$$

where $f \in C^b(X_A)$ and $y \in Y_B$ (note that the domain of ψ^* contains Y_B). Then ψ^b induces a $*$-homomorphism $\psi^\beta : C(\beta X_A) \to C(\beta Y_B)$, which has the property that $\psi^\beta(f)|_{Y_B} = \psi(f)|_{Y_B}$ for all $f \in C_0(X)$ by construction.

To see that (Ψ^β, ψ^β) is indeed a morphism, first note that for any $c \in A^\beta$, $f \in C(\beta X_A)$ and $y \in Y_B$, the definitions of Ψ^β and ψ^β ensure that

$$\begin{aligned}[\Psi^\beta(\mu_{A^\beta}(g)c)](y) &= \Psi_y[g(\psi^*(y))c(\psi^*(y))] = (\psi^\beta(g)(y))(\Psi^\beta(c)(y)) \\ &= [\mu_{B^\beta}(\psi^\beta(g))\Psi^\beta(c)](y).\end{aligned}$$

In other words,

$$[\Psi^\beta(\mu_{A^\beta}(g)c) - \mu_{B^\beta}(\psi^\beta(g))\Psi^\beta(c)](y) = 0$$

for $y \in Y_B$. Since $(B^\beta, \beta Y, \mu_{B^\beta})$ is a compactification of (B, Y, μ_B), Proposition 2.12(ii) shows that any $d \in (B, K, \mu_B)^\beta$ has $\|d\| = \sup_{y \in Y_B} \|d(y)\|$, so we must have

$$\Psi^\beta(\mu_{A^\beta}(g)c) = \mu_{B^\beta}(\psi^\beta(g))\Psi^\beta(c),$$

which shows that (Ψ^β, ψ^β) is indeed a morphism.

For the final assertion, we note first that the identity morphism $(\mathrm{Id}_A, \mathrm{Id}_{C_0(X)})$ satisfies

$$(\mathrm{Id}_{A^\beta}, \mathrm{Id}_{C_0(X)}^\beta) = (\mathrm{Id}_{A^\beta}, \mathrm{Id}_{C(\beta X_A)}),$$

which is the identity morphism on $(A^\beta, \beta X_A, \mu_{A^\beta})$. Moreover, given morphisms $(\Psi, \psi) : (A, X, \mu_A) \to (B, Y, \mu_B)$ and $(\Phi, \phi) : (B, Y, \mu_B) \to (C, Z, \mu_C)$ satisfying (4.1), it is clear that the composition $(\Phi, \phi) \circ (\Psi, \psi) = (\Phi \circ \Psi, \phi \circ \psi)$ also satisfies (4.1), and that $(\phi \circ \psi)^* = \psi^* \circ \phi^*$. This implies that $(\phi \circ \psi)^\beta(f) = (\phi^\beta \circ \psi^\beta)(f)$ for all $f \in C(\beta X_A)$.

For $z \in Z_C$, letting $y = \phi^*(z)$ and $x = \psi^*(\phi^*(z))$, a straightforward but tedious calculation shows that the $*$-homomorphisms $\Psi_y : A_x \to B_y$, $\Phi_z : B_y \to C_z$ and $(\Phi \circ \Psi)_z : A_x \to C_z$ satisfy

$$(\Phi \circ \Psi)_z = \Phi_z \circ \Psi_y,$$

so that for all $a \in A^\beta$,

$$[(\Phi \circ \Psi)^\beta(a)](z) = (\Phi_z \circ \Psi_y)(a(x)) = \Phi_z([\Psi^\beta(a)](y)) = [(\Phi^\beta \circ \Psi^\beta)(a)](z).$$

Hence the morphisms $((\Phi \circ \Psi)^\beta, (\phi \circ \psi)^\beta)$ and $(\Phi^\beta \circ \Psi^\beta, \phi^\beta \circ \psi^\beta)$ agree. ∎

Note that the commutative C^*-algebra $A = C_0(X)$ is a trivial $C_0(X)$-algebra with fibre $\mathbb{C}$, and that $A^\beta = C^b(X)$ is a trivial $C(\beta X)$-algebra with fibre $\mathbb{C}$. It is natural to ask whether or not the same is true for a trivial $C_0(X)$-algebra A of the form $A = C_0(X, B)$ for some C^*-algebra B. It was shown in [39] that it is not true in general that every $f \in C^b(X, B)$ extends to a continuous function $f : \beta X \to B$. In particular, we cannot expect $C^b(X, B)$ and $C(\beta X, B)$ to be isomorphic in general.

Consider now the usual identification of $f \in C^b(X, B)$ with the corresponding cross section $X \to \coprod_{x \in X} B$ of the trivial bundle over X with fibre B. Corollary 4.7 below shows that Theorems 3.4 and 4.2 give rise to an extension $\overline{f} : \beta X \to \coprod_{y \in \beta X} A^\beta_y$ of f to a continuous section of the $C(\beta X)$-algebra $(A^\beta, \beta X, \mu_{A^\beta})$.

Recall that a locally compact Hausdorff space X is said to be *pseudocompact* if every continuous function $f : X \to \mathbb{C}$ is necessarily bounded.

COROLLARY 4.7. *Let B be a C^*-algebra and let (A, X, μ_A) be the trivial $C_0(X)$-algebra defined by $A = C_0(X, B)$. Then*

(i) $\Gamma^b(A) = C^b(X, B)$, *hence*

(ii) *every $a \in C^b(X, B)$ (regarded as a cross-section of the trivial bundle over X with fibre B) extends uniquely to a continuous cross-section $\overline{a} : \beta X \to \coprod_{y \in \beta X} A^\beta_y$ of the bundle associated with the continuous $C(\beta X)$-algebra $(A^\beta, \beta X, \mu_{A^\beta})$.*

Moreover, the following are equivalent:

(iii) *A^β is canonically isomorphic to $C(\beta X, B)$,*

(iv) *either B is finite dimensional or X is pseudocompact.*

Proof. (i) Let $a \in C^b(X, B)$ and $x \in X$. If U is a neighbourhood of x in X with compact closure then there is $f \in C_0(X)$ with $f|_{\overline{U}} \equiv 1$, so that $f \cdot a \in A$. Moreover, for all $y \in U$ we have $\|(f \cdot a - a)(y)\| = 0$, hence $a \in \Gamma^b(A)$.

Conversely, let $c \in \Gamma^b(A), x \in X$ and $\varepsilon > 0$. Then there is some $a \in A$ and a neighbourhood U of x such that $\|c(y) - a(y)\| < \varepsilon/2$ for all $y \in U$. Hence

$$\|c(y) - c(x)\| \leqslant \|c(y) - a(y)\| + \|a(x) - c(x)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so that $y \mapsto c(y)$ is continuous at x.

(ii) By Theorem 3.4(iii), $(A^\beta, \beta X, \mu_{A^\beta})$ is a continuous $C(\beta X)$-algebra. The fact that every $a \in C^b(X, B)$ admits a continuous extension to $\overline{a} \in \Gamma^b((A^\beta, \beta X, \mu_{A^\beta}))$ follows from Theorem 4.2.

The equivalence of (iii) and (iv) follows from Corollary 2 of [39], which shows that the natural embedding $C(\beta X, B) \hookrightarrow C^b(X, B)$ is surjective if and only if either B is finite dimensional or X is pseudocompact. ∎

EXAMPLE 4.8. Let (A, X, μ_A) be the $C_0(\mathbb{N})$-algebra $C_0(\mathbb{N}, B) = c_0(B)$. Then $A^\beta = C^b(\mathbb{N}, B) = \ell^\infty(B)$, which defines a $C(\beta X)$-algebra with respect to the natural multiplication of sequences by functions in $C(\beta X)$. For $y \in \beta X \backslash X$, the fibre algebras A^β_y are given by ultrapowers of B.

REMARK 4.9. Let $A = C_0(X, B)$ be a trivial $C_0(X)$-algebra. Then Corollary 4.7 shows that $A^\beta = C^b(X, B)$ defines a trivial $C(\beta X)$-algebra if and only if either B is finite dimensional or X is pseudocompact. However, for general B, it is clear that A admits a trivial compactification over βX, namely $C(\beta X, B)$.

Consider the case of a locally trivial $C_0(X)$-algebra (A, X, μ_A) with constant fibre C. If $C = M_n$ for some n, then A is an n-homogeneous C^*-algebra and $\mathrm{Prim}(A)$ is homeomorphic to X. Moreover, in this case we have $A^\beta = M(A)$, and the following are shown to be equivalent in Proposition 2.9 of [35]:

(i) A is of finite type, i.e., there exists a finite open cover $\{U_i : 1 \leqslant i \leqslant m\}$ of X such that $\mu_A(C_0(U_i)) \cdot A \cong C_0(U_i, C)$ for all i;

(ii) $(A^\beta, \beta X, \mu_{A^\beta})$ is a locally trivial $C(\beta X)$-algebra;

(iii) there exists a locally trivial compactification (B, K, μ_B) of (A, X, μ_A) over some compactification K of X.

It would be interesting to know whether or not the equivalence of (i) and (iii) still holds in the case of a locally trivial $C_0(X)$-algebra (A, X, μ_A) with infinite dimensional fibre C. By considering the trivial case, Corollary 4.7 shows that we cannot expect property (ii) to be equivalent to (i) and (iii) in general.

5. THE FIBRE ALGEBRAS OF $(A^\beta, \beta X_A, \mu_{A^\beta})$

In this section, we consider the question of characterising, for a compactification $(A^\beta, K, \mu^K_{A^\beta})$ of a $C_0(X)$-algebra (A, X, μ_A), the set of points of K for which the fibre algebras A^β_y of $(A^\beta, K, \mu^K_{A^\beta})$ are nonzero. By analogy with the subset X_A of X defined in (1.3), we denote this space by K_{A^β}. In other words, we consider the question of whether or not K_{A^β} is a compactification of X_A, and in particular, whether or not $(\beta X_A)_{A^\beta}$ is the Stone–Čech compactification of X_A.

We show in Theorem 5.5 that this is the case for a large class of $C_0(X)$-algebras (A, X, μ_A) (including all σ-unital continuous $C_0(X)$-algebras). Moreover, we show in Theorem 5.9 that for all σ-unital $C_0(X)$-algebras, the set $K_{A^\beta} \backslash X_A$ is at least dense in $K \backslash X_A$.

In Example 5.10 however, we exhibit a separable $C_0(X)$-algebra (A, X, μ_A) and a point $y \in \beta X_A \backslash X_A$ for which $A^\beta_y = \{0\}$. The key observation here is the fact that there exist remote points in the Stone–Čech remainder of X_A.

A key technique in establishing the above uses a deep technical result of Archbold and Somerset ([5], Theorem 2.5), applied in a similar manner to Theorem 3.3 of [7]. We shall not need the full strength of this result here. Proposition 5.1 below establishes a significant consequence needed for our purposes.

PROPOSITION 5.1. *Let (A, X, μ_A) be a σ-unital $C_0(X)$-algebra and u a strictly positive element of A with $\|u\| = 1$. If $f \in C^b(X_A)$ with $0 < f(x) \leqslant 1$ for all $x \in X_A$, then there is an element $b \in A^\beta$ with $\|b(x)\| \geqslant 1$ whenever $f(x) \leqslant \|u(x)\|$.*

Proof. Let $b \in M(A)$ be the element constructed from u and f in Theorem 2.5 of [5]. Then since f is everywhere nonzero in X_A, we have $b \in A + H_x$ for all $x \in X_A$ by Theorem 2.5(ii) of [5], i.e., $b \in A^\beta$.

For each $x \in X_A$, let $g_{f(x)} : [0, 1] \to [0, 1]$ be the piecewise linear function with $g_{f(x)}(t) = 0$ for $0 \leqslant t \leqslant (1/2)f(x)$, $g_{f(x)}(t) = 1$ for $f(x) \leqslant t \leqslant 1$, and $g_{f(x)}$ linear on the interval $[(1/2)f(x), f(x)]$. Then it is shown in Theorem 2.5(i) of [5] that

$$\widetilde{\pi}_x(b) = g_{f(x)}(\widetilde{\pi}_x(u))$$

for all $x \in X_A$, where $\widetilde{\pi}_x : M(A) \to M(A_x)$ is the strictly continuous extension of the quotient homomorphism $\pi_x : A \to A_x$ to $M(A)$ of (1.5). Since u is positive, the norm of $\widetilde{\pi}_x(u)$ is equal to its spectral radius. Hence if $f(x) \leqslant \|u(x)\|$ $(= \|\widetilde{\pi}_x(u)\|)$, then $g_{f(x)}(\|\widetilde{\pi}_x(u)\|) = 1$, and so the norm of the restriction of $g_{f(x)}$ to the spectrum of $\widetilde{\pi}_x(u)$ is equal to 1. In particular, $\|\widetilde{\pi}_x(b)\| = 1$.

Finally, using the fact that $\ker(\widetilde{\pi}_x) \subseteq H_x$ for all $x \in X_A$, we see that

$$\|b(x)\| \geqslant \|\widetilde{\pi}_x(b)\| = 1$$

whenever $f(x) \leqslant \|u(x)\|$. ∎

Let (A, X, μ_A) be a σ-unital $C_0(X)$-algebra and fix a strictly positive element $u \in A$ with $\|u\| = 1$. Since norm functions $x \mapsto \|a(x)\|$ vanish at infinity on X for all $a \in A$, for each real number α with $0 < \alpha \leqslant 1$ the set

$$K(\alpha) := \{x \in X : \|u(x)\| \geqslant \alpha\} \tag{5.1}$$

is compact (note that $K(\alpha) \subseteq X_A$). Let $\{c_n\}$ be a decreasing sequence, with $0 < c_n \leqslant 1$ for all n and c_n converging to 0 as $n \to \infty$. Then since u is strictly positive, $u(x) \neq 0$ for all $x \in X_A$ and so

$$\bigcup_{n=1}^{\infty} K(c_n) = X_A.$$

For convenience we fix some notation for the remainder of this section;

$$K_n = K\Big(\frac{1}{n}\Big) = \Big\{x \in X : \|u(x)\| \geqslant \frac{1}{n}\Big\}, \tag{5.2}$$

so that $\bigcup_{n=1}^{\infty} K_n = X_A$ as before.

The following proposition is essentially an extension of the method of Theorem 3.3 in [7].

PROPOSITION 5.2. *Let (A, X, μ_A) be a σ-unital $C_0(X)$-algebra and u a strictly positive element of A with $\|u\| = 1$. Let K be a compactification of X_A, $(A^\beta, K, \mu^K_{A^\beta})$ the compactification of (A, X, μ_A) defined in Theorem 3.4(ii), and $y \in K \backslash X_A$.*

Suppose that there is a (relatively) closed subset $F \subseteq X_A$ and a strictly decreasing sequence $\{c_n\}$ with $0 < c_n \leqslant 1$ and $c_n \to 0$, such that:

(i) *$y \in \mathrm{cl}_K(F)$ and*

(ii) *for all $n \in \mathbb{N}$, $F \cap K(c_n) \subseteq \mathrm{Int}_F(F \cap K(c_{n+1}))$.*

Then there is $b \in A^\beta$ such that the section $b : K \to \coprod\limits_{y \in K} A^\beta_y$ satisfies

(a) *$\|b(x)\| = 1$ for all $x \in F$, and hence*

(b) *$\|b(y)\| = 1$.*

In particular, $A^\beta_y \neq \{0\}$.

Proof. For each $n \in \mathbb{N}$ let $F_n = F \cap K(c_n)$ (so that $F = \bigcup\limits_{n=1}^{\infty} F_n$) and let $f_n : F \to [0, 1]$ be a continuous function with $f_n|_{F_n} \equiv 1$ and $f_n|_{F \backslash \mathrm{Int}_F(F_{n+1})} \equiv 0$ (note that the assumption of (b) implies that the sets F_n and $F \backslash \mathrm{Int}(F_{n+1})$ are disjoint and closed. Let $f = \sum\limits_{n=1}^{\infty} 2^{-n}(c_n f_n)$, so that f is continuous on F, $0 \leqslant f \leqslant 1$ and $f(x) > 0$ for all $x \in F$.

We claim that $f(x) \leqslant \|u(x)\|$ for all $x \in F$. Indeed, any $x \in F$ belongs to $F_{k+1} \backslash F_k$ for some k, and hence $f_n(x) = 0$ for $1 \leqslant n \leqslant k - 1$. It follows that

$$f(x) = \sum_{n=k}^{\infty} 2^{-n}(c_n) f_n(x) \leqslant c_k \sum_{n=k}^{\infty} 2^{-n} f_n(x) \leqslant c_k 2^{-k+1} \leqslant c_k < \|u(x)\|,$$

the final inequality holding since $x \notin K(c_k)$ by assumption.

Now, since X_A is normal and F is closed, f has a continuous extension to $\overline{f} : X_A \to [0, 1]$. Suppose that $Z(\overline{f})$ is non-empty, then since $Z(\overline{f})$ and F are disjoint closed subsets of X_A, there is a continuous function $k : X_A \to [0, 1]$ such that $k|_{Z(\overline{f})} \equiv 1$ and $k|_F \equiv 0$. If $Z(\overline{f}) = \varnothing$, then set $k = 0$.

Finally, let $g = \min(\overline{f} + k, 1)$, so that $g : X_A \to [0, 1]$, g is continuous, $g(x) > 0$ for all $x \in X_A$ and $g|_F = f$.

Using Proposition 5.1 applied to g, we get $b \in A^\beta$ with $\|b(x)\| = 1$ for all $x \in F$. Since $y \in \mathrm{cl}_K(F)$, this implies that for all neighbourhoods W of y in K there is $x \in W \cap X_A$ with $\|b(x)\| = 1$. Moreover, as

$$\|b(y)\| = \inf_W \sup_{x \in W \cap X_A} \|b(x)\|,$$

(where W ranges over all neighbourhoods of y in K) by Proposition 2.12(iii), it follows that $\|b(y)\| = 1$. In particular, $A^\beta_y \neq \{0\}$. ∎

Proposition 5.2 will be our main technique for constructing points $y \in K\backslash X_A$ for which A_y^β is nonzero. In Theorem 5.5 we shall show that in certain cases (such as that of continuous $C_0(X)$-algebras), we may take $F = X_A$, so that $A_y^\beta \neq \{0\}$ for all such y. One of these cases arises when the base space X is the so-called *Glimm space* of A, a particular space constructed from $\mathrm{Prim}(A)$.

For a C^*-algebra A, define an equivalence relation $\approx$ on $\mathrm{Prim}(A)$ as follows: for $P, Q \in \mathrm{Prim}(A)$, $P \approx Q$ if and only if $f(P) = f(Q)$ for all $f \in C^b(\mathrm{Prim}(A))$. As a set, we define $\mathrm{Glimm}(A)$ as the quotient space $\mathrm{Prim}(A)/\approx$, and we denote by $\rho_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ the quotient map. For $f \in C^b(\mathrm{Prim}(A))$, define f^ρ on $\mathrm{Glimm}(A)$ via $f^\rho([P]) = f(p)$, where $[P]$ is the $\approx$ equivalence class of P in $\mathrm{Prim}(A)$ (note that f^ρ is well-defined by construction). We then equip $\mathrm{Glimm}(A)$ with the topology τ_{cr} induced by the functions $\{f^\rho : f \in C^b(\mathrm{Prim}(A))\}$. With this topology, $\mathrm{Glimm}(A)$ is a completely regular (Hausdorff) space (the *complete regularisation* of $\mathrm{Prim}(A)$). For more details of this construction, we refer the reader to [3], Chapter 3 of [20], [31] and Chapter 2 of [32].

It is clear that if $\mathrm{Glimm}(A)$ is locally compact, the continuous map ρ_A : $\mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ gives rise to a $C_0(\mathrm{Glimm}(A))$-algebra $(A, \mathrm{Glimm}(A), \mu_A)$. In general however, $\mathrm{Glimm}(A)$ may fail to be locally compact, e.g. [14]. Nonetheless, if ρ_A is regarded as a map $\mathrm{Glimm}(A) \to \beta\mathrm{Glimm}(A)$, we get a $C(\beta\mathrm{Glimm}(A))$-algebra $(A, \beta\mathrm{Glimm}(A), \mu_A)$.

The space $\mathrm{Glimm}(A)$ is in some ways more tractable as a base space over which to represent a given C^*-algebra A as a $C_0(X)$-algebra. Not every completely regular space Y arises as $\mathrm{Glimm}(A)$ for some C^*-algebra A, indeed, Lazar and Somerset have recently given a complete characterisation (for separable A) of those spaces Y that do [29]. By contrast, every completely regular σ-compact space Y arises as X_A for some $C_0(X)$-algebra (A, X, μ_A) ([5], Section 2).

LEMMA 5.3. *Let A be a σ-unital C^*-algebra such that* $\mathrm{Glimm}(A)$ *is locally compact. Let $u \in A$ be a strictly positive element of norm* 1. *Then there is an increasing sequence $\{n_j\}$ such that:*

(i) $\bigcup_{j=1}^{\infty} K_{n_j} = \mathrm{Glimm}(A)$, *and*

(iii) *for each j we have*

$$K_{n_j} \subset \mathrm{Int} K_{n_{j+1}}.$$

Proof. Note that since u is strictly positive it is evident that $\bigcup_{n=1}^{\infty} K_n = \mathrm{Glimm}(A)$. Moreover, since $\mathrm{Glimm}(A)$ is locally compact, each $x \in \mathrm{Glimm}(A)$ has an open neighbourhood U_x in $\mathrm{Glimm}(A)$ such that $\overline{U}_x$ is compact.

By Theorem 2.1 of [28], for each compact $K \subseteq \mathrm{Glimm}(A)$ there is some $\alpha > 0$ such that

$$K \subseteq \{y \in \mathrm{Glimm}(A) : \|u(y)\| \geqslant \alpha\}.$$

In particular, for each $x \in \mathrm{Glimm}(A)$ there is $m_x \in \mathbb{N}$ such that $\overline{U}_x \subseteq K_{m_x}$.

We define the sequence $\{n_j\}$ inductively. Let $n_1 = 1$. For $j \geqslant 1$, note that the collection $\{U_x : x \in K_{n_j+1}\}$ is an open cover of K_{n_j+1}, so by compactness there are $x_1, x_2, \ldots, x_r \in K_{n_j+1}$ such that

$$K_{n_j+1} \subseteq \bigcup_{i=1}^{r} U_{x_i}.$$

By the previous paragraph, there are $m_{x_i} \in \mathbb{N}$ with $\overline{U}_{x_i} \subseteq K_{m_{x_i}}$ for $1 \leqslant i \leqslant r$. Set $n_{j+1} = \max\{n_{x_i} : 1 \leqslant i \leqslant r\}$. Then we have the following and properties (i) and (ii) are then immediate:

$$K_{n_j+1} \subseteq \bigcup_{i=1}^{r} U_{x_i} \subseteq \bigcup_{i=1}^{r} \overline{U}_{x_i} \subseteq K_{n_{j+1}}. \quad \blacksquare$$

When X_A is not $\mathrm{Glimm}(A)$, the conclusion of Lemma 5.3 can fail, even when X_A is compact and A commutative, as Example 5.4 shows. The reason that this situation does not arise in the case where $X_A = \mathrm{Glimm}(A)$ is the result of Lazar [27], which shows that the usual topology on $\mathrm{Glimm}(A)$ is precisely the quotient topology induced by the canonical surjection $\mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ when A is σ-unital.

EXAMPLE 5.4. Let $A = c_0 = C_0(\mathbb{N})$, $X = \{0\} \cup \{1/n :\in \mathbb{N}\}$ with the subspace topology from $\mathbb{R}$. Let $\phi_A : \mathbb{N} \to X$ the continuous surjection defined by $\phi(1) = 0$ and $\phi(n) = 1/(n-1)$ for $n \geqslant 2$. Then we get a $C(X)$-algebra (A, X, μ_A) with base map ϕ_A such that $X_A = X$. To avoid ambiguity we will write elements $a \in A$ as sequences $\{a_n\}$.

Let $u \in A$ be the strictly positive element $u_n = 1/n$. Then $u(1/n) = u_{n+1} = 1/(n+1)$ for $n \in \mathbb{N}$, and $u(0) = u_1 = 1$. It follows that $K_1 = \{0\}$ and

$$K_n = \{0\} \cup \left\{ \frac{1}{m} : 1 \leqslant m \leqslant n-1 \right\}$$

for $n \geqslant 2$. Note that the point 0 is not an interior point of any set K_n.

THEOREM 5.5. *Let (A, X, μ_A) be a $C_0(X)$-algebra, and suppose that one of the following conditions hold:*

(i) *there is a strictly positive element $u \in A$ with $x \mapsto \|u(x)\|$ continuous on X_A (e.g. if (A, X, μ_A) is a σ-unital, continuous $C_0(X)$-algebra);*

(ii) *for all $x \in X_A$ we have*

$$\mu_A(C_0(X)) \cap A \not\subseteq J_x;$$

(iii) *$(A, X, \mu_A) = (A, \mathrm{Glimm}(A), \mu_A)$, where $\mathrm{Glimm}(A)$ is locally compact and A σ-unital.*

Then there is $b \in A^\beta$ with $\|b(x)\| = 1$ for all $x \in X_A$. Hence if K is any compactification of X_A, the fibre algebras A^β_y of the compactification $(A^\beta, K, \mu^K_{A^\beta})$ of (A, X, μ_A) are nonzero for all $y \in K$.

Proof. Suppose that A satisfies (i), then we may assume with no loss of generality that $\|u\| = 1$. For each $n \in \mathbb{N}$, let K_n be the subset of X_A defined by the norm function of u as in (5.2). Since $x \mapsto \|u(x)\|$ is continuous on X, for all $n \in \mathbb{N}$ the set

$$O_{n+1} := \Big\{ x \in X_A : \|u(x)\| > \frac{1}{n+1} \Big\}$$

is open, and by definition we have $K_n \subseteq O_{n+1} \subseteq K_{n+1}$. Moreover, since u is strictly positive, $X_A = \bigcup_{n=1}^{\infty} K_n$. Hence we may apply Proposition 5.2 with $F = X_A$ and $c_n = 1/n$ for all n.

If (ii) holds, then by Proposition 3.2(i), we have $A^\beta = M(A)$. In particular, this implies that $\mathrm{Prim}(A^\beta)$ is compact, and hence its continuous image K_{A^β} is again compact. Since X_A is dense in K, it follows that $K_{A^\beta} = K$, i.e. $A^\beta_y \neq \{0\}$ for all $y \in K$, the conclusion follows. Note that we may take $b = 1_{M(A)} \in A^\beta$ in this case.

In case (iii), again let u be a strictly positive element of A with $\|u\| = 1$. Let $\{n_j\}_{j \geqslant 1}$ be the sequence constructed in Lemma 5.3, so that again, $K_{n_j} \subseteq \mathrm{Int}_{X_A}(K_{n_{j+1}})$ for all $j \geqslant 1$ and $X_A = \bigcup_{j=1}^{\infty} K_{n_j}$. Deleting repetitions where necessary, we get a strictly decreasing sequence c_n, $0 < c_n \leqslant 1$, converging to 0 such that $K(c_n) \subseteq \mathrm{Int}_K(c_{n+1})$ for all n and $X_A = \bigcup_{n=1}^{\infty} K(c_n)$. Applying Proposition 5.2 with $F = X_A$ yields the required result. ∎

REMARK 5.6. Conditions (i), (ii) and (iii) of Theorem 5.5 are closely related, though no two of them are equivalent in general.

Indeed, for σ-unital A, (ii) implies (i). To see this, we first observe that if (A, X, μ_A) satisfies (ii), then $A^\beta = M(A)$ (Proposition 3.2(i)), each A_x is unital and $\mu_A(f)(x) = f(x) \cdot 1_{A_x}$ for all $f \in C_0(X)$ and $x \in X_A$ ([4], Proposition 2.2(ii)). Identifying A with $\Gamma_0(A)$ and A^β with $\Gamma^b(A)$, we see that $\mu_A(f) \in A$ if and only if $x \mapsto \|\mu_A(f)(x)\| = |f(x)|$ vanishes at infinity on X_A, which occurs if and only if $f \in C_0(X_A)$. Since (ii) holds, this shows that for any $x \in X_A$ we may choose $f \in C_0(X_A)$ with $f(x) \neq 0$, hence X_A is locally compact.

If A is moreover σ-unital, then X_A is σ-compact ([5], Section 2), and so there is a countable increasing approximate identity $\{f_n\}$ in $C_0(X_A)$. Then $\{\mu_A(f_n)\}$ is a countable approximate identity for A, and so

$$u := \sum_{n=1}^{\infty} 2^{-n} \mu_A(f_n) = \mu_A\Big(\sum_{n=1}^{\infty} 2^{-n} f_n \Big)$$

is a strictly positive element of A with $x \mapsto \|u(x)\|$ continuous on X_A (being the image under μ_A of an element of $C_0(X_A)$). This shows that (ii) implies (i) when A is σ-unital.

In the special case that $X_A = \mathrm{Glimm}(A)$, (i) implies (iii), since if u is strictly positive with $x \mapsto \|u(x)\|$ continuous on $\mathrm{Glimm}(A)$, then for each $x \in \mathrm{Glimm}(A)$ the set $\{y \in \mathrm{Glimm}(A) : \|u(y)\| \geqslant \|u(x)\|/2\}$ is a compact neighbourhood of x in $\mathrm{Glimm}(A)$.

Clearly neither (i) nor (iii) imply (ii); for example, the $C_0((0,1))$-algebra $C_0((0,1), K(H))$ (where H is a separable infinite dimensional Hilbert space) satisfies (i) and (iii) but not (ii).

Example 5.7 gives a $C(X)$-algebra (A, X, μ_A) (with X compact and X_A dense in X) which fails to satisfy any of the conditions (i), (ii) or (iii) of Theorem 5.5, yet for which $A_y^\beta \neq \{0\}$ for all $y \in X \backslash X_A$.

We remark also that Example 5.8 (which is a small modification of Example 5.7), shows that condition (i) of Theorem 5.5 is strictly weaker than continuity.

EXAMPLE 5.7. Let $A = C_0(\mathbb{N}, K(H))$ and identify $\mathrm{Prim}(A)$ with $\mathbb{N}$ in the usual way. Let $X = [0,1]$ and let $\phi_A : \mathbb{N} \to \mathbb{Q} \cap (0,1)$ be a bijection. Then (A, X, μ_A) defines a $C(X)$-algebra with base map ϕ_A. We shall show that the fibre algebras A_y^β of the compactification $(A^\beta, X, \mu^X_{A^\beta})$ of (A, X, μ_A) are nonzero for all $y \in X$.

In this case, $\mathrm{Glimm}(A) = \mathrm{Prim}(A)$ and so $X_A = \mathbb{Q} \cap (0,1)$ is not homeomorphic to $\mathrm{Glimm}(A)$, so that condition (iii) of Theorem 5.5 does not apply. Since $Z(A) = \{0\}$, we have

$$\mu_A(C_0(X)) \cap A \subseteq ZM(A) \cap A = Z(A) = \{0\} \subseteq J_x$$

for all $x \in X_A$, so that (ii) does not apply either.

If $a \in A$ is any nonzero element, then $x \mapsto \|a(x)\|$ is discontinuous on X_A. Indeed, suppose that $\|a(y)\| \neq 0$ for some $y \in X_A$ and that $x \mapsto \|a(x)\|$ were continuous at y. Then the set $\{x \in X_A : \|a(x)\| > (1/2)\|a(y)\|\}$ would be open in X_A and contained in the compact subset $\{x \in X_A : \|a(x)\| \geqslant (1/2)\|a(y)\|\}$. The latter would then be a compact subset of $\mathbb{Q}$ with non-empty interior, which is a contradiction. In particular, condition (i) cannot apply.

Nonetheless, let $y \in X \backslash X_A$, and let q_n be a sequence of distinct rationals in $(0,1)$ converging to y. Then $F := \{q_n : n \in \mathbb{N}\}$ is (relatively) closed and discrete in X_A. Since K_n is compact for each n, each $K_n \cap F$ is finite, and so we may apply Proposition 5.2 to F to conclude that $A_y^\beta \neq \{0\}$.

EXAMPLE 5.8. Let (A, X, μ_A) be as in Example 5.7 and let B be the C^*-subalgebra $A + \mu_A(C_0(0,1))$ of $M(A)$. Note that $M(B) = M(A)$ and that (B, X, μ_B) is a $C(X)$-algebra with $\mu_B = \mu_A$. Let $\{f_n\} \in C_0((0,1))$ be an increasing approximate identity for $C_0(0,1)$ with $\|f_n\| = 1$ for all n. Then $\mu_A(f_n)$ is an approximate identity for B and so $f := 2^{-n} f_n$ is a strictly positive element of B. Clearly $y \mapsto \|f(y)\|$ is continuous on $X = X_B$, while as in Example 5.7, no element a of the subalgebra A of B has $y \mapsto \|a(y)\|$ continuous on X.

In fact, the techniques used in Example 5.7 may be used to obtain a much stronger result, as Theorem 5.9 shows.

THEOREM 5.9. *Let (A, X, μ_A) be a σ-unital $C_0(X)$-algebra, K a compactification of X_A and $(A^\beta, K, \mu^K_{A^\beta})$ the compactification of (A, X, μ_A) given in Theorem* 3.4(ii)*. Denoting by $\{A^\beta_y : y \in K\}$ the fibres of $(A^\beta, K, \mu^K_{A^\beta})$, then the follwing set is dense in $K\backslash X_A$:*

$$\{y \in K\backslash X_A : A^\beta_y \neq \{0\}\}.$$

Proof. Let $w \in K\backslash X_A$ and let W be a compact neighbourhood of w in K. We shall show that there is a point $y \in W \cap (K\backslash X_A)$ such that $A^\beta_y \neq 0$. As $w \in \mathrm{cl}_K(X_A)$, $W \cap X_A \neq \emptyset$ and has the following properties:

(i) $W \cap X_A$ is closed, and hence σ-compact;

(ii) since $w \in \mathrm{Int}_K(W) \cap \mathrm{cl}_K(X_A)$, it follows that $w \in \mathrm{cl}_K(W \cap X_A)$; hence $W \cap X_A$ is not closed in K and in particular, is non-compact;

(iii) since a σ-compact, completely regular space is pseudocompact if and only if it is compact ([20], 8.2 and 8A), we conclude from (i) and (ii) that $W \cap X_A$ is not pseudocompact.

By Corollary 1.21 of [20], $W \cap X_A$ contains a countably infinite, discrete subset $F = \{x_n : n \geqslant 1\}$ which is C-embedded in $W \cap X_A$. Suppose $x \in W \cap X_A$ lies in the closure of F in $W \cap X_A$, and let $f : F \to \mathbb{R}$ be given by $f(x_n) = n$. Since F is C-embedded in $W \cap X_A$, f has a continuous extension to $\overline{f} \in C(W \cap X_A)$. Thus if x were not in F, every neighbourhood of x in $W \cap X_A$ would contain a subset on which f were unbounded, contradicting the fact that $\overline{f}$ is continuous, real-valued and extends f. Hence $x \in F$ and so F is closed in $W \cap X_A$, and moreover, closed in X_A.

Since $F \subseteq W \cap X_A$ and F is non-compact, we have

$$\mathrm{cl}_K(F) = \mathrm{cl}_W(F) \supsetneq F,$$

and hence $\mathrm{cl}_W(F)\backslash F$ is nonempty and contained in $W\backslash X_A$. Let $y \in \mathrm{cl}_W(F)\backslash F$.

Note that for each $n \in \mathbb{N}$, $F \cap K_n$ must be finite (since F is closed and discrete and K_n is compact). In particular, for all n we have $F \cap K_n \subseteq \mathrm{Int}_F(F \cap K_{n+1})$. By Proposition 5.2, $A^\beta_y \neq \{0\}$. ∎

The proof of Theorem 5.9 shows that $A^\beta_y \neq \{0\}$ whenever $y \in K\backslash X_A$ lies in the closure in K of a relatively closed, discrete subset of X_A (countability is in fact ensured by σ-compactness of X_A). In general, however, there do exist points of $K\backslash X_A$ that do not lie in the closure of any subset of this form. In the context of Stone–Čech remainders such points are called *far points*, and are a particular case of *remote points*.

There exist remote points of $\beta\mathbb{R}$ and $\beta\mathbb{Q}$ [38]. In the case of $\beta\mathbb{R}$, this was originally shown (assuming the continuum hypothesis) by Fine and Gillman in [19], and later (without this assumption) by van Douwen [38].

Example 5.10 exhibits a $C_0(X)$-algebra (A, X, μ_A) with $A^{\beta}_y = \{0\}$ for every remote point $y \in \beta X_A \backslash X_A$.

EXAMPLE 5.10. Let $A = C_0(\mathbb{N}, K(H))$ and identify $\mathrm{Prim}(A)$ with $\mathbb{N}$ in the usual way. Let $X = \beta\mathbb{Q}$ and let $\phi_A : \mathbb{N} \to \mathbb{Q}$ be a bijection. Then (A, X, μ_A) defines a $C(X)$-algebra with base map ϕ_A and $X_A = \mathbb{Q}$.

We claim that for any $b \in A^{\beta}$ and $\varepsilon > 0$, the set

$$\{q \in \mathbb{Q} : \|b(x)\| \geqslant \varepsilon\}$$

is discrete. Writing $a = (a_n)$ for an element of A (i.e. a sequence of elements of $K(H)$), the set

$$\{n \in \mathbb{N} : \|a_n\| \geqslant \varepsilon\}$$

is finite, and hence its image

$$K(a, \varepsilon) := \{q \in \mathbb{Q} : \|a(q)\| \geqslant \varepsilon\} \subset \mathbb{Q}$$

is again finite.

Now, if $b \in \Gamma^b(A)$ and $q \in \mathbb{Q}$, there is $a \in A$ and a neighbourhood U of q in $\mathbb{Q}$ such that $\|b(x) - a(x)\| < (1/2)\varepsilon$ for all $x \in U$. Moreover, since $K(a, \varepsilon)$ is finite, there is a neighbourhood $V \subseteq U$ of q with $\|a(x)\| < (1/2)\varepsilon$ for all $x \in V\backslash\{q\}$.

In particular, for all $x \in V\backslash\{q\}$, we have

$$\|b(x)\| = \|b(x) - a(x) + a(x)\| \leqslant \|b(x) - a(x)\| + \|a(x)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since by Theorem 4.2, A^{β} is isomorphic to $\Gamma^b(A)$, it follows that the following set is discrete for all $\varepsilon > 0$:

$$\{q \in \mathbb{Q} : \|b(q)\| \geqslant \varepsilon\}.$$

Now suppose that $y \in \beta\mathbb{Q}\backslash\mathbb{Q}$ is a remote point, and $b \in A^{\beta}$. By Proposition 2.12(iii), we have

$$\|b(y)\| = \inf_{W} \sup_{x \in W \cap \mathbb{Q}} \|b(x)\|,$$

as W ranges over all neighbourhoods of y in $\beta\mathbb{Q}$. Since y is remote, for all $\varepsilon > 0$ there is a neighbourhood W of y such that W is disjoint from the countable, discrete set $\{q \in \mathbb{Q} : \|b(x)\| \geqslant \varepsilon\}$. Hence $\sup_{x \in W \cap \mathbb{Q}} \|b(x)\| < \varepsilon$, and so $\|b(y)\| = 0$ for all $b \in A^{\beta}$.

Note that any set closed subset $F \subseteq X_A$ satisfying the hypothesis of Proposition 5.2 (for any choice of decreasing sequence $\{c_n\}$) must be discrete (F is necessarily countable since X_A is).

Indeed, for any such F we have $F_n := F \cap K(c_n)$ finite. Since $F_n \subseteq \mathrm{Int}_F(F_{n+1})$, it is clear that each element of F_n has a neighbourhood disjoint from $F\backslash F_{n+1}$. Moreover, if $x \in F_n\backslash F_{n-1}$, then since $\bigcup_{m=1}^{n} F_m$ is finite, x has a neighbourhood disjoint from $\left(\bigcup_{m=1}^{n} F_m\right)\backslash\{x\}$ also.

REMARK 5.11. If A and ϕ_A are as in Example 5.10, then regarding ϕ_A as a map into $\mathbb{R}$, $(A, \mathbb{R}, \mu_A)$ is a $C_0(\mathbb{R})$-algebra, and $\beta\mathbb{R}$ is a compactification of X_A. For each $b \in A^\beta$ and $\varepsilon > 0$, we have

$$\{x \in \mathbb{R} : \|b(x)\| \geqslant \varepsilon\} = \{q \in \mathbb{Q} : \|b(q)\| \geqslant \varepsilon\}$$

and hence this set is discrete. Since there exist remote points of $\beta\mathbb{R}$, the $C(\beta\mathbb{R})$-algebra $(A^\beta, \beta\mathbb{R}, \mu_{A^\beta})$ has fibres $A^\beta_y = \{0\}$ at these points.

6. MAXIMALITY AND UNIQUENESS OF $(A^\beta, \beta X_A, \mu_{A^\beta})$

For a completely regular space X, βX may be described as the unique compactification of X satisfying the following maximality condition: given any compactification K of X, the inclusion $\iota : X \to K$ has a unique extension to a continuous surjection $\bar{\iota} : \beta X \to K$ ([20], Theorem 6.12). Equivalently, given any such K, there is a unital, injective $*$-homomorphism $C(K) \to C(\beta X)$. In this section we study the problem of generalising the latter property to the $C(\beta X_A)$-algebra $(A^\beta, \beta X_A, \mu_{A^\beta})$ associated with a $C_0(X)$-algebra (A, X, μ_A).

Indeed, suppose that $(\Psi, \psi) : (A, X, \mu_A) \to (B, K, \mu_B)$ is a compactification of the $C_0(X)$-algebra (A, X, μ_A). Theorem 6.1 below shows that the $C(K)$-algebra $(A^\beta, K, \mu^K_{A^\beta})$ satisfies a similar maximality condition, namely, that there is an injective morphism $(B, K, \mu_B) \to (A^\beta, K, \mu^K_{A^\beta})$ which is the identity on (A, X, μ_A).

If in addition (B, K, μ_B) is a continuous $C(K)$-algebra (note that a necessary condition for this to occur is that (A, X, μ_A) be a continuous $C_0(X)$-algebra), then we get an injective morphism $(B, K, \mu_B) \to (A^\beta, \beta X_A, \mu_{A^\beta})$. This can fail without continuity of (B, K, μ_B), as we shall see in Example 6.2.

THEOREM 6.1. *Let (B, K, μ_B) be a compactification of (A, X, μ_A).*

(i) *There is an injective $*$-homomorphism $B \to A^\beta$ extending the identity on A.*

(ii) *Let $(A^\beta, K, \mu^K_{A^\beta})$ denote the compactification of (A, X, μ_A) given in Theorem* 3.4(ii). *Then $B \to A^\beta$ is $C(K)$-linear and hence gives rise to an injective morphism $(B, K, \mu_B) \to (A^\beta, K, \mu^K_{A^\beta})$.*

(iii) *The pair of $*$-homomorphisms $(B \to A^\beta, C(K) \to C(\beta X_A))$ define a morphism $(B, K, \mu_B) \to (A^\beta, \beta X_A, \mu_{A^\beta})$. If in addition (B, K, μ_B) is a continuous $C(K)$-algebra, then $(B \to A^\beta, C(K) \to C(\beta X_A))$ is an injective morphism.*

Proof. (i) Since A is an essential ideal of B we may regard B as a C^*-subalgebra of $M(A)$ containing A. Hence it suffices to prove that $B \subseteq A^\beta$. But then since (B, K, μ_B) is a compactification of (A, X, μ_A), we may identify B with a C^*-subalgebra of cross-sections in $\Gamma^b(A)$ by Proposition 2.12(ii). But then A^β is isomorphic to $\Gamma^b(A)$ under the same map by Theorem 4.2(i), so that $B \subseteq A^\beta$ as required.

(ii) Since $(A^\beta, K, \mu^K_{A^\beta})$ and (B, K, μ_B) are both compactifications of (A, X, μ_A), the embeddings $A \to A^\beta$ and $A \to B$ are both $C(K)$-linear, i.e. for all $f \in C(K)$ and $a \in A$ we have $\mu^K_A(f)a = \mu^K_{A^\beta}(f)a$ and $\mu^K_A(f)a = \mu_B(f)a$, hence $\mu_B(f)a = \mu^K_{A^\beta}(f)a$. As A is an essential ideal of B, it follows that $\mu_B(f)b = \mu^K_{A^\beta}(f)b$ for all $f \in C(K)$ and $b \in B$, so that the embedding of B into A^β is also $C(K)$-linear.

Finally,

$$(B \hookrightarrow A^\beta, \mathrm{id}_{C(K)}) : (B, K, \mu_B) \to (A^\beta, K, \mu^K_{A^\beta})$$

is injective by Proposition 2.6.

(iii) Since by part (ii), $(B \to A^\beta, \mathrm{id}_{C(K)})$ is a morphism it is clear that $(B \to A^\beta, C(K) \to C(\beta X_A))$ is again a morphism.

Suppose now that (B, K, μ_B) is a continuous $C(K)$-algebra. For clarity, we shall denote the given $*$-homomorphisms by $\Psi : B \to A^\beta$ and $\psi : C(K) \to C(\beta X_A)$. Then the dual of ψ is $\psi^* : \beta X_A \to K$, the canonical continuous map extending the identity on X_A.

Since $(A^\beta, \beta X_A, \mu_{A^\beta})$ and (B, K, μ_B) are both compactifications of A, the induced $*$-homomorphisms $\Psi_x : B_x \to A^\beta_x$ are in fact $*$-isomorphisms for all $x \in X_A$. Thus it remains to show that the $*$-homomorphisms $\Psi_y : B_{\psi^*(y)} \to A^\beta_y$ are injective for all $y \in \beta X_A \backslash X$.

Indeed, given such an y, let (x_α) be a net in X_A converging to y in βX_A, so that, regarding X_A as a subspace of K, the same net (x_α) converges to $\psi^*(y)$. Suppose for a contradiction that Ψ_y were not injective. Then there would be some $b \in B$ with $\|b(\psi^*(y))\| = 1$ for which $\Psi_y(b(\psi^*(y))) = (\Psi(b))(y) = 0$. Since (B, K, μ_B) is a continuous $C(K)$-algebra, $t \mapsto \|b(t)\|$ is lower-semicontinuous on K, so that for any $\varepsilon > 0$ there is an index α_0 with $\|b(x_\alpha)\| > 1 - \varepsilon$ whenever $\alpha \geqslant \alpha_0$.

On the other hand, since $y \mapsto \|\Psi(b)(y)\|$ is upper-semicontinuous on βX_A, there is some index α_1 with $\|\Psi(b)(x_\alpha)\| < \varepsilon$ for all $\alpha \geqslant \alpha_1$. But then $\|b(x)\| = \|\Psi(b)(x)\|$ for all $x \in X_A$. Choosing $\alpha \geqslant \max\{\alpha_0, \alpha_1\}$ yields a contradiction. ∎

The assumption in Theorem 6.1(ii) that (B, K, μ_B) is a continuous $C(K)$-algebra cannot be dropped in general, as is easily seen from the commutative case.

EXAMPLE 6.2. Let (A, X, μ_A) be the $C_0(\mathbb{N})$-algebra defined by $C_0(\mathbb{N})$, and let $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$. Let $B = C^b(\mathbb{N})$, and equip B with the structure of a $C(\widehat{\mathbb{N}})$-algebra with respect to the obvious action of $C(\widehat{\mathbb{N}})$ on $C^b(\mathbb{N})$ by multiplication (equivalently, define the base map $\mathrm{Prim}(C^b(\mathbb{N})) \cong \beta\mathbb{N} \to \widehat{\mathbb{N}}$ to be the canonical surjection). Then $(B, \widehat{\mathbb{N}}, \mu_B)$ is a discontinuous $C(\widehat{\mathbb{N}})$-algebra, and is clearly a compactification of (A, X, μ_A).

In this case $(A^\beta, \beta X, \mu_{A^\beta})$ is canonically isomorphic to $C(\beta\mathbb{N})$. The fibre algebras of B are $B_n \cong \mathbb{C}$ for $n \in \mathbb{N}$ and $B_\infty \cong (\ell^\infty / c_0)$, while the fibre algebras of $(A^\beta, \beta X, \mu_{A^\beta})$ are isomorphic to $\mathbb{C}$ for all $y \in \beta\mathbb{N}$. Hence B_∞ does not embed into

A^{β}_y for any $y \in \beta\mathbb{N}\backslash\mathbb{N}$. In this case the homomorphisms $\Psi_y : B_\infty \to A^{\beta}_y, y \in \beta\mathbb{N}\backslash\mathbb{N}$, coincide with the point-evaluations.

If $C = C(\widehat{\mathbb{N}})$ is regarded as a bundle compactification of (A, X, μ_A), then C is a continuous $C(\widehat{\mathbb{N}})$-algebra and is identified with the subalgebra of $C(\beta\mathbb{N})$ consisting of those functions that are constant on $\beta\mathbb{N}\backslash\mathbb{N}$. In this case the fibre maps send $C_\infty \cong \mathbb{C} \to A^{\beta}_y \cong \mathbb{C}$ canonically for $y \in \beta\mathbb{N}\backslash\mathbb{N}$.

LEMMA 6.3. *Let (A, X, μ_A) be a σ-unital, continuous $C_0(X)$-algebra and let $f \in C^b(X_A)$ be real-valued and non-negative. Then there is some $c \in A^{\beta}$ with $\|c(x)\| = f(x)$ for all $x \in X_A$.*

Proof. Let b be the element of A^{β} with $\|b(x)\| = 1$ for all $x \in X_A$ obtained from Theorem 5.9. Then f extends to $\overline{f} \in C(\beta X_A)$. Setting $c = \mu_{A^{\beta}}(\overline{f})b$, we have $c \in A^{\beta}$ and $\|c(x)\| = \|f(x)b(x)\| = |f(x)|\|b(x)\| = f(x)$, for all $x \in X_A$. ∎

Theorem 6.4 considers the question of whether $(A^{\beta}, \beta X_A, \mu_{A^{\beta}})$ is unique amongst compactifications (B, K, μ_B) of A having the property that $\Gamma(B) \cong \Gamma^b(A)$. Note that by Theorem 4.2, we cannot expect this in general (as X_A often admits more than one compactification).

If in addition (A, X, μ_A) is continuous, then it turns out that $(A^{\beta}, \beta X_A, \mu_{A^{\beta}})$ is the unique continuous compactification of (A, X, μ_A) with the required extension property.

THEOREM 6.4. *Let (A, X, μ_A) be a $C_0(X)$-algebra and (B, K, μ_B) a compactification of (A, X, μ_A). Suppose that (B, K, μ_B) has the property that for every $a \in \Gamma^b(A)$ there is $b \in \Gamma(B)$ such that $b|_{X_A} = a$. Then*

(i) *the injective morphism $(B, K, \mu_B) \to (A^{\beta}, K, \mu^K_{A^{\beta}})$ of Theorem* 6.1 *is an isomorphism.*

If in addition, (A, X, μ_A) is continuous and σ-unital, then

(ii) *(B, K, μ_B) is continuous if and only if K is canonically homeomorphic to βX_A. Moreover, when this occurs, the injective morphism $(B, K, \mu_B) \to (A^{\beta}, \beta X_A, \mu_{A^{\beta}})$ is an isomorphism.*

Proof. (i) By Proposition 2.6, it suffices to show that the injective $*$-homomorphism $B \to A^{\beta}$ is surjective. If any $a \in \Gamma^b(A)$ extends to $b \in \Gamma(B)$, it then follows that the injective $*$-homomorphism $B \to \Gamma^b(A)$ sending $b \mapsto b|_{X_A}$ of Proposition 2.12(ii) is a $*$-isomorphism. In the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & A^{\beta} \\ \downarrow & & \downarrow \\ \Gamma(B) & \xrightarrow[b \mapsto b|_{X_A}]{} & \Gamma^b(A), \end{array}$$

the vertical arrows are isomorphisms by Theorems 1.6 and 4.2. Since $b \mapsto b|_X$ is a $*$-isomorphism, it must be the case that $B \to A^\beta$ is also an isomorphism.

(ii) Suppose first that (B, K, μ_B) is continuous.

If $f \in C^b(X_A)_+$ then by Lemma 6.3, there is $c \in \Gamma^b(A)$ with $f(x) = \|c(x)\|$ on X_A. By assumption, the extension property of (B, K, μ_B) would then imply that there is an element $b \in B$ with $b(x) = c(x)$ for all $x \in X_A$. Then since (B, K, μ_B) is a continuous $C(K)$-algebra, $y \mapsto \|b(y)\|$ is a continuous function on K extending f. Setting $\overline{f}(y) = \|b(y)\|$ for $y \in K$, it follows that every $f \in C^b(X_A)_+$ has an extension to a continuous function $\overline{f}$ in $C(K)$.

For a general real-valued function $g \in C^b(X_A)$, write $g = g_+ - g_-$ where g_+ and g_- are the positive and negative parts of g respectively. Then $\overline{g} = \overline{g}_+ - \overline{g}_-$ gives the required extension. Thus X_A is C^*-embedded in K and hence K is canonically homeomorphic to βX_A.

Conversely, suppose that K is canonically homeomorphic to βX_A. By Theorem 3.4(iii), (B, K, μ_B) is continuous.

The final assertion then follows from Proposition 2.6. ∎

Note that one choice of compactification of X_A is given by taking $K = \mathrm{cl}_{\beta X}(X_A)$.

COROLLARY 6.5. *Let (A, X, μ_A) be a σ-unital, continuous $C_0(X)$-algebra. If $K = \mathrm{cl}_{\beta X}(X_A)$ then $(A^\beta, K, \mu^K_{A^\beta})$ is a continuous $C(K)$-algebra if and only if X_A is C^*-embedded in X.*

Proof. If X_A is C^*-embedded in X then K is canonically homeomorphic to βX_A. It then follows that $(A^\beta, K, \mu^K_{A^\beta})$ is canonically isomorphic to $(A^\beta, \beta X_A, \mu_{A^\beta})$ and hence is continuous by Theorem 3.4(iii). Conversely, suppose that $(A^\beta, K, \mu^K_{A^\beta})$ is a continuous $C(K)$-algebra. Then by Theorem 6.4(ii), K is canonically homeomorphic to βX_A, and so X_A is C^*-embedded in βX and moreover, C^*-embedded in X. ∎

Since for any compact space K it is clear that $\beta K = K$, the operation of constructing βX from X is a "closure" operation in the sense that $\beta(\beta X) = \beta X$. It is natural to ask whether or not constructing $(A^\beta, K, \mu^K_{A^\beta})$ from (A, X, μ_A) also has this property. Indeed, applying Theorem 3.4 to $(A^\beta, K, \mu^K_{A^\beta})$ gives rise to a compactification $(A^{\beta\beta}, K, \mu^K_{A^{\beta\beta}})$ of $(A^\beta, K, \mu^K_{A^\beta})$, where A is $C(K)$-linearly embedded into $A^{\beta\beta}$ as an essential ideal, hence $(A^{\beta\beta}, K, \mu^K_{A^{\beta\beta}})$ is also a compactification of (A, X, μ_A).

Certainly we have $A^\beta = A$ whenever X_A is compact by Proposition 3.2. However, this alone is not sufficient to show that $A^{\beta\beta} = A^\beta$, since as we have seen in Section 5, the set of nonzero fibres of $(A^\beta, K, \mu^K_{A^\beta})$ need not be compact. Nonetheless, the maximality of $(A^\beta, K, \mu^K_{A^\beta})$ obtained in Theorem 6.4 ensures that we do indeed have $A^{\beta\beta} = A^{\beta}$, without any additional assumptions on (A, X, μ_A), as Corollary 6.6 shows.

COROLLARY 6.6. *Let (A, X, μ_A) be a $C_0(X)$-algebra and consider the compactification $(A^{\beta}, K, \mu^K_{A^{\beta}})$ of (A, X, μ_A), where K is any compactification of X_A. Then there is a natural isomorphism $(A^{\beta\beta}, K, \mu^K_{A^{\beta\beta}}) \to (A^{\beta}, K, \mu^K_{A^{\beta}})$.*

Proof. Since A^{β} is a compactification of A and $A^{\beta\beta}$ is a compactification of A^{β}, the natural embedding of A into $A^{\beta\beta}$ is $C(K)$-linear, A is an essential ideal in $A^{\beta\beta}$, and for $x \in X_A$ the fibre algebras satisfy $A^{\beta\beta}_x = A^{\beta}_x = A_x$. Hence the $C(K)$-linear embedding of A into $A^{\beta\beta}$ is a compactification of (A,X, μ_A), and so by Theorem 6.1(ii) there is an injective morphism $(A^{\beta\beta}, K, \mu^K_{A^{\beta\beta}}) \to (A^{\beta}, K, \mu^K_{A^{\beta}})$ which extends the identity on (A, K, μ^K_A).

Applying Theorem 4.2(iii) twice, we see that every $a \in \Gamma^b(A)$ has a unique extension to $b \in \Gamma(A^{\beta})$ which in turn extends uniquely to $c \in \Gamma(A^{\beta\beta})$. Hence by Theorem 6.4(i), the injective morphism $(A^{\beta\beta}, K, \mu^K_{A^{\beta\beta}}) \to (A^{\beta}, K, \mu^K_{A^{\beta}})$ is an isomorphism. ∎

REMARK 6.7. Note that Proposition 3.2 can not be used here to conclude that $(\beta X_A)_{A^{\beta}}$ is a compactification of X_A, since A^{β} will not in general be σ-unital. Indeed, let (A, X, μ_A) be the $C(X)$-algebra of Example 5.10. Then since $X_A \subseteq X_{A^{\beta}} \subsetneq X$ and X_A is dense in X, it follows that $X_{A^{\beta}}$ cannot be compact. Nonetheless, Corollary 6.6 shows that $A^{\beta\beta} = A^{\beta}$. In particular, the equivalence of Theorem 3.3 of [7] may fail when the $C_0(X)$-algebra under consideration is not σ-unital.

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