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ALGEBRAIC PAIRS OF PURE COMMUTING ISOMETRIES WITH FINITE MULTIPLICITY

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ABSTRACT. An algebraic isopair is a commuting pair of pure isometries that is annihilated by a polynomial. The notion of the rank of a pure algebraic isopair with finite bimultiplicity is introduced as an s-tuple $\alpha = (\alpha_1, \dots, \alpha_s)$ of natural numbers. A pure algebraic isopair of finite bimultiplicity with rank α, acting on a Hilbert space, is nearly $\max\{\alpha_1, \dots, \alpha_s\}$-cyclic and there is a finite codimensional invariant subspace such that the restriction to that subspace is $\max\{\alpha_1, \dots, \alpha_s\}$-cyclic.

KEYWORDS: *Commuting isometries, algebraic isopairs, cyclic operators, rational inner functions, distinguished varieties.*

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1. INTRODUCTION

Given a polynomial $p \in \mathbb{C}[z, w]$ (or in $\mathbb{C}[z]$) let $Z(p)$ denote its zero set. We say p is *square free* if q^2 does not divide p for every non-constant polynomial $q(z, w) \in \mathbb{C}[z, w]$. We say $q \in \mathbb{C}[z, w]$ is the *square free version* of p if q is the polynomial with smallest degree such that q divides p and $Z(p) = Z(q)$. The square free version is unique up to multiplication by a nonzero constant.

Let $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{E}$ denote the open unit disk, the boundary of the unit disk and complement of the closed unit disk in $\mathbb{C}$, respectively. In [2] the notion of an inner toral polynomial is introduced. (See also [5], [6], [9], [11].) A polynomial $q \in \mathbb{C}[z, w]$ is *inner toral* if

$$Z(q) \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2.$$

In other words, if $(z, w) \in Z(q)$ then either $|z|, |w| < 1$ or $|z| = 1 = |w|$ or $|z|, |w| > 1$. A *distinguished variety* in $\mathbb{C}^2$ is the zero set of an inner toral polynomial.

Let V be an isometry defined on a Hilbert space H. By the Wold decomposition, there exist two reducing subspaces for V, say K and L, such that $H =$

$K \oplus L$ and $S = V|_K$ is a shift operator and $U = V|_L$ is a unitary operator. We say V is *pure*, if there is no unitary part. An isometry V is pure if and only if $\bigcap_{j=1}^{\infty} V^j(H) = \{0\}$. A subspace $\mathcal{W}$ of H is called a wandering subspace for V if $V^n(\mathcal{W}) \perp V^m(\mathcal{W})$ for $n \neq m$ and $H = \bigoplus_{n=0}^{\infty} V^n(\mathcal{W})$. If V is a pure isometry and $\mathcal{W} = H \ominus V(H) = \ker(V^*)$, then $\ker(V^*)$ is a wandering subspace for V. Moreover, if V is a pure isometry then $V \cong M_z$ on the Hilbert–Hardy space $H^2_{\mathcal{W}}$ of $\mathcal{W}$-valued functions for a Hilbert space $\mathcal{W}$ with dimension $\dim(\ker(V^*))$. The *multiplicity* of a pure isometry V is defined as $\mathrm{mult}(V) = \dim(\ker(V^*))$.

A *pure isopair* is a pair of commuting pure isometries. A pure isopair $V = (S, T)$ is a *pure algebraic isopair* if there is a nonzero polynomial $q \in \mathbb{C}[z, w]$ such that $q(S, T) = 0$ and is also referred to as *pure q-isopair*. The study of pure algebraic isopairs was initiated in [2] and also discussed in [10]. Among the many results in [2] it is shown (see Theorem 1.20) if $V = (S, T)$ is a pure algebraic isopair, then there is a square free inner toral polynomial $\boldsymbol{p}$ such that $\boldsymbol{p}(S, T) = 0$ that is minimal in the sense if $q(S, T) = 0$, then $\boldsymbol{p}$ divides q. We call this polynomial $\boldsymbol{p}$ the *minimal polynomial* of V. The minimal polynomial of V is unique up to multiplication by a nonzero constant. Moreover, in [2] the notion of a *nearly cyclic* pure isopair is introduced. Here we fix a square free inner toral polynomial $\boldsymbol{p}$ and consider nearly multi-cyclic pure isopairs with the minimal polynomial $\boldsymbol{p}$.

An isopair $V = (S, T)$ acting on a Hilbert space H is called *at most nearly k-cyclic* if there exist distinct $f_1, \ldots, f_k \in H$ such that the closure of

$$\Big\{ \sum_{j=1}^{k} q_j(S, T) f_j : q_j \in \mathbb{C}[z, w] \text{ for } j = 1, 2, \ldots, k \Big\} \tag{1.1}$$

is of finite codimension in H. It is called *at least nearly k-cyclic* if the closure of

$$\Big\{ \sum_{j=1}^{l} q_j(S, T) f_j : q_j \in \mathbb{C}[z, w] \text{ for } j = 1, 2, \ldots, l \Big\}$$

is not of finite codimension in H for any $l < k$ and for any set of $f_1, \ldots, f_l \in H$. We say $V = (S, T)$ is *nearly k-cyclic* if it is both at most nearly k-cyclic and at least nearly k-cyclic. Moreover, $V = (S, T)$ is called *k-cyclic* if it is nearly k-cyclic and the span given in (1.1) is dense in H.

Given a pair of isometries $V = (S, T)$, define the *bimultiplicity* of V by

$$\mathrm{bimult}(V) = (\mathrm{mult}(S), \mathrm{mult}(T)).$$

It is a well known fact that we can view pure isopairs as pairs of multiplication operators. In particular, if $V = (S, T)$ is a pure $\boldsymbol{p}$-isopair of finite multiplicity (M, N), then there exists an $M \times M$ matrix-valued rational inner function Φ with its poles in $\mathbb{E}$, such that V is unitarily equivalent to (M_z, M_Φ) on $H^2_{\mathbb{C}^M}$ and

$\boldsymbol{p}(M_z, M_\Phi) = 0$ (see [2]). Moreover

$$\boldsymbol{p}(\lambda, \Phi(\lambda)) = 0 \quad \text{for } \lambda \in \overline{\mathbb{D}}. \tag{1.2}$$

DEFINITION 1.1. We say a point $(\lambda, \mu) \in \mathbb{C}^2$ is a *regular point for* $\boldsymbol{p}$ if $(\lambda, \mu) \in Z(\boldsymbol{p})$, but

$$\nabla \boldsymbol{p}(\lambda, \mu) = \Big(\frac{\partial \boldsymbol{p}}{\partial z}, \frac{\partial \boldsymbol{p}}{\partial w}\Big)|_{(\lambda,\mu)} \neq 0.$$

Let $\boldsymbol{p}$ be a square free inner toral polynomial. Write $\boldsymbol{p} = \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_s$ as a product of (distinct) irreducible factors. Then each $\boldsymbol{p}_j$ is inner toral. In other words, each $Z(\boldsymbol{p}_j)$ is a distinguished variety. The zero set of $\boldsymbol{p}$ is the union of the zero sets of $\boldsymbol{p}_j$. Let

$$\mathfrak{V}(\boldsymbol{p}_j) = Z(\boldsymbol{p}_j) \cap \mathbb{D}^2, \quad \mathfrak{V}(\boldsymbol{p}) = Z(\boldsymbol{p}) \cap \mathbb{D}^2 = \bigcup_{j=1}^{s} \mathfrak{V}(\boldsymbol{p}_j).$$

Let $\mathbb{N}$ denote the nonnegative integers and $\mathbb{N}_+$ denote the positive integers.

PROPOSITION 1.2. *Let $V = (S, T)$ be a pure $\boldsymbol{p}$-isopair of finite bimultiplicity with minimal polynomial $\boldsymbol{p}$ and suppose $\boldsymbol{p} = \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_s$, a product of distinct irreducible factors. For each j and $(\lambda, \mu) \in \mathfrak{V}(\boldsymbol{p}_j)$ that is a regular point for $\boldsymbol{p}$, the dimension of the intersection of* $\ker(S - \lambda)^*$ *and* $\ker(T - \mu)^*$ *is a nonzero constant.*

DEFINITION 1.3. Let $V = (S, T)$ be a pure $\boldsymbol{p}$-isopair of finite bimultiplicity with minimal polynomial $\boldsymbol{p}$ and suppose $\boldsymbol{p} = \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_s$, a product of distinct irreducible factors. The rank of V is a s-tuple, $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_+^s$, denoted by $\operatorname{rank}(V)$, where

$$\alpha_j = \dim(\ker(S - \lambda)^* \cap \ker(T - \mu)^*),$$

and $(\lambda, \mu) \in \mathfrak{V}(\boldsymbol{p}_j)$ and a regular point for $\boldsymbol{p}$.

THEOREM 1.4. *Suppose $V = (S, T)$ is a pure $\boldsymbol{p}$-isopair of finite bimultiplicity with minimal polynomial $\boldsymbol{p}$ and write $\boldsymbol{p} = \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_s$ as a product of distinct irreducible factors. If V has rank $(\alpha_1, \alpha_2, \ldots, \alpha_s)$, then V is nearly* $\max\{\alpha_1, \ldots, \alpha_s\}$*-cyclic.*

REMARK 1.5. Compare Theorem 1.4 with the results in [2].

We prove Theorem 1.4 in section 5. An important ingredient in the proof of Theorem 1.4 is a representation for a pure $\boldsymbol{p}$-isopair as a pair of multiplication operators on a reproducing kernel Hilbert space over $\mathfrak{V}(\boldsymbol{p})$ in the case $\boldsymbol{p}$ is irreducible. Representations of this type already appear in the literature, (Theorem D.14 of [7] for instance). Here we provide additional information. See Theorems 4.1 and 4.9.

REMARK 1.6. The concept of nearly multi-cyclic isopairs was introduced in [2]. A discussion on multicyclicity of a bundle shift given in terms of its multiplicities can be found in [1]. In [13], the article presents a way to realize a Riemann surface with a distinguished variety.

2. PRELIMINARIES

PROPOSITION 2.1. *Suppose $p, q \in \mathbb{C}[z,w]$.*

(i) *$Z(p) \cap Z(q)$ is a finite set if and only if p and q are relatively prime.*

(ii) *If p and q are relatively prime, then the ideal $I \subset \mathbb{C}[z,w]$ generated by p and q has finite codimension in $\mathbb{C}[z,w]$; i.e. there is a finite dimensional subspace $\mathcal{R}$ of $\mathbb{C}[z,w]$ such that for each $\psi \in \mathbb{C}[z,w]$ there exist polynomials $s, t \in \mathbb{C}[z,w]$ and $r \in \mathcal{R}$ such that*

$$\psi = sp + tq + r.$$

Proof. Bezout's theorem says that if two algebraic curves, say described by $p = 0$ and $q = 0$, do not have any common factors, then they have only finitely many points in common. In particular if p and q do not have any common factors, then $Z(p)$ and $Z(q)$ have only finitely many points in common. In particular, for the ideal I generated by p and q, the affine variety $V(I) = Z(p) \cap Z(q)$ is finite. The *Finiteness theorem* of [7], page 13, says that if $V(I)$ is finite then the quotient ring $\mathbb{C}[z,w]/I$ has a finite dimension. Hence the ideal I has finite codimension in $\mathbb{C}[z,w]$. ∎

For $p \in \mathbb{C}[z,w]$ and $\lambda \in \mathbb{D}$, let $p_\lambda(w) = p(\lambda, w)$.

LEMMA 2.2. *Suppose $\boldsymbol{p}$ is square free and inner toral and write $\boldsymbol{p} = \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_s$ as a product of irreducible factors. Let q be a nonzero polynomial.*

(i) *If q vanishes on a countably infinite subset of $\mathfrak{V}(\boldsymbol{p}_j)$, then $\boldsymbol{p}_j$ divides q.*

(ii) *If q vanishes on a cofinite subset of $\mathfrak{V}(\boldsymbol{p})$, then $\boldsymbol{p}$ divides q.*

(iii) *If $Z(q) \cap Z(\boldsymbol{p}) \cap \mathbb{D}^2$ is finite, then q and $\boldsymbol{p}$ are relatively prime.*

(iv) *The polynomial $\frac{\partial \boldsymbol{p}}{\partial w}$ has only finitely many zeros in $\mathfrak{V}(\boldsymbol{p})$.*

(v) *If $q\frac{\partial \boldsymbol{p}}{\partial w}$ is zero on a cofinite subset of $\mathfrak{V}(\boldsymbol{p})$, then $\boldsymbol{p}$ divides q.*

(vi) *If Λ is the set of all $\lambda \in \mathbb{D}$ for which $\boldsymbol{p}_\lambda(w)$ has distinct zeros, then $\Lambda \subset \mathbb{D}$ is cofinite.*

Proof. The proof of item (i) follows from Proposition 2.1 item (i) and by the fact that $\boldsymbol{p}_j$ is irreducible. By item (i), each $\boldsymbol{p}_j$ divides q. Since the $\boldsymbol{p}_j$'s are distinct, their product divides q, proving item (ii). If q and $\boldsymbol{p}$ have a common factor, then because $\boldsymbol{p}$ is inner toral, $Z(q)$ and $Z(\boldsymbol{p})$ have infinitely many common points in $\mathbb{D}^2$, proving (iii).

Let $q = \frac{\partial \boldsymbol{p}}{\partial w}$ and suppose q has infinitely many zeros in $\mathfrak{V}(\boldsymbol{p})$. In this case there is a j such that q has infinitely many zeros in $\mathfrak{V}(\boldsymbol{p}_j)$. Hence by (i), q vanishes on $\mathfrak{V}(\boldsymbol{p}_j)$. Therefore, either $\frac{\partial \boldsymbol{p}_j}{\partial w}$ has infinitely many zeros in $\mathfrak{V}(\boldsymbol{p}_j)$ or there is an ℓ such that $\boldsymbol{p}_\ell$ has infinitely many zeros in $\mathfrak{V}(\boldsymbol{p}_j)$ and thus, by part (i), $\boldsymbol{p}_j$ divides $\frac{\partial \boldsymbol{p}_j}{\partial w}$ or $\boldsymbol{p}_j$ divides $\boldsymbol{p}_\ell$, a contradiction. Item (v) follows from item (ii). To prove item (vi), if Λ is not cofinite, then $\frac{\partial \boldsymbol{p}}{\partial w}$ has infinitely many zeros in $Z(\boldsymbol{p})$. Since $\boldsymbol{p}$ is inner toral, $\frac{\partial \boldsymbol{p}}{\partial w}$ has infinitely many zeros in $\mathfrak{V}(\boldsymbol{p})$, a contradiction to item (iv) and hence Λ is cofinite. ∎

PROPOSITION 2.3. *Suppose $p \in \mathbb{C}[z,w]$ is a square free polynomial and write $p = p_1 p_2 \cdots p_s$ as a product of irreducible factors $p_j \in \mathbb{C}[z,w]$. If $q \in \mathbb{C}[z,w]$ and $Z(p) \subseteq Z(q)$, then there exist $\gamma = (\gamma_1, \dots, \gamma_s) \in \mathbb{N}_+^s$ and an $r \in \mathbb{C}[z,w]$ such that p_j and r are relatively prime and*

$$q = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_s^{\gamma_s} r.$$

The proof is an application of Bezout's theorem.

REMARK 2.4. If p and q are inner toral polynomials, then we may replace the condition $Z(p) \subseteq Z(q)$ with $\mathfrak{V}(p) \subseteq Z(q)$.

3. RESULTS FOR GENERAL $\boldsymbol{p}$

In this section $\boldsymbol{p} = \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_s$ is a general square free inner toral polynomial with (distinct) irreducible factors $\boldsymbol{p}_j$. Let (n_j, m_j) be the bidegree of $\boldsymbol{p}_j(z,w)$.

In [2] it is proven that any nearly cyclic pure $\boldsymbol{p}$-isopair is unitarily equivalent to a cyclic pure $\boldsymbol{p}$-isopair restricted to a finite codimensional invariant subspace (see Proposition 3.6 in [2]). Next proposition is a more generalized version of this result.

PROPOSITION 3.1. *Suppose $V = (S,T)$ is a pure $\boldsymbol{p}$-isopair of finite bimultiplicity (M,N) acting on the Hilbert space K. If H is a finite codimension V-invariant subspace of K and W is the restriction of V to H, then there exists a finite codimension subspace L of H such that V is unitarily equivalent to the restriction of W to L.*

REMARK 3.2. In case the codimension of H is one, the codimension of L (in H) can be chosen as $N-1$ (or as $M-1$). In general, the proof yields a relation between the codimensions of H in K and L in H (or in K).

COROLLARY 3.3. *Suppose $V = (S,T)$ is a pure $\boldsymbol{p}$-isopair of finite bimultiplicity (M,N) acting on the Hilbert space K. If there exists a finite codimension V-invariant subspace H of K such that the restriction of V to H is β-cyclic, then there exists a β-cyclic pure isopair W acting on a Hilbert space L and on a finite codimension W-invariant subspace F of L such that $W|_F$ is unitarily equivalent to V.*

Proof of Proposition 3.1. Following the argument in Proposition 3.6 of [2], let $F = K \ominus H$ and write, with respect to the decomposition $K = H \oplus F$,

$$V = (S,T) = \begin{pmatrix} W = (S,T)|_H & (X,Y) \\ 0 & (A,B) \end{pmatrix}. \tag{3.1}$$

In particular A (and likewise B) is a contraction on a finite dimensional Hilbert space. Because V is pure and A is a contraction, A has spectrum in the open disc $\mathbb{D}$. Choose a (finite) Blaschke u such that $u(A) = 0$. Note that $u(S)$ is an isometry on K and moreover the codimension of the range of $u(S)$ (equal to the dimension of the kernel of $u(S)^*$) in K is (at most) dM, where d is the degree (number of zeros) of u. Further, since

$$u(S) = \begin{pmatrix} u(S|_H) & X' \\ 0 & u(A) = 0 \end{pmatrix},$$

the range $L = u(S)K$ of $u(S)$ is a subspace of H of finite codimension. Since $u(S)V = Wu(S)$ it follows that L is invariant for W and V is unitarily equivalent to W restricted to L.

To prove the remark, note that if A is a scalar (equivalently H has codimension one in K), then u can be chosen to be a single Blaschke factor. In which case the codimension of L is N in K and hence $N-1$ in H. In general, if d is the degree of the Blaschke u, then the codimension of L in K is dN. By reversing the roles of S and T one can replace N with M, the multiplicity of the shift T. ∎

PROPOSITION 3.4. *Let (M_z, M_Φ) be a pure isopair of finite bimultiplicity (M,N) with minimal polynomial $\boldsymbol{p}$, where $\Phi(z)$ is an $M \times M$ matrix-valued rational inner function. There exists an $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_+^s$ such that:*

(i) *for $\lambda \in \mathbb{D}$, the characteristic polynomial $f_\lambda(w)$ of $\Phi(\lambda)$ satisfies*

$$f_\lambda(w) = \det(w - \Phi(\lambda)) = c(\lambda)\boldsymbol{p}_{1,\lambda}^{\alpha_1}(w) \cdots \boldsymbol{p}_{s,\lambda}^{\alpha_s}(w), \tag{3.2}$$

for a constant (in w) $c(\lambda)$;

(ii) *for each λ such that $\boldsymbol{p}_\lambda$ has m distinct zeros, $\Phi(\lambda)$ is diagonalizable and similar to*

$$\bigoplus_{j=1}^{s} \bigoplus_{\mu_j \in Z(\boldsymbol{p}_{j,\lambda})} \mu_j I_{\alpha_j};$$

(iii) *if $(\lambda, \mu) \in Z(\boldsymbol{p}_j)$ and $\frac{\partial \boldsymbol{p}}{\partial w}|_{(\lambda,\mu)} \neq 0$, then*

$$\dim\ker(\Phi(\lambda) - \mu) = \alpha_j.$$

Proof. First note that, by equation (1.2), for all $\lambda \in \overline{\mathbb{D}}$

$$\boldsymbol{p}_\lambda(\Phi(\lambda)) = \boldsymbol{p}(\lambda, \Phi(\lambda)) = 0. \tag{3.3}$$

In particular, the spectrum, $\sigma(\Phi(\lambda))$, is a subset of $Z(\boldsymbol{p}_\lambda)$.

Note that $\det(wI_m - \Phi(z))$ is a rational function whose denominator $d(z)$ (a polynomial in z alone) does not vanish in $\overline{\mathbb{D}}$. Let $q(z,w) = d(z)\det(wI_m - \Phi(z))$, the numerator of $\det(wI_m - \Phi(z))$. For fixed $z \in \mathbb{D}$, let

$$q_z(w) = d(z)\det(wI_m - \Phi(z)) = \sum_{j=0}^{M} q_j(z) w^j.$$

By Cayley–Hamilton theorem, $q_z(\Phi(z)) = \sum_{j=0}^{M} q_j(z)\Phi(z)^j = 0$ and therefore $q(z, \Phi(z)) = 0$ for all $z \in \mathbb{D}$. Now for $\gamma \in \mathbb{C}^M$ and $\lambda \in \mathbb{D}$,

$$q(M_z, M_\Phi(z))^* \gamma s_\lambda = \overline{q(\lambda, \Phi(\lambda))} \gamma s_\lambda = 0.$$

Therefore, $q(M_z, M_\Phi) = 0$. Since $\boldsymbol{p}$ is the minimal polynomial for (M_z, M_Φ), $\mathfrak{V}(\boldsymbol{p})$ is a subset of $Z(q)$. Hence there exist an $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_+^s$ and a polynomial r such that $\boldsymbol{p}_j$ does not divide r for each j and

$$d(z)\det(w - \Phi(z)) = q(z,w) = \boldsymbol{p}_1^{\alpha_1}(z,w) \cdots \boldsymbol{p}_s^{\alpha_s}(z,w)\, r(z,w). \tag{3.4}$$

For $(\lambda, \mu) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$, μ is in the spectrum of $\Phi(\lambda)$ if and only if $q(\lambda, \mu) = 0$. In particular, $q(z,w)$ is a polynomial whose zero set in $\mathbb{D} \times \mathbb{C}$ is the set $\{(z,w) : z \in \mathbb{D},\, w \in \sigma(\Phi(z))\} \subseteq \mathfrak{V}(\boldsymbol{p})$. Observe $Z(r) \cap [\mathbb{D} \times \mathbb{C}] \subseteq Z(q) \cap [\mathbb{D} \times \mathbb{C}] \subseteq \mathfrak{V}(\boldsymbol{p})$. On the other hand, r can have only finitely many zeros in $\mathfrak{V}(\boldsymbol{p})$ as otherwise r has infinitely many zeros on some $\mathfrak{V}(\boldsymbol{p}_j)$ and, by Lemma 2.2 item (i) $\boldsymbol{p}_j$ divides r. Hence $r(z,w)$ has only finitely many zeros in $\mathbb{H} = \mathbb{D} \times \mathbb{C}$. We conclude there are only finitely many $z \in \mathbb{D}$ such that $r_z(w) = r(z,w)$ has a zero and consequently r depends on z only so that $r(z,w) = r(z)$. Thus, for $\lambda \in \mathbb{D}$, the characteristic polynomial $f_\lambda(w)$ of $\Phi(\lambda)$ satisfies

$$f_\lambda(w) = \det(w - \Phi(\lambda)) = c(\lambda)\boldsymbol{p}_{1,\lambda}^{\alpha_1}(w) \cdots \boldsymbol{p}_{s,\lambda}^{\alpha_s}(w), \tag{3.5}$$

for a constant (in w) $c(\lambda)$.

Let Λ be the set of all $\lambda \in \mathbb{D}$ for which $\boldsymbol{p}_\lambda$ has $\sum_{j=1}^{s} m_j$ distinct zeros. By Lemma 2.2 item (vi), $\Lambda \subseteq \mathbb{D}$ is cofinite. For $\lambda \in \Lambda$, the polynomial $\boldsymbol{p}_\lambda$ has distinct zeros and by (3.3), $\boldsymbol{p}_\lambda(\Phi(\lambda)) = 0$. Hence, $\Phi(\lambda)$ is diagonalizable and, for given $\mu_j \in Z(\boldsymbol{p}_{j,\lambda})$, the dimension of the eigenspace of $\Phi(\lambda)$ at μ_j is α_j. Thus $\Phi(\lambda)$ is similar to

$$\bigoplus_{j=1}^{s} \bigoplus_{\mu_j \in Z(\boldsymbol{p}_{j,\lambda})} \mu_j I_{\alpha_j}.$$

Let $(\lambda, \mu) \in Z(\boldsymbol{p}_j)$ be such that $\frac{\partial p}{\partial w}|_{(\lambda,\mu)} \neq 0$. The minimal polynomial for $\Phi(\lambda)$ has a zero of multiplicity 1 at μ, since it divides $\boldsymbol{p}_\lambda$. Hence $\Phi(\lambda)$ is similar to $\mu I_{\alpha_j} \oplus J$ where the spectrum of J does not contain μ. Therefore, the kernel of $\Phi(\lambda) - \mu$ has dimension α_j. ∎

PROPOSITION 3.5. *Let $V = (S, T)$ be a pure $\boldsymbol{p}$-isopair of finite bimultiplicity and suppose $\boldsymbol{p} = \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_s$ a product of distinct irreducible factors. For each j and $(\lambda, \mu) \in \mathfrak{V}(\boldsymbol{p}_j)$ such that $\frac{\partial p}{\partial w}|_{(\lambda,\mu)} \neq 0$, the dimension of the intersection of* $\ker(S - \lambda)^*$ *and* $\ker(T - \mu)^*$ *is a nonzero constant.*

Proof. By the standard model theory for pure isopairs with finite bimultiplicity, there exists an $M \times M$ matrix-valued rational inner function Φ such that $V = (S, T)$ is unitarily equivalent to (M_z, M_Φ) on $H^2_{\mathbb{C}^M}$ and $\boldsymbol{p}(M_z, M_\Phi) = 0$. Let $(\lambda, \mu) \in \mathfrak{V}(\boldsymbol{p}_j)$ be a regular point for $\boldsymbol{p}$. Observe that for any $\gamma \in \ker(\Phi(\lambda) - \mu)^*$, both $(S - \lambda)^* s_\lambda \gamma = 0$ and $(T - \mu)^* s_\lambda \gamma = 0$. Hence $s_\lambda \gamma \in \ker(S - \lambda)^* \cap \ker(T - \mu)^*$. Now suppose $f \in \ker(S - \lambda)^* \cap \ker(T - \mu)^*$. Since $(S - \lambda)^* f = 0$, there is a vector $\gamma \in \mathbb{C}^N$ such that $f = s_\lambda \gamma$. Thus, $0 = (T - \mu)^* s_\lambda \gamma = s_\lambda (\Phi(\lambda)^* - \mu^*)\gamma$.

Hence

$$s_\lambda \ker(\Phi(\lambda)-\mu)^* = \ker(S-\lambda)^* \cap \ker(T-\mu)^*.$$

Since $\dim\ker(\Phi(\lambda)-\mu)^* = \dim\ker(\Phi(\lambda)-\mu)$, we have

$$\dim[\ker(S-\lambda)^* \cap \ker(T-\mu)^*] = \dim\ker(\Phi(\lambda)-\mu), \tag{3.6}$$

and hence by Proposition 3.4 item (iii), $\dim[\ker(S-\lambda)^* \cap \ker(T-\mu)^*] = \alpha_j$. ∎

COROLLARY 3.6. *Let $V = (S,T)$ be a pure $\boldsymbol{p}$-isopair of finite bimultiplicity and suppose $\boldsymbol{p} = \boldsymbol{p}_1\boldsymbol{p}_2\cdots\boldsymbol{p}_s$ a product of distinct irreducible factors. For each j and $(\lambda,\mu) \in \mathfrak{V}(\boldsymbol{p}_j)$ such that $\frac{\partial \boldsymbol{p}}{\partial z}|_{(\lambda,\mu)} \neq 0$, dimension of the intersection of* $\ker(S-\lambda)^*$ *and* $\ker(T-\mu)^*$ *is a nonzero constant.*

The proof is immediate from the symmetry of S and T and Proposition 3.5.

Proof of Proposition 1.2. Let $(\lambda,\mu) \in \mathfrak{V}(\boldsymbol{p}_j)$. If $\frac{\partial \boldsymbol{p}}{\partial w}|_{(\lambda,\mu)} \neq 0$, then by Proposition 3.5, there exists a non zero constant $\alpha_j \in \mathbb{N}^+$ such that

$$\dim(\ker(S-\lambda)^* \cap \ker(T-\mu)^*) = \alpha_j.$$

If $\frac{\partial \boldsymbol{p}}{\partial z}|_{(\lambda,\mu)} \neq 0$, then by Corollary 3.6, there exists a non zero constant $\beta_j \in \mathbb{N}^+$ such that

$$\dim(\ker(S-\lambda)^* \cap \ker(T-\mu)^*) = \beta_j.$$

Note that, since $\boldsymbol{p}$ is square free, so is $\boldsymbol{p}_j$ and hence there are infinitely many points in $\mathfrak{V}(\boldsymbol{p}_j)$ such that both partial derivatives $\frac{\partial \boldsymbol{p}}{\partial z}|_{(z_0,w_0)}$ and $\frac{\partial \boldsymbol{p}}{\partial w}|_{(z_0,w_0)}$ do not vanish. If (λ,μ) is a regular point for $\boldsymbol{p}$ such that $\frac{\partial \boldsymbol{p}}{\partial z}|_{(\lambda,\mu)} \neq 0$ and $\frac{\partial \boldsymbol{p}}{\partial w}|_{(\lambda,\mu)} \neq 0$, then $\alpha_j = \beta_j$. Therefore, if $(\lambda,\mu) \in \mathfrak{V}(\boldsymbol{p}_j)$ is a regular point for $\boldsymbol{p}$, then the dimension of the intersection of $\ker(S-\lambda)^*$ and $\ker(T-\mu)^*$ is a nonzero constant. ∎

COROLLARY 3.7. *If (S,T) is a pure $\boldsymbol{p}$-isopair of finite bimultiplicity (M,N) with rank $\alpha = (\alpha_1,\ldots,\alpha_s) \in \mathbb{N}^s_+$, then*

$$M = \sum_{j=1}^{s} m_j\alpha_j \quad \textit{and} \quad N = \sum_{j=1}^{s} n_j\alpha_j. \tag{3.7}$$

Proof. First, view (S,T) as (M_z, M_Φ) where $\Phi(z)$ is an $M\times M$ matrix-valued rational inner function. By Proposition 3.4 item (i), for $\lambda \in \mathbb{D}$,

$$\det(w-\Phi(\lambda)) = c(\lambda)\boldsymbol{p}_{1,\lambda}^{\alpha_1}(w)\ \cdots\ \boldsymbol{p}_{s,\lambda}^{\alpha_s}(w)$$

for a constant (in w) $c(\lambda)$. Comparing the degree in w on the left and the right, for all but finitely many λ, we have

$$M = \sum_{j=1}^{s} \alpha_j m_j.$$

To see the relation on N, view $\boldsymbol{p}$ as $\boldsymbol{p}(w,z)$ a polynomial of bidegree (m,n). Note that each factor $\boldsymbol{p}_j = \boldsymbol{p}_j(w,z)$ has bidegree (m_j,n_j). Moreover $\boldsymbol{p}(T,S) = 0$

and (T,S) has bimultiplicity (N,M). Model (T,S) as $(M_w, M_{\Psi(w)})$, where $\Psi(w)$ is an $N\times N$ matrix valued ration inner function. By Proposition 3.4, item (i), there exists $(\beta_1,\beta_2,\dots,\beta_s)\in\mathbb{N}_+^s$ such that for $\mu\in\mathbb{D}$,

$$\det(z-\Psi(\mu)) = c'(\mu)\boldsymbol{p}_{1,\mu}^{\beta_1}(z)\ \cdots\ \boldsymbol{p}_{s,\mu}^{\beta_s}(z) \tag{3.8}$$

for a constant (in z) $c'(\mu)$. By Proposition 3.4 item (iii), for $(\mu,\lambda)\in Z(\boldsymbol{p}_j)$ that is a regular point for $\boldsymbol{p}$,

$$\dim\ker(\Psi(\mu)-\lambda)=\beta_j.$$

Now by equation (3.6),

$$\dim[\ker(S-\lambda)^*\cap\ker(T-\mu)^*]=\beta_j.$$

Since (S,T) has rank α, we get $\beta_j=\alpha_j$ for $j=1,\dots,s$ and by comparing the degree in z on the left and the right of (3.8), for all but finitely many μ, we have

$$N=\sum_{j=1}^{s}\alpha_j n_j. \quad \blacksquare$$

PROPOSITION 3.8. *If $V=(S,T)$ is a finite bimultiplicity k-cyclic pure $\boldsymbol{p}$-isopair acting on the Hilbert space K, then for each $(\lambda,\mu)\in\mathfrak{V}(\boldsymbol{p})$,*

$$\dim(\ker(S-\lambda)^*\cap\ker(T-\mu)^*)\leqslant k.$$

In particular, if $\boldsymbol{p}$ is the minimal polynomial for V and if V has rank α, then $k\geqslant\max\{\alpha_1,\dots,\alpha_s\}$.

Proof. Let $\{f_1,\dots,f_k\}$ be a cyclic set for (S,T). For any $q(z,w)\in\mathbb{C}[z,w]$, $f\in\ker(S-\lambda)^*\cap\ker(T-\mu)^*$ and $1\leqslant j\leqslant k$,

$$\langle q(S,T)f_j,f\rangle=\langle f_j,q(S,T)^*f\rangle=\langle f_j,q(\lambda,\mu)^*f\rangle=q(\lambda,\mu)\langle f_j,f\rangle.$$

If $\dim(\ker(S-\lambda)^*\cap\ker(T-\mu)^*)>k$, then there exists a non zero vector $f\in\ker(S-\lambda)^*\cap\ker(T-\mu)^*$ perpendicular to f_j for all j. Thus $\langle q(S,T)f_j,f\rangle=0$ for all j and for any q, and hence $\langle g,f\rangle=0$ for any $g\in\Big\{\sum_{j=1}^{k}q_j(S,T)f_j : q_j\in\mathbb{C}[z,w]\Big\}$, a contradiction. Therefore, $\dim(\ker(S-\lambda)^*\cap\ker(T-\mu)^*)\leqslant k$. The last statement of the proposition follows from the definition of the rank. $\blacksquare$

PROPOSITION 3.9. *Suppose $V=(S,T)$ is a finite bimultiplicity pure $\boldsymbol{p}$-isopair with minimal polynomial $\boldsymbol{p}$ and with rank $\alpha=(\alpha_1,\dots,\alpha_s)\in\mathbb{N}_+^s$ acting on a Hilbert space K. If H is a finite codimension V-invariant subspace of K, then $W=V|_H$ has rank α too.*

Proof. Write $W=V|_H=(S_0,T_0)$. Let $F=K\ominus H$. Thus F has finite dimension and $K=H\oplus F$. With respect to this decomposition, write

$$S^*=\begin{pmatrix}S_0^* & 0\\ X^* & A^*\end{pmatrix},\quad T^*=\begin{pmatrix}T_0^* & 0\\ Y^* & B^*\end{pmatrix}.$$

Observe that $\sigma(A) \times \sigma(B)$ is a finite set since A and B act on a finite dimensional space. Fix $1 \leqslant j \leqslant s$. Let Γ be the set of all $(\lambda, \mu) \in \mathfrak{V}(\boldsymbol{p}_j)$ such that the dimension of $\ker(S-\lambda)^* \cap \ker(T-\mu)^*$ is α_j and $(\lambda, \mu) \notin \sigma(A) \times \sigma(B)$. Hence by Proposition 1.2, Γ contains the cofinite set of all regular points. Since also the set $\sigma(A) \times \sigma(B)$ is finite, Γ is a cofinite subset of $\mathfrak{V}(\boldsymbol{p}_j)$. Fix $(\lambda, \mu) \in \Gamma$ and let

$$L = \ker(S-\lambda)^* \cap \ker(T-\mu)^* \quad \text{and} \quad L_0 = \ker(S_0-\lambda)^* \cap \ker(T_0-\mu)^*.$$

Let $\mathcal{P} \subseteq H$ be the projection of L onto H. Given $f \in L$, write $f = f_1 \oplus f_2$, where $f_1 \in H$ and $f_2 \in F$. Since $f \in L$, the kernel of $(S_0-\lambda)^*$ contains f_1. Likewise the kernel of $(T_0-\lambda)^*$ contains f_1. Therefore, $\mathcal{P} \subseteq L_0$. If $\dim(L_0) < \alpha_j$, then, since $\dim(L) = \alpha_j$, there exists a non zero vector of the form $0 \oplus v$ in L and hence $\ker(A-\lambda)^* \cap \ker(B-\mu)^*$ is non-empty. But, $\ker(A-\lambda)^* \cap \ker(B-\mu)^*$ is empty by the choice of (λ, μ). Thus $\dim(L_0) = \alpha_j$ for almost all (λ, μ) in $\mathfrak{V}(\boldsymbol{p}_j)$. Therefore W also has rank α. ∎

COROLLARY 3.10. *Suppose $V = (S, T)$ is a finite bimultiplicity pure $\boldsymbol{p}$-isopair with minimal polynomial $\boldsymbol{p}$ and with rank $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_+^s$ acting on a Hilbert space K. If H is a finite codimension V-invariant subspace of K, then $W = V|_H$ is at least $\beta = \max\{\alpha_1, \dots, \alpha_s\}$-cyclic. Hence V is at least nearly β-cyclic.*

Proof. By Proposition 3.9, W has rank α. By Proposition 3.8, W is at least β-cyclic. Thus, each restriction of V to a finite codimension invariant subspace is at least β-cyclic and hence V is at least nearly β-cyclic. ∎

4. THE CASE $\boldsymbol{p}$ IS IRREDUCIBLE

In this section $\boldsymbol{p}$ is an irreducible square free inner toral polynomial of bidegree (n, m).

A *rank α-admissible kernel* $\mathcal{K}$ over $\mathfrak{V}(\boldsymbol{p})$ consists of an $\alpha \times m\alpha$ matrix polynomial Q and an $\alpha \times n\alpha$ matrix polynomial P such that

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{1 - z\zeta^*} = \mathcal{K}((z,w),(\zeta,\eta)) = \frac{P(z,w)P(\zeta,\eta)^*}{1 - w\eta^*}, \quad (z,w),(\zeta,\eta) \in \mathfrak{V}(\boldsymbol{p})$$

where Q and P have full rank α at some point in $\mathfrak{V}(\boldsymbol{p})$. In particular, at some point $x \in \mathfrak{V}(\boldsymbol{p})$ the matrix $\mathcal{K}(x,x)$ has full rank α [8]. An $\alpha \times \alpha$ matrix-valued kernel on a set Ω has *full rank* at $x \in \Omega$, if $\mathcal{K}(x,x)$ has full rank α. We refer to $(\mathcal{K}, P, Q)$ as an *α-admissible triple*.

Let $H^2(\mathcal{K})$ denote the Hilbert space associated to the rank α admissible kernel $\mathcal{K}$. For a point $y \in \mathfrak{V}(\boldsymbol{p})$, denote by $\mathcal{K}_y$ the $\alpha \times \alpha$ matrix function on $\mathfrak{V}(\boldsymbol{p})$ defined by $\mathcal{K}_y(x) = \mathcal{K}(x,y)$. Elements of $H^2(\mathcal{K})$ are $\mathbb{C}^\alpha$ vector-valued functions on $\mathfrak{V}(\boldsymbol{p})$ and the linear span of $\{\mathcal{K}_y\gamma : y \in \mathfrak{V}(\boldsymbol{p}), \gamma \in \mathbb{C}^\alpha\}$ is dense in $H^2(\mathcal{K})$. Note that the operators X and Y determined densely on $H^2(\mathcal{K})$ by

$X\mathcal{K}_{(\lambda,\mu)}\gamma = \lambda^*\mathcal{K}_{(\lambda,\mu)}\gamma$ and $Y\mathcal{K}_{(\lambda,\mu)}\gamma = \mu^*\mathcal{K}_{(\lambda,\mu)}\gamma$ are contractions. By Theorem 4.1 item (i) below, X^* is a bounded operator on $H^2(\mathcal{K})$. Further for $f \in H^2(\mathcal{K})$, $\langle X^*f, \mathcal{K}_{\lambda,\mu}\gamma\rangle = \lambda\langle f(\lambda,\mu),\gamma\rangle$. Hence X^* is the operator of multiplication by z on $H^2(\mathcal{K})$. Likewise, Y^* is a bounded operator on $H^2(\mathcal{K})$ and it is the multiplication by w on $H^2(\mathcal{K})$.

THEOREM 4.1. *If $\mathcal{K}$ is a rank α-admissible kernel over $\mathfrak{V}(\boldsymbol{p})$, then*
(i) *X is bounded on the linear span of $\{\mathcal{K}_y\gamma : y \in \mathfrak{V}(\boldsymbol{p}),\ \gamma \in \mathbb{C}^\alpha\}$;*
(ii) *for each $1 \leqslant j \leqslant m\alpha$ and each positive integer n, the vector z^nQe_j (Qe_j is the j-th column of Q) lies in $H^2(\mathcal{K})$;*
(iii) *the span of $\{s_\lambda Q(\lambda,\mu)^*\gamma : (\lambda,\mu) \in \mathfrak{V}(\boldsymbol{p}), \gamma \in \mathbb{C}^\alpha\}$ is dense in $H^2_{\mathbb{C}^{m\alpha}}$;*
(iv) *the set $\mathscr{B} = \{z^nQe_j : n \in \mathbb{N}, 1 \leqslant j \leqslant m\alpha\}$ is an orthonormal basis for $H^2(\mathcal{K})$; and*
(v) *the operators S and T densely defined on $\mathscr{B}$ by $Sf = zf$ and $Tf = wf$ extend to a pair of pure isometries on $H^2(\mathcal{K})$.*

Proof. For a finite set of points $(\lambda_1,\mu_1),\dots,(\lambda_n,\mu_n) \in \mathfrak{V}(\boldsymbol{p})$, and $\gamma_1,\dots,\gamma_n \in \mathbb{C}^\alpha$, observe that

$$\begin{aligned}\Big\langle (I-X^*X)\sum_{j=1}^n \mathcal{K}_{(\lambda_j,\mu_j)}\gamma_j, \sum_{k=1}^n \mathcal{K}_{(\lambda_k,\mu_k)}\gamma_k\Big\rangle &= \sum_{j,k=1}^n \langle (1-\lambda_k\overline{\lambda}_j)\mathcal{K}_{(\lambda_j,\mu_j)}(\lambda_k,\mu_k)\gamma_j,\gamma_k\rangle \\ &= \sum_{j,k=1}^n \langle Q(\lambda_k,\mu_k)Q^*(\lambda_j,\mu_j)\gamma_j,\gamma_k\rangle \\ &= \Big\langle \sum_{j=1}^n Q^*(\lambda_j,\mu_j)\gamma_j, \sum_{k=1}^n Q^*(\lambda_k,\mu_k)\gamma_k\Big\rangle \geqslant 0.\end{aligned}$$

Therefore, X is bounded on the linear span of $\{\mathcal{K}_y\gamma : y \in \mathfrak{V}(\boldsymbol{p}),\ \gamma \in \mathbb{C}^\alpha\}$.

To prove item (ii), note that by Theorem 4.15 of [12], if f is a $\mathbb{C}^\alpha$ valued function defined on $\mathfrak{V}(\boldsymbol{p})$ and if $\mathcal{K}((z,w),(\zeta,\eta)) - f(z,w)f(\zeta,\eta)^*$ is a (positive semidefinite) kernel function then $f \in H^2(\mathcal{K})$. Since

$$\begin{aligned}&\mathcal{K}((z,w),(\zeta,\eta)) - (z\zeta^*)^nQ(z,w)Q^*(\zeta,\eta) \\ &\qquad = \sum_{j=1}^{n-1}(z\zeta^*)^jQ(z,w)Q^*(\zeta,\eta) + (z\zeta^*)^{n+1}\mathcal{K}((z,w),(\zeta,\eta))\end{aligned}$$

is positive semidefinite, it follows that $z^nQe_j \in H^2(\mathcal{K})$.

By a result in Lemma 4.1 of [8], there exists a cofinite subset $\Lambda \subset \mathbb{D}$ such that for each $\lambda \in \Lambda$ there exist distinct points $\mu_1,\dots,\mu_m \in \mathbb{D}$ such that $(\lambda,\mu_j) \in \mathfrak{V}(\boldsymbol{p})$ and the $m\alpha \times m\alpha$ matrix,

$$R(\lambda) := \begin{pmatrix} Q(\lambda,\mu_1)^* & \cdots & Q(\lambda,\mu_m)^*\end{pmatrix}$$

has full rank. Define a map U from $H^2(\mathcal{K})$ to $H^2_{\mathbb{C}^{m\alpha}}$ by

$$U\mathcal{K}_{(\lambda,\mu)}(z,w)\gamma = s_\lambda(z)Q(\lambda,\mu)^*\gamma.$$

Observe that for $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathbb{D}^2$ and $\gamma, \delta \in \mathbb{C}^\alpha$,

$$\begin{aligned}\langle U\mathcal{K}_{(\lambda_1,\mu_1)}(z,w)\gamma, U\mathcal{K}_{(\lambda_2,\mu_2)}(z,w)\delta\rangle &= \langle s_{\lambda_1}(z)Q(\lambda_2,\mu_2)Q^*(\lambda_1,\mu_1)\gamma, s_{\lambda_2}(z)\delta\rangle \\ &= \delta^* Q(\lambda_2,\mu_2)Q^*(\lambda_1,\mu_1)\gamma\langle s_{\lambda_1}(z), s_{\lambda_2}(z)\rangle \\ &= \frac{\delta^* Q(\lambda_2,\mu_2)Q^*(\lambda_1,\mu_1)\gamma}{1-\overline{\lambda}_1\lambda_2} \\ &= \delta^*\mathcal{K}((\lambda_2,\mu_2),(\lambda_1,\mu_1))\gamma \\ &= \langle \mathcal{K}_{(\lambda_1,\mu_1)}(z,w)\gamma, \mathcal{K}_{(\lambda_2,\mu_2)}(z,w)\delta\rangle.\end{aligned}$$

Therefore, U is an isometry and hence a unitary onto its range. Given $\lambda \in \mathbb{D}$, the span of

$$\{U\mathcal{K}_{(\lambda,\mu_j)}\gamma : \mu_j \in Z(\boldsymbol{p}_\lambda),\ \gamma \in \mathbb{C}^\alpha\}$$

is equal to s_λ times the span of

$$\{Q(\lambda,\mu_j)^* e_k : 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant \alpha\} \subseteq \mathbb{C}^{m\alpha}.$$

If $\lambda \in \Lambda$, then $R(\lambda)$ has full rank. Thus for such λ, the span of $\{Q(\lambda,\mu)^*\gamma : \mu$ such that $(\lambda,\mu) \in \Gamma, \gamma \in \mathbb{C}^\alpha\}$ is all of $\mathbb{C}^{m\alpha}$. Since $\Lambda \subseteq \mathbb{D}$ is cofinite, $\{s_\lambda\mathbb{C}^{m\alpha} : \lambda \in \Lambda\}$ is dense in $H^2_{\mathbb{C}^{m\alpha}}$. Since,

$$\{s_\lambda\mathbb{C}^{m\alpha} : \lambda \in \Lambda\} \subseteq \operatorname{span}\{s_\lambda Q(\lambda,\mu)^*\gamma : (\lambda,\mu) \in \mathfrak{V}(\boldsymbol{p}), \gamma \in \mathbb{C}^\alpha\},$$

the span of $\{s_\lambda Q(\lambda,\mu)^*\gamma : (\lambda,\mu) \in \mathfrak{V}(\boldsymbol{p}), \gamma \in \mathbb{C}^\alpha\}$ is also dense in $H^2_{\mathbb{C}^{m\alpha}}$, proving item (iii). Moreover, it proves that U is onto and hence unitary.

Let q_k denote the k-th column of Q. Thus $q_k = Qe_k$. Note that, for any $a \in \mathbb{N}$ and $1 \leqslant j \leqslant m\alpha$,

$$\begin{aligned}\langle U^* z^a e_j(\zeta,\eta), e_k\rangle &= \langle U^* z^a e_j, \mathcal{K}_{(\zeta,\eta)} e_k\rangle = \langle z^a e_j, U\mathcal{K}_{(\zeta,\eta)} e_k\rangle \\ &= \sum_{i=1}^{m\alpha}\langle z^a e_j, (s_\zeta q_i^*(\zeta,\eta)e_k)e_i\rangle = \langle q_j(\zeta,\eta)\zeta^a, e_k\rangle = \langle (z^a q_j)(\zeta,\eta), e_k\rangle\end{aligned}$$

and hence it follows that $U^* z^a e_j = z^a q_j$ and $Uz^a q_j = z^a e_j$. In particular, $\{z^a q_j : a \in \mathbb{N}, 1 \leqslant j \leqslant m\alpha\}$ is an orthonormal basis for $H^2(\mathcal{K})$ completing the proof of item (iv).

To prove item (v), observe that $M_z U = US$ on $\mathscr{B}$ and then extending to $H^2(\mathcal{K})$, it is true on $H^2(\mathcal{K})$ too. It is now evident that S is a pure isometry of multiplicity $m\alpha$ with wandering subspace $\{Q\gamma : \gamma \in \mathbb{C}^{m\alpha}\}$ (the span of the columns of Q). Likewise for T by symmetry. ∎

PROPOSITION 4.2 ([3]). *Suppose Φ is an $M \times M$ matrix-valued rational inner function and the pair (M_z, M_Φ) of multiplication operators on $H^2_{\mathbb{C}^M}$. If the rank of the projection $I - M_\Phi M_\Phi^*$ is N, then there exists a unitary matrix U of size $(M+N) \times (M+N)$,*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{matrix} M, \\ N \end{matrix} \qquad \text{(columns } M \;\; N\text{)}$$

such that

$$\Phi(z) = A + zB(I - zD)^{-1}C.$$

PROPOSITION 4.3. *If $V = (S, T)$ is a finite bimultiplicity (M, N) pure $\boldsymbol{p}$-isopair of rank α, modeled as (M_z, M_Φ) on $H^2_{\mathbb{C}^M}$, where Φ is an $M \times M$ matrix-valued rational inner function, then $M = m\alpha$ and*

(i) *there exists an $\alpha \times m\alpha$ matrix polynomial Q such that $Q(z, w)$ has full rank at almost all points of $\mathfrak{V}(\boldsymbol{p})$;*

(ii) *for $(z, w) \in \mathfrak{V}(\boldsymbol{p})$*

$$Q(z, w)(\Phi(z) - w) = 0;$$

(iii) *there exists an $\alpha \times n\alpha$ matrix polynomial P such that $P(z, w)$ has full rank at almost all points of $\mathfrak{V}(\boldsymbol{p})$ and an α-admissible kernel $\mathcal{K}$ such that*

$$\frac{Q(z, w)Q(\zeta, \eta)^*}{1 - z\zeta^*} = \mathcal{K}((z, w), (\zeta, \eta)) = \frac{P(z, w)P(\zeta, \eta)^*}{1 - w\eta^*} \quad \text{on } \mathfrak{V}(\boldsymbol{p}) \times \mathfrak{V}(\boldsymbol{p}).$$

REMARK 4.4. The triple $(\mathcal{K}, P, Q)$ in Proposition 4.3 is a rank α-admissible triple.

Proof. Applying Corollary 3.7 to irreducible $\boldsymbol{p}$ gives $M = m\alpha$. Let Λ denote the set of $\lambda \in \mathbb{D}$ such that $\boldsymbol{p}_\lambda$ has m distinct zeros. By Lemma 2.2 item (vi) Λ is cofinite. Let

$$\Gamma = \{(\lambda, \mu) : \lambda \in \Lambda,\ \mu \in Z(\boldsymbol{p}_\lambda)\}.$$

By Proposition 3.4 item (ii), for each $(\lambda, \mu) \in \Gamma$, the matrix $\Phi(\lambda)$ is diagonalizable and $\Phi(\lambda) - \mu$ has an α dimensional kernel. Now fix $(\lambda_0, \mu_0) \in \Gamma$. Hence there exist unitary matrices Π and Π_* such that

$$\Pi_*(\Phi(\lambda_0) - \mu_0)\Pi = \begin{pmatrix} 0_\alpha & 0 \\ 0 & A \end{pmatrix},$$

where A is $(m-1)\alpha \times (m-1)\alpha$ and invertible. Let

$$\Sigma(z, w) = \Pi_*(\Phi(z) - w)\Pi.$$

For $(\lambda, \mu) \in \Gamma$, the matrix $\Sigma(z, w)$ has an α dimensional kernel. Write,

$$\Sigma(z, w) = \begin{pmatrix} E(z) - w & G(z) \\ H(z) & L(z) - w \end{pmatrix},$$

where E is $\alpha \times \alpha$ and L is of size $(m-1)\alpha \times (m-1)\alpha$. By construction $L(z) - w$ is invertible at (λ_0, μ_0) and the other entries are 0 there. In particular, $L(\lambda) - \mu$ is invertible for almost all points $(\lambda, \mu) \in \mathfrak{V}(\boldsymbol{p})$. Moreover, if $L(z) - w$ is invertible, then

$$\Sigma(z, w) = \begin{pmatrix} I & G(z) \\ 0 & L(z) - w \end{pmatrix} \begin{pmatrix} \Psi(z, w) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ (L(z) - w)^{-1}H(z) & I \end{pmatrix},$$

where

$$\Psi(z, w) = E(z) - w - G(z)(L(z) - w)^{-1}H(z).$$

Thus, on the cofinite subset of $\mathfrak{V}(\boldsymbol{p})$ where $L(\lambda)-\mu$ is invertible and $\Sigma(\lambda,\mu)$ has an α dimensional kernel, $\Psi(\lambda,\mu)=0$ and moreover,

$$\begin{pmatrix} I_\alpha & -G(\lambda)(L(\lambda)-\mu)^{-1} \end{pmatrix} \Pi_*(\Phi(\lambda)-\mu)=0.$$

Let

$$\mathcal{Q}(z,w)=\begin{pmatrix} I_\alpha & -G(z)(L(z)-w)^{-1} \end{pmatrix} \Pi_*.$$

It follows that

$$\mathcal{Q}(z,w)(\Phi(z)-w)=0$$

for almost all points in $\mathfrak{V}(\boldsymbol{p})$. After multiplying $\mathcal{Q}$ by an appropriate scalar polynomial we obtain an $\alpha\times m\alpha$ matrix polynomial $Q(z,w)$ that has full rank at almost all points of $\mathfrak{V}(\boldsymbol{p})$ and satisfies

$$Q(z,w)(\Phi(z)-w)=0$$

for all $(z,w)\in\mathfrak{V}(\boldsymbol{p})$.

Since T has multiplicity N, the operator M_Φ also has multiplicity N and hence the projection $I-M_\Phi M_\Phi^*$ has rank N. By Theorem 4.2, there exists a unitary matrix U of size $(M+N)\times(M+N)$,

$$U=\begin{array}{c} \begin{matrix} M & N \end{matrix} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \begin{matrix} M, \\ N \end{matrix}$$

such that

$$\Phi(z)=A+zB(I-zD)^{-1}C.$$

Define P by $P(z,w)=Q(z,w)B(I-zD)^{-1}$ and verify, for $(z,w)\in\mathfrak{V}(\boldsymbol{p})$,

$$\begin{pmatrix} Q & zP \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\begin{pmatrix} wQ & P \end{pmatrix} \quad \text{on } \mathfrak{V}(\boldsymbol{p}).$$

It follows that, for $(\zeta,\eta)\in\mathfrak{V}(\boldsymbol{p})$,

$$Q(z,w)Q(\zeta,\eta)^*+z\zeta^*P(z,w)P(\zeta,\eta)^*=w\eta^*Q(z,w)Q(\zeta,\eta)^*+P(z,w)P(\zeta,\eta)^*.$$

Rearranging gives

$$\frac{Q(z,w)Q(\zeta,\eta)^*}{1-z\zeta^*}=\mathcal{K}((z,w),(\zeta,\eta))=\frac{P(z,w)P(\zeta,\eta)^*}{1-w\eta^*} \quad \text{on } \mathfrak{V}(\boldsymbol{p})\times\mathfrak{V}(\boldsymbol{p}).$$

Finally, if $(\zeta,\eta)\in\mathfrak{V}(\boldsymbol{p})$ is such that $Q(\zeta,\eta)$ has full rank α, then $P(\zeta,\eta)P(\zeta,\eta)^*$ also has full rank α. Therefore, $P(\zeta,\eta)$ also has full rank α and hence $\mathcal{K}$ is a rank α-admissible kernel. ∎

THEOREM 4.5. *If $V=(S,T)$ is a finite bimultiplicity (M,N) pure $\boldsymbol{p}$-isopair with rank α, then there exists a rank α-admissible triple $(\mathcal{K},P,Q)$ such that V is unitarily equivalent to the operators of multiplication by z and w on $H^2(\mathcal{K})$.*

Proof. Note that (S,T) is unitarily equivalent to (M_z, M_Φ) on $H^2_{\mathbb{C}^M}$, where Φ is an $M \times M$ matrix-valued rational inner function. By Proposition 4.3, there exists a rank α-admissible triple $(\mathcal{K}, P, Q)$ such that

$$Q(z,w)(\Phi(z) - w) = 0 \tag{4.1}$$

for all $(z,w) \in \mathfrak{V}(\boldsymbol{p})$. Define

$$U : H^2_{\mathbb{C}^M} \to H^2(\mathcal{K})$$

on the span of

$$\mathcal{B} = \{s_\zeta Q^*(\zeta,\eta)\gamma : (\zeta,\eta) \in \mathfrak{V}(\boldsymbol{p}),\ \gamma \in \mathbb{C}^\alpha\} \subseteq H^2_{\mathbb{C}^M}$$

by

$$Us_\zeta(z)Q^*(\zeta,\eta)\gamma = \mathcal{K}_{(\zeta,\eta)}(z,w)\gamma.$$

For $(\zeta,\eta) \in \mathfrak{V}(\boldsymbol{p})$ and $\gamma_j \in \mathbb{C}^\alpha$ for $1 \leqslant j \leqslant 2$,

$$\begin{aligned}
\langle Us_{\zeta_1}(z)Q^*(\zeta_1,\eta_1)\gamma_1, &Us_{\zeta_2}(z)Q^*(\zeta_2,\eta_2)\gamma_2\rangle \\
&= \langle \mathcal{K}_{(\zeta_1,\eta_1)}(z,w)\gamma_1, \mathcal{K}_{(\zeta_2,\eta_2)}(z,w)\gamma_2\rangle \\
&= \langle \mathcal{K}_{(\zeta_1,\eta_1)}(\zeta_2,\eta_2)\gamma_1, \gamma_2\rangle \\
&= \langle s_{\zeta_1}(\zeta_2)Q(\zeta_2,\eta_2)Q^*(\zeta_1,\eta_1)\gamma_1, \gamma_2\rangle \\
&= \langle s_{\zeta_1}(z)Q^*(\zeta_1,\eta_1)\gamma_1, s_{\zeta_2}(z)Q^*(\zeta_2,\eta_2)\gamma_2\rangle.
\end{aligned}$$

Hence U is an isometry. By Theorem 4.1 item (iii) the span of $\mathcal{B}$ is dense in $H^2_{\mathbb{C}^M}$. Moreover, the range of U is dense in $H^2(\mathcal{K})$. Thus, U is a unitary. Rewrite (4.1) as

$$w^*Q^*(z,w) = \Phi^*(z)Q^*(z,w). \tag{4.2}$$

Let $\widetilde{M}_z$ and $\widetilde{M}_w$ be the operators of multipliction by z and w on $H^2(\mathcal{K})$, respectively. For $(\zeta,\eta) \in \mathfrak{V}(\boldsymbol{p})$ and $\gamma \in \mathbb{C}^\alpha$, using (4.2), observe that

$$\begin{aligned}
\widetilde{M}_w^* U(s_\zeta(z)Q^*(\zeta,\eta)\gamma) &= \widetilde{M}_w^*(\mathcal{K}_{(\zeta,\eta)}(z,w)\gamma) = \overline{\eta}\mathcal{K}_{(\zeta,\eta)}(z,w)\gamma \\
&= \overline{\eta}U(s_\zeta Q^*(\zeta,\eta)\gamma) = U(s_\zeta(z)\overline{\eta}Q^*(\zeta,\eta)\gamma) \\
&= U(s_\zeta(z)\Phi(\zeta)^*Q^*(\zeta,\eta)\gamma) = UM_\Phi^*(s_\zeta(z)Q^*(\zeta,\eta)\gamma).
\end{aligned}$$

Similarly,

$$\widetilde{M}_z^* U(s_\zeta(z)Q^*(\zeta,\eta)\gamma) = UM_z^*(s_\zeta(z)Q^*(\zeta,\eta)\gamma).$$

Therefore, $UM_z^* = \widetilde{M}_z^* U$ and $UM_\Phi^* = \widetilde{M}_w^* U$ on the span of $\mathcal{B}$, and hence on $H^2_{\mathbb{C}^M}$. Thus our original (S,T) is unitarily equivalent to $(\widetilde{M}_w, \widetilde{M}_w)$ on $H^2(\mathcal{K})$. ∎

DEFINITION 4.6. If $\mathcal{B}$ is a subspace of vector space $\mathcal{X}$, then the *codimension* of $\mathcal{B}$ in $\mathcal{X}$ is the dimension of the quotient space $\mathcal{X}/\mathcal{B}$.

LEMMA 4.7. *Suppose $\mathcal{X}$ is a vector space (over $\mathbb{C}$) and $\mathcal{Q}$ and $\mathcal{B}$ are subspaces of $\mathcal{X}$. If $\mathcal{Q} \subset \mathcal{B}$ and $\mathcal{Q}$ has finite codimension in $\mathcal{X}$, then $\mathcal{Q}$ has finite codimension in $\mathcal{B}$.*

LEMMA 4.8. *Suppose K is a Hilbert space and $\mathcal{Q} \subset \mathcal{B} \subset K$ are linear subspaces (thus not necessarily closed) and let $\overline{\mathcal{Q}}$ denote the closure of $\mathcal{Q}$. If $\mathcal{Q}$ has finite codimension in $\mathcal{B}$ and if $\mathcal{B}$ is dense in K, then there exists a finite dimensional subspace $\mathcal{D}$ of K such that $K = \overline{\mathcal{Q}} \oplus \mathcal{D}$.*

THEOREM 4.9. *If $\mathcal{K}$ is a rank α admissible kernel function defined on $\mathfrak{V}(\boldsymbol{p})$ and $S = M_z$, $T = M_w$ are the operators of multiplication by z and w, respectively on $H^2(\mathcal{K})$, then the pair (S, T) is nearly α-cyclic.*

Proof. Since $\mathcal{K}$ is a rank α admissible kernel, there exist matrix polynomials Q and P of size $\alpha \times m\alpha$ and $\alpha \times n\alpha$ respectively, such that

$$\mathcal{K}((z,w),(\zeta,\eta)) = \frac{Q(z,w)Q^*(\zeta,\eta)}{1 - z\overline{\zeta}} = \frac{P(z,w)P^*(\zeta,\eta)}{1 - w\overline{\eta}}, \quad (z,w),(\zeta,\eta) \in \mathfrak{V}(\boldsymbol{p})$$

and Q and P have full rank α at some point in $\mathfrak{V}(\boldsymbol{p})$. Fix $(\zeta,\eta) \in \mathfrak{V}(\boldsymbol{p})$ so that $Q(\zeta,\eta)$ has full rank α. By the definition of $\mathcal{K}$ and Lemma 3.3 of [8], $\mathcal{K}((z,w),(\zeta,\eta))$ has full rank α at almost all points in $\mathfrak{V}(\boldsymbol{p})$. Let

$$Q_0 = Q_0(z,w) = Q(z,w)Q^*(\zeta,\eta).$$

Then $Q_0 e_j = (1 - S\overline{\zeta})\mathcal{K}_{(\zeta,\eta)} e_j$. By Theorem 4.1 item (ii), $Q_0 e_j$, the j-th column of Q_0, is also in $H^2(\mathcal{K})$. Letting $\widetilde{q} = \widetilde{q}(z,w)$ to be the determinant of Q_0, since $\mathcal{K}((z,w),(\zeta,\eta))$ has full rank α at almost all points in $\mathfrak{V}(\boldsymbol{p})$, $\widetilde{q}$ is nonzero except for finitely many points in $\mathfrak{V}(\boldsymbol{p})$. Thus, $\boldsymbol{p}$ and $\widetilde{q}$ have only finitely many common zeros in $\mathfrak{V}(\boldsymbol{p})$. By Lemma 2.2 item (iii), $\boldsymbol{p}$ and q are relatively prime. Let I be the ideal generated by $\boldsymbol{p}$ and $\widetilde{q}$. By Proposition 2.1 item (ii), $\mathbb{C}[z,w]/I$ is finite dimensional. Observe that

$$\widetilde{q}_j = \widetilde{q} e_j = Q_0 \mathrm{Adj}(Q_0) e_j = \sum_{k=1}^{\alpha} b_{kj} Q_0 e_k \in H^2(\mathcal{K}),$$

where b_{kj} is the (k,j)-entry of $\mathrm{Adj}(Q_0)$. If $\vec{r}$ is an $\alpha \times 1$ matrix polynomial with entries r_j, then

$$\vec{r}\widetilde{q} = \sum_{j=1}^{\alpha} r_j \, \mathrm{Adj}(Q_0) Q_0 e_j \in H^2(\mathcal{K}). \tag{4.3}$$

Since $\mathbb{C}[z,w]/I$ is finite dimensional, there is a finite dimensional subspace $\mathscr{S} \subseteq \mathbb{C}[z,w]$ such that

$$\{r\widetilde{q} + s\boldsymbol{p} + t : r, s \in \mathbb{C}[z,w], t \in \mathscr{S}\} = \mathbb{C}[z,w].$$

Therefore

$$\left\{\vec{r}\widetilde{q} + \vec{s}\boldsymbol{p} + \vec{t} : \vec{r}, \vec{s} \text{ are vector polynomials }, \vec{t} \in \bigoplus_1^{\alpha} \mathscr{S}\right\} = \bigoplus_1^{\alpha} \mathbb{C}[z,w],$$

and hence the span $\mathcal{Q}$ of $\{r_1\widetilde{q}_1, \ldots, r_\alpha\widetilde{q}_\alpha \ : r_1, \ldots, r_\alpha \in \mathbb{C}[z,w]\}$ is of finite codimension in $\bigoplus_1^{\alpha} \mathbb{C}[z,w]$.

Let $\mathcal{B} = \bigvee\{z^n Q e_j : n \in \mathbb{N}, 1 \leqslant j \leqslant m\alpha\} \subseteq \bigoplus_1^\infty \mathbb{C}[z,w]$. By equation (4.3) $\mathcal{Q} \subset \mathcal{B}$. By Lemma 4.7, $\mathcal{Q}$ has finite codimension in $\bigoplus_1^\alpha \mathbb{C}[z,w]$. Moreover, $\mathcal{B}$ is dense in $H^2(\mathcal{K})$ by Theorem 4.1 item (iv). Hence by Lemma 4.8, the closure of $\mathcal{Q}$ in $H^2(\mathcal{K})$ has finite codimension in $H^2(\mathcal{K})$. Equivalently, the closure of $\left\{\sum_{j=1}^{\alpha} r_j(S,T)\widetilde{q}_j : r_j \in \mathbb{C}[z,w]\right\}$ is of finite codimension in $H^2(\mathcal{K})$. Thus (S,T) is α-cyclic on $\overline{\mathcal{Q}}$ and hence at most nearly α-cyclic in $H^2(\mathcal{K})$.

Moreover, by Corollary 3.10, (S,T) has rank at most α. For $(\zeta,\eta) \in \mathfrak{V}(\boldsymbol{p})$ and for $\gamma \in \mathbb{C}^\alpha$, note that

$$\mathcal{K}_{(\zeta,\eta)}\gamma \in \ker(M_z - \zeta)^* \cap \ker(M_w - \eta)^*.$$

Hence, if $(\zeta,\eta) \in \mathfrak{V}(\boldsymbol{p})$ is such that $\mathcal{K}_{(\zeta,\eta)}$ has full rank α, then $\ker(M_z - \zeta)^* \cap \ker(M_w - \eta)^*$ has dimension at least α. Therefore, (S,T) has rank at least α. Thus (S,T) has rank α. By Corollary 3.10, (S,T) is at least nearly α-cyclic and hence (S,T) is nearly α-cyclic on $H^2(\mathcal{K})$. ∎

PROPOSITION 4.10. *If $V = (S,T)$ is a finite bimultiplicity pure $\boldsymbol{p}$-isopair of rank α acting on the Hilbert space K, then there exists a finite codimension V invariant subspace H of K such that the restriction of V to H is α-cyclic.*

For the proof combine Theorems 4.5 and 4.9.

5. DECOMPOSITION OF FINITE RANK ISOPAIRS

PROPOSITION 5.1. *Suppose p_1, $p_2 \in \mathbb{C}[z,w]$ are relatively prime square free inner toral polynomials, but not necessarily irreducible. If $V_j = (S_j, T_j)$ are β_j-cyclic p_j-pure isopairs, then $V = V_1 \oplus V_2$ is a p_1p_2-isopair and is at most nearly* $\max\{\beta_1,\beta_2\}$*-cyclic.*

Proof. Clearly,

$$p_1p_2(V) = (p_1(V_1) \oplus p_1(V_2))(p_2(V_1) \oplus p_2(V_2)) = (0 \oplus p_1(V_2))(p_2(V_1) \oplus 0) = 0.$$

Let I be the ideal generated by p_1 and p_2. By Proposition 2.1 item (ii), I has finite codimension in $\mathbb{C}[z,w]$. Hence there exists a finite dimension subspace $\mathcal{R}$ of $\mathbb{C}[z,w]$ such that, for each $\psi \in \mathbb{C}[z,w]$, there exist $s_1, s_2 \in \mathbb{C}[z,w]$ and $r \in \mathcal{R}$ such that

$$\psi = s_1p_1 + s_2p_2 + r.$$

Let K denote the Hilbert space that V acts upon. Let $\beta = \max\{\beta_1,\beta_2\}$ and suppose without loss of generality $\beta_1 = \beta_2 = \beta$. For $j = 1,2$, choose cyclic sets $\Gamma_j = \{\gamma_{j,1},\ldots,\gamma_{j,\beta}\}$ for V_j. (In the case where $\beta_1 < \beta_2$ we can set Γ_1 to be $\{\gamma_{1,1},\ldots,\gamma_{1,\beta_1},0,0,\ldots,0\}$, so that this new Γ_1 has $\beta = \beta_2$ vectors.) Let $K_0 = \{\psi_1(V_1)\gamma_{1,k} \oplus \psi_2(V_2)\gamma_{2,k} : 1 \leqslant k \leqslant \beta, \psi_j \in \mathbb{C}[z,w]\}$. By the hypothesis, K_0 is dense in K. For given polynomials $\psi_1, \psi_2 \in \mathbb{C}[z,w]$, there exist $s_1, s_2 \in \mathbb{C}[z,w]$ and $r \in \mathcal{R}$ such that

$$\psi_1 - \psi_2 = -s_1 p_1 + s_2 p_2 + r.$$

Rearranging gives,

$$\psi_1 + s_1 p_1 = \psi_2 + s_2 p_2 + r.$$

Let $\varphi = \psi_1 + s_1 p_1$. It follows that

$$\varphi = \psi_2 + s_2 p_2 + r.$$

Consequently,

$$\varphi(V)[\gamma_{1,k} \oplus \gamma_{2,k}] = \varphi(V_1)\gamma_{1,k} \oplus \varphi(V_2)\gamma_{2,k} = \psi_1(V_1)\gamma_{1,k} \oplus (\psi_2(V_2)\gamma_{2,k} + r(V_2)\gamma_{2,k}).$$

Let H_0 denote the span of $\{\varphi(V)[\gamma_{1,k} \oplus \gamma_{2,k}] : 1 \leqslant k \leqslant \beta,\ \psi \in \mathbb{C}[z,w]\}$ and H be the closure of H_0. Let $\mathcal{L}$ denote the span of $\{0 \oplus r(V_2)\gamma_{2,k} : 1 \leqslant k \leqslant \beta, r \in \mathcal{R}\}$. Note that $\mathcal{L}$ is finite dimensional since $\mathcal{R}$ is and hence $\mathcal{L}$ is closed. Moreover,

$$K_0 = H_0 + \mathcal{L}.$$

Hence H_0 has finite codimension in K_0. By Lemma 4.8, H has finite codimension in K. Evidently H is V invariant and the restriction of V to H is at most β-cyclic. Therefore, V is at most nearly β-cyclic. ∎

PROPOSITION 5.2. *If $V_j = (S_j, T_j)$ are finite bimultiplicity pure $\boldsymbol{p}_j$-isopairs with rank α_j acting on Hilbert spaces K_j, where $\boldsymbol{p}_j$ are irreducible and relatively prime inner toral polynomials for $1 \leqslant j \leqslant s$, then $\bigoplus_{j=1}^{s} V_j$ is nearly* $\max\{\alpha_1, \alpha_2, \dots, \alpha_s\}$*-cyclic on $\bigoplus_{j=1}^{s} K_j$.*

Proof. First suppose $s = 2$. By Proposition 4.10, each V_j is α_j-cyclic on some finite codimensional invariant subspace H_j of K_j. By Proposition 5.1, $V_1|_{H_1} \oplus V_2|_{H_2}$ is at most nearly $\max\{\alpha_1, \alpha_2\}$-cyclic on $H_1 \oplus H_2$. Since each H_j has finite codimension in K_j, it follows that $V = V_1 \oplus V_2$ is at most nearly $\max\{\alpha_1, \alpha_2\}$-cyclic on $K_1 \oplus K_2$. On the other hand, V has rank (α_1, α_2) and hence, by Corollary 3.10, is at least $\max\{\alpha_1, \alpha_2\}$-cyclic. Thus V is nearly $\max\{\alpha_1, \alpha_2\}$-cyclic.

Arguing by induction, suppose the result is true for $0 \leqslant j - 1 < s$. Thus $V' = V_1 \oplus \cdots \oplus V_{j-1}$ is nearly $\beta = \max\{\alpha_1, \alpha_2, \dots, \alpha_{j-1}\}$-cyclic on $K' = K_1 \oplus K_2 \oplus \cdots \oplus K_{j-1}$. Hence there exists a finite codimensional invariant subspace H' of K' such that the restriction of V' to H' is β-cyclic. Since V_j is a finite bimultiplicity $\boldsymbol{p}_j$ isopair with rank α_j, by Proposition 4.10, there exists a finite codimensional invariant subspace H_j of K_j such that $V_j|_{H_j}$ is α_j-cyclic. Note that $\boldsymbol{p}_1 \cdots \boldsymbol{p}_{j-1}$ and $\boldsymbol{p}_j$ are relatively prime. Applying Proposition 5.1 to $V'|_{H'}$ and $V_j|_{H_j}$, it follows that $V'|_{H'} \oplus V_j|_{H_j}$ is at most nearly $\gamma = \max\{\beta, \alpha_j\}$-cyclic on $H' \oplus H_j$. Since H' and H_j have finite codimension in K' and K_j respectively, $W = V_1 \oplus V_2 \oplus \cdots \oplus V_j$ is at most nearly γ-cyclic on $K_1 \oplus K_2 \oplus \cdots \oplus K_j$. On the other hand, W has rank

$(\alpha_1, \dots, \alpha_j)$ and is therefore at least nearly γ-cyclic by Corollary 3.10. Thus W is nearly $\gamma = \max\{\alpha_1, \dots, \alpha_j\}$-cyclic. ∎

Proof of Theorem 1.4. By Theorem 2.1 of [2], there exist a finite codimension subspace H of K that is invariant for V and pure $\boldsymbol{p}_j$-isopairs V_j such that

$$W = V|_H = V_1 \oplus V_2 \oplus \cdots \oplus V_s.$$

By Proposition 3.9, W has rank α. Hence V_j has rank α_j. By Proposition 5.2, there is a finite codimension invariant subspace L of H such that the restriction of W to L is $\beta = \max\{\alpha_1, \alpha_2, \dots, \alpha_s\}$-cyclic. Thus L is a finite codimensional subspace of K such that $V|_L$ is β-cyclic. Hence V is at most nearly β-cyclic. By Corollary 3.10, V is at least nearly β-cyclic. Therefore, V is nearly $\max\{\alpha_1, \alpha_2, \dots, \alpha_s\}$ cyclic. ∎

COROLLARY 5.3. *Suppose $V = (S, T)$ is a pure $\boldsymbol{p}$-isopair of finite bimultiplicity with minimal polynomial $\boldsymbol{p}$ and write $\boldsymbol{p} = \boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_s$ as a product of distinct irreducible factors. If V has rank α and $\beta = \max\{\alpha_1, \dots, \alpha_s\}$, then*

(i) *there exists a finite codimension invariant subspace H for V such that the restriction of V to H is β-cyclic;*

(ii) *V is not k-cyclic for any $k < \beta$; and*

(iii) *there exists a β-cyclic pure $\boldsymbol{p}$-isopair V' and an invariant subspace K for V' such that V is the restriction of V' to K.*

Proof. Proofs of items (i) and (ii) follow from Theorem 1.4 and the definition of nearly k-cyclic isopairs. The proof of item (iii) is an application of item (i) and Corollary 3.3. ∎

5.1. EXAMPLE. In this section we discuss an example of pure $\boldsymbol{p}$-isopairs of finite rank to illustrate the connection of the rank of a pure $\boldsymbol{p}$-isopair to nearly cyclicity and to the representation as direct sums.

Consider the irreducible, square free inner toral polynomial, $\boldsymbol{p} = z^3 - w^2$. The distinguished variety, $\mathcal{V}$, defines by $\boldsymbol{p}$ is called *Neil parabola* [8]. The triple $(\mathcal{K}_1, Q_1, P_1)$ given by

$$Q_1(z,w) = \begin{pmatrix} 1 & w \end{pmatrix}, \quad P_1(z,w) = \begin{pmatrix} 1 & z & z^2 \end{pmatrix}$$

and the corresponding kernel function

$$\frac{1 + w\overline{\eta}}{1 - z\overline{\zeta}} = \mathcal{K}_1((z,w),(\zeta,\eta)) = \frac{1 + z\overline{\zeta} + z^2\overline{\zeta}^2}{1 - w\overline{\eta}},$$

is a 1-admissible triple. Likewise for the choice of

$$Q_2(z,w) = \begin{pmatrix} z & w \end{pmatrix}, \quad P_2(z,w) = \begin{pmatrix} w & z & z^2 \end{pmatrix}$$

and the corresponding kernel function

$$\frac{z\overline{\zeta} + w\overline{\eta}}{1 - z\overline{\zeta}} = \mathcal{K}_2((z,w),(\zeta,\eta)) = \frac{w\overline{\eta} + z\overline{\zeta} + z^2\overline{\zeta}^2}{1 - w\overline{\eta}},$$

the triple $(\mathcal{K}_2, Q_2, P_2)$ is also a 1-admissible triple. For $j = 1,2$, let V_j be the pair (M_z, M_w) defined on $H^2(\mathcal{K}_j)$. Now each V_j is a pure $\boldsymbol{p}$-isopair or rank 1 and each V_j is nearly 1-cyclic.

Let $Q = Q_1 \oplus Q_2$, $P = P_1 \oplus P_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Observe that the triple $(\mathcal{K}, Q, P)$ is a 2-admissible triple and $V = (M_z, M_w)$ defined on $H^2(\mathcal{K})$ is a pure $\boldsymbol{p}$-isopair and nearly 2-cyclic. In fact V is a pure $\boldsymbol{p}$-isopair of rank 2 that can be written as a direct sum of two pure $\boldsymbol{p}$-isopairs, V_1 and V_2.

However this is not true in general. In other words, there exist pure $\boldsymbol{p}$-isopairs of finite rank (say $\alpha \in \mathbb{N}$), that cannot be expressed as a direct sum of α number of pure $\boldsymbol{p}$-isopairs. For instance, let

$$H' = \{f \in H^2(\mathcal{K}) : \langle f, (1 \quad -z)^\top \rangle = 0\}$$

and $V' = V|_{H'}$. Observe that H' is a finite codimensional subspace of $H^2(\mathcal{K})$ and H' is invariant under V. By the stability of the rank V' has rank 2 and hence nearly 2-cyclic.

Moreover, the collection of vectors of the form:

$$\begin{pmatrix} 1/\sqrt{2} \\ z/\sqrt{2} \end{pmatrix}, \begin{pmatrix} z^n \\ 0 \end{pmatrix}_{n\geqslant 1}, \begin{pmatrix} wz^n \\ 0 \end{pmatrix}_{n\geqslant 0}, \begin{pmatrix} 0 \\ z^n \end{pmatrix}_{n\geqslant 2}, \begin{pmatrix} 0 \\ wz^n \end{pmatrix}_{n\geqslant 0},$$

forms an orthonormal basis for H'. Hence the reproducing kernel , $\widetilde{\mathcal{K}}$, for H' has the form

$$\widetilde{\mathcal{K}}((z,w),(\zeta,\eta)) = \begin{pmatrix} \frac{1}{2} + \frac{z\overline{\zeta} + w\overline{\eta}}{1 - z\overline{\zeta}} & \frac{\overline{\zeta}}{2} \\ \frac{z}{2} & z\overline{\zeta}\left(\frac{1}{2} + \frac{z\overline{\zeta} + w\overline{\eta}}{1 - z\overline{\zeta}}\right) \end{pmatrix}.$$

Since $\widetilde{\mathcal{K}}((z,w),(0,0))$ is not diagonalizable, $\widetilde{\mathcal{K}}$ is not diagonalizable. Consequently, H' and V' are not direct sums. In otherwords, V' is a pure $\boldsymbol{p}$-isopair of rank 2 that cannot be expressed as a direct sum of two other pure $\boldsymbol{p}$-isopairs.

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