J. OPERATOR THEORY
80:1(2018), 167–185
doi: 10.7900/jot.2017sep11.2185

THE ORBIT SPACES OF GROUPOIDS WHOSE C^*-ALGEBRAS ARE GCR

DANIEL W. VAN WYK

Communicated by Marius Dădârlat

ABSTRACT. Let G be a second countable locally compact Hausdorff groupoid with a continuous Haar system. We remove the assumption of amenability in a theorem by Clark about GCR groupoid C^*-algebras. We show that if the groupoid C^*-algebra of G is GCR then the orbit space of G is a T_0 topological space.

KEYWORDS: *Groupoids, C^*-algebras, type I representations, GCR, postliminal.*

MSC (2010): 20L05, 46L45, 46L51, 46L55.

INTRODUCTION

C^*-algebras can be divided into classes based on their representation theory. Two such classes of well-behaved C^*-algebras are GCR and CCR C^*-algebras. Let $\mathcal{K}(\mathcal{H})$ denote the compact operators on a Hilbert space $\mathcal{H}$. A C^*-algebra $\mathcal{A}$ is called CCR, if for every irreducible representation $\pi : \mathcal{A} \to B(\mathcal{H}_\pi)$ we have $\pi(\mathcal{A}) = \mathcal{K}(\mathcal{H}_\pi)$. $\mathcal{A}$ is called GCR if for every irreducible representation π we have $\pi(\mathcal{A}) \supset \mathcal{K}(\mathcal{H}_\pi)$.

We investigate the orbit spaces of groupoids whose C^*-algebras are GCR or CCR. The techniques used in the GCR and CCR cases are quite different. In this paper, the first of two, we treat the classes of groupoids whose C^*-algebras are GCR, or equivalently type I. In [3] Clark gives the following characterization for groupoids whose C^*-algebras are GCR.

THEOREM (Clark). *Let G be a second-countable locally compact Hausdorff groupoid with a Haar system. Suppose that all the stability subgroups of G are amenable. Then $C^*(G)$ is* GCR *if and only if the orbit space is T_0 and the stability subgroups of G are* GCR.

Clark's theorem generalizes a theorem for C^*-algebras of transformation groups by Gootman [11]. However, Gootman does not assume that the stabilizers are amenable. Due to the lack of an amenability assumption in Gootman's

GCR characterization, Clark conjectures that the amenability hypothesis in the groupoid characterization is unnecessary [3]. We provide an affirmative answer to Clark's conjecture.

Clark defines a map from the orbit space of the groupoid into the spectrum $C^*(G)^\wedge$ of the groupoid C^*-algebras $C^*(G)$, and requires amenable stabilizers to show that this map is continuous. Clark's GCR proof only uses the continuity of this map, and thus amenability, to prove that "if $C^*(G)$ is GCR then the orbit space is T_0". Therefore, to remove the amenability assumption from Clark's GCR characterization we only need to show that if $C^*(G)$ is GCR, then the orbit space is T_0.

Clark uses a different approach than Gootman. We show that Gootman's approach can be adapted to the groupoid setting. In a second countable locally compact Hausdorff groupoid the stabilizers always vary measurably, even when the stabilizers do not vary continuously, ([20], Lemma 1.6). With an appropriate measure we construct a direct integral representation of $C^*(G)$ from representations that are induced from stabilizers. We prove a groupoid version of Lemma 4.2 in [8] by Effros, that imposes a condition on the measure which ensures that the direct integral is a type I representation. Then we prove the contrapositive: if the orbit space is not T_0, then Ramsay's Mackey–Glimm dichotomy for groupoids ([18], Theorem 2.1) ensures we have a non-trivial ergodic measure on the unit space. Then by our groupoid version of Effros' lemma, if the measure is non-trivially ergodic, then the direct integral representation cannot be type I. Since a C^*-algebra is GCR if and only if it is type I ([10], [14], [21]), the result follows.

1. PRELIMINARIES

Throughout G is a second-countable locally compact Hausdorff groupoid, with a continuous Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ (see [19] for these definitions). Let r and s denote the range and source maps, respectively, from G onto the unit space $G^{(0)}$. For $u \in G^{(0)}$ we let $G^u := r^{-1}(u)$, $G_u := s^{-1}(u)$ and the stabilizer or stability subgroup at u is $G^u_u := r^{-1}(u) \cap s^{-1}(u)$. For $x \in G$, the map $R(x) := (r(x), s(x))$ defines an equivalence relation $\sim$ on $G^{(0)}$. For $u \in G^{(0)}$ we let $[u] := \{v \in G^{(0)} : u \sim v\}$ denote the orbit of u. The orbit space $G^{(0)}/G$ is the quotient space for the equivalence relation $\sim$ on $G^{(0)}$.

Throughout $C_c(X)$ denotes the continuous compactly supported functions from the topological space X into $\mathbb{C}$. If $f, g \in C_c(G)$, then

$$f * g(x) := \int_G f(y)g(y^{-1}x)\,\mathrm{d}\lambda^{r(x)}(y) \quad \text{and} \quad f^*(x) := \overline{f(x^{-1})},$$

define convolution and involution operations on $C_c(G)$, respectively. With these operations $C_c(G)$ is a $*$-algebra. Let $\mathcal{H}$ be a separable Hilbert space and $B(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$. A *representation* of $C_c(G)$ is a $*$-homomorphism $\pi : C_c(G) \to B(\mathcal{H})$ such that $\|\pi(f)\| \leqslant \|f\|_I$, where $\|f\|_I$ is the I-norm on $C_c(G)$ (see [19] for the I-norm). Then $C^*(G)$ is the completion of $C_c(G)$ in the norm

$$\|f\| := \{\sup \|\pi(f)\| : \pi \text{ is a representation of } C_c(G)\}.$$

We assume all representations are non-degenerate.

We use representations of $C^*(G)$ which are induced from the trivial representations of stabilizers. Fix $u \in G^{(0)}$, $f \in C_c(G)$ and $\phi, \psi \in C_c(G_u)$. The trivial representation 1_u of G_u^u is given by $1_u(t) = 1$ for every $t \in G_u^u$. The integrated form of 1_u is the representation $\pi_{1_u} : C_c(G_u^u) \to \mathbb{C}$ given by

$$\pi_{1_u}(a) := \int_{G_u^u} a(t)\, \Delta_u(t)^{-1/2}\, \mathrm{d}\beta^u(t),$$

which extends to give a representation of $C^*(G_u^u)$. Note: the modular function in the integrand above is due to the fact that we view the stabilizers as *subgroupoids*. Clark [3] and Ionescu and Williams [12] show that we can induce π_{1_u} to get a representation

$$\operatorname{Ind}_{G_u^u}^G \pi_{1_u} : C^*(G) \to B(\mathcal{H}_u).$$

We briefly describe how the induced representation is constructed and introduce some notation. Clark and Ionescu and Williams use induction via Hilbert modules. They show that $C_c(G_u)$ is a right $C^*(G_u^u)$-pre-Hilbert module, where the right inner product on $C_c(G_u)$ is given by

$$\langle \psi, \phi \rangle_{C_c(G_u^u)}(t) = \psi^* * \phi(t). \tag{1.1}$$

The completion of $C_c(G_u)$ in the inner product (1.1) gives a right $C^*(G_u^u)$-Hilbert module $\mathcal{X}$. Furthermore, for all $\gamma \in G_u$,

$$f \cdot \phi(\gamma) := \int_G f(\eta)\phi(\eta^{-1}\gamma)\, \mathrm{d}\lambda^{r(\gamma)}(\eta) = f * \phi(\gamma) \tag{1.2}$$

defines an action of $C_c(G)$ on $C_c(G_u)$ as adjointable operators, which extends to an action of $C^*(G)$ on $\mathcal{X}$. Before the induced representation, we first consider the representation space. Define an inner product on $C_c(G_u)$ by

$$(\phi \mid \psi)_u = \pi_{1_u}(\langle \psi, \phi \rangle_{C^*(G_u^u)}) = \pi_{1_u}(\psi^* * \phi) = \int_{G_u^u} \psi^* * \phi(t) \Delta_u(t)^{-1/2}\, \mathrm{d}\beta^u(t). \tag{1.3}$$

Denote the completion of $C_c(G_u)$ in the inner product in (1.3) by $\mathcal{H}_u$. Then the induced representation $\operatorname{Ind}_{G_u^u}^G \pi_{1_u} : C_c(G) \to C_c(G_u)$ is defined by

$$\operatorname{Ind}_{G_u^u}^G \pi_{1_u}(f)(\phi) = f * \phi. \tag{1.4}$$

By Proposition 2.66 of [17], $\operatorname{Ind}_{G_u^u}^G \pi_{1_u}$ extends to give a representation of $C^*(G)$ as bounded linear operators on the Hilbert space $\mathcal{H}_u$. To simplify notation we write

$$l^u := \operatorname{Ind}_{G_u^u}^G \pi_{1_u},$$

for any $u \in G^{(0)}$. Finally, each representation $l^u, u \in G^{(0)}$, is an irreducible representation of $C^*(G)$ [4], [12].

GCR GROUPOID C^*-ALGEBRAS. Section 2 addresses the measurability of a map on the units space which is used to construct a direct integral representation. In Section 3 we construct a Borel–Hilbert bundle and a direct integral representation. The direct integral representation acts on the Hilbert space of square integrable sections of the Borel–Hilbert bundle. We also prove Proposition 3.10, a groupoid version of a result by Effros, giving a condition on the measure used for the direct integral representation to be type I. Section 4 contains our main GCR result, Theorem 4.2.

2. A BOREL MAP ON THE UNIT SPACE

For the construction of the direct integral representation, we need Proposition 2.1 below, which gives the measurability of certain maps on the unit space. These maps have the form of an integral, where the group with respect to which we integrate and its Haar measure depend on the particular unit in $G^{(0)}$. This dependence is fine, since Renault shows ([20], Lemma 1.5) that the stabilizer map $u \mapsto G_u^u$ from $G^{(0)}$ to the space of all closed subgroups with the Fell topology, always varies measurably. However, the integrand also has a modular function that depends on the unit in $G^{(0)}$. We show that despite the modular function these maps are still Borel. Specifically we show the following proposition.

PROPOSITION 2.1. *Let $u \in G^{(0)}$ and $f \in C_c(G)$. There exists a Haar measure β^u on G_u^u with associated modular function Δ_u such that the following map is Borel:*

$$u \mapsto \int_{G_u^u} f(t)\, \Delta_u(t)^{-1/2}\, \mathrm{d}\beta^u(t).$$

For a locally compact Hausdorff space X let $\mathscr{B}(X)$ be the Borel σ-algebra on X. A *Borel measure* is a positive Radon measure on $\mathscr{B}(X)$.

Let

$$\mathrm{S} := \bigcup_{u \in G^{(0)}} G_u^u$$

denote the stability subgroupoid of G. Then the range and source maps agree on S and $\mathrm{S}^{(0)} = G^{(0)}$. Because $G^{(0)}$ is Hausdorff, S is a closed subset of G.

Let $\mathscr{C}(\mathrm{S})$ be the set of all closed subsets of S with the Fell topology. Then $\mathscr{C}(\mathrm{S})$ is a compact Hausdorff space, ([22], Proposition H.3). Since S is second countable, so is $\mathscr{C}(\mathrm{S})$. Let

$$\Sigma := \{H \in \mathscr{C}(\mathrm{S}) : H \textit{ is a closed subgroup of } \mathrm{S}\}.$$

Give

$$\Sigma * \mathrm{S} := \{(H, \gamma) \in \Sigma \times \mathrm{S} : H \in \Sigma, \gamma \in H\}$$

the relative topology inherited from the product topology on $\Sigma \times \mathrm{S}$. Note: $\Sigma * \mathrm{S}$ is a group bundle groupoid, and its unit space is identified with Σ. We show that $\Sigma * \mathrm{S}$ is locally compact Hausdorff.

LEMMA 2.2. *The groupoid $\Sigma * \mathrm{S}$ is second-countable, locally compact and Hausdorff.*

Proof. We first show that $\Sigma \times \mathrm{S}$ is second-countable, locally compact and Hausdorff. Then we show that $\Sigma * \mathrm{S}$ is closed in $\Sigma \times \mathrm{S}$.

As a subspace S is automatically second countable and Hausdorff. Since S is closed in G, it is locally compact. As a subspace of $\mathscr{C}(\mathrm{S})$, Σ is second-countable and Hausdorff. Since $\Sigma \cup \{\emptyset\}$ is compact in $\mathscr{C}(\mathrm{S})$ and $\mathscr{C}(\mathrm{S})$ is Hausdorff, we have that $\Sigma \cup \{\emptyset\}$ is closed in $\mathscr{C}(\mathrm{S})$. Also, $\mathscr{C}(\mathrm{S})$ Hausdorff implies $\{\emptyset\}$ is closed in $\mathscr{C}(\mathrm{S})$. Thus, $\Sigma = (\Sigma \cup \{\emptyset\}) \backslash \{\emptyset\}$ is open in $\mathscr{C}(\mathrm{S})$, and hence locally compact. Since both S and Σ are second-countable locally compact and Hausdorff, so is $\Sigma \times \mathrm{S}$.

We show that $\Sigma * \mathrm{S}$ is closed in $\Sigma \times \mathrm{S}$. Suppose that $\{(H_i, \gamma_i)\}$ is a sequence in $\Sigma * \mathrm{S}$ converging to some (H, γ) in $\Sigma \times \mathrm{S}$. Then $H_i \to H$ in $\mathscr{C}(\mathrm{S})$, $\gamma_i \in H_i$ for every i, and $\gamma_i \to \gamma$. Thus $\gamma \in H$, by the characterization of convergence in $\mathscr{C}(\mathrm{S})$ ([22], Lemma H.2). Then $(H, \gamma) \in \Sigma * \mathrm{S}$, which shows that $\Sigma * \mathrm{S}$ is closed in $\Sigma \times \mathrm{S}$. Since $\Sigma * \mathrm{S}$ is closed in $\Sigma \times \mathrm{S}$, it is locally compact. Second-countability and Hausdorffness are automatic for subspaces. Thus $\Sigma * \mathrm{S}$ is second-countable, locally compact and Hausdorff. ∎

The next lemma is used in the proof of Proposition 2.1. It shows that we can associate with every $f \in C_c(G)$ a function in $C_c(\Sigma * \mathrm{S})$.

LEMMA 2.3. *Suppose that $f \in C_c(G)$. For all $(H, \gamma) \in \Sigma * \mathrm{S}$ define*

$$F(H, \gamma) := f(\gamma).$$

Then $F \in C_c(\Sigma * \mathrm{S})$.

Proof. Note that F is just f composed with the projection onto the second coordinate. Since both f and the projection are continuous, it follows that F is also continuous.

We show that F has compact support. Let $\{(H_i, \gamma_i)\}$ be a sequence in the support of F. Then $\{\gamma_i\}$ is a sequence in the support of f. Since f has compact support, $\{\gamma_i\}$ has a convergent subsequence such that (after relabelling) $\gamma_j \to \gamma$ in $\mathrm{supp}(f)$. Then $\{H_j\}$ is a sequence in the compact space $\Sigma \cup \{\emptyset\}$. Thus $\{H_j\}$ has a convergent subsequence, such that (after relabelling) $H_k \to H$ in $\Sigma \cup \{\emptyset\}$. Since $\gamma_k \in H_k$ and $\gamma_k \to \gamma$, the characterization of convergent sequences in $\mathscr{C}(\mathrm{S})$ ([22], Lemma H.2) implies that $\gamma \in H$. Thus (H_i, γ_i) has a convergent subsequence (H_k, γ_k) converging to (H, γ) in $\mathrm{supp}(F)$. Thus $\mathrm{supp}(F)$ is compact, and so $F \in C_c(\Sigma * \mathrm{S})$. ∎

In [20], Renault deals with possibly non-Hausdorff groupoids. He therefore introduces locally conditionally compact groupoids. We use a result from [20] in Proposition 2.1 to claim that $\Sigma * \mathrm{S}$ has a Haar system. So we need to know that $\mathrm{S} = \bigcup_{u \in G^{(0)}} G_u^u$ is locally conditionally compact.

A set L in a groupoid G is called *left (respectively right) conditionally compact* if for every compact set $K \subset G^{(0)}$, the set $KL = L \cap r^{-1}(K)$ (respectively $LK = L \cap s^{-1}(K)$) is compact. The set L is called *conditionally compact* if it is both left and right conditionally compact. If every point in the groupoid has a conditionally compact neighbourhood, then G is *locally conditionally compact*.

LEMMA 2.4. *Let G be a locally compact Hausdorff groupoid. Then $\mathrm{S} = \bigcup_{u \in G^{(0)}} G_u^u$ is locally conditionally compact.*

Proof. Since S is closed in G, it is locally compact and Hausdorff in the relative topology from G. Let $\gamma \in \mathrm{S}$. Then γ has a compact neighbourhood $L \subset \mathrm{S}$. Let K be any compact set in $G^{(0)}$. Since $G^{(0)}$ is Hausdorff, K is closed in $G^{(0)}$. Since $r_{\mathrm{S}} := r|_{\mathrm{S}}$ is continuous, $r_{\mathrm{S}}^{-1}(K)$ is closed in S. The intersection of the compact set L and the closed set $r_{\mathrm{S}}^{-1}(K)$ is compact in S. Hence S is left conditionally compact. Put $s_{\mathrm{S}} := s|_{\mathrm{S}}$. Since $r_{\mathrm{S}} = s_{\mathrm{S}}$ on S, it follows that $L \cap s^{-1}(K)$ is also compact. Thus S is right conditionally compact. Since every $\gamma \in G$ has a conditionally compact neighbourhood, S is locally conditionally compact. ∎

Now we prove Proposition 2.1.

Proof of Proposition 2.1. Recall that $\Sigma * \mathrm{S}$ is a group-bundle groupoid with unit space Σ. By Lemma 2.4, S is locally conditionally compact. Hence, by Corollary 1.4 of [20], the groupoid $\Sigma * \mathrm{S}$ has a continuous Haar system $\{\beta^H\}_{H \in \Sigma}$.

Set $\beta^u := \beta^{G_u^u}$ for each $u \in G^{(0)}$. We claim that each β^u is a Haar measure on G_u^u. From the definition of a Haar system, each β^u is a non-zero Radon measure such that $\operatorname{supp}(\beta^u) = \operatorname{supp}(\beta^{G_u^u}) = G_u^u$. With $f \in C_c(G)$, we set $F(H, \gamma) := f(\gamma)$. Then $F \in C_c(\Sigma * \mathrm{S})$ by Lemma 2.3. Suppose that $x, \gamma \in G_u^u$. Then the left invariance of the Haar system $\{\beta^H\}_{H \in \Sigma}$ gives

$$\begin{aligned}\int_{G_u^u} f(x\gamma) \mathrm{d}\beta^u(\gamma) &= \int_{G_u^u} F(G_u^u, x\gamma) \, \mathrm{d}\beta^{G_u^u}(\gamma) = \int_{G_u^u} F[(G_u^u, x)(G_u^u, \gamma)] \, \mathrm{d}\beta^{G_u^u}(\gamma) \\ &= \int_{G_u^u} F(G_u^u, \gamma) \, \mathrm{d}\beta^{G_u^u}(\gamma) = \int_{G_u^u} f(\gamma) \, \mathrm{d}\beta^u(\gamma).\end{aligned}$$

Hence every β^u is a non-zero left invariant Radon measure on G_u^u, that is, β^u is a Haar measure on G_u^u.

For every $H \in \Sigma$, let Δ_H denote the modular function of the group H corresponding to a Haar measure β^H. For the particular case where $H = G_u^u$ for some $u \in G^{(0)}$, we write $\Delta_u := \Delta_{G_u^u}$. By Lemma 5.3 of [3] the map $D : \Sigma * \mathrm{S} \to \mathbb{R}$

given by $D(H,\gamma) = \Delta_H(\gamma)$ is continuous. Hence the pointwise product $F \cdot D^{-1/2}$ belongs to $C_c(\Sigma * \mathrm{S})$. Since $\{\beta^H\}_{H\in\Sigma}$ is a Haar system, the map

$$G_u^u \mapsto \int_{G_u^u} F(G_u^u,\gamma)D(G_u^u,\gamma)^{-1/2}\,\mathrm{d}\beta^{G_u^u}(\gamma) \tag{2.1}$$

is continuous. Also, by Lemma 1.5 of [20] the stabilizer map $u \mapsto G_u^u$ is Borel. By composing the stabilizer map with (2.1), we get that

$$u \mapsto \int_{G_u^u} F(G_u^u,\gamma)D(G_u^u,\gamma)^{-1/2}\,\mathrm{d}\beta^{G_u^u}(\gamma) = \int_{G_u^u} f(\gamma)\Delta_u(\gamma)^{-1/2}\,\mathrm{d}\beta^u(\gamma)$$

is Borel. ∎

3. A DIRECT INTEGRAL REPRESENTATION OF $C^*(G)$

In this section we construct a direct integral representation of $C^*(G)$, and give a condition on the measure which ensures that the direct integral representation is type I. The idea is to associate with every $u \in G^{(0)}$ the irreducible representation l^u of $C^*(G)$. Then with an appropriate measure on $G^{(0)}$, we combine all the l^u's and their representations spaces in a measurable way to form a new representation of $C^*(G)$. We construct our direct-integral Hilbert space via a Borel–Hilbert bundle, as defined Appendix F.2 of [22]. Thus our first goal is to construct a Borel–Hilbert bundle.

A *Polish space* is a topological space which is homeomorphic to a separable complete metric space. A subset E in a Polish space X is *analytic* if there is a Polish space Y and a continuous map $f : Y \to X$ such that $f(Y) = E$.

Suppose that $\{\mathcal{H}_x\}_{x\in X}$ is a family of non-zero Hilbert spaces indexed by a set X. Let

$$X * \mathcal{H} := \{(x,h) : x \in X, h \in \mathcal{H}_x\}$$

be the disjoint union and $\rho : X * \mathcal{H} \to X$ the projection onto the first coordinate. A *section* is a function $f : X \to X * \mathcal{H}$ such that $\rho \circ f(x) = x$. So a section has the form $f(x) = (x, \widehat{f}(x))$, with $\widehat{f}(x) \in \mathcal{H}_x$. As is common in the literature, we do not always make a distinction between f and $\widehat{f}$.

We recall the definition of a Borel–Hilbert bundle.

DEFINITION 3.1 ([22], Definition F.1). Let $\mathcal{H} = \{\mathcal{H}_x\}_{x\in X}$ be a family of separable Hilbert spaces indexed by an analytic Borel space X. Then $(X * \mathcal{H}, \rho)$ is a *Borel–Hilbert bundle* if $X * \mathcal{H}$ has a Borel structure such that:

(i) ρ is a Borel map;

(ii) there is a sequence $\{f_n\}$ of sections such that

(a) the maps $\widetilde{f}_n : X * \mathcal{H} \to \mathbb{C}$, defined by

$$\widetilde{f}_n(x,h) := (\widehat{f}_n(x) \mid h)_{\mathcal{H}_x},$$

are Borel for each n;

(b) for every m and n,

$$x \mapsto (\widehat{f_n}(x) \mid \widehat{f_m}(x))_{\mathcal{H}_x}$$

is Borel; and

(c) the functions $\{\widetilde{f_n}\} \cup \{\rho\}$ separate points of $X * \mathcal{H}$.

The sequence $\{f_n\}$ is called a *fundamental sequence* for $(X * \mathcal{H}, \rho)$. A *Borel section* is a section f of $(X * \mathcal{H}, \rho)$ such that

$$x \mapsto (\widehat{f}(x) \mid \widehat{f_n}(x))_{\mathcal{H}_x}$$

is Borel for all n. Let $\mathcal{B}(X * \mathcal{H})$ be the set of all Borel sections.

Suppose that $u \in G^{(0)}$. Recall that l^u denotes the irreducible representation of $C^*(G)$ that acts on the completion $\mathcal{H}_u$ of $C_c(G_u)$. We show that

$$G^{(0)} * \mathcal{H} := \{(u, h) : u \in G^{(0)}, h \in \mathcal{H}_u\}$$

is a Borel–Hilbert bundle by invoking the following proposition.

PROPOSITION 3.2 ([22], Proposition F.8). *Suppose that X is an analytic Borel space and that $\mathcal{H} = \{\mathcal{H}_x\}_{x \in X}$ is a family of separable Hilbert spaces. Suppose that $\{f_n\}$ is a countable family of sections of $X * \mathcal{H}$ such that conditions* (b) *and* (c) *of Definition* 3.1 *are satisfied. Then there is a unique analytic Borel structure on $X * \mathcal{H}$ such that $(X * \mathcal{H}, \rho)$ becomes an analytic Borel–Hilbert bundle and $\{f_n\}$ is a fundamental sequence.*

To apply Proposition 3.2 we need candidates for sections that satisfy conditions (b) and (c) of Definition 3.1. We use a sequence of functions from $C_c(G)$ that is dense in the inductive limit topology, and such that their restrictions to $C_c(G_u)$ are dense in $\mathcal{H}_u$, for every $u \in G^{(0)}$. It is almost certainly well-known to experts that $C_c(G)$ is separable in the inductive limit topology. We still give a proof as the construction is used to show that the sequence is also dense in each $\mathcal{H}_u$ when restricted to $C_c(G_u)$. When considering the inductive limit topology, it will suffice to know that if $f_i \to f$ uniformly and the $\operatorname{supp}(f_i)$ is eventually contained in some compact set, then $f_i \to f$ in the inductive limit topology. Note: the converse of this statement is false (see Example D.9 of [17] for a counter example).

LEMMA 3.3. *There is a countable sequence of functions $\{f_i\}$ in $C_c(G)$ which is dense in $C_c(G)$ in the inductive limit topology. Moreover, the restrictions $\{f_i|_{G_u}\}$ are dense in $\mathcal{H}_u$ for every $u \in G^{(0)}$.*

Proof. Suppose that U is an open set with compact closure in G. Then every $f \in C_0(U)$ extends to $C_c(G)$ by putting $f(x) = 0$ if $x \notin U$. In this way we view $C_0(U)$ as a $*$-subalgebra of $C_c(G)$ consisting of functions which vanish outside of the compact set $\overline{U}$. Since G is second-countable and locally compact, we can write G as the union of a sequence of open sets $\{U_i\}$ such that $U_i \subset U_{i+1}$ and $\overline{U}_i$ is compact. The set U_i is second-countable for every $i \in \mathbb{N}$. Thus $C_0(U_i)$ is a separable Banach space in the uniform norm $\|\cdot\|_\infty$. For each $i \in \mathbb{N}$, let $\{f_{i_k}\}_k$ be a countable dense set in $C_0(U_i)$, which we view as a subset of $C_c(G)$.

Let $f \in C_c(G)$. We show that f can be approximated by functions of the form $\{f_{i_k}\}_k$ in the inductive limit topology. Since $\{U_i\}$ is an increasing sequence of open sets with compact closure, it follows that $\operatorname{supp}(f) \subset U_i$ for some i. Then $f \in C_0(U_i)$ and there is a subsequence $\{f_{i_{k(j)}}\}_j$ of $\{f_{i_k}\}_k$ such that $f_{i_{k(j)}} \to f$ uniformly in $C_0(U_i)$ as $j \to \infty$. Viewing each $f_{i_{k(j)}}$ as an element of $C_c(G)$ we have $\operatorname{supp}(f_{i_{k(j)}}) \subset \overline{U}_i$ for all j. Thus $f_{i_{k(j)}}$ converges to f in $C_c(G)$ in the inductive limit topology. Since f was arbitrary, it follows that $\{f_{i_k}\}_{i,k}$ is countable and dense in $C_c(G)$ in the inductive limit topology.

Fix $u \in G^{(0)}$. We show that $\{f_{i_k}\}_{i,k}$ restricted to G_u is dense in $\mathcal{H}_u$. Suppose that $h \in \mathcal{H}_u$ and let $\|\cdot\|_u$ denote the norm defined by the inner product on $\mathcal{H}_u$. Let $\varepsilon > 0$. Then there is an $f \in C_c(G_u)$ such that

$$\|h - f\|_u < \frac{\varepsilon}{2}. \tag{3.1}$$

Since G_u is closed in G, the support of f is compact in G. By Lemma 1.42 of [22], we extend f to $C_c(G)$ (using the same notation f for the extension). Since $\{U_i\}$ is an increasing chain of relatively compact sets, there is an i such that $\operatorname{supp}(f) \subset U_i$. Then (after relabelling) there is a subsequence $\{f_j\}$ of $\{f_{i_k}\}_{i,k}$ such that $f_j \to f$ uniformly and $\operatorname{supp}(f_j) \subset \overline{U}_i$ for every j.

We now consider two cases. First suppose that $\overline{U}_i \cap G_u^u = \emptyset$. Then $\operatorname{supp}(f - f_j) \cap G_u^u = \emptyset$, and thus

$$\begin{aligned}\|f-f_j\|_u &= [(f-f_j, f - f_j)_u]^{1/2} = \Big[\int_{G_u^u} ((f-f_j)^* * (f-f_j))(t)\, \Delta_u(t)^{-1/2}\, \mathrm{d}\beta^u(t)\Big]^{1/2} \\ &= \Big[\int_{G_u^u}\int_{G_u^u} (f - f_j)^*(s)(f - f_j)(s^{-1}t)\, \mathrm{d}\beta^u(s)\, \Delta_u(t)^{-1/2}\, \mathrm{d}\beta^u(t)\Big]^{1/2} = 0. \end{aligned} \tag{3.2}$$

Second, suppose that $\overline{U}_i \cap G_u^u \neq \emptyset$. Put

$$M^2 := \Big(\sup\{\Delta_u^{-1/2}(t) : t \in \overline{U}_i \cap G_u^u\}\Big)(\beta^u(\overline{U}_i \cap G_u^u))^2.$$

Then $M^2 < \infty$, since Δ_u is continuous in t and $\overline{U}_i \cap G_u^u$ is compact. Then we have that

$$\begin{aligned}\|f-f_j\|_u &= [(f-f_j, f-f_j)_u]^{-1/2} = \Big[\int_{G_u^u} ((f-f_j)^* * (f-f_j))(t)\, \Delta_u(t)^{-1/2}\, \mathrm{d}\beta^u(t)\Big]^{1/2} \\ &= \Big[\int_{G_u^u}\int_{G_u^u} (f - f_j)^*(s)(f - f_j)(s^{-1}t)\, \mathrm{d}\beta^u(s)\, \Delta_u(t)^{-1/2}\, \mathrm{d}\beta^u(t)\Big]^{1/2}\end{aligned}$$

$$\leqslant \left[\|f - f_j\|_\infty^2 \sup_{t \in \overline{U}_i \cap G_u^u} \{\Delta_u(t)^{-1/2}\}\, \beta^u(\overline{U}_i \cap G_u^u)^2\right]^{1/2} = \|f - f_j\|_\infty M. \tag{3.3}$$

Since $f_j \to f$ uniformly and $\mathrm{supp}(f_j) \subset \overline{U}_i$ for every j, it follows that there is a $j_0 \in \mathbb{N}$ such that if $j > j_0$, then

$$\|f - f_j\|_\infty < \frac{\varepsilon}{2M}.$$

Fix any $j > j_0$. Then it follows from (3.1), (3.2) and (3.3) that

$$\|h - f_j\|_u \leqslant \|h - f\|_u + \|f - f_j\|_u \leqslant \|h - f\|_u + \|f - f_j\|_\infty M < \varepsilon.$$

Thus every neighborhood of h contains some $f_j \in \{f_{i_k}\}_{i,k}$, which shows that $\{f_{i_k}\}_{i,k}$ is dense in $\mathcal{H}_u$. ∎

Next we show that $G^{(0)} * \mathcal{H}$ is a Borel–Hilbert bundle.

PROPOSITION 3.4. *There is a sequence $\{f_i\}$ of functions that is dense in $C_c(G)$ in the inductive limit topology. For every i and every $u \in G^{(0)}$ put*

$$\widehat{g}_i(u) := f_i|_{G_u}.$$

Then $G^{(0)} * \mathcal{H}$ is a Borel–Hilbert bundle with fundamental sequence $\{g_i\}$ given by

$$g_i(u) := (u, \widehat{g}_i(u)).$$

Proof. By Lemma 3.3 there is a sequence $\{f_i\} \subset C_c(G)$ which is dense in $C_c(G)$ in the inductive limit topology.

We continue our convention of writing $g_i(u)$ to mean $\widehat{g}_i(u)$ if no confusion is possible. We show that the conditions of Proposition 3.2 are satisfied for $G^{(0)} * \mathcal{H}$ and $\{g_i\}$. Because G is Hausdorff, $G^{(0)}$ is closed in G. Thus $G^{(0)}$ is also second-countable, locally compact and Hausdorff. Hence $G^{(0)}$ is Polish ([22], Lemma 6.5), and thus an analytic Borel space.

Next we show that the sequence of sections $\{g_i\}$ satisfy conditions (b) and (c) of Definition 3.1. Fix $m, n \in \mathbb{N}$. Then

$$u \mapsto (g_n(u) \mid g_m(u))_u = \int_{G_u^u} (f_m^* * f_n)(t)\, \Delta_u(t)^{-1/2}\, \mathrm{d}\beta^u(t),$$

which is Borel by Proposition 2.1. Thus (b) is satisfied.

Let $\rho : G^{(0)} * \mathcal{H} \to G^{(0)}$ be the projection onto the first coordinate. Let $\widetilde{g}_i$ be as in Definition 3.1(a). We show that $\{\widetilde{g}_i\} \cup \{\rho\}$ separate the points of $G^{(0)} * \mathcal{H}$. Suppose that $\{\widetilde{g}_i\} \cup \{\rho\}$ do not separate points. Then there exist distinct points (u, h) and (v, k) in $G^{(0)} * \mathcal{H}$ such that, for every $\phi \in \{\widetilde{g}_i\} \cup \{\rho\}$, we have $\phi(u, h) = \phi(v, k)$. First notice that if $\phi = \rho$, then $\rho(u, h) = \rho(v, k)$ implies $u = v$. Thus $k \in \mathcal{H}_u$. Then, besides ρ, we have $\widetilde{g}_i(u, h) = \widetilde{g}_i(u, k)$ for every $i \in \mathbb{N}$. That is,

$$(g_i(u) \mid h) = (g_i(u) \mid k),$$

or

$$0 = (g_i(u) \mid h - k) = (f_i|_{G_u} \mid h - k).$$

Thus $h-k$ is in the orthogonal complement $\{f_i|_{G_u} : i \in \mathbb{N}\}^\perp$ in $\mathcal{H}_u$. By Lemma 3.3 the set $\{f_i|_{G_u}\}$ is dense in $\mathcal{H}_u$. Thus $\{f_i|_{G_u} : i \in \mathbb{N}\}^\perp = \{0\}$, implying $h = k$. So $(u,h) = (v,k)$, which contradicts the assumption that these points are distinct. So condition (c), and hence all the conditions of Proposition 3.2 are satisfied, showing that $G^{(0)} * \mathcal{H}$ is a Borel–Hilbert bundle with fundamental sequence $\{g_i\}$. ∎

Next we form an L^2-space with sections of $\mathcal{B}(G^{(0)} * \mathcal{H})$. To form this L^2-space we use a quasi-invariant measure (see for example Definition 3.1 in [19]). Since we work in second-countable, locally compact Hausdorff spaces, all Borel measures are σ-finite. The class of measures equivalent to any such σ-finite measure contains a finite measure. For groupoids the notions of quasi-invariance and ergodicity depend on the measure class. So we may assume without loss of generality that our quasi-invariant measure is a probability measure. Also, if G is second-countable, locally compact and Hausdorff, then there always exists a quasi-invariant measure on $G^{(0)}$ ([19], Proposition 3.6).

Let μ be a quasi-invariant measure on $G^{(0)}$ and let

$$\mathscr{L}^2(G^{(0)} * \mathcal{H}, \mu) := \{f \in \mathcal{B}(G^{(0)} * \mathcal{H}) : u \mapsto \|f(u)\|_u^2 \text{ is } \mu\text{-integrable}\}.$$

Let $L^2(G^{(0)} * \mathcal{H}, \mu)$ be the vector space formed by taking the quotient of $\mathscr{L}^2(G^{(0)} * \mathcal{H}, \mu)$, where sections agreeing μ-a.e. are equivalent. As is common in the literature, we use the same symbol f for the class of sections in $L^2(G^{(0)} * \mathcal{H}, \mu)$ to which f belongs. Let $f, g \in L^2(G^{(0)} * \mathcal{H}, \mu)$. The functions $u \mapsto \|f(u)\|_u^2$ and $u \mapsto \|g(u)\|_u^2$ belong to $L^2(G^{(0)}, \mu)$. By Hölder's inequality $u \mapsto \|f(u)\|_u \, \|g(u)\|_u$ is in $L^1(G^{(0)}, \mu)$. Then, by the Cauchy–Schwarz inequality

$$\begin{aligned}\Big| \int_{G^{(0)}} (f(u) \mid g(u))_u \mathrm{d}\mu(u) \Big| &\leqslant \int_{G^{(0)}} |(f(u) \mid g(u))_u| \, \mathrm{d}\mu(u) \\ &\leqslant \int_{G^{(0)}} \|f(u)\|_u \|g(u)\|_u \, \mathrm{d}\mu(u) < \infty.\end{aligned}$$

Thus $u \mapsto (f(u) \mid g(u))_u$ is μ-integrable, and so

$$(f \mid g) := \int_{G^{(0)}} (f(u) \mid g(u))_u \, \mathrm{d}\mu(u)$$

defines an inner product on $L^2(G^{(0)} * \mathcal{H}, \mu)$. With this inner product $L^2(G^{(0)} * \mathcal{H}, \mu)$ is a Hilbert space. The Hilbert space $L^2(G^{(0)} * \mathcal{H}, \mu)$ is what is known as the Hilbert space direct integral, also denoted by $\int_{G^{(0)}}^{\oplus} \mathcal{H}_u \, \mathrm{d}\mu(u)$ in the literature.

We turn our attention to the direct integral representation, which will act on $L^2(G^{(0)} * \mathcal{H}, \mu)$. Fix $a \in C^*(G)$. For every $f \in L^2(G^{(0)} * \mathcal{H}, \mu)$ and $u \in G^{(0)}$, define $(L(a)f)(u) := (u, l^u(a)(f(u)))$. We write

$$(L(a)f)(u) = l^u(a)(f(u)) \tag{3.4}$$

to shorten notation.

PROPOSITION 3.5. *Let $a \in C^*(G)$. Then $L(a)$ is a bounded linear operator on $L^2(G^{(0)} * \mathcal{H}, \mu)$, and the map $a \mapsto L(a)$ defines a representation of $C^*(G)$ on $L^2(G^{(0)} * \mathcal{H}, \mu)$.*

Proof. Note that the linearity of $L(a)$ follows from the linearity of each $l^u(a)$. To show that $L(a)$ is a bounded linear operator on $L^2(G^{(0)} * \mathcal{H}, \mu)$, we first show that $L(a)$ maps Borel sections to Borel sections. Let $k \in \mathcal{B}(G^{(0)} * \mathcal{H})$. We claim that $L(a)k \in \mathcal{B}(G^{(0)} * \mathcal{H})$. To see this, let $\{g_i\}$ be the fundamental sequence given by Proposition 3.4. Recall, $g_i(u) = (u, g_i(u))$ with $g_i(u) = f_i|_{G_u}$, and where $\{f_i\}$ is a countable sequence in $C_c(G)$ which is dense in the inductive limit topology. We must show that

$$u \mapsto ((L(a)k)(u) \mid g_n(u))_u = (l^u(a)(k(u)) \mid g_n(u))_u$$

is Borel for every n. Since $\{g_i\}$ is a fundamental sequence, it is sufficient to show that

$$u \mapsto (l^u(a)g_n(u) \mid g_m(u))_u \tag{3.5}$$

is Borel for all m and n ([5], Proposition 1). First we show that (3.5) is Borel for functions in the dense subspace $C_c(G)$ of $C^*(G)$, and then for an arbitrary $a \in C^*(G)$. Let $h \in C_c(G)$ and fix m and n. Then

$$\begin{aligned}(l^u(h)g_n(u) \mid g_m(u))_u &= \int\limits_{G_u^u} [f_m^* * (l^u(h)f_n)](t)\Delta_u(t)^{-1/2}\mathrm{d}\beta^u(t)\\ &= \int\limits_{G_u^u} [f_m^* * (h * f_n)](t)\Delta_u(t)^{-1/2}\mathrm{d}\beta^u(t)\\ &= \int\limits_{G_u^u} (f_m^* * h * f_n)(t)\Delta_u(t)^{-1/2}\mathrm{d}\beta^u(t).\end{aligned}$$

Since $f_m^* * h * f_n \in C_c(G)$, it follows from Proposition 2.1 that

$$u \mapsto \int\limits_{G_u^u} (f_m^* * h * f_n)(t)\Delta_u(t)^{-1/2}\mathrm{d}\beta^u(t) \tag{3.6}$$

is Borel. Now, let $a \in C^*(G)$ be arbitrary. Then there is a sequence $\{h_i\}$ in $C_c(G)$ such that $h_i \to a$ in the C^*-norm. Put $\phi_i(u) = (l^u(h_i)g_n(u) \mid g_m(u))_u$ and $\phi(u) = (l^u(a)g_n(u) \mid g_m(u))_u$. By (3.6), $\{\phi_i\}$ is a sequence of Borel measurable functions on $G^{(0)}$ for all m and n. Since inner products and the l^u's are continuous, $\phi_i(u) \to \phi(u)$ for every $u \in G^{(0)}$. This pointwise convergence implies that ϕ is Borel measurable. Hence the map (3.5) is Borel and it follows that $L(a)f \in \mathcal{B}(G^{(0)} * \mathcal{H})$.

Next we show that $L(a)$ maps $L^2(G^{(0)} * \mathcal{H}, \mu)$ into $L^2(G^{(0)} * \mathcal{H}, \mu)$. That is, we show that for every $k \in L^2(G^{(0)} * \mathcal{H}, \mu)$ and $a \in C^*(G)$, the map $u \mapsto \|(L(a)k)(u)\|_u^2$ is μ-integrable. In this case we have

$$\begin{aligned}\int\limits_{G^{(0)}} \|(L(a)k)(u)\|_u^2 \,\mathrm{d}\mu(u) &= \int\limits_{G^{(0)}} \|l^u(a)(k(u))\|_u^2 \,\mathrm{d}\mu(u) \\ &\leqslant \|a\|^2 \int\limits_{G^{(0)}} \|k(u)\|_u^2 \,\mathrm{d}\mu(u) = \|a\|^2 \|k\|_2^2 < \infty. \end{aligned} \tag{3.7}$$

Thus the map $u \mapsto \|(L(a)k)(u)\|_u^2$ is μ-integrable, showing that $L(a)k \in L^2(G^{(0)} * \mathcal{H}, \mu)$.

The boundedness of $L(a)$ follows from (3.7), since

$$\|L(a)k\|_2^2 = \int\limits_{G^{(0)}} \|l^u(a)k(u)\|_u^2 \,\mathrm{d}\mu(u) \leqslant \|a\|^2 \|k\|_2^2.$$

Hence $L(a) \in B(L^2(G^{(0)} * \mathcal{H}, \mu))$ for every $a \in C^*(G)$.

Lastly, that $a \mapsto L(a)$ is a representation of $C^*(G)$ follows from (3.4) and the fact that every l^u is a representation of $C^*(G)$. ∎

The representation L of (3.4) is a *direct integral representation* ([6], Definition 8.1.3), and is denoted by

$$L := \int_{G^{(0)}}^{\oplus} l^u \,\mathrm{d}\mu(u).$$

The operators $L(a), a \in C^*(G)$, are called *decomposable operators*, and are denoted by

$$L(a) := \int_{G^{(0)}}^{\oplus} l^u(a) \,\mathrm{d}\mu(u).$$

We need a few last remarks on von Neumann algebras and some lemmas which we use to prove a groupoid version of Effros' lemma for transformation groups.

If $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ a self-adjoint subset of $B(\mathcal{H})$, then we denote by $\mathcal{M}'$ the *commutant* of $\mathcal{M}$. We say $\mathcal{M}$ is a von Neumann algebra if $\mathcal{M} = \mathcal{M}''$. The *centre* of a von Neumann algebra $\mathcal{M}$ is the abelian von Neumann algebra

$$\mathcal{Z}(\mathcal{M}) := \mathcal{M}' \cap \mathcal{M}.$$

LEMMA 3.6. *Let $\mathcal{M}$ be a von Neumann algebra and suppose that $\mathcal{N}$ is a maximal abelian von Neumann subalgebra of $\mathcal{M}'$, in the sense that $\mathcal{N}$ is not properly contained in any other abelian von Neumann subalgebra of $\mathcal{M}'$. Then $\mathcal{Z}(\mathcal{M}) \subset \mathcal{N}$.*

Proof. The structure of the proof is as follows: we first show that the von Neumann algebra $\overline{K}^{\mathrm{sot}}$ generated by $\mathcal{Z}(\mathcal{M})$ and $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}'$. Then we show that $\overline{K}^{\mathrm{sot}}$ is an abelian von Neumann algebra. Lastly, we show that $\overline{K}^{\text{sot}}$ contains both $\mathcal{Z}(\mathcal{M})$ and $\mathcal{N}$. By the maximality of $\mathcal{N}$, we then have $\mathcal{Z}(\mathcal{M}) \subset \mathcal{N}$.

Suppose that $z \in \mathcal{Z}(\mathcal{M})$ and $n \in \mathcal{N}$. Then $z \in \mathcal{M}$ and $n \in \mathcal{M}'$. Thus $zn = nz$. Hence any product formed from elements of $\mathcal{Z}(\mathcal{M})$ and $\mathcal{N}$ are of the form zn, with $z \in \mathcal{Z}(\mathcal{M})$ and $n \in \mathcal{N}$. Let $K = \operatorname{span}\{zn : z \in \mathcal{Z}(\mathcal{M}), n \in \mathcal{N}\}$. Then K is an abelian $*$-subalgebra of $\mathcal{M}'$, since $\mathcal{M}'$ is itself a von Neumann algebra and contains both $\mathcal{Z}(\mathcal{M})$ and $\mathcal{N}$. Then von Neumann's double commutant theorem ([2], Theorem 2.4.11) implies the strong operator closure $\overline{K}^{\text{sot}}$ of K is a von Neumann subalgebra of $\mathcal{M}'$. Also, $\overline{K}^{\text{sot}}$ contains both $\mathcal{Z}(\mathcal{M})$ and $\mathcal{N}$.

We claim that $\overline{K}^{\text{sot}}$ is abelian. First suppose that $S \in K$ and $T \in \overline{K}^{\text{sot}}$. Let (T_α) be a net in K converging to T in the strong operator topology. The maps $T \to ST$ and $T \to TS$ are continuous in the strong operator topology (for the fixed S). Thus $T_\alpha S \to TS$. Since K is an abelian $*$-algebra we also have that $T_\alpha S = ST_\alpha \to ST$. Since the strong operator topology is Hausdorff, it follows that

$$ST = TS. \tag{3.8}$$

Now let $S, T \in \overline{K}^{\text{sot}}$, and let $\{S_\alpha\}$ be a sequence in K such that $S_\alpha \to S$ in the strong operator topology. Applying equation (3.8), we have that $S_\alpha T = TS_\alpha$ for every α. Taking the limit now shows that $ST = TS$.

We showed that $\overline{K}^{\text{sot}}$ is an abelian von Neumann subalgebra of $\mathcal{M}'$ which contains both $\mathcal{Z}(\mathcal{M})$ and $\mathcal{N}$. But, $\mathcal{N}$ is maximal abelian in $\mathcal{M}'$. Thus $\overline{K}^{\text{sot}} = \mathcal{N}$ and $\mathcal{Z}(\mathcal{M}) \subset \mathcal{N}$. ∎

DEFINITION 3.7. Let $\mathscr{B}(G^{(0)})$ be the Borel subsets of $G^{(0)}$. A *projection-valued measure* E for $L^2(G^{(0)} * \mathcal{H}, \mu)$ is a function from $\mathscr{B}(G^{(0)})$ into the set of orthogonal projections on $L^2(G^{(0)} * \mathcal{H}, \mu)$ such that:

(i) $E(G^{(0)}) = 1$;

(ii) $E(A \cap B) = E(A)E(B)$, for $A, B \in \mathscr{B}(G^{(0)})$; and

(iii) $E(\bigcup A_i) = \sum E(A_i)$ for pairwise disjoint Borel subsets A_i.

For $A \subset G^{(0)}$ let 1_A denote the characteristic function on A. Then, for every $f \in L^2(G^{(0)} * \mathcal{H}, \mu)$,

$$(E_A f)(u) := 1_A(u) f(u) \tag{3.9}$$

defines a projection-valued measure called the *canonical projection-valued measure*. Note that for a fixed $A \subset G^{(0)}$, the projection E_A is the decomposable operator $E_A = \int_{G^{(0)}}^{\oplus} 1_A(u)\, I_u \,\mathrm{d}\mu(u)$.

Applying Corollary IV.12 of [9] to our specific direct integral representation $L = \int_{G^{(0)}}^{\oplus} l^u \mathrm{d}\mu(u)$ of $C^*(G)$ gives the following proposition.

PROPOSITION 3.8 ([9], Corollary IV.12). *The representations l^u in the direct integral representation $L = \int_{G^{(0)}}^{\oplus} l^u \mathrm{d}\mu(u)$ are μ-almost all irreducible if and only if the range of the canonical projection valued measure for $L^2(G^{(0)} * \mathcal{H}, \mu)$ is a maximal abelian algebra of projections in $L(C^*(G))'$.*

A set $A \subset G^{(0)}$ is *invariant* if $r(s^{-1}(A)) = A$. If μ is a quasi-invariant measure on $G^{(0)}$, then μ is *ergodic* if $\mu(A) = 0$ or $\mu(G^{(0)} \backslash A) = 0$ for all invariant sets $A \subset G^{(0)}$. If, in addition, μ is concentrated on an orbit, then μ is *trivially ergodic*.

LEMMA 3.9. *Let $A \subset G^{(0)}$ be a Borel set and μ a Borel measure on $G^{(0)}$. Then $r(s^{-1}(A))$ is measurable.*

Proof. The continuity of r and s implies that they are Borel measurable. Hence $s^{-1}(A)$ is a Borel subset of G and thus analytic. Then $r(s^{-1}(A))$ is an analytic set in $G^{(0)}$ by Theorem 3.3.4 and Corollary 1 of [1]. Hence

$$r(s^{-1}(A))$$

is μ-measurable by Theorem 3.2.4 of [1]. ∎

Suppose that π is a representation of a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Then π is a *factor representation* if the centre of the von Neumann algebra $\pi(\mathcal{A})''$ consists of scalar multiples of the identity operator on $\mathcal{H}$. If the von Neumann algebra $\pi(\mathcal{A})''$ is a type I von Neumann algebra then π is a *type I representation*. The C^*-algebra $\mathcal{A}$ is type I if every representation of $\mathcal{A}$ is type I.

We now adapt Effros' proof of Lemmma 4.2 in [8] from transformation groups to groupoids. We only need one direction of Effros' "if and only if" statement, that is, type I implies trivially ergodic.

PROPOSITION 3.10. *Let μ be an ergodic Borel measure on $G^{(0)}$. Then*

(i) *the direct integral representation $L = \int_{G^{(0)}}^{\oplus} l^u \mathrm{d}\mu(u)$ is a factor representation of $C^*(G)$; and*

(ii) *if L is a type* I *factor representation, then μ is trivially ergodic.*

We give a brief overview of the structure of the proof of (i) in an attempt to make the proof easier to read. We prove the contrapositive of (i), and split the proof into three main parts.

Existence of a direct integral projection. We show that if L is not a factor representation, then there is a projection P and a Borel set $B \in \mathscr{B}(G^{(0)})$ such that $P = \int_{G^{(0)}} 1_B(u) I_u \mathrm{d}\mu(u)$ with $P \neq I$ and $P \neq 0$.

Convergence of integrands off a null set N. We show that there is sequence $\{a_n\} \subset C^*(G)$ and a null set $N \in \mathscr{B}(G^{(0)})$ such that $l^u(a_{n_k}) \to 1_B(u) I_u$ strongly for every $u \in G^{(0)} \backslash N$.

An invariant set neither null nor conull. Lastly, we show that $C := r(s^{-1}(B \setminus N))$ is invariant, and is neither null nor conull under μ. That is, μ is not ergodic.

Proof. (i) *Existence of a direct integral projection*. Suppose that L is not a factor representation of $C^*(G)$. Then the centre $L(C^*(G))' \cap L(C^*(G))''$ has elements other than multiples of the identity. Since von Neumann algebras are generated by their projections ([16], Theorem 4.1.11) and the centre is an abelian von Neumann algebra, there exists a projection $P \in L(C^*(G))' \cap L(C^*(G))''$ with $P \neq 0$ and $P \neq I$ (I being the identity operator in $B(L^2(G^{(0)} * \mathcal{H}, \mu))$). Let E denote the canonical projection-valued measure from $\mathscr{B}(G^{(0)})$ into the set of orthogonal projections in $B(L^2(G^{(0)} * \mathcal{H}, \mu))$. Since every l^u is irreducible, it follows from Proposition 3.8 that the range $E(\mathscr{B}(G^{(0)}))$ of the canonical projection-valued measure is a maximal abelian algebra in $L(C^*(G))'$. Thus the set of projections contained in the image of the canonical projection-valued measure generates a maximal abelian von Neumann subalgebra $E(\mathscr{B}(G^{(0)}))''$ of $L(C^*(G))'$. Then $L(C^*(G))' \cap L(C^*(G))'' \subset E(\mathscr{B}(G^{(0)}))''$, by Lemma 3.6. Hence $P \in E(\mathscr{B}(G^{(0)}))''$. Because $E(\mathscr{B}(G^{(0)}))''$ is generated by its projections $E(\mathscr{B}(G^{(0)}))$, it follows that $P \in E(\mathscr{B}(G^{(0)}))$. Hence there is a Borel set B in $\mathscr{B}(G^{(0)})$ such that $0 \neq \mu(B) \neq \mu(G^{(0)})$ and $P = E_B = \int_{G^{(0)}} 1_B(u) I_u \mathrm{d}\mu(u)$ (where 1_B is the indicator function of B).

Convergence of integrands off a null set N. By Lemmma 3.1 of [7] L is a non-degenerate representation because each l^u is non-degenerate. By von Neumann's double commutant theorem ([2], Theorem 2.4.11) $L(C^*(G))$ is dense in $L(C^*(G))''$ in the strong operator topology. By Kaplansky's density theorem the unit ball of $L(C^*(G))$ is strongly dense in the unit ball of $L(C^*(G))''$ ([13], Theorem 5.3.5). With Kaplansky's density theorem and because the unit ball of $L(C^*(G))''$ is metrizable in the strong operator topology ([5], Proposition 1) we may replace nets with sequences in the strong closure of the unit ball of $L(C^*(G))$. Thus, since $\|E_B\| = 1$, there is a sequence $\{a_n\} \subset C^*(G)$ such that $L(a_n)$ is in the unit ball of $L(C^*(G))''$, and

$$L(a_n) = \int_{G^{(0)}}^{\oplus} l^u(a_n)\, \mathrm{d}\mu(u) \to E_B = \int_{G^{(0)}} 1_B(u) I_u \, \mathrm{d}\mu(u)$$

in the strong operator topology. Now, due to the strong convergence of the sequence of direct integrals above ([5], Proposition 4 of Part II, Chapter 2) tells us there is a subsequence a_{n_k} such that for μ-a.e. $u \in G^{(0)}$

$$l^u(a_{n_k}) \to 1_B(u) I_u \tag{3.10}$$

strongly. That is, there is a Borel set $N \subset G^{(0)}$ such that $\mu(N) = 0$ and $l^u(a_{n_k}) \to 1_B(u) I_u$ strongly for every $u \in G^{(0)} \setminus N$.

An invariant set neither null nor conull. Let $C := r(s^{-1}(B \setminus N))$. Then C is an invariant subset of $G^{(0)}$. Moreover C is measurable by Lemma 3.9. It will suffice to show that $\mu(C) = \mu(B)$, because then $\mu(C) \neq 0$ and $\mu(G^{(0)} \setminus C) =$

$\mu(G^{(0)}) - \mu(C) = \mu(G^{(0)}) - \mu(B) \neq 0$, which shows that μ is not ergodic. Note, since $B\backslash N \subset C$, it follows that $\mu(B\backslash N) \leqslant \mu(C)$. Then, since $\mu(N) = 0$, we get

$$\mu(B\backslash N) = \mu(B) - \mu(N) = \mu(B). \tag{3.11}$$

Thus $\mu(B) \leqslant \mu(C)$. Similarly, to get the reverse inequality we show that $C\backslash N \subset B$. Suppose that $w \in C\backslash N$. Since $C = \{u \in G^{(0)} : u \in [v] \text{ and } v \in B\backslash N\}$, there is a $v \in B\backslash N$ such that v is equivalent to w. Lemma 5.1 of [3] shows that the map $[u] \mapsto [l^u]$ from the orbit space $G^{(0)}/G$ into the spectrum $C^*(G)^\wedge$ is well-defined. Lemma 5.5 of [3] shows that this map $[u] \mapsto [l^u]$ is injective. Hence v is equivalent to w if and only if l^v is unitarily equivalent to l^w. So since $v \sim w$ there is a unitary operator $U : \mathcal{H}_v \to \mathcal{H}_w$ such that $l^w(a) = Ul^v(a)U^*$, for every $a \in C^*(G)$. Since $w \notin N$, we apply (3.10) to l^w, that is,

$$l^w(a_{n_k}) \to 1_B(w)I_w.$$

Since $v \in B\backslash N$, it follows that $1_B(v) = 1$, and we can also apply (3.10) to l^v. Then

$$l^w(a_{n_k}) = Ul^v(a_{n_k})U^* \to U1_B(v)I_vU^* = UI_vU^* = I_w.$$

Limits are unique in the strong operator topology. Thus $1_B(w)I_w = I_w$, which implies that $1_B(w) = 1$. Hence $w \in B$. That is, $C\backslash N \subset B$. Now a similar computation to (3.11) shows that $\mu(C) \leqslant \mu(B)$. Hence $\mu(C) = \mu(B)$, proving (i).

(ii) Suppose that L is a factor representation of type I. Then by Theorem 2.7 of [15], almost all l^u are unitarily equivalent. That is, there is a conull set $A \subset G^{(0)}$ such that $u \sim v$ for all $u, v \in A$. Suppose that $u \in A$. Then $A \subset \{w \in G^{(0)} : w \sim u\}$ and $\mu(\{w \in G^{(0)} : w \sim u\}) \neq 0$. On the other hand, $\mu(\{v \in G^{(0)} : v \nsim u\}) = 0$. Thus

$$\begin{aligned} \mu(G^{(0)}) &= \mu(\{w \in G^{(0)} : w \sim u\} \cup (G^{(0)}\backslash\{w \in G^{(0)} : w \sim u\})) \\ &= \mu(\{w \in G^{(0)} : w \sim u\}) + \mu(\{v \in G^{(0)} : v \nsim u\}) \\ &= \mu(\{w \in G^{(0)} : w \sim u\}). \end{aligned}$$

Hence μ is concentrated on an orbit, and is thus trivially ergodic. ∎

4. CHARACTERIZING GCR GROUPOID C^*-ALGEBRAS

After one last lemma we prove Theorem 4.2 which says that if $C^*(G)$ is type I (or equivalently GCR) then $G^{(0)}/G$ is T_0.

LEMMA 4.1. *Let G be a second-countable, locally compact and Hausdorff groupoid. Let $R : G \to G^{(0)} \times G^{(0)}$, defined by $R(\gamma) := (r(\gamma), s(\gamma))$, be the equivalence relation induced on $G^{(0)}$. Then $R(G)$ is an F_σ subset in $G^{(0)} \times G^{(0)}$.*

Proof. Since G is second-countable and locally compact we can express G in form $G = \bigcup_{i=1}^{\infty} U_i$, where each U_i is a neighborhood with compact closure. Since the range and source maps are continuous and $G^{(0)}$ is Hausdorff, it follows that $r(\overline{U}_i)$ and $s(\overline{U}_i)$ are compact for every i, and thus closed in $G^{(0)} \times G^{(0)}$. So $R(G) = R\Big(\bigcup_{i=1}^{\infty} \overline{U}_i\Big) = \bigcup_{i=1}^{\infty} R(\overline{U}_i)$ is an F_σ set in $G^{(0)} \times G^{(0)}$. ∎

THEOREM 4.2. *Let G be a second-countable locally compact and Hausdorff groupoid with a Haar system. If $C^*(G)$ is type* I *then $G^{(0)}/G$ is T_0.*

Proof. We prove the contrapositive. Suppose that $G^{(0)}/G$ is not T_0. By Lemma 4.1 the hypotheses of Theorem 2.1 in [18] are satisfied. So there exists a non-trivial ergodic measure μ on $G^{(0)}$. By Proposition 3.10 the direct integral representation $L = \int l^u \mathrm{d}\mu$ is a non-type I factor representation. Hence $C^*(G)$ is not type I, which concludes the proof. ∎

Combining Theorem 4.2 and Clark's Theorem 7.1 in [3] we can formulate a refined characterization of GCR groupoids C^*-algebras without amenability.

THEOREM 4.3. *Let G be a second-countable, locally compact and Hausdorff groupoid with a Haar system. Then $C^*(G)$ is* GCR *if and only if the stability subgroups of G are* GCR *and $G^{(0)}/G$ is T_0.*

Acknowledgements. This paper forms part of work done in the author's doctoral thesis. I thank my advisors Astrid an Huef and Lisa Orloff Clark for their guidance, support and endless patience throughout my studies.

REFERENCES

[1] W. ARVESON, *An Invitation to C^*-Algebras*, Grad. Texts Math., vol. 39, Springer-Verlag, New York-Heidelberg 1976.

[2] O. BRATTELI, D.W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics*, Vol. 1, C- and W-Algebras, Symmetry Groups, Decomposition of States, Second edition, Texts Monogr. Phys., Springer-Verlag, New York 1987.

[3] L.O. CLARK, CCR and GCR groupoid C^*-algebras, *Indiana Univ. Math. J.* **56**(2007), 2087–2110.

[4] L.O. CLARK, Classifying the types of principal groupoid C^*-algebras, *Indiana Univ. Math. J.* **57**(2007), 251–266.

[5] J. DIXMIER, *Von Neumann Algebras*, North-Holland Publ. Co., Amsterdam 1981.

[6] J. DIXMIER, *C^*-Algebras*, North-Holland Publ. Co., Amsterdam 1982.

[7] E. EFFROS, A decomposition theory for representations of C^*-algebras, *Trans. Amer. Math. Soc.* **107**(1963), 83–106.

[8] E. EFFROS, Transformation groups and C^*-algebras, *Ann. Math.* **81**(1965), 38–55.

[9] R.C. FABEC, *Fundamentals of Infinite Dimensional Representation Theory*, Chapman and Hall/CRC Monogr. Surveys Pure Appl. Math., vol. 114, Chapman and Hall/CRC, Boca Raton, FL 2000.

[10] J. GLIMM, Type I C^*-algebras, *Ann. Math.* **3**(1961), 572–612.

[11] E.C. GOOTMAN, The type of some C^*-algebras and W^*-algebras associated with transformation groups, *Pacific J. Math.* **48**(1973), 93–106.

[12] M. IONESCU, D.P. WILLIAMS, Irreducible representations of groupoid C^*-algebras, *Proc. Amer. Math. Soc.* **137**(2009), 1323–1332.

[13] R.V. KADISON, J.R. RINGROSE, *Fundamentals of the Theory of Operator Algebras*, Vol. I, *Elementary Theory*, Amer. Math. Soc., Providence, RI 1997.

[14] I. KAPLANSKY, The structure of certain operator algebras, *Trans. Amer. Math. Soc.* **2**(1951), 219–255.

[15] G.W. MACKEY, Induced representations of locally compact groups. II, *Ann. Math.* **58**(1953), 193–221.

[16] G.J. MURPHY, *C^*-Algebras and Operator Theory*, Academic Press. Inc., Boston 1990.

[17] I. RAEBURN, D.P. WILLIAMS, *Morita Equivalence and Continuous-Trace C^*-Algebras*, Amer. Math. Soc., Providence, RI 1998.

[18] A. RAMSAY, The Mackey–Glimm dichotomy for foliations and other Polish groupoids. II, *J. Funct. Anal.* **94**(1990), 358–374.

[19] J. RENAULT, *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Math., vol. 793, Springer, Berlin 1980.

[20] J. RENAULT, The ideal structure of groupoid crossed product C^*-algebras, *J. Operator Theory* **25**(1991), 3–36.

[21] S. SAKAI, On a characterization of type I C^*-algebra, *Bull. Amer. Math. Soc.* **72**(1966), 508–512.

[22] D.P. WILLIAMS, *Crossed Products of C^*-Algebras*, Amer. Math. Soc., Providence, RI 2007.

DANIEL W. VAN WYK, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF OTAGO, DUNEDIN, 9010, NEW ZEALAND
E-mail address: dwvanwyk79@gmail.com

Received September 11, 2017; revised April 9, 2018.