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HYPERCYCLIC SHIFT FACTORIZATIONS FOR UNILATERAL WEIGHTED BACKWARD SHIFT OPERATORS

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ABSTRACT. We show that every unilateral weighted backward shift T on ℓ^p, where $1 \leqslant p < \infty$, has the factorization $T = AB$ with two hypercyclic operators A and B, one of which is a unilateral weighted backward shift and the other one is a bilateral weighted shift.

KEYWORDS: *Weighted shift, hypercyclic operator, chaotic operator, factorization.*

MSC (2010): Primary 47A16, 47B37; Secondary 47A68, 46A15.

1. INTRODUCTION

A bounded linear operator $T : H \to H$ on a separable, infinite-dimensional, complex Hilbert space H is *cyclic* if there exists a vector h in H such that the set $\{p(T)h : p \text{ is a polynomial}\}$ is dense in H. Since the vector $p(T)h$ takes the form $p(T)h = a_0h + a_1Th + a_2T^2h + \cdots + a_nT^nh$, where $a_i \in \mathbb{C}$, we see that $p(T)h \in \operatorname{span}\{h\} + \operatorname{ran} T$. Thus the dimension of $(\operatorname{ran} T)^\perp$ is at most 1. In other words, if the dimension of $(\operatorname{ran} T)^\perp = k \geqslant 2$, then T cannot be cyclic. Nevertheless Wu ([18], Theorem 2.12) showed that such an operator T is indeed the product of at most $k + 2$ cyclic operators. Switching from the product to the sum, Wu ([18], Theorem 1.5) further showed that every bounded linear operator $T : H \to H$ is the sum of two cyclic operators.

In the present paper, we focus on a property of an operator, called hypercyclicity, which is stronger than cyclicity. By the definition, a bounded linear operator $T : X \to X$ on a separable, infinite-dimensional, complex Banach space X is *hypercyclic* if there exists a vector x in X whose orbit $\{T^nx : n \geqslant 0\}$ is dense in X. As a hypercyclic generalization of one of Wu's results, Grivaux [10] showed that every bounded linear operator on the Hilbert space H is the sum of two hypercyclic operators.

Along the line of sums and products of cyclic or hypercyclic operators, we show in the present paper how to factor a unilateral weighted backward shift on the sequence Banach space ℓ^p as the product of two hypercyclic weighted shifts on ℓ^p. Weighted shifts, as defined in (1.1) through (1.4) below, often serve as a fundamental testing ground for many theories about operators; see Shields [17]. In fact, the first example of a hypercyclic operator on a Banach space is a unilateral weighted backward shift offered by Rolewicz [13]. Later, Salas [15] provided a necessary and sufficient condition for a unilateral weighted backward shift to be hypercyclic, and also a necessary and sufficient condition for a bilateral weighted shift to be hypercyclic. The two conditions are given in terms of the weight sequences of the weighted shifts. They are used in our present work to show a weighted shift is hypercyclic.

To precisely describe the results in the present paper, we need definitions of weighted shifts on the Banach space ℓ^p, with $1 \leqslant p < \infty$. For that we let $\mathbb{Z}^+$ denote the set of all nonnegative integers and let $\{e_i : i \in \mathbb{Z}^+\} = \{e_0, e_1, e_2, \dots\}$ be the canonical basis of ℓ^p, where $e_0 = (1,0,0,0,\dots), e_1 = (0,1,0,0,\dots), e_2 = (0,0,1,0,\dots), \dots$ etc. With respect to the canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ every vector $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ in ℓ^p is represented as a convergent series

$$\alpha = \sum_{i=0}^{\infty} \alpha_i e_i.$$

The canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ can be rearranged and relabelled, via a bijection $\sigma : \mathbb{Z} \to \mathbb{Z}^+$, as a two-sided sequence

$$\{e_{\sigma(i)} : i \in \mathbb{Z}\} = \{\dots, e_{\sigma(-2)}, e_{\sigma(-1)}, e_{\sigma(0)}, e_{\sigma(1)}, e_{\sigma(2)}, \dots\}.$$

Correspondingly every vector α in ℓ^p can be represented as a two-sided convergent series

$$\alpha = (\dots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots) = \sum_{i=-\infty}^{\infty} \alpha_i e_{\sigma(i)},$$

whose norm becomes

$$\|\alpha\| = \Big(\sum_{i=-\infty}^{\infty} |\alpha_i|^p \Big)^{1/p}.$$

No matter whether a vector α in ℓ^p is represented as a one-sided convergent series with respect to the canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ or as a two-sided convergent series via a bijection $\sigma : \mathbb{Z} \to \mathbb{Z}^+$, its norm $\|\alpha\|$ stays the same.

In the present paper, we always use ℓ^p to denote the same sequence Banach space, regardless whether the canonical basis is represented as a one-sided sequence or a two-sided sequence. Furthermore, we allow a permutation, or a reordering, of a given canonical basis to define a weighted shift. To make our definitions precise, we take the one-sided canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ of ℓ^p, where $1 \leqslant p < \infty$.

DEFINITION 1.1. A bounded linear operator $T : \ell^p \to \ell^p$ is said to be a *unilateral weighted backward shift*, or *unilateral backward shift* for the sake of brevity, if there are a bijection $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ and a bounded positive weight sequence

$\{w_i : i \in \mathbb{N}\}$ such that $Te_{\sigma(0)} = 0$ and $Te_{\sigma(i)} = w_i e_{\sigma(i-1)}$ whenever $i \in \mathbb{N}$. In other words, for any vector $\alpha = \sum_{i=0}^{\infty} \alpha_i e_{\sigma(i)}$ in ℓ^p, we have

$$T\alpha = T\Big(\sum_{i=0}^{\infty} \alpha_i e_{\sigma(i)}\Big) = \sum_{i=1}^{\infty} w_i \alpha_i e_{\sigma(i-1)}. \tag{1.1}$$

When the canonical basis is shifted in the opposite direction, we have the following definition.

DEFINITION 1.2. A bounded linear operator $T : \ell^p \to \ell^p$ is a *unilateral weighted forward shift*, or *unilateral forward shift* for the sake of brevity, if there exist a bijection $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ and a bounded positive weight sequence $\{w_i : i \in \mathbb{Z}^+\}$ such that $Te_{\sigma(i)} = w_i e_{\sigma(i+1)}$, whenever $i \in \mathbb{Z}^+$. In other words, for any vector $\alpha = \sum_{i=0}^{\infty} \alpha_i e_{\sigma(i)}$ in ℓ^p, we have

$$T\alpha = T\Big(\sum_{i=0}^{\infty} \alpha_i e_{\sigma(i)}\Big) = \sum_{i=0}^{\infty} w_i \alpha_i e_{\sigma(i+1)}. \tag{1.2}$$

With the same one-sided canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ of ℓ^p, we use a bijection $\sigma : \mathbb{Z} \to \mathbb{Z}^+$ to rearrange it as a two-sided canonical basis $\{e_{\sigma(i)} : i \in \mathbb{Z}\}$ of ℓ^p and define the following two types of shift operators.

DEFINITION 1.3. A bounded linear operator $T : \ell^p \to \ell^p$ is a *bilateral weighted backward shift*, or *bilateral backward shift* for the sake of brevity, if there exist a bijection $\sigma : \mathbb{Z} \to \mathbb{Z}^+$ and a bounded positive weight sequence $\{w_i : i \in \mathbb{Z}\}$ such that $Te_{\sigma(i)} = w_i e_{\sigma(i-1)}$ whenever $i \in \mathbb{Z}$. In other words, for any vector $\alpha = \sum_{i=-\infty}^{\infty} \alpha_i e_{\sigma(i)}$ in ℓ^p, we have

$$T\alpha = T\Big(\sum_{i=-\infty}^{\infty} \alpha_i e_{\sigma(i)}\Big) = \sum_{i=-\infty}^{\infty} w_i \alpha_i e_{\sigma(i-1)}. \tag{1.3}$$

Similar to the unilateral case, the two-sided sequence $\{e_{\sigma(i)} : i \in \mathbb{Z}\}$ may be shifted in the forward direction.

DEFINITION 1.4. A bounded linear operator $T : \ell^p \to \ell^p$ is a *bilateral weighted forward shift*, or *bilateral forward shift* for the sake of brevity, if there exist a bijection $\sigma : \mathbb{Z} \to \mathbb{Z}^+$ and a bounded positive weight sequence $\{w_i : i \in \mathbb{Z}\}$ such that $Te_{\sigma(i)} = w_i e_{\sigma(i+1)}$ whenever $i \in \mathbb{Z}$. In other words, for any vector $\alpha = \sum_{i=-\infty}^{\infty} \alpha_i e_{\sigma(i)}$ in ℓ^p,

$$T\alpha = T\Big(\sum_{i=-\infty}^{\infty} \alpha_i e_{\sigma(i)}\Big) = \sum_{i=-\infty}^{\infty} w_i \alpha_i e_{\sigma(i+1)}. \tag{1.4}$$

Obviously, a bilateral forward shift becomes a bilateral backward shift, if we replace the bijection σ in its definition (1.3) with the bijection $\sigma_0(i)$ where

$\sigma_0(i) = \sigma(-i)$. Thus we simply use the term *bilateral weighted shift*, or *bilateral shift* for the sake of brevity, without referring to the direction, forward or backward, when the direction does not play a role in our discussion. More generally, we use the term *shift* to mean any of the above four types of unilateral and bilateral shifts. In any case, one can easily verify that whenever T is a shift corresponding to a bounded positive weight sequence $\{w_i\}$, we have

$$\|T\| = \sup\{w_i\}.$$

In the present paper, we only consider shifts T with positive weight sequences $\{w_i\}$. However, the assumption on the positivity of the weight sequence does not cause any loss of generality in the results of our paper, because a shift with a complex weight sequence is indeed similar to a shift with a positive weight sequence; see Shields ([17], Corollary 1). Furthermore, hypercyclicity of an operator is preserved under similarity of operators.

As mentioned earlier, we are interested in hypercyclic factorizations of a unilateral backward shift T on ℓ^p; that is, factoring T as the product $T = AB$, where A and B are two hypercyclic shifts on ℓ^p. In order to have the factorization work, we need to determine the allowable types of shifts A and B within the factorization. First, A and B cannot be two bilateral shifts because $ABe_i \neq 0$ for all integers i while there is an integer j such that $Te_j = 0$.

Second, A and B cannot be two unilateral backward shifts. To see this by the way of contradiction, without loss of generality suppose T is given by (1.1) and in particular $Te_0 = 0$, and suppose A and B are two such shifts given by two bijections $\rho : \mathbb{Z}^+ \to \mathbb{Z}^+$ and respectively $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$. Then $Be_{\sigma(0)} = Ae_{\rho(0)} = 0$. Since $Te_{\sigma(0)} = ABe_{\sigma(0)} = 0$, we have $\sigma(0) = 0$. Furthermore, there is an integer j with $j \geqslant 1$ such that $Be_{\sigma(j)}$ is a nonzero multiple of $e_{\rho(0)}$, say $Be_{\sigma(j)} = \alpha e_{\rho(0)}$. Since $j \neq 0$, we have $\sigma(j) \neq 0$ and $Te_{\sigma(j)} = ABe_{\sigma(j)} = \alpha Ae_{\rho(0)} = 0$, which is a contradiction.

Lastly, neither of A, B can be a unilateral forward shift. To show this, observe that for any such a shift, by (1.2) one of the vectors in the canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ cannot be in its range. On one hand, if there is a vector e_n such that $e_n \notin \operatorname{ran} A$, then $e_n \notin \operatorname{ran} AB = \operatorname{ran} T$, which is a contradiction. On the other hand, suppose there is a vector e_m such that $e_m \notin \operatorname{ran} B$. Since the vector Ae_m is a nonzero multiple of a canonical basis vector with $Ae_m \notin \operatorname{ran} AB = \operatorname{ran} T$, we have again a contraction.

On the positive side, in Section 2 we show that every unilateral backward shift $T : \ell^p \to \ell^p$ can be factored as $T = UB$, where U is a hypercyclic unilateral backward shift and B is a hypercyclic bilateral shift; see Theorem 2.3 below. In Section 3, we show that T can also be factored in the reversed order; that is, $T = B'U'$ where B' is a hypercyclic bilateral shift and U' is a hypercyclic unilateral backward shift; see Theorem 3.1 below.

One special case of interest to us is when the weight sequence $\{w_i : i \in \mathbb{N}\}$ of the unilateral backward shift T in Theorems 2.3 and 3.1 is bounded away from 0. In that case, the factors B, U, B', U' obtained from both theorems can have additional properties, for which we need the following definitions. For a bounded linear operator $S : X \to X$ on a Banach space X, a vector x in X is said to be a *periodic point* of S if there is a positive integer n such that $S^n x = x$. The operator S is said to be *chaotic* if S is hypercyclic and has a dense set of periodic points. Next, an operator $S : X \to X$ is said to be *mixing* if for any two nonempty open subsets $\mathcal{O}_1, \mathcal{O}_2$ of X, there exists a positive integer N such that $T^n(\mathcal{O}_1) \cap \mathcal{O}_2 \neq \emptyset$, whenever $n \geqslant N$. It is known that S is hypercyclic if S is mixing. Lastly, S is said to be *frequently hypercyclic* if there exists a vector x in X such that for every nonempty open subset $\mathcal{O}$ of X, the set $E = \{n \geqslant 1 : T^n x \in \mathcal{O}\}$ has a positive lower density given by

$$\underline{\text{dens}}(E) = \liminf_{N \to \infty} \frac{\text{card}(E \cap [1, N])}{N},$$

where card denotes the cardinality.

When the weight sequence of a unilateral backward shift T on ℓ^p is bounded away from 0, we show with both factorizations $T = UB$ and $T = B'U'$ given by Theorem 2.3 and Theorem 3.1, that all factors B, U, B' and U' can be selected so that they are chaotic, mixing, and frequently hypercyclic (Corollaries 2.5 and 3.2).

To conclude our discussion in this section, we remark that if T is a unilateral forward shift, then $T \neq AB$, for any two hypercyclic operators A, B. This is obvious because if A, B are two hypercyclic operators, then they both have a dense range, and so does their product AB. Nevertheless, by (1.2) the closure of the range ran T of T has codimension 1, and so it is not dense. Indeed, with this dense-range argument, it is obvious that T cannot be factored as the product of two unilateral backward shifts, or two bilateral shifts, or one unilateral backward shift and one bilateral shift, all of which possess a dense range.

2. A FACTORIZATION FOR UNILATERAL WEIGHTED BACKWARD SHIFTS

In this section and the next section, we show that every unilateral backward shift T can be factored as a product of two hypercyclic shifts. More specifically, in Theorem 2.3 of this section, we show that T can be factored as $T = UB$, where B is a hypercyclic bilateral shift and U is a hypercyclic unilateral backward shift. Then in Theorem 3.1 of the next section, we show that T can also be factored in the reversed order as $T = B'U'$, where B', U' are respectively, a hypercyclic bilateral shift and a hypercyclic unilateral backward shift.

For a unilateral backward shift to be hypercyclic, we quote the following necessary and sufficient condition in terms of its weight sequence, established by Salas ([15], Corollary 2.9).

THEOREM 2.1. *Let $1 \leqslant p < \infty$ and $\{w_i : i \in \mathbb{N}\}$ be a bounded sequence of positive weights. Suppose $T : \ell^p \to \ell^p$ is the unilateral backward shift given by $Te_0 = 0$*

and $Te_i = w_i e_{i-1}$ *for all positive integers i. Then T is hypercyclic if and only if*

$$\sup\{w_1 w_2 \cdots w_n : n \geqslant 1\} = \sup\{\|T^n e_n\| : n \geqslant 1\} = \infty.$$

For a bilateral backward shift T on ℓ^p to be hypercyclic, Salas ([15], Theorem 2.1) offered the following necessary and sufficient condition in terms of a two-sided canonical basis $\{f_i : i \in \mathbb{Z}\}$ of ℓ^p: *let T be given by* $Tf_i = w_i f_{i-1}$ *for all integers i. Then T is hypercyclic if and only if for any positive ε and any positive integer N, there exists an arbitrarily large n such that whenever* $|j| \leqslant N$,

$$\prod_{i=0}^{n-1} w_{j-i} < \varepsilon, \quad \text{and} \quad \prod_{i=1}^{n} w_{j+i} > \frac{1}{\varepsilon}.$$

The above necessary and sufficient condition is more general than what we need in the present paper. As an immediate corollary, the following simple sufficient condition is helpful for the arguments presented in this paper.

THEOREM 2.2. *Let* $1 \leqslant p < \infty$. *Suppose* $T : \ell^p \to \ell^p$ *is a bilateral backward shift with* $Tf_i = w_i f_{i-1}$, *whenever* $i \in \mathbb{Z}$. *Then T is hypercyclic if there exist positive scalars* a, b *and integers M, N with* $N > M$ *such that whenever* $i \geqslant N > M > j$, *we have*

$$0 < w_j \leqslant a < 1 < b \leqslant w_i \leqslant \|T\|.$$

Using Theorems 2.1 and 2.2, we now proceed to obtain Theorem 2.3 that factors a unilateral backward shift as the product of two hypercyclic shifts. Since both shifts are hypercyclic, their norms must be larger than 1. On the positive side, the theorem guarantees that their norms are kept well under control.

THEOREM 2.3. *Let* $1 \leqslant p < \infty$. *If* $T : \ell^p \to \ell^p$ *is a unilateral weighted backward shift, then there exist a hypercyclic unilateral weighted backward shift* $U : \ell^p \to \ell^p$, *and a hypercyclic bilateral weighted shift* $B : \ell^p \to \ell^p$ *such that*

$$T = UB.$$

Moreover, for any positive ε, the shifts U and B can be chosen so that their norms $\|U\|, \|B\|$ *are no larger than* $(1+\varepsilon)\max\{1, \|T\|\}$.

Proof. Without loss of generality, by reordering the canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ of ℓ^p if necessary, we can assume T is given by

$$Te_i = \begin{cases} 0 & \text{if } i = 0, \\ w_i e_{i-1} & \text{if } i \geqslant 1, \end{cases} \tag{2.1}$$

where each weight w_i satisfies $0 < w_i \leqslant \|T\|$. For any given positive ε, select two positive scalars a, b such that

$$b = (1+\varepsilon)\max\{1, \|T\|\} \quad \text{and} \quad b^{-1} < a < 1. \tag{2.2}$$

Next, let $j_0 = 0$, and inductively select a strictly increasing sequence $\{j_k : k \geqslant 0\}$ of nonnegative even integers that satisfy the following two inequalities:

$$(2.3) \qquad j_{k+1} \geqslant 2j_k + 4 \quad \text{and} \quad b^{j_{k+1}/2} \geqslant \frac{kb^{k+1}}{b^{j_k/2} a^k w_{j_1} w_{j_2} \cdots w_{j_k}}, \quad \text{whenever } k \geqslant 0.$$

Using the sequence $\{j_k : k \geqslant 0\}$, we construct the hypercyclic bilateral shift B and a hypercyclic unilateral backward shift U such that $T = UB$, in the following two separated steps.

Step I. *Construction of the shift B.* First, we need to define a bijection $\sigma : \mathbb{Z} \to \mathbb{Z}^+$ to reorder the members in the canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ as a two-sided sequence $\{e_{\sigma(i)} : i \in \mathbb{Z}\}$ listed in (2.4) below, in which $e_0 = e_{j_0}$ is placed in the 0-th position of the two-sided sequence; that is, $\sigma(0) = j_0 = 0$. In addition, the right-hand side of e_0 consists of all e_{j_k} and the left-hand side of e_0 consists of the remaining vectors e_j in ascending order of the index j as we go to the left:

$$(2.4) \qquad \ldots, e_{2+j_k}, \overbrace{e_{1+j_k}, e_{-1+j_k}}^{\text{missing } e_{j_k}\ (k \geqslant 1)}, e_{-2+j_k}, \ldots, e_3, e_2, \overbrace{e_1}^{e_{\sigma(-1)}}, \overbrace{e_0}^{e_{\sigma(0)}}, \overbrace{e_{j_1}}^{e_{\sigma(1)}}, e_{j_2}, e_{j_3}, \ldots.$$

Claim 1. Corresponding to the rearrangement in (2.4), the bijection $\sigma : \mathbb{Z} \to \mathbb{Z}^+$ is given by the formula:

$$\sigma(i) = \begin{cases} j_i & \text{if } i \geqslant 0, \\ |i| + k & \text{if } (k+1) - j_{k+1} \leqslant i \leqslant (k-1) - j_k \text{ and } k \geqslant 0. \end{cases}$$

Proof of Claim 1. From the rearrangement of the canonical basis in (2.4), we clearly have $\sigma(i) = j_i$ for all nonnegative integers i.

Focusing on $\sigma(i)$ for the negative integers i, we observe that when $k = 0$ the formula in our claim gives $\sigma(i) = |i|$ if $1 - j_1 \leqslant i \leqslant -1$. That is, $\sigma(-1) = 1, \sigma(-2) = 2, \ldots, \sigma(1 - j_1) = -1 + j_1$, which give the rearrangement in (2.4) for integers i satisfying $1 - j_1 \leqslant i \leqslant -1$.

Inductively suppose $\sigma(i) = |i| + k$ for $(k+1) - j_{k+1} \leqslant i \leqslant (k-1) - j_k$. In particular, $\sigma((k+1) - j_{k+1}) = -1 + j_{k+1}$. It follows from the omission of the vector $e_{j_{k+1}}$ between $e_{-1+j_{k+1}}$ and $e_{1+j_{k+1}}$ in (2.4) that $\sigma(k - j_{k+1}) = 1 + j_{k+1} = |k - j_{k+1}| + (k+1)$, and hence $\sigma(i) = |i| + (k+1)$ whenever $(k+2) - j_{k+2} \leqslant i \leqslant k - j_{k+1}$. This finishes the proof of Claim 1 by mathematical induction. ∎

Using the bijection σ, define the bilateral shift $B : \ell^p \to \ell^p$ by

$$(2.5) \qquad Be_{\sigma(i)} = \begin{cases} a^{-1} e_{\sigma(i-1)} & \text{if } i \geqslant 1, \\ e_{\sigma(-1)} = e_1 & \text{if } i = 0, \\ b^{-1} w_{\sigma(i)} e_{\sigma(i-1)} & \text{if } i \leqslant -1. \end{cases}$$

From the selection of the scalars a, b in (2.2), the weights of the shift B satisfy

$$b^{-1} w_{\sigma(i)} \leqslant b^{-1} \|T\| \leqslant (1+\varepsilon)^{-1} < 1 < a^{-1} < b.$$

Thus $\|B\| \leqslant b$, giving us the desired upper bound on the norm of B. Furthermore, since

$$b^{-1} w_{\sigma(i)} \leqslant (1+\varepsilon)^{-1} < 1 < a^{-1},$$

the shift B is hypercyclic by a direct application of Theorem 2.2.

Step II. *Construction of the shift U.* Moving on to defining the unilateral backward shift U, we need a bijection $\rho : \mathbb{Z}^+ \to \mathbb{Z}^+$ to reorder the members of the canonical basis $\{e_i : i \in \mathbb{Z}^+\}$. For that we define $\rho(i)$ for integers i in the interval $[j_{k-1}, j_k - 1]$ using an inductive process on the integer $k \geqslant 1$.

The first step is to reorder those vectors e_i with indices i in the interval $[0, j_1 - 1] = [j_0, j_1 - 1]$. We put all vectors e_i with odd indices i in the interval $[j_0, j_1 - 1]$ before all vectors e_i with even indices i in the same interval $[j_0, j_1 - 1]$ as follows:

$$\overbrace{e_1, e_3, \ldots, e_{-3+j_1}, e_{-1+j_1}}^{e_i \text{ with odd indices } i}, \overbrace{e_0, e_2, \ldots, e_{-4+j_1}, e_{-2+j_1}}^{e_i \text{ with even indices } i}. \tag{2.6}$$

Since $\rho : \mathbb{Z}^+ \to \mathbb{Z}^+$ provides the above reordering of $e_0, e_1, \ldots, e_{-1+j_1}$, we must have $\rho(0) = \rho(j_0) = 1, \rho(1) = 3, \ldots$, and lastly $\rho(-1+j_1) = -2+j_1$. Furthermore, we have

$$\rho\Big(\frac{j_1 + j_0}{2}\Big) = 0. \tag{2.7}$$

To see this, we first observe that there are altogether $j_1 - j_0 = j_1$ vectors in the list (2.6), exactly half of which e_i have odd indices i. Since $\rho(0) = 1$, the vector $e_0 = e_{j_0}$ with the first even index in (2.6) must be at the $(j_0 + (j_1 - j_0)/2)$-th position, which establishes (2.7).

All vectors listed in (2.6) precede all vectors e_i with indices i in the interval $[j_1, -1+j_2]$, which are reordered in the second step. Those reordered in the second step in turn precede all vectors e_i with indices i in the interval $[j_2, -1+j_3]$, which are reordered in the third step and so on.

In the $(k+1)^{\text{st}}$ step, where $k \geqslant 0$, the vectors e_i with indices i in the interval $[j_k, -1+j_{k+1}]$ are reordered as follows:

$$\underbrace{\overbrace{e_{1+j_k}, e_{3+j_k}, \ldots, e_{-1+j_{k+1}}}^{e_i \text{ with odd indices } i}, \overbrace{e_{j_k}, e_{2+j_k}, \ldots, e_{-2+j_{k+1}}}^{e_i \text{ with even indices } i}}_{(k+1)^{\text{st}} \text{ step of the rearrangement}}. \tag{2.8}$$

Correspondingly $\rho(j_k) = 1 + j_k$ and

$$\rho\Big(\frac{j_k + j_{k+1}}{2}\Big) = j_k, \tag{2.9}$$

and lastly $\rho(-1+j_{k+1}) = -2+j_{k+1}$. Inductively we complete the rearrangement of the canonical basis $\{e_i : i \in \mathbb{Z}^+\}$. Summarizing the above rearrangements provided by (2.6) and (2.8), we provide the following formula for the bijection $\rho : \mathbb{Z}^+ \to \mathbb{Z}^+$, written in terms of odd and even values of $\rho(i)$:

$$(2.10)\quad \rho(i) = \begin{cases} (2m+1)+j_k & \text{if } i = m + j_k, \text{ for } k \geqslant 0, 0 \leqslant m \leqslant \frac{j_{k+1}-j_k}{2} - 1, \\ 2m + j_k & \text{if } i = m + \frac{j_{k+1}+j_k}{2}, \text{ for } k \geqslant 0, 0 \leqslant m \leqslant \frac{j_{k+1}-j_k}{2} - 1. \end{cases}$$

In terms of ρ, define the unilateral backward shift $U : \ell^p \to \ell^p$ by

$$(2.11)\qquad Ue_{\rho(i)} = \begin{cases} 0 & \text{if } i = 0, \\ aw_{j_{k+1}} e_{\rho(i-1)} & \text{if } \rho(i) = j_k, \text{ for some } k \geqslant 0, \\ be_{\rho(i-1)} & \text{if } \rho(i) \neq j_k, \text{ for any } k \geqslant 0. \end{cases}$$

To show $\|U\| \leqslant b$ as stated in the theorem, we simply observe that by (2.2) we get $aw_{j_{k+1}} \leqslant a\|T\| < \|T\| < b$.

Claim 2. U is hypercyclic.

Proof of Claim 2. For each integer $k \geqslant 1$, set

$$n_k = \frac{j_{k+1} + j_k}{2} - 1 = m + j_k, \quad \text{where } m = \frac{j_{k+1} - j_k}{2} - 1.$$

Observe that if $1 \leqslant i \leqslant n_k$, then by (2.10), $\rho(i) = j_\ell$ for some integer ℓ exactly when $i = (j_{\ell+1} + j_\ell)/2$, for $\ell = 0, 1, 2, \ldots, k-1$. Thus $\rho(i) = j_\ell$ for exactly k values of ℓ. Therefore by (2.11),

$$U^{n_k} e_{\rho(n_k)} = a^k w_{j_1} \cdots w_{j_k} b^{-k+n_k} e_{\rho(0)}.$$

It follows from the definition of n_k and the selection of the sequence $\{j_k\}$ in (2.3) that

$$\|U^{n_k} e_{\rho(n_k)}\| = a^k w_{j_1} \cdots w_{j_k} b^{-k+n_k} = a^k w_{j_1} \cdots w_{j_k} b^{(j_{k+1}+j_k)/2-k-1} \geqslant k.$$

Thus by Theorem 2.1, the unilateral shift U is hypercyclic, finishing the proof of Claim 2 and our construction of U in Step II. ∎

Having constructed the shifts B and U, we now complete the proof of Theorem 2.3 with the final step.

Step III. *To verify* $T = UB$. By (2.1) we need to verify that $UBe_{\sigma(0)} = Te_{\sigma(0)} = 0$ and

$$UBe_{\sigma(i)} = Te_{\sigma(i)} = w_{\sigma(i)} e_{\sigma(i)-1},$$

for each positive integer i.

First, when $i = 0$, by (2.5), (2.10), and (2.11) we have $UBe_{\sigma(0)} = Ue_1 = Ue_{\rho(0)} = 0$. Second, for all positive integers i, we see from Claim 1 that $\sigma(i) = j_i$ and $\sigma(i-1) = j_{i-1}$. Furthermore it follows from (2.9) that if $\ell = (j_i + j_{i-1})/2$ then $\rho(\ell) = j_{i-1} = \sigma(i-1)$. Letting

$$m = \frac{j_i - j_{i-1}}{2} - 1,$$

we have $\ell - 1 = m + j_{i-1}$, and so by (2.10) we have $\rho(\ell - 1) = (2m+1) + j_{i-1} = -1 + j_i = \sigma(i) - 1$. Hence by (2.5) and (2.11),

$$UBe_{\sigma(i)} = a^{-1}Ue_{\sigma(i-1)} = a^{-1}Ue_{j_{i-1}} = a^{-1}Ue_{\rho(\ell)} = w_{j_i}e_{\rho(\ell-1)} = w_{\sigma(i)}e_{\sigma(i)-1},$$

finishing the proof for the second case.

Third, suppose the integer $i \leqslant -1$ and $\sigma(i) = -1 + j_k$ for some integer $k \geqslant 1$. Since the definition of σ gives the listing in (2.4), we see that $\sigma(i-1) = 1 + j_k$. Similarly, since the definition of ρ gives the listing in (2.8), we see that if ℓ is the integer satisfying $\rho(\ell) = 1 + j_k$ then $\rho(\ell - 1) = -2 + j_k = \sigma(i) - 1$. Hence by (2.5) and (2.11), we have

$$\begin{aligned} UBe_{\sigma(i)} &= b^{-1}w_{\sigma(i)}Ue_{\sigma(i-1)} = b^{-1}w_{\sigma(i)}Ue_{1+j_k} \\ &= b^{-1}w_{\sigma(i)}Ue_{\rho(\ell)} = w_{\sigma(i)}e_{\rho(\ell-1)} = w_{\sigma(i)}e_{\sigma(i)-1}, \end{aligned}$$

finishing the proof for the third case.

Lastly, suppose the integer $i \leqslant -1$ and $\sigma(i) \neq -1 + j_k$ for any integer $k \geqslant 0$. Since the definition of σ gives the listing in (2.4), we see that $\sigma(i-1) = \sigma(i) + 1 \neq j_m$ and $\sigma(i) + 1 \neq 1 + j_m$ for any integer $m \geqslant 0$. Therefore, if ℓ is the integer with $\rho(\ell) = \sigma(i-1)$ then $\rho(\ell) \neq j_m$ and $\rho(\ell) \neq 1 + j_m$ for any integer $m \geqslant 0$. Thus by (2.8), $\rho(\ell - 1) = -2 + \rho(\ell) = -2 + \sigma(i-1) = \sigma(i) - 1$. Hence, by (2.5) and (2.11) we have

$$UBe_{\sigma(i)} = b^{-1}w_{\sigma(i)}Ue_{\sigma(i-1)} = b^{-1}w_{\sigma(i)}Ue_{\rho(\ell)} = w_{\sigma(i)}e_{\sigma(i)-1},$$

which finishes Step III, and hence the whole proof of Theorem 2.3. ∎

Continuing with the argument in the above proof, we can show that the shifts U, B in the factorization $T = UB$ can be chaotic, mixing, and frequently hypercyclic under an additional assumption on T; see Corollary 2.5. For that, we need to quote the *frequent hypercyclicity criterion*, which is a sufficient condition introduced by Bayart and Grivaux [3] to show that an operator is frequently hypercyclic.

THEOREM 2.4. *A bounded linear operator $T : X \to X$ on a Banach space X is chaotic, mixing, and frequently hypercyclic if there exist a dense subset X_0 of X and a mapping $S : X_0 \to X_0$ such that for any $x \in X_0$,*

(i) $\sum_{k=0}^{\infty} T^k x$ *converges unconditionally,*

(ii) $\sum_{k=0}^{\infty} S^k x$ *converges unconditionally, and*

(iii) $TSx = x$.

Theorem 2.4 can be applied to the factors U and B in Theorem 2.3 when the weight sequence $\{w_i : i \in \mathbb{N}\}$ of the unilateral backward shift T is *bounded away from 0;* that is, there is a scalar $\delta > 0$ such that $w_i > \delta > 0$ for all i.

COROLLARY 2.5. *Let $1 \leqslant p < \infty$. Suppose $T : \ell^p \to \ell^p$ is a unilateral weighted backward shift, and $U, B : \ell^p \to \ell^p$ are two shifts that satisfy the conclusion of Theorem* 2.3. *If the weight sequence of T is bounded away from* 0, *then both U and B can be selected to be chaotic, mixing, and frequently hypercyclic, with weight sequences bounded away from 0. In particular, B is invertible.*

Proof. Without loss of generality, by rearranging and relabelling the canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ of ℓ^p if necessary, assume T is given by $Te_0 = 0$ and $Te_i = w_i e_{i-1}$ whenever $i \geqslant 1$. Also let δ be a real scalar such that $0 < \delta < w_i \leqslant \|T\|$ for all positive integers i.

For any given positive ε, select the same positive scalars a, b satisfying inequalities in (2.2) as in the proof of Theorem 2.3. Next select a real scalar C such that

$$C \geqslant \max\{1, a^{-1}\delta^{-1}\}. \tag{2.12}$$

In the rest of the proof, we follow all other notations and definitions in the proof of Theorem 2.3, except that we require the strictly increasing sequence $\{j_k\}$ of nonnegative even integers in that proof to satisfy (2.3), along with the following additional condition:

$$b^{j_k} > \left(1 + \frac{\varepsilon}{2}\right)^{j_k} C^k b^k. \tag{2.13}$$

The above inequality (2.13) is possible because $b \geqslant 1 + \varepsilon$.

To show the shifts U and B given by (2.11) and (2.5) satisfy the frequent hypercyclicity criterion in Theorem 2.4, we set X_0 in the statement of the theorem to be $X_0 = \operatorname{span}\{e_j : j \in \mathbb{Z}^+\}$.

Claim 3. Define a map $R : X_0 \to X_0$ by taking

$$Re_{\rho(i)} = \begin{cases} a^{-1} w_{j_{k+1}}^{-1} e_{\rho(i+1)} & \text{if } i \geqslant 0 \text{ and } \rho(i+1) = j_k, \text{ for some } k \geqslant 0, \\ b^{-1} e_{\rho(i+1)} & \text{if } i \geqslant 0 \text{ and } \rho(i+1) \neq j_k, \text{ for any } k \geqslant 0, \end{cases}$$

and extending linearly to X_0. Then the shifts U and R satisfy conditions (i), (ii) and (iii) of Theorem 2.4.

Proof of Claim 3. First, for the unilateral backward shift U, we have $U^{n+1}e_{\rho(n)} = 0$ whenever $n \geqslant 0$. Hence by the linearity of U, condition (i) of Theorem 2.4 is obviously satisfied for any vector x in X_0.

It is clear from (2.11) that $URe_{\rho(i)} = e_{\rho(i)}$ for any nonnegative integer i, and hence by linearity, $URx = x$ for any $x \in X_0$, and so condition (iii) of Theorem 2.4 is satisfied.

To finish the proof of our claim, it remains to show that R satisfies condition (ii) of Theorem 2.4 for any vector x in X_0. By linearity and the triangle inequality, it suffices to show that $\sum\limits_{n=1}^{\infty} \|R^n e_{\rho(i)}\| < \infty$, for any nonnegative integer i. Even more, by the definition of R, each $e_{\rho(i)}$ is a nonzero multiple of $R^i e_{\rho(0)}$. Thus it suffices to only show that

$$\sum_{n=1}^{\infty} \|R^n e_{\rho(0)}\| < \infty. \tag{2.14}$$

To estimate the norm $\|R^n e_{\rho(0)}\|$ in the infinite sum, note from (2.9) that $\rho(i+1) = j_\ell$, for some integer ℓ precisely when $i+1 = (j_{\ell+1}+j_\ell)/2$. Thus the estimation of $\|R^n e_{\rho(0)}\|$ using the definition of R depends on the number of integers ℓ such that $(j_{\ell+1}+j_\ell)/2 \leqslant n$. For that reason, we first calculate $R^m e_{\rho(j_\ell)}$, where $\ell \geqslant 0$ and $0 \leqslant m \leqslant j_{\ell+1} - j_\ell$. Note that from (2.10), $\rho(j_\ell) = 1 + j_\ell$. Also from (2.8), there are $(j_{\ell+1} - j_\ell)/2$ vectors e_i with odd indices i. Hence by the definition of R and (2.8), we have

$$R^m e_{\rho(j_\ell)} = \begin{cases} b^{-m} e_{\rho(m+j_\ell)} & \text{if } 0 \leqslant m \leqslant \frac{j_{\ell+1}-j_\ell}{2} - 1, \\ b^{-m+1} a^{-1} w_{j_{\ell+1}}^{-1} e_{\rho(m+j_\ell)} & \text{if } \frac{j_{\ell+1}-j_\ell}{2} \leqslant m \leqslant j_{\ell+1} - j_\ell. \end{cases} \tag{2.15}$$

Use (2.15) repeatedly with integers $\ell = 0, 1, \ldots, k-1$. Corresponding to each value of ℓ, take $m = j_{\ell+1} - j_\ell$ and obtain

$$\begin{aligned} R^{j_k} e_{\rho(0)} &= R^{j_k - j_1}(R^{j_1 - j_0} e_{\rho(j_0)}) = b^{-(j_1-j_0)+1} a^{-1} w_{j_1}^{-1} R^{j_k - j_1} e_{\rho(j_1)} \\ &= b^{1-j_1} a^{-1} w_{j_1}^{-1} R^{j_k-j_2}(R^{j_2-j_1} e_{\rho(j_1)}) = b^{2-j_2} a^{-2} w_{j_1}^{-1} w_{j_2}^{-1} R^{j_k-j_2} e_{\rho(j_2)} \\ &= \cdots = b^{k-j_k} a^{-k} w_{j_1}^{-1} w_{j_2}^{-1} \cdots w_{j_k}^{-1} e_{\rho(j_k)}. \end{aligned} \tag{2.16}$$

Using (2.16), we proceed to estimate $R^n e_{\rho(0)}$ for any integer $n \geqslant 1$. We first obtain the integer $k \geqslant 0$ such that $j_k \leqslant n < j_{k+1}$, and so

$$\begin{aligned} \|R^n e_{\rho(0)}\| &= \|R^{n-j_k}(R^{j_k} e_{\rho(0)})\| \\ &\leqslant b^{k-j_k} a^{-k} w_{j_1}^{-1} \cdots w_{j_k}^{-1} \|R^{n-j_k} e_{\rho(j_k)}\| \quad \text{(by (2.16))} \\ &\leqslant b^{k-j_k} C^k \|R^{n-j_k} e_{\rho(j_k)}\| \quad \text{(by (2.12))} \\ &< \Big(1 + \frac{\varepsilon}{2}\Big)^{-j_k} \|R^{n-j_k} e_{\rho(j_k)}\| \quad \text{(by (2.13))}. \end{aligned} \tag{2.17}$$

To estimate the norm $\|R^{n-j_k} e_{\rho(j_k)}\|$ in the above inequality, we observe that $0 \leqslant n - j_k < j_{k+1} - j_k$, and so by (2.15),

$$\begin{aligned} \|R^{n-j_k} e_{\rho(j_k)}\| &\leqslant \max\{b^{-n+j_k}, b^{-n+1+j_k} a^{-1} w_{j_{1+k}}^{-1}\} \\ &\leqslant bCb^{-n+j_k} \leqslant bC(1+\varepsilon)^{-n+j_k} \quad \text{(by (2.2))}. \end{aligned}$$

Combining the above inequality with (2.17) yields

$$\begin{aligned} \|R^n e_{\rho(0)}\| &\leqslant bC\Big(1 + \frac{\varepsilon}{2}\Big)^{-j_k} (1+\varepsilon)^{-n+j_k} \\ &\leqslant bC\Big(1 + \frac{\varepsilon}{2}\Big)^{-j_k} \Big(1 + \frac{\varepsilon}{2}\Big)^{-n+j_k} = bC\Big(1 + \frac{\varepsilon}{2}\Big)^{-n}. \end{aligned}$$

Hence $\sum_{n=1}^{\infty} \|R^n e_{\rho(0)}\| < \infty$, which concludes the proof of Claim 3. ∎

We proceed to show the shift B satisfies the frequent hypercyclicity criterion as well.

Claim 4. Define a linear map $S : X_0 \to X_0$ by taking

$$Se_{\sigma(i)} = \begin{cases} ae_{\sigma(i+1)} & \text{if } i \geqslant 0, \\ e_{\sigma(0)} & \text{if } i = -1, \\ bw^{-1}_{\sigma(i+1)} e_{\sigma(i+1)} & \text{if } i \leqslant -2, \end{cases}$$

and extending linearly to X_0. Then the shifts B and S satisfy conditions (i), (ii), and (iii) in Theorem 2.4.

Proof of Claim 4. To show B satisfies condition (i) of Theorem 2.4, by linearity it suffices to show

$$\sum_{n=0}^{\infty} B^n e_{\sigma(j)}$$

converges unconditionally for any integer j. Since B is a bilateral backward shift, for any integer j there is a nonnegative integer n such that $j - n$ is a negative integer, and $B^n e_{\sigma(j)}$ is a nonzero multiple of $e_{\sigma(j-n)}$. Hence we need only to focus on a negative integer j in showing the unconditional convergence of $\sum B^n e_{\sigma(j)}$. We have by (2.5) and our choice of b in (2.2), $Be_{\sigma(j)} = b^{-1} w_{\sigma(j)} e_{\sigma(j-1)}$ where $b^{-1} w_{\sigma(j)} \leqslant b^{-1} \|T\| \leqslant (1+\varepsilon)^{-1}$. Thus, by repeating the argument, we have $\|B^n e_{\sigma(j)}\| \leqslant (1+\varepsilon)^{-n}$. Hence for any integer i we have $\sum\limits_{n=1}^{\infty} \|B^n e_{\sigma(i)}\| < \infty$, showing that B satisfies condition (i).

To show B and S satisfy condition (iii) in Theorem 2.4, we use (2.5) to verify that $BSe_{\sigma(i)} = e_{\sigma(i)}$ for all integers i, and hence by linearity, $BSx = x$ for any vector x in X_0.

To show that S satisfies condition (ii) in Theorem 2.4, we establish that $\sum\limits_{n=1}^{\infty} \|S^n x\| < \infty$ whenever x is a vector in X_0. However, by the linearity of S, it suffices for us to show $\sum\limits_{n=1}^{\infty} \|S^n e_{\sigma(i)}\| < \infty$ for any integer i. Since there is a nonnegative integer n so that $n + i$ is a positive integer and $S^n e_{\sigma(i)}$ is a nonzero multiple of $e_{\sigma(n+i)}$. Thus it suffices to show

$$\sum_{n=1}^{\infty} \|S^n e_{\sigma(i)}\| < \infty \quad \text{for the case that the integer } i \geqslant 1. \tag{2.18}$$

In that case, we have $Se_{\sigma(i)} = ae_{\sigma(i+1)}$, and so by repeating the argument we get $\|S^n e_{\sigma(i)}\| = a^n$ which leads to $\sum\limits_{n=1}^{\infty} \|S^n e_{\sigma(i)}\| < \infty$ by (2.2), completing the proof of our claim. ∎

To finish the proof of our corollary, we note the weights of the shift U, as defined in (2.11), are given by b and $aw_{j_{k+1}}$, which satisfy $b \geqslant \|T\| > a\|T\| \geqslant aw_{j_{k+1}} > a\delta$. Thus $\|U\| \leqslant b$ and the weights of U are greater than $a\delta$.

Similarly for the shift B defined in (2.5), using (2.2) we see that its weights $a^{-1}, 1, b^{-1}w_{\sigma(i)}$, satisfy $b > a^{-1} > 1 > b^{-1}\|T\| \geqslant b^{-1}w_{\sigma(i)} > b^{-1}\delta$. Thus $\|B\| \leqslant b$, and the weights of B are greater than $b^{-1}\delta$, which makes the bilateral shift B invertible. This finishes the proof of our corollary. ∎

To conclude this section, we observe that Claim 4 in the proof of Corollary 2.5 is established without using the hypothesis that the weight sequence of T is bounded away from 0. Hence the exact same proof of Claim 4 can be applied in the proof of Theorem 2.3 and show that the bilateral shift B in the factorization $T = UB$ given by Theorem 2.3 is indeed chaotic, mixing and frequently hypercyclic.

3. ANOTHER FACTORIZATION FOR UNILATERAL WEIGHTED BACKWARD SHIFTS

After successfully factoring in the previous section a unilateral backward shift T as $T = UB$, where U is a hypercyclic unilateral backward shift and B is a hypercyclic bilateral shift, in this section we show that the order of the factorization can be reversed.

THEOREM 3.1. *Let $1 \leqslant p < \infty$. If $T : \ell^p \to \ell^p$ is a unilateral weighted backward shift, then there exist a hypercyclic bilateral weighted shift $B : \ell^p \to \ell^p$, and a hypercyclic unilateral weighted backward shift $U : \ell^p \to \ell^p$ such that*

$$T = BU.$$

Moreover, for any given positive ε, the shifts B, U can be chosen so that both norms $\|B\|, \|U\|$ are no larger than $(1+\varepsilon)\max(1, \|T\|)$.

Proof. The constructions of the desired shifts U, B are similar to those in the proof of Theorem 2.3, but with a different twist. Starting the same way as in the proof of Theorem 2.3, without loss of generality, by reordering the canonical basis if necessary, assume T is given by

$$Te_i = \begin{cases} 0 & \text{if } i = 0, \\ w_i e_{i-1} & \text{if } i \geqslant 1. \end{cases}$$

where each w_i satisfies $0 < w_i \leqslant \|T\|$.

For any positive ε, select two positive numbers a, b for which

$$b = (1+\varepsilon)\max\{1, \|T\|\} \quad \text{and} \quad b^{-1} < a < 1. \tag{3.1}$$

Next let $j_0 = 0$ and inductively select a strictly increasing sequence $\{j_k : j \geqslant 0\}$ of nonnegative even integers that satisfy the following two inequalities:

$$(3.2)\quad j_{k+1} \geqslant 2j_k + 4 \quad \text{and} \quad b^{j_{k+1}/2} > \frac{kb^k}{b^{j_k/2}a^k w_{1+j_0} w_{1+j_1} \cdots w_{1+j_{k-1}}}, \quad \text{for } k \geqslant 1.$$

Using the sequence $\{j_k : k \geqslant 0\}$, we construct the hypercyclic bilateral shift B and a hypercyclic unilateral backward shift U such that $T = BU$, in the following two separated steps.

Step I. *Construction of the shift B.* To define the bilateral shift B, use the same bijection $\sigma : \mathbb{Z} \to \mathbb{Z}^+$ giving (2.4) to rearrange the canonical basis as follows:

$$(3.3)\qquad \ldots, e_{2+j_k}, \overbrace{e_{1+j_k}, e_{-1+j_k}}^{\text{missing } e_{j_k}\,(k \geqslant 1)}, e_{-2+j_k}, \ldots, e_3, e_2, \overbrace{e_1}^{e_{\sigma(-1)}}, \overbrace{e_0}^{e_{\sigma(0)}}, \overbrace{e_{j_1}}^{e_{\sigma(1)}}, e_{j_2}, e_{j_3}, \ldots$$

That is, $\sigma : \mathbb{Z} \to \mathbb{Z}^+$ is given by the same formula as in Claim 1 of the proof of Theorem 2.3:

$$(3.4)\qquad \sigma(i) = \begin{cases} j_i & \text{if } i \geqslant 0, \\ |i| + k & \text{if } (k+1) - j_{k+1} \leqslant i \leqslant (k-1) - j_k, \text{ and } k \geqslant 0. \end{cases}$$

Define the bilateral shift $B : \ell^p \to \ell^p$ on $\{e_{\sigma(i)} : i \in \mathbb{Z}\}$ by

$$(3.5)\qquad Be_{\sigma(i)} = \begin{cases} a^{-1} e_{\sigma(i-1)} & \text{if } i \geqslant 1, \\ b^{-1} w_{1+\sigma(i-1)} e_{\sigma(i-1)} & \text{if } i \leqslant 0. \end{cases}$$

The weights of B satisfy

$$(3.6)\qquad b^{-1} w_{1+\sigma(i-1)} \leqslant b^{-1}\|T\| \leqslant (1+\varepsilon)^{-1} < 1 < a^{-1} < b, \quad \text{(by (2.2))}.$$

Therefore $\|B\| < b$, giving the desired upper bound for the norm of the shift B. Furthermore by (3.6), B is hypercyclic by a direct application of Theorem 2.2.

Step II. *Construction of the shift U.* Moving on to defining the unilateral backward shift U, we recall that the corresponding unilateral backward shift in the proof of Theorem 2.3 is constructed by grouping the canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ into blocks in each of which all vectors e_i with odd indices i are placed before all vectors e_i with even indices i; see (2.8). For the unilateral weighted shift $U : \ell^p \to \ell^p$ that works for our current proof, we reverse the roles of the even and odd indices.

To be precise, define a bijection $\rho : \mathbb{Z}^+ \to \mathbb{Z}^+$ inductively for blocks of integers $[1 + j_k, j_{k+1}]$, where $k \geqslant 0$. We begin by keeping e_0 in its original zeroth position; that is, setting $\rho(0) = 0$. Then in the first step, we reorder the vectors $e_1, e_2, \ldots, e_{j_1}$ by placing all vectors e_i with even indices i in $[1, j_1] = [1 + j_0, j_1]$ before all vectors e_i with odd indices i in $[1 + j_0, j_1]$. That is, we reorder the first $1 + j_1$ vectors in the canonical basis as

$$(3.7)\qquad \overbrace{e_0}^{e_{\rho(0)}}, \overbrace{e_2, e_4, \ldots, e_{-2+j_1}, e_{j_1}}^{e_i \text{ with even indices } i}, \overbrace{e_1, e_3, \ldots, e_{-3+j_1}, e_{-1+j_1}}^{e_i \text{ with odd indices } i}.$$

Since $\rho : \mathbb{Z}^+ \to \mathbb{Z}^+$ defines this reordering, we have $\rho(1) = \rho(1 + j_0) = 2 = 2 + j_0$, and furthermore, $\rho(2) = 4, \ldots$, and lastly $\rho(j_1) = -1 + j_1$. Since there are an equal number $(j_0 + j_1)/2$ of even and odd indices in $[1 + j_0, j_1]$, it follows that

$$\rho\Big(\frac{j_0 + j_1}{2}\Big) = j_1 \quad \text{and} \quad \rho\Big(\frac{j_0 + j_1}{2} + 1\Big) = 1 = 1 + j_0.$$

Repeat this procedure for all vectors e_i with indices i in the interval $[1 + j_1, j_2]$ in the second step, and then for all vectors e_i with indices i in the interval $[1 + j_2, j_3]$ in the third step, and so on. For the general $(k+1)^{\text{st}}$ step with $k \geqslant 0$, we reorder the vectors e_i with indices i in the interval $[1 + j_k, j_{k+1}]$ as follows:

$$\underbrace{\overbrace{e_{2+j_k}, e_{4+j_k}, \ldots, e_{j_{k+1}}}^{e_i \text{ with even indices } i}, \overbrace{e_{1+j_k}, e_{3+j_k}, \ldots, e_{-1+j_{k+1}}}^{e_i \text{ with odd indices } i},}_{(k+1)^{\text{st}} \text{ step of the rearrangement}} \tag{3.8}$$

In particular, note that

$$\rho(1 + j_k) = 2 + j_k, \quad \text{and} \quad \rho(j_{k+1}) = -1 + j_{k+1}, \tag{3.9}$$

and furthermore,

$$\rho\Big(\frac{j_k + j_{k+1}}{2}\Big) = j_{k+1}, \quad \text{and} \quad \rho\Big(\frac{j_k + j_{k+1}}{2} + 1\Big) = 1 + j_k. \tag{3.10}$$

Summarizing the above computations, we provide the following formula for the bijection $\rho : \mathbb{Z}^+ \to \mathbb{Z}^+$, written in terms of even and odd values of $\rho(i)$:

$$\rho(i) = \begin{cases} 0 & \text{if } i = j_0 = 0, \\ 2m + j_k & \text{if } i = m + j_k, \text{ with } k \geqslant 0 \text{ and } 1 \leqslant m \leqslant \frac{j_{k+1} - j_k}{2}, \\ (2m-1) + j_k & \text{if } i = m + \frac{j_{k+1} + j_k}{2} \text{ with } k \geqslant 0 \text{ and } 1 \leqslant m \leqslant \frac{j_{k+1} - j_k}{2}. \end{cases} \tag{3.11}$$

Using the bijection $\rho : \mathbb{Z}^+ \to \mathbb{Z}^+$, define the unilateral backward shift $U : \ell^p \to \ell^p$ by

$$Ue_{\rho(i)} = \begin{cases} 0 & \text{if } i = 0, \\ aw_{1+j_k} e_{\rho(i-1)} & \text{if } \rho(i) = 1 + j_k \text{ for some } k \geqslant 0, \\ be_{\rho(i-1)} & \text{if } \rho(i) \neq 1 + j_k \text{ for any } k \geqslant 0. \end{cases} \tag{3.12}$$

Obviously $\|U\| = b$, because the weights of U satisfy $aw_{1+j_k} \leqslant a\|T\| < \|T\| < b$, which establishes the desired norm estimate for U.

Claim 5. The unilateral backward shift U is hypercyclic.

Proof of Claim 5. Set $n_k = (j_{k+1} + j_k)/2$ for each positive integer k. Note $n_k = m + j_k$, where $m = (j_{k+1} - j_k)/2$. Thus (3.11) gives $\rho(n_k) = j_{k+1}$. Hence it follows from (3.11), or (3.8), that for integers i with $1 \leqslant i \leqslant n_k$, we have $\rho(i) = 1 + j_\ell$ for some integer ℓ, precisely when $i = 1 + (j_{\ell+1} + j_\ell)/2$, for $\ell = 0, 1, 2, \ldots, k - 1$.

Thus $\rho(i) = 1 + j_\ell$ for exactly k values of ℓ. Hence for each positive integer k, by the definition of U in (3.12) we have

$$\begin{aligned}\|U^{n_k} e_{\rho(n_k)}\| &= \|a^k w_{1+j_0} w_{1+j_1} \cdots w_{1+j_{k-1}} b^{-k+n_k} e_0\| \\ &= \frac{b^{j_{k+1}/2} b^{j_k/2} a^k w_{1+j_0} \cdots w_{1+j_{k-1}}}{b^k} \\ &> k, \quad \text{(by (3.2))}.\end{aligned}$$

Therefore by a direct application of Theorem 2.1, the shift U is hypercyclic. ∎

To complete the proof of Theorem 3.1, it remains to check the following factorization.

Step III. *To verify* $T = BU$. Clearly from (3.12), $BUe_{\rho(0)} = BUe_0 = 0 = Te_{\rho(0)}$. Thus in the rest of this step, it suffices to focus on showing $BUe_{\rho(i)} = Te_{\rho(i)} = w_{\rho(i)} e_{-1+\rho(i)}$ for all positive integers i.

First, consider the case when $\rho(i) = 1 + j_k$ for some integer $k \geqslant 0$. By (3.10) we get

$$i = 1 + \frac{j_{k+1} + j_k}{2} \quad \text{and} \quad \rho(i-1) = \rho\Big(\frac{j_{k+1} + j_k}{2}\Big) = j_{k+1}.$$

Thus by (3.12),

$$Ue_{\rho(i)} = aw_{1+j_k} e_{\rho(i-1)} = aw_{\rho(i)} e_{j_{k+1}}. \tag{3.13}$$

Now by (3.4), $\sigma(k+1) = j_{k+1}$ and $\sigma(k) = j_k = -1 + \rho(i)$. By the definition of B in (3.5), we have

$$Be_{j_{k+1}} = Be_{\sigma(k+1)} = a^{-1} e_{\sigma(k)} = a^{-1} e_{-1+\rho(i)}. \tag{3.14}$$

Combining (3.13) and (3.14) yields

$$BUe_{\rho(i)} = w_{\rho(i)} e_{-1+\rho(i)},$$

finishing the proof for the first case.

Second, consider the case when $\rho(i) = 2 + j_k$ for some integer $k \geqslant 0$. If $k = 0$, then by (3.7), $i = 1$, $\rho(i) = 2$, and $\rho(i-1) = \rho(0) = 0$. In addition, by (3.12), (3.5), and (3.3), we have

$$BUe_{\rho(1)} = BUe_2 = bBe_0 = bBe_{\sigma(0)} = w_{1+\sigma(-1)} e_{\sigma(-1)} = w_2 e_1 = w_{\rho(1)} e_{-1+\rho(1)}.$$

Next suppose $k \geqslant 1$ and $\rho(i) = 2 + j_k$. Hence by (3.9), we have $i = 1 + j_k$ and also $\rho(i-1) = \rho(j_k) = -1 + j_k$. Thus by (3.12),

$$Ue_{\rho(i)} = be_{\rho(i-1)} = be_{-1+j_k}. \tag{3.15}$$

Now if $\sigma(\ell) = -1 + j_k$ for some negative integer ℓ, by the reordering given in (3.3) we get $\sigma(\ell - 1) = 1 + j_k$. Thus by (3.5),

$$Be_{-1+j_k} = Be_{\sigma(\ell)} = b^{-1} w_{1+\sigma(\ell-1)} e_{\sigma(\ell-1)} = b^{-1} w_{2+j_k} e_{1+j_k} = b^{-1} w_{\rho(i)} e_{-1+\rho(i)}. \tag{3.16}$$

Combining (3.15) and (3.16) yields

$$BUe_{\rho(i)} = w_{\rho(i)} e_{-1+\rho(i)},$$

finishing the proof for the second case.

Lastly, consider the case when $\rho(i) \neq 1 + j_k$ and $\rho(i) \neq 2 + j_k$, for any integer $k \geqslant 0$. Thus by (3.12),

$$Ue_{\rho(i)} = be_{\rho(i-1)}. \tag{3.17}$$

From the reordering given by (3.8), we see that $\rho(i-1) = -2 + \rho(i)$, and hence $\rho(i-1) \neq -1 + j_k$ or j_k. It follows from the reordering given in (3.3) that if $\sigma(\ell) = \rho(i-1)$ then $\ell \leqslant -1$ and $\sigma(\ell - 1) \neq 1 + j_k$, and thus $\sigma(\ell-1) = 1 + \sigma(\ell)$. Consequently by (3.5),

$$\begin{aligned} Be_{\rho(i-1)} &= Be_{\sigma(\ell)} = b^{-1} w_{1+\sigma(\ell-1)} e_{\sigma(\ell-1)} \\ &= b^{-1} w_{2+\sigma(\ell)} e_{1+\sigma(\ell)} = b^{-1} w_{\rho(i)} e_{-1+\rho(i)}. \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18) yields $BUe_{\rho(i)} = w_{\rho(i)} e_{-1+\rho(i)}$, which completes Step III and concludes the proof of our theorem. ∎

Parallel to Corollary 2.5, we can show that the two shifts B, U in the conclusion of Theorem 3.1 can have additional properties if the weights of the unilateral backward shift T are bounded away from zero.

COROLLARY 3.2. *Let $1 \leqslant p < \infty$. Suppose $T : \ell^p \to \ell^p$ is a unilateral weighted backward shift, and $B, U : \ell^p \to \ell^p$ are two shifts that satisfy the conclusion of Theorem* 3.1. *If the weight sequence of T is bounded away from* 0, *then both B and U can be selected to be chaotic, mixing, and frequently hypercyclic, with weight sequences bounded away from 0. In particular, B is invertible.*

Proof. Without loss of generality, by reordering the canonical basis if necessary, let T be given by $Te_0 = 0$ and $Te_i = w_i e_{i-1}$ whenever $i \geqslant 1$. Let δ be a real scalar such that $0 < \delta < w_i \leqslant \|T\|$, for each integer $i \geqslant 1$.

For any given positive ε, select the same positive scalars a, b that satisfy (3.1) as in the proof of Theorem 3.1. Next select a real scalar C such that

$$C \geqslant \max\{1, a^{-1}\delta^{-1}\}. \tag{3.19}$$

In the rest of the proof, we follow all other notations and definitions in the proof of Theorem 3.1, except that we require the sequence $\{j_k\}$ of nonnegative even integers to satisfy (3.2), along with the following additional condition:

$$b^{j_k} > \left(1 + \frac{\varepsilon}{2}\right)^{j_k} C^k b^k. \tag{3.20}$$

Such a selection of $\{j_k\}$ is possible because $b \geqslant 1 + \varepsilon$.

To prove our corollary, we show that the shifts B, U given by (3.5) and (3.12) satisfy the frequently hypercyclicity criterion in Theorem 2.4. Set X_0 in the theorem to be $X_0 = \text{span}\,\{e_i : i \in \mathbb{Z}^+\}$.

Claim 6. Define a map $R : X_0 \to X_0$ by taking

$$Re_{\rho(i)} = \begin{cases} a^{-1}w_{1+j_k}^{-1}e_{\rho(i+1)} & \text{if } \rho(i+1) = 1 + j_k \text{ for some } k \geqslant 0, \\ b^{-1}e_{\rho(i+1)} & \text{if } \rho(i+1) \neq 1 + j_k \text{ for any } k \geqslant 0, \end{cases}$$

and extending linearly to X_0. Then the shifts U and R satisfy conditions (i), (ii), (iii) in Theorem 2.4

Proof of Claim 6. For the unilateral backward shift U, and any positive integer n, we have $U^{n+1}e_{\rho(n)} = 0$. Hence U satisfies condition (i).

It is easy to use (3.12) to check that $URe_{\rho(i)} = e_{\rho(i)}$, and hence by linearity $URx = x$ whenever $x \in X_0$, and so condition (iii) is satisfied.

To show R satisfies condition (ii), we note that each $e_{\rho(n)}$ is a nonzero multiple of $R^n e_{\rho(0)}$. Hence it suffices to check that

$$\sum_{n=1}^{\infty} \|R^n e_{\rho(0)}\| < \infty.$$

To estimate $\|R^n e_{\rho(0)}\|$ in the infinite sum, note from (3.10) that $\rho(i+1) = 1 + j_\ell$ for some integer $\ell \geqslant 0$, precisely when $i = (j_\ell + j_{\ell+1})/2$. Thus the estimation of $\|R^n e_{\rho(0)}\|$ using the definition of R depends on the number of integers ℓ so that $(j_\ell + j_{\ell+1})/2 \leqslant n$. For that reason, we first calculate $R^m e_{\rho(j_\ell)}$, where $\ell \geqslant 0$ and $0 \leqslant m < j_{\ell+1} - j_\ell$. Note that from (3.11), $\rho(j_\ell) = j_\ell - 1$. Thus by the definition of R and (3.8),

$$R^m e_{\rho(j_\ell)} = \begin{cases} b^{-m}e_{\rho(m+j_\ell)} & \text{if } 0 \leqslant m \leqslant \frac{j_{\ell+1}-j_\ell}{2}, \\ b^{-m+1}a^{-1}w_{1+j_\ell}^{-1}e_{\rho(m+j_\ell)} & \text{if } \frac{j_{\ell+1}-j_\ell}{2} + 1 \leqslant m \leqslant j_{\ell+1} - j_\ell. \end{cases} \tag{3.21}$$

Use (3.21) repeatedly for integers $\ell = 0, 1, \ldots, k-1$, and take $m = j_{\ell+1} - j_\ell$ corresponding to each value of ℓ. We get

$$\begin{aligned} R^{j_k}e_{\rho(0)} &= R^{j_k-j_1}(R^{j_1-j_0}e_{\rho(j_0)}) = b^{-(j_1-j_0)+1}a^{-1}w_{1+j_0}^{-1}R^{j_k-j_1}e_{\rho(j_1)} \\ &= b^{1-j_1}a^{-1}w_{1+j_0}^{-1}R^{j_k-j_2}(R^{j_2-j_1}e_{\rho(j_1)}) = b^{2-j_2}a^{-2}w_{1+j_0}^{-1}w_{1+j_1}^{-1}R^{j_k-j_2}e_{\rho(j_2)} \\ &= \cdots = b^{k-j_k}a^{-k}w_{1+j_0}^{-1}w_{1+j_1}^{-1}\cdots w_{1+j_{k-1}}^{-1}e_{\rho(j_k)}. \end{aligned} \tag{3.22}$$

Using (3.22), we proceed to estimate $R^n e_{\rho(0)}$ for any integer $n \geqslant 1$. We first obtain the integer $k \geqslant 0$ such that $j_k \leqslant n < j_{k+1}$, and so by (3.22),

$$\begin{aligned} \|R^n e_{\rho(0)}\| &= \|R^{n-j_k}R^{j_k}e_{\rho(0)}\| = b^{-j_k+k}a^{-k}w_{1+j_0}^{-1}w_{1+j_1}^{-1}\cdots w_{1+j_{k-1}}^{-1}\|R^{n-j_k}e_{\rho(j_k)}\| \\ &\leqslant b^{-j_k}b^k C^k \|R^{n-j_k}e_{\rho(j_k)}\|, \quad \text{(by (3.19))} \\ &< \left(1 + \frac{\varepsilon}{2}\right)^{-j_k}\|R^{n-j_k}e_{\rho(j_k)}\|, \quad \text{(by (3.20))} \\ &\leqslant bC\left(1 + \frac{\varepsilon}{2}\right)^{-j_k}(1+\varepsilon)^{-n+j_k}, \quad \text{(by (3.21), (3.19) and (3.1))} \end{aligned}$$

$$\leqslant bC\Big(1+\frac{\varepsilon}{2}\Big)^{-n}.$$

Hence we have

$$\sum_{n=1}^{\infty} \|R^n e_{\rho(0)}\| < \infty,$$

which concludes the proof of Claim 6. ∎

We now proceed to show B satisfies the hypotheses in Theorem 2.4.

Claim 7. Define $S : \ell^p \to \ell^p$ by taking

$$Se_{\sigma(i)} = \begin{cases} ae_{\sigma(i+1)} & \text{if } i \geqslant 0, \\ bw^{-1}_{1+\sigma(i)} e_{\sigma(i+1)} & \text{if } i \leqslant -1, \end{cases}$$

and extending linearly to X_0. Then the shifts B and S satisfy conditions (i), (ii), and (iii) of Theorem 2.4.

Proof of Claim 7. Clearly it follows from (3.5) that $BSe_{\sigma(i)} = e_{\sigma(i)}$ for all integers i and hence by linearity $BSx = x$ whenever x is in X_0, showing condition (iii) of Theorem 2.4.

To show S satisfies condition (ii), we observe that if $n > 0$ then $S^n e_{\sigma(0)}$ is a nonzero multiple of $e_{\sigma(n)}$. Furthermore if $n < 0$, then $e_{\sigma(0)}$ is a nonzero multiple of $S^{|n|} e_{\sigma(n)}$. Thus to show $\sum_{n=1}^{\infty} \|S^n x\| < \infty$ for each $x \in X_0$, it suffices to show, by linearity and the triangle inequality, that $\sum_{n=1}^{\infty} \|S^n e_{\sigma(0)}\| < \infty$. This is obvious because it directly follows from the definition of S that $\|S^n e_{\sigma(0)}\| = a^n$.

To finish the proof of our claim, it remains to show B satisfies condition (i). For the exact same reason as for S above, it suffices to show that $\sum_{n=1}^{\infty} \|B^n e_{\sigma(0)}\| < \infty$. Hence it follows from the definition of B in (3.5) that

$$\|B^n e_{\sigma(0)}\| = b^{-n} w_{1+\sigma(-1)} w_{1+\sigma(-2)} \cdots w_{1+\sigma(-n)} \leqslant \Big(\frac{\|T\|}{b}\Big)^n \leqslant (1+\varepsilon)^{-n},$$

by (3.1). Hence we have established Claim 7. ∎

To finish the proof of our corollary, we remark that the weights of the shift B, as defined in (3.5), are given in the form of a^{-1} and $b^{-1} w_{1+\sigma(i-1)}$, which by (3.1) satisfy $b > a^{-1} > 1 > b^{-1}\|T\| \geqslant b^{-1} w_{1+\sigma(i-1)} > b^{-1}\delta$. Thus $\|B\| \leqslant b$, and the weights of B are greater than $b^{-1}\delta$. Since the shift B is bilateral, it is invertible.

Similarly for the shift U defined in (3.12), its weights are in the form of b and aw_{1+j_k}, which satisfy $b > \|T\| > a\|T\| \geqslant aw_{1+j_k} > a\delta$. Thus $\|U\| \leqslant b$ and the weights of U are greater than $a\delta$. ∎

We remark that by Corollary 3.2, we have $T = BU$ and B is an invertible hypercyclic bilateral shift, and so $B^{-1}T = U$. Note that B^{-1} is hypercyclic because B is hypercyclic by a result of Kitai ([11], Corollary 2.2). Thus there is a hypercyclic bilateral shift $A = B^{-1}$ such that $AT = U$, which is a mixing, chaotic, frequently hypercyclic unilateral backward shift. Similarly we can apply the exact same argument to the result of Corollary 2.5 and summarize as follows.

COROLLARY 3.3. *Let $1 \leqslant p < \infty$, Suppose $T : \ell^p \to \ell^p$ is a unilateral weighted backward shift whose weight sequence is bounded away from 0. Then there exist invertible hypercyclic bilateral weighted shifts A_1 and A_2 such that A_1T and TA_2 are unilateral weighted backward shifts which are mixing, chaotic and frequently hypercyclic.*

To conclude our discussion in this section, we remark the pair of factors U and B in the factorization given by Theorem 2.3 must be different from the pair given by Theorem 3.1, because any such pair do not commute. In other words, if there are a unilateral backward shift U and a bilateral shift B such that UB is a unilateral backward shift T, then $T \neq BU$. To prove that, without loss of generality, by relabeling the canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ if necessary, assume

$$Te_i = \begin{cases} 0 & \text{if } i = 0, \\ w_i e_{i-1} & \text{if } i \geqslant 1, \end{cases}$$

where $0 < w_i \leqslant \|T\|$ for each integer $i \geqslant 1$. By way of contradiction, suppose there exist a unilateral backward shift U and a bilateral shift B such that $T = UB = BU$.

We now show that the unilateral backward shift $U : \ell^p \to \ell^p$ is given by

$$Ue_i = \begin{cases} 0 & \text{if } i = 0, \\ u_i e_{i-1} & \text{if } i \geqslant 1, \end{cases} \tag{3.23}$$

where $0 < u_i \leqslant \|U\|$ for each integer $i \geqslant 1$. For that, we observe that there is a bijection $\rho : \mathbb{Z}^+ \to \mathbb{Z}^+$ and a bounded positive weight sequence $\{u_i : i \in \mathbb{Z}^+\}$ for which $Ue_{\rho(0)} = 0$ and $Ue_{\rho(i)} = u_{\rho(i)}e_{\rho(i-1)}$ for all positive integers i. To this end, we show that $\rho(i) = i$.

Since U and B commute, for each positive integer i we have $0 = T^i e_{i-1} = B^i U^i e_{i-1}$. Since the bilateral shift B is one-to-one, we have $U^i e_{i-1} = 0$. In other words

$$e_{i-1} \in \{e_{\rho(0)}, e_{\rho(1)}, \ldots, e_{\rho(i-1)}\}, \quad \text{whenever } i \geqslant 1.$$

Putting $i = 1, 2, 3, \ldots$ in respective order, we get $\rho(i) = i$, establishing (3.23).

It follows from (3.23) that if $i \geqslant 0$ then

$$Be_i = u_{i+1}^{-1} BUe_{i+1} = u_{i+1}^{-1} Te_{i+1} = u_{i+1}^{-1} w_{i+1} e_i,$$

which contradicts that B is a bilateral shift.

4. FINAL REMARKS

In this last section we provide a few remarks on future research directions. First, Theorem 2.3 and Theorem 3.1 naturally lead to the question whether we can factor a bilateral shift $T : \ell^p \to \ell^p$ as the product of two hypercyclic shifts. If one of the two hypercyclic shifts is a unilateral backward shift, then their product must have a nontrivial kernel, which is in the form of span$\{e_i\}$ for some vector e_i of the canonical basis, and hence the product cannot be the bilateral shift T. Nevertheless, it has been proved in [9] that every bilateral shift T is the product of two hypercyclic bilateral shifts. Indeed, if T is invertible, then both bilateral shifts can also be selected to be invertible.

Second, there are many applications for operator decompositions such as the polar decomposition of a Hilbert space operator, the LU decomposition for a matrix and Cholesky decomposition for a Hermitian positive definite matrix. Since the hypercyclic factorizations given in Theorem 2.3 and Theorem 3.1 are examples of operator decompositions, we raise the following question.

QUESTION 4.1. Are there applications of the factorization in Theorem 2.3 or Theorem 3.1?

Third, we observe that in both proofs of Theorem 2.3 and Theorem 3.1, the canonical basis is carefully rearranged to define the hypercyclic unilateral backward shift U and the hypercyclic bilateral shift B. In the next set of four questions, we ask whether we can make further arrangements on the canonical basis to allow the two shifts U and B to have other properties often exhibited in hypercyclic operators.

We begin with the property of having a *hypercyclic subspace*, which by definition, is an infinite dimensional, closed subspace of ℓ^p that consists entirely, except for the zero vector, of hypercyclic vectors. It was proved by León-Saavedra and Montes-Rodríguez [12] that every hypercyclic bilateral shift has a hypercyclic subspace. In addition, they also proved that for a unilateral backward shift U defined by (1.1), U has a hypercyclic subspace if and only if

$$\sup_n w_1 \cdots w_n = \infty, \quad \text{and} \quad \sup_n \left(\inf_k w_{k+1} w_{k+2} \cdots w_{k+n} \right)^{1/n} \leqslant 1.$$

A subspace is said to be a *common hypercyclic subspace* of a family of operators, if it is a hypercyclic subspace for each operator in the family. With the techniques established by Aron, Bès, León and Peris [1], and Bayart [2] for showing the existence of a common hypercyclic subspace, one may ask the following question.

QUESTION 4.2. Can the two hypercyclic shifts U and B in Theorem 2.3 or Theorem 3.1 be chosen to have a common hypercyclic subspace?

We now turn our attention to the next property of dual hypercyclicity. By definition, a bounded linear operator $T : X \to X$ on a separable, infinite dimensional Banach space X with a separable dual space X^* is said to be *dual hypercyclic* if both T and its adjoint T^* are hypercyclic. Indeed whenever X^* is separable, Salas [16] proved that there is a dual hypercyclic operator T on X. Back to the setting of our present paper, the adjoint of a unilateral backward shift is a unilateral forward shift, which cannot be hypercyclic. In the case when the underlying Banach sequence space is ℓ^p with $p > 1$, its dual space is separable. We may try to make the bilateral shift B in Theorem 2.3 and Theorem 3.1 dual hypercyclic, by modifying the definitions of $\{j_k\}$ in (2.3) and (3.2) and arrange some of the vectors e_{j_k} on the left hand side of e_0 in (2.4) and (3.3). To this end, the techniques of Chan [7] or Salas [14] may be helpful in producing a dual hypercyclic bilateral shift B.

For the case of ℓ^1, its dual space ℓ^∞ is not separable, and so no operator on the dual space ℓ^∞ can be hypercyclic. In this case, with the weak* topology, a shift T on ℓ^∞ can be hypercyclic; that is, there is a vector x in ℓ^∞ so that $\{x, Tx, T^2x, \ldots\}$ is weak-star dense in ℓ^∞. For the details of weak-star hypercyclicity of a shift on ℓ^∞, one may refer to the work of Bès, Chan and Sanders [4]. To summarize the above two cases of different values of p, we raise the following question.

QUESTION 4.3. On the sequence space ℓ^p with $1 < p < \infty$, can the bilateral shift B in Theorem 2.3 or Theorem 3.1 be chosen to be dual hypercyclic? When $p = 1$, can the hypercyclic bilateral shift B in Theorem 2.3 or Theorem 3.1 be chosen so that its adjoint $B^* : \ell^\infty \to \ell^\infty$ is weak-star hypercyclic?

Suppose Question 4.3 has a positive answer for the bilateral shift B in Theorem 2.3 for the case when $1 < p < \infty$. Since the adjoint of a unilateral forward shift $F : \ell^p \to \ell^p$ is a unilateral backward shift F^*, we can write by Theorem 2.3

$$F^* = UB,$$

where U is a hypercyclic unilateral backward shift, and B is a dual hypercyclic bilateral shift. Then by taking the adjoint on both sides of the equation, we obtain

$$F = B^*U^*.$$

Thus we conclude that every unilateral forward shift is the product of a dual hypercyclic bilateral shift and a unilateral forward shift. Similarly, if Question 4.3 has a positive answer for the shift B in Theorem 3.1, or for the case $p = 1$, we have an analogous conclusion.

Moving on to the next property of hypercyclicity relative to the weak topology of a Banach space, we say that a bounded linear operator $T : X \to X$ on a separable, infinite dimensional Banach space X is *weakly hypercyclic* if there is a vector $x \in X$ whose orbit $\{x, Tx, T^2x, \ldots\}$ is weakly dense in X. Of course, every hypercyclic operator is a weakly hypercyclic operator, but it was shown by Chan and Sanders [8] that there exists a bilateral shift that is weakly hypercyclic but fails to be hypercyclic. Thus we have the following question.

QUESTION 4.4. Can the hypercyclic shift B in Theorem 2.3 or Theorem 3.1 be chosen to be weakly hypercyclic but not hypercyclic?

We now switch our focus to disjoint hypercyclicity for a finite number of bounded linear operators $T_1, T_2, \ldots, T_N$ with $N \geqslant 2$ on a separable, infinite dimensional Banach space X. We say the operators $T_1, T_2, \ldots, T_N$ are *disjoint hypercyclic* if there exists a vector $x \in X$ for which the N-tuple $(x, x, \ldots, x)$ is a hypercyclic vector for the direct sum operator

$$T_1 \oplus T_2 \oplus \cdots \oplus T_N.$$

For disjoint hypercyclic operators, the dynamics exhibited in certain orbits are so distinct that the same power for each of the operators $T_1, T_2, \ldots, T_N$ can be used to approximate completely different vectors. Disjoint hypercyclicity for shifts was well-studied by Bès and Peris in [6] and by Bès, Martin and Sanders in [5]. It naturally follows to ask the following question.

QUESTION 4.5. Since the factorization $T = UB$ in Theorem 2.3 is not unique for a unilateral backward shift T, do there exist two factorizations $T = U_1B_1 = U_2B_2$ such that U_1, U_2 are disjoint hypercyclic and B_1, B_2 are disjoint hypercyclic? A similar question can be asked for the factorization $T = BU$ in Theorem 3.1.

A positive answer to Question 4.5 shows that different hypercyclic shift factorizations of the same shift T may exhibit drastically different dynamics.

Lastly, we raise two questions about operators that shift vectors outside the original canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ of ℓ^p, in relation to the work of Grivaux [10] who showed that every operator T on a separable, infinite dimensional Hilbert space is the sum of two hypercyclic operators. In light of Theorem 2.3 and Theorem 3.1, we wonder whether we can additively decompose a unilateral backward shift T, as the sum of a hypercyclic unilateral backward shift U and a hypercyclic bilateral shift B. It is easy to see that such an additive decomposition does not work if we strictly follow the definitions in (1.1) and (1.3), because if we have $Te_i = w_ie_{i-1}$, then it follows that Ue_i and Be_i are in span$\{e_{i-1}\}$. Thus it cannot happen that B is a bilateral shift and U is a unilateral backward shift at the same time. However, we can consider another basis $\{f_j : j \in \mathbb{Z}\}$ of ℓ^p, for instance each vector f_j in span $\{e_0, e_1, e_2, \ldots\}$, so that B is of the form $Bf_j = b_jf_{j-1}$ for some positive scalar b_j. In other words, B is a bilateral shift on a basis that is different from the original canonical basis $\{e_i : i \in \mathbb{Z}^+\}$. Similarly, we can also have the unilateral backward shift on a different basis as well.

QUESTION 4.6. For any unilateral backward shift $T : \ell^p \to \ell^p$ given by $Te_0 = 0$ and $Te_i = w_ie_{i-1}$ for $i \geqslant 1$, do there exist a hypercyclic unilateral backward shift U and a hypercyclic bilateral shift B on bases of ℓ^p that are different from the original canonical basis $\{e_i : i \in \mathbb{Z}^+\}$ such that $T = U + B$?

Similarly, if we allow to have shifts on different bases of ℓ^p, we also have the following question.

QUESTION 4.7. For any unilateral backward shift $T : \ell^p \to \ell^p$ given by $Te_0 = 0$ and $Te_i = w_ie_{i-1}$ for $i \geqslant 1$, do there exist hypercyclic unilateral backward shifts U_1 and U_2 on bases of ℓ^p that are different from the original canonical basis $\{e_i : i \geqslant 0\}$ such that $T = U_1 + U_2$?

To conclude this section, we remark that we cannot write the shift T in Question 4.7 as the sum $T = U_1 + U_2$ of two hypercyclic unilateral backward shifts U_1, U_2, if we strictly follow the definition (1.1) without using a different basis $\{f_i : i \geqslant 0\}$. This is easy to see because if $Te_i = w_i e_{i-1}$ for some positive weight w_i where $i \geqslant 1$, then it follows that for such U_1 and U_2, we must have $U_1 e_i = u_{1,i} e_{i-1}$ and $U_2 e_i = u_{2,i} e_{i-1}$ for some weights $u_{1,i}, u_{2,i} > 0$. Hence,

$$w_i = u_{1,i} + u_{2,i}. \tag{4.1}$$

Thus if $w_i < 1$ for all positive integers i, then both $u_{1,i}, u_{2,i} < 1$, and so U_1 and U_2 cannot be hypercyclic. Despite the above nonhypercyclicity of U_1 and U_2, if we allow $u_{1,i}$ and $u_{2,i}$ to be negative weights, then we can modify Theorem 2.1 to work for complex weights so (4.1) presents no obstacle for U_1 and U_2 to be hypercyclic operators.

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