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SUBMODULES OF THE HARDY MODULE IN INFINITELY MANY VARIABLES

HUI DAN, KUNYU GUO, and HANSONG HUANG

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ABSTRACT. This paper is concerned with polynomially generated submodules of the Hardy module $H^2(\mathbb{D}_2^\infty)$. Since the polynomial ring $\mathcal{P}_\infty$ in infinitely many variables is not Noetherian, some standard tricks for finitely many variables fail to work. Therefore, we need to introduce new techniques to the situation of infinitely many variables. It is shown that some classical results of $H^2(\mathbb{D}^n)$ remain valid for infinitely many variables. However, some new phenomena indicate that the Hardy module $H^2(\mathbb{D}_2^\infty)$ diverges considerably from the case in finitely many variables.

KEYWORDS: *The Hardy module, infinitely many variables, closed ideals, Ahern–Clark's theorem.*

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1. INTRODUCTION

To begin with, we introduce some notations. Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$, and as in [18], set $\mathbb{C}^\infty = \{(\lambda_1, \lambda_2, \ldots) : \lambda_n \in \mathbb{C} \text{ for every } n\}$. Denote by $\mathbb{D}_2^\infty$ the Hilbert's multidisk; that is,

$$\mathbb{D}_2^\infty = \Big\{\lambda = (\lambda_n) \in \mathbb{C}^\infty : \sum_{n=1}^{\infty} |\lambda_n|^2 < +\infty \text{ and } |\lambda_n| < 1 \text{ for each } n\Big\}.$$

We write $z = (z_1, z_2, \ldots)$, and let $\mathbb{Z}_+^\infty$ denote the set of all finitely supported sequences of non-negative integers, that is, $\mathbb{Z}_+^\infty = \bigcup\limits_{n=1}^{\infty} \mathbb{Z}_+^n$, which is an additive semigroup. For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots)$ in $\mathbb{Z}_+^\infty$, define $z^\alpha = \prod\limits_{k=1}^{n} z_k^{\alpha_k}$. Denote by $H^2(\mathbb{D}_2^\infty)$ the Hardy space over the Hilbert multidisk $\mathbb{D}_2^\infty$ consisting of formal power series $f(z) = \sum_{\alpha\in\mathbb{Z}_+^\infty} c_\alpha z^\alpha$ satisfying

$$\|f\|^2 = \sum_{\alpha\in\mathbb{Z}_+^\infty} |c_\alpha|^2 < +\infty.$$

To simplify notation, rewrite $H^2(\mathbb{D}_2^\infty) = H^2_\infty$. The space H^2_∞ is a reproducing kernel Hilbert space on $\mathbb{D}_2^\infty$ with the kernel $K_\lambda(z) = \prod_{n=1}^{\infty} 1/(1-\overline{\lambda}_n z_n)$ for $\lambda = (\lambda_n) \in \mathbb{D}_2^\infty$, where the infinite product $\prod_{n=1}^{\infty} 1/(1-\overline{\lambda}_n z_n)$ converges because $\{\overline{\lambda}_n z_n\}$ is a sequence in l^1. This space was extensively used in the study of Dirichlet series, and the Riesz basis problem and the completeness problem of the standard Lebesgue space $L^2(0,1)$ [15], [18]. Also as is shown in [18], there is a close connection between cyclic vectors of H^2_∞ and the Riemann's hypothesis. We refer the reader to the references [3], [4], [5], [6], [7], [19] for more information.

In this paper, we are mainly concerned with invariant subspaces of H^2_∞. A closed subspace M is called an invariant subspace of H^2_∞ if $z_i M \subseteq M$ for $i = 1, 2, \ldots$. This is equivalent to $pM \subseteq M$ for each polynomial p. A cyclic vector f of H^2_∞ is such that $[f] = H^2_\infty$, where $[f]$ denotes the invariant subspace generated by f. As indicated by Helson [16], such invariant subspaces are not yet well understood. Building on ideas of Beurling and Bohr [7], [8], below we briefly describe the background of the research on invariant subspaces and cyclic vectors of the space H^2_∞.

Let H be a separable Hilbert space with a given orthonormal basis $\{e_1, e_2, \ldots\}$. For each natural number n, define an isometric operator on H by $T_n e_k = e_{nk}, k = 1, 2, \ldots$. In essence, the operator semigroup (T_n) is independent of choices of orthonormal bases of H. Then the map $n \mapsto T_n$ is a representation of the multiplicative semigroup of natural numbers $\mathbb{N}$ on H. We are interested in invariant subspaces and cyclic vectors of the semigroup (T_n). By an invariant subspace of (T_n), we mean a closed subspace $M \subseteq H$ such that $T_n M \subseteq M$ for each natural number n, and a cyclic vector h is such that $[h] = H$, where $[h] = \overline{\operatorname{span}}\{T_n h : n \in \mathbb{N}\}$. Set $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ to be consecutive prime numbers. For $n \in \mathbb{N}$, by applying the fundamental theorem of arithmetic, it follows that there exists a unique decomposition $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$. Define the map $\alpha : \mathbb{N} \to \mathbb{Z}_+^\infty$ by $\alpha(n) = (k_1, k_2, \ldots, k_m, 0, \ldots)$, then α is a semigroup isomorphism from $\mathbb{N}$ onto $\mathbb{Z}_+^\infty$. For $h = \sum_n a_n e_n$, the Bohr transform ([8], [15]) is

$$\mathbf{B} : H \to H^2_\infty, \quad \mathbf{B}h(z) = \sum_n a_n z^{\alpha(n)},$$

which is a unitary transform. It is easy to verify that

$$\mathbf{B} T_n \mathbf{B}^{-1} f(z) = z^{\alpha(n)} f(z), \quad f \in H^2_\infty,$$

and hence $\mathbf{B}\mathrm{Lat}(T_n) = \mathrm{Lat}(M_{z_n})$, that is, M is an invariant subspace of (T_n) if and only if $\mathbf{B}M$ is an invariant subspace of H^2_∞. In particular, a vector $h \in H$ is cyclic for the semigroup (T_n) if and only if $\mathbf{B}h$ is cyclic in H^2_∞.

Let us mention three well known examples to illuminate the significance of the research on invariant subspaces and cyclic vectors of the space H^2_∞. The first is the standard Lebesgue space $L^2(0,1)$. For each ϕ in $L^2(0,1)$, ϕ is considered as a function on the whole real line by extending ϕ to an odd periodic function of period 2. The space $L^2(0,1)$ has a canonical orthonormal basis $\{\phi_k(x) = \sqrt{2}\sin(k\pi x) : k \in \mathbb{N}\}$. Then the operator T_n defined above is a dilation operator $D_n : L^2(0,1) \to L^2(0,1)$ defined by $D_n\phi(x) = \phi(nx)$. As presented in [7], Beurling's completeness problem asks for which functions $\phi \in L^2(0,1)$, the dilation system $\{\phi(nx)\}_{n\geqslant 1}$ is a complete sequence in $L^2(0,1)$, that is, the linear span of $\{\phi(nx)\}_{n\geqslant 1}$ is dense in $L^2(0,1)$. As mentioned above, this is equivalent to the characterization of cyclic vectors of H^2_∞. A problem closely related to the Beurling's completeness problem, is the Riesz basis problem of $L^2(0,1)$: for which functions $\phi \in L^2(0,1)$, the dilation system $\{\phi(nx)\}_{n\geqslant 1}$ is a Riesz basis of $L^2(0,1)$ (an orthonormal basis with respect to an equivalent norm). This problem was discussed in [15] in great detail, and has a complete answer. Translated in the language of H^2_∞, the answer reads as follows: the dilation system $\{\phi(nx)\}_{n\geqslant 1}$ is a Riesz basis of $L^2(0,1)$ if and only if $\mathbf{B}\phi$ is an invertible multiplier of H^2_∞.

The second example is the case of Dirichlet series. A Dirichlet series is a formal series $f(s) = \sum\limits_{n=1}^{\infty} a_n n^{-s}$ involving a complex variable s. With the norm of

$$\|f\| = \Big(\sum_n |a_n|^2\Big)^{1/2} < \infty,$$

such Dirichlet series form, in a natural way, a Hilbert space of analytic functions on the half-plane $\mathcal{R}s > 1/2$, and denoted by $\mathcal{H}$. Note that the space $\mathcal{H}$ has a canonical orthonormal basis $\{e_n = n^{-s} : n = 1, 2, \ldots\}$. For each natural number m, define an isometric operator on $\mathcal{H}$ by $M_m e_k = e_{mk}, k = 1, 2, \ldots$. Then for a function $f(s) = \sum\limits_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}$, $M_m f(s) = m^{-s} f(s)$ for $m = 1, 2, \ldots$. This says that invariant subspaces of (M_m) are just those for multipliers of $\mathcal{H}$. Picking an $s \in \mathbb{C}$ with $\mathcal{R}s > 1/2$, then $\mathcal{M}_s = \{f \in \mathcal{H} : f(s) = 0\}$ is an invariant subspace for multipliers with codimension 1. By the Bohr transform, $\mathbf{B}\mathcal{M}_s$ is an invariant subspace for H^2_∞ with codimension 1. By applying Theorem 3.2 of this paper, we show that this invariant subspace is the closure of a maximal ideal of $\mathcal{P}_\infty$ with its zero point in $\mathbb{D}^\infty_2$. This implies that for $\phi \in \mathcal{H}$, if $\mathbf{B}\phi$ has no zero point in $\mathbb{D}^\infty_2$, then ϕ has necessarily no zero point in the half-plane $\mathcal{R}s > 1/2$. This helps us to study which Dirichlet series it have no zero point in the half-plane $\mathcal{R}s > 1/2$. Some examples are presented in Section 3.

The third is the Hardy space $H^2(\mathbb{D})$ on the unit disk. Set

$$H^2_0 = \{f \in H^2(\mathbb{D}) : \widehat{f}(0) = f(0) = 0\},$$

with a canonical orthonormal basis $\{z, z^2, \dots\}$. For each natural number n, as in [18], define a power dilation operator on $H_0^2(\mathbb{D})$ by $P_n f(z) = f(z^n)$. The study of invariant subspaces and cyclic vectors of the semigroup (P_n) is related to the zeros of the Riemann zeta function, see pp. 1605–1608 of [18]. Therefore, the characterization of invariant subspaces and cyclic vectors of the semigroup (P_n) is interesting and challenging in itself.

We now come back to the situation of invariant subspaces of H^2_∞. Intuitively, the study of invariant subspaces of H^2_∞ is analogous to the situation of the Hardy space $H^2(\mathbb{D}^n)$. In the past forty years, the theory of Hilbert modules developed by Douglas and Paulsen [10] has provided some useful methods to approach the study of invariant subspaces of the Hardy space $H^2(\mathbb{D}^n)$. Set $\mathcal{P}_n = \mathbb{C}[z_1, \dots, z_n]$, the polynomial ring of n-complex variables. Following [11], let X be a reproducing Banach space on some domain Ω in $\mathbb{C}^n$, and we call X a reproducing $\mathcal{P}_n$-module on Ω if $p \cdot X$ is contained in X for every $p \in \mathcal{P}_n$. Then $H^2(\mathbb{D}^n)$ is a reproducing $\mathcal{P}_n$-module on $\mathbb{D}^n$, where the module action is defined by multiplications by polynomials, and a submodule of $H^2(\mathbb{D}^n)$ is just a closed subspace invariant under multiplications by polynomials. The submodules of $H^2(\mathbb{D}^n)$ studied in this context were the ones that are the closures of ideals of polynomials. The techniques involved come mainly from commutative algebra and algebraic geometry. For an ideal $\mathcal{I}$ of $\mathcal{P}_n$, we write $[\mathcal{I}]$ for the closure of $\mathcal{I}$ in $H^2(\mathbb{D}^n)$. Then $[\mathcal{I}]$ is a finitely generated submodule of $H^2(\mathbb{D}^n)$ since $\mathcal{P}_n$ is a Noetherian ring [2], [11]. The ring $\mathcal{P}_n$ has a natural topology induced by the norm of $H^2(\mathbb{D}^n)$. Then an ideal $\mathcal{I}$ is closed in this topology if and only if $[\mathcal{I}] \cap \mathcal{P}_n = \mathcal{I}$. Then these closed ideals can be put in one-to-one correspondence with the submodules generated by them, and hence they can be used to “label” these submodules. It follows that the study of these submodules reduces to the characterization for closed ideals. Along this line, a remarkable result was obtained by Douglas, Paulsen, Sah and Yan [11]. They proved the following theorem.

THEOREM 1.1. *If each algebraic component of the zero variety $Z(\mathcal{I})$ of $\mathcal{I}$ has a nonempty intersection with $\mathbb{D}^n$, then $\mathcal{I}$ is closed.*

Let us mention that in their paper the authors called these ideals contracted. Douglas and Paulsen conjectured that the opposite direction is also true [10]. For $n = 2$, Gelca gave an affirmative answer to the Douglas–Paulsen's conjecture [9], [12]. However, it remains unknown whether this holds in general. The above result indicates the study of closedness of ideals is closely connected to geometry of zero varieties of ideals. When $Z(\mathcal{I})$ is finite and lies in $\mathbb{D}^n$, this reduces to a remarkable algebraic reduction theorem for finite codimensional submodules studied in [1] by Ahern and Clark. This theorem is the following one.

THEOREM 1.2. *Suppose M is a submodule of $H^2(\mathbb{D}^n)$ of finite codimension. Then $M \cap \mathcal{P}_n$ is a closed ideal of the ring $\mathcal{P}_n$, and*

(i) *$M \cap \mathcal{P}_n$ is dense in M;*

(ii) $\dim\, \mathcal{P}_n / M \cap \mathcal{P}_n = \dim\, H^2(\mathbb{D}^n)/M$;
(iii) $Z(M \cap \mathcal{P}_n)$ *is finite and lies in* $\mathbb{D}^n$.
Conversely, if $\mathcal{I}$ *is an ideal of the polynomial ring* $\mathcal{P}_n$, *and* $Z(\mathcal{I}) \subseteq \mathbb{D}^n$, *then the ideal* $\mathcal{I}$ *is closed, and* $\dim\, \mathcal{P}_n/\mathcal{I} = \dim\, H^2(\mathbb{D}^n)/[\mathcal{I}] < \infty$.

In this paper we partially generalize the above stated results to the situation of submodules of H^2_∞. Putting $\mathcal{P}_\infty = \bigcup_{n=1}^{\infty} \mathcal{P}_n$, then $\mathcal{P}_\infty$ is a polynomial ring in infinitely many variables for which each polynomial only depends on finitely many variables. Then the Hardy space H^2_∞ is a reproducing $\mathcal{P}_\infty$-module on $\mathbb{D}_2^\infty$. It is shown that the Ahern–Clark Theorem 1.2 remains true for a finite codimensional submodule M of H^2_∞. However, in general the latter part of the Ahern–Clark theorem is not valid in the case of infinitely many variables. Since the polynomial ring $\mathcal{P}_\infty$ is not Noetherian, some rather standard tricks in dealing with submodules of finitely many variables fail to work. Therefore, we need to introduce some new techniques in commutative algebra for the study of submodules in infinitely many variables. By applying the Hilbert Nullstellensatz in infinitely many variables, one obtains that every ideal of $\mathcal{P}_\infty$ with finitely many zeros can be uniquely decomposed as a finite intersection of primary ideals for which each has exactly one zero point. This enables us to characterize closedness and density of those ideals with finitely many zeros in the topology of H^2_∞. Applying the characteristic space technique, we establish an inequality to link multiplicities of zeros and codimensions of submodules.

This paper is arranged as follows. Section 2 provides some preliminaries for the commutative algebra in infinitely many variables. Section 3 discusses the version of the Ahern–Clark theorem in infinitely many variables. Some examples are presented to show that the case of H^2_∞ diverges considerably from the situation of finitely many variables. This section also establishes an inequality to link multiplicities of zeros and codimensions of submodules. Section 4 considers the closedness and density of a class of ideals of $\mathcal{P}_\infty$.

2. SOME PRELIMINARIES FOR THE POLYNOMIAL RING IN INFINITELY MANY VARIABLES

This section establishes some preliminaries for the commutative algebra in infinitely many variables, especially for primary ideals and radical ideals for $\mathcal{P}_\infty$, the ring of polynomials in infinitely many variables.

It is well known that for a positive integer n each maximal ideal $\mathfrak{m}$ of $\mathcal{P}_n$ is generated by $z_1 - \mu_1, \ldots, z_n - \mu_n$, where $\mu_1, \ldots, \mu_n$ are complex numbers (depending on $\mathfrak{m}$). For $\lambda \in \mathbb{C}^\infty$, denote by $\mathcal{M}_\lambda$ the ideal of all polynomials that vanish at λ. The following proposition is a consequence of the main result of [17]. For completeness, we present a direct proof which applies a well known argument.

PROPOSITION 2.1. *Suppose $\mathcal{I}$ is a proper ideal of $\mathcal{P}_\infty$. The following are equivalent:*

(i) *$\mathcal{I}$ is maximal;*

(ii) $\dim \mathcal{P}_\infty/\mathcal{I} = 1$;

(iii) *there exists some $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{C}^\infty$ such that $\mathcal{I} = \mathcal{M}_\lambda$; that is, $\mathcal{I}$ is generated by $\{z_n - \lambda_n : n \geqslant 1\}$.*

Proof. (i) $\Rightarrow$ (ii) Denote by K the quotient ring $\mathcal{P}_\infty/\mathcal{I}$ of $\mathcal{P}_\infty$ modulo $\mathcal{I}$. Then K is an extension field of $\mathbb{C}$. To prove the field extension $K/\mathbb{C}$ is algebraic, assume, on the contrary, that there exists a transcendental element t in K over $\mathbb{C}$. Then $\{1/(t-a)\}_{a\in\mathbb{C}}$ are linearly independent, which implies $\dim_{\mathbb{C}} K \geqslant \aleph_1$, where $\dim_{\mathbb{C}}$ denotes the algebraic dimension of a complex vector space over $\mathbb{C}$. On the other hand, $\dim_{\mathbb{C}} K \leqslant \dim_{\mathbb{C}} \mathcal{P}_\infty = \aleph_0$, a contradiction. Thus, $K/\mathbb{C}$ is algebraic. Since $\mathbb{C}$ is algebraically closed, we have $K = \mathbb{C}$.

(ii) $\Rightarrow$ (iii) follows directly from $\mathcal{P}_\infty = \mathcal{I} + \mathbb{C}$.

(iii) $\Rightarrow$ (i) For each polynomial p in $\mathcal{P}_\infty$, we have $p - p(\lambda) \in \mathcal{I}$. Thus $\mathcal{I}$ is maximal. ∎

By Zorn's lemma, each proper ideal is contained in a maximal ideal. Then Proposition 2.1 immediately gives the Hilbert Nullstellensatz for infinitely many variables.

COROLLARY 2.2. *For each proper ideal $\mathcal{I}$ of $\mathcal{P}_\infty$, $Z(\mathcal{I})$ is nonempty.*

For an ideal $\mathcal{I}$, its radical ideal $r(\mathcal{I})$ is defined to be the ideal of all polynomials p such that $p^k \in \mathcal{I}$ for some positive integer k [2]. The following proposition comes from [17].

PROPOSITION 2.3. *For an ideal $\mathcal{I}$ of $\mathcal{P}_\infty$, we have $r(\mathcal{I}) = \{p \in \mathcal{P}_\infty : p|_{Z(\mathcal{I})} = 0\}$.*

Following the definition in [2], an ideal $\mathcal{I}$ of $\mathcal{P}_\infty$ is called primary if for any two polynomials p and q, the conditions $pq \in \mathcal{I}$ and $p \notin \mathcal{I}$ imply the existence of an integer k such that $q^k \in \mathcal{I}$. In this case, $\mathcal{I}$ is called $r(\mathcal{I})$-primary. When $\mathcal{I}$ is primary, its radical ideal $r(\mathcal{I})$ is prime, i.e., if $pq \in r(\mathcal{I})$, then either p is in $r(\mathcal{I})$, or q is in $r(\mathcal{I})$.

Recall that $\mathcal{P}_n$ is a Noetherian ring, and the Lasker–Noether decomposition theorem states that each ideal of $\mathcal{P}_n$ admits a finite primary decomposition [24]. However, in general this is not true for $\mathcal{P}_\infty$ (Example 4.8). The following shows that every ideal of $\mathcal{P}_\infty$ with finitely many zeros has a finite primary decomposition.

PROPOSITION 2.4. *Suppose $\mathcal{I}$ is an ideal of $\mathcal{P}_\infty$ with finitely many zeros $\lambda^{(1)}$, $\dots, \lambda^{(k)}$. Then $\mathcal{I}$ has a unique primary decomposition*

$$\mathcal{I} = \bigcap_{i=1}^{k} \mathfrak{p}^{(i)},$$

where $\mathfrak{p}^{(i)}$ are primary ideals satisfying $r(\mathfrak{p}^{(i)}) = \mathcal{M}_{\lambda^{(i)}}$.

Proof. By Proposition 2.3,

$$r(\mathcal{I}) = \{p \in \mathcal{P}_\infty : p(\lambda^{(i)}) = 0, i = 1, \ldots, k\} = \bigcap_{i=1}^{k} \mathcal{M}_{\lambda^{(i)}}.$$

For each positive integer n, put $\mathfrak{m}_n^{(i)} = \{p \in \mathcal{P}_n : p(\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) = 0\}$ and set $\mathcal{I}_n = \mathcal{I} \cap \mathcal{P}_n$, an ideal of $\mathcal{P}_n$. Then

$$r_n(\mathcal{I}_n) = r(\mathcal{I}) \cap \mathcal{P}_n = \Big(\bigcap_{i=1}^{k} \mathcal{M}_{\lambda^{(i)}}\Big) \cap \mathcal{P}_n = \bigcap_{i=1}^{k} (\mathcal{M}_{\lambda^{(i)}} \cap \mathcal{P}_n) = \bigcap_{i=1}^{k} \mathfrak{m}_n^{(i)},$$

where $r_n(\mathcal{I}_n)$ is the radical ideal of $\mathcal{I}_n$ in $\mathcal{P}_n$. Let N be the minimal integer such that

$$(\lambda_1^{(i)}, \ldots, \lambda_N^{(i)}) \neq (\lambda_1^{(j)}, \ldots, \lambda_N^{(j)})$$

when $i \neq j$. For each integer $n \geqslant N$, $\mathcal{I}_n$ has an irredundant primary decomposition in $\mathcal{P}_n$

$$\mathcal{I}_n = \bigcap_{j=1}^{m} \mathfrak{q}_j.$$

Then for each $j = 1, \ldots, m$,

$$\bigcap_{i=1}^{k} \mathfrak{m}_n^{(i)} = r_n(\mathcal{I}_n) = \bigcap_{l=1}^{m} r_n(\mathfrak{q}_l) \subseteq r_n(\mathfrak{q}_j).$$

Since $r_n(\mathfrak{q}_j)$ is prime, there is an integer $i_j \in \{1, \ldots, k\}$ such that

$$\mathfrak{m}_n^{(i_j)} = r_n(\mathfrak{q}_j).$$

Combining the equality

$$\bigcap_{i=1}^{k} \mathfrak{m}_n^{(i)} = \bigcap_{j=1}^{m} r_n(\mathfrak{q}_j)$$

with the uniqueness of $\{r_n(\mathfrak{q}_1), \ldots, r_n(\mathfrak{q}_m)\}$, we have $m = k$. Hence, without a loss of generality, we let $\mathfrak{p}_n^{(i)} = \mathfrak{q}_i$ for $i = 1, \ldots, k$ such that $\mathfrak{m}_n^{(i)} = r_n(\mathfrak{p}_n^{(i)})$ for each i.

Next we will show that for $n \geqslant N$,

$$\mathfrak{p}_n^{(i)} = \mathfrak{p}_{n+1}^{(i)} \cap \mathcal{P}_n.$$

For this, note that

$$r_n(\mathfrak{p}_{n+1}^{(i)} \cap \mathcal{P}_n) = r_{n+1}(\mathfrak{p}_{n+1}^{(i)}) \cap \mathcal{P}_n = \mathfrak{m}_{n+1}^{(i)} \cap \mathcal{P}_n = \mathfrak{m}_n^{(i)}. \tag{2.1}$$

It is known that if the radical ideal of an ideal $\mathfrak{p}$ is maximal, then $\mathfrak{p}$ is primary ([2], p. 51, Proposition 4.2). Therefore $\mathfrak{p}_{n+1}^{(i)} \cap \mathcal{P}_n$ is a primary ideal in $\mathcal{P}_n$. Since

$\mathcal{I}_{n+1} = \bigcap\limits_{i=1}^{k} \mathfrak{p}_{n+1}^{(i)}$,

$$\mathcal{I}_n = \mathcal{I}_{n+1} \cap \mathcal{P}_n = \bigcap_{i=1}^{k} (\mathfrak{p}_{n+1}^{(i)} \cap \mathcal{P}_n)$$

is an irredundant primary decomposition of $\mathcal{I}_n$. Besides, by (2.1)

$$r_n(\mathfrak{p}_n^{(i)}) = \mathfrak{m}_n^{(i)} = r_n(\mathfrak{p}_{n+1}^{(i)} \cap \mathcal{P}_n),$$

and note that for each i ($1 \leqslant i \leqslant k$) $\mathfrak{m}_n^{(i)}$ is a minimal prime ideal associated with $\mathcal{I}_n$. Then by the corollary of the second uniqueness theorem ([2], p. 54, Corollary 4.11), $\mathfrak{p}_n^{(i)} = \mathfrak{p}_{n+1}^{(i)} \cap \mathcal{P}_n$. For $1 \leqslant i \leqslant k$, set

$$\mathfrak{p}^{(i)} = \bigcup_{n=N}^{\infty} \mathfrak{p}_n^{(i)}.$$

Then $\mathfrak{p}^{(i)}$ is an ideal of $\mathcal{P}_\infty$. For each positive integer n, if $n \geqslant N$, then

$$z_n - \lambda_n^{(i)} \in \mathfrak{m}_n^{(i)} = r_n(\mathfrak{p}_n^{(i)}) \subseteq r(\mathfrak{p}^{(i)});$$

if $n < N$, $z_n - \lambda_n^{(i)} \in \mathfrak{m}_N^{(i)} = r_N(\mathfrak{p}_N^{(i)}) \subseteq r(\mathfrak{p}^{(i)})$. Then $r(\mathfrak{p}^{(i)}) = \mathcal{M}_{\lambda^{(i)}}$, and $\mathfrak{p}^{(i)}$ is $\mathcal{M}_{\lambda^{(i)}}$-primary. Since for $n \geqslant N$,

$$\mathcal{I}_n = \bigcap_{i=1}^{k} \mathfrak{p}_n^{(i)} = \bigcap_{i=1}^{k} (\mathfrak{p}^{(i)} \cap \mathcal{P}_n) = \Big(\bigcap_{i=1}^{k} \mathfrak{p}^{(i)}\Big) \cap \mathcal{P}_n,$$

it follows that $\mathcal{I} = \bigcap\limits_{i=1}^{k} \mathfrak{p}^{(i)}$. Again by the corollary of the second uniqueness theorem [2], it is immediate to see that the primary decomposition of $\mathcal{I}$ is unique. ∎

3. FINITE CODIMENSIONAL SUBMODULES OF H^2_∞

The purpose of this section is to present a generalization of Ahern and Clark's algebraic reduction theorem to the situation of H^2_∞. In what follows we endow the ring $\mathcal{P}_\infty$ with the topology of H^2_∞. Then an ideal $\mathcal{I}$ is closed in this topology if and only if $[\mathcal{I}] \cap \mathcal{P}_\infty = \mathcal{I}$, where $[\mathcal{I}]$ is the closure of $\mathcal{I}$ in H^2_∞. For $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{C}^\infty$, let $\mathcal{M}_\lambda$ be the ideal of all polynomials that vanish at λ.

The following is needed in the sequel.

LEMMA 3.1. *For $\lambda \in \mathbb{C}^\infty$, $\mathcal{M}_\lambda$ is dense in H^2_∞ if and only if $\lambda \notin \mathbb{D}_2^\infty$. Equivalently, $\mathcal{M}_\lambda$ is closed if and only if $\lambda \in \mathbb{D}_2^\infty$.*

Proof. For $\lambda \in \mathbb{D}_2^\infty$, let K_λ denote the reproducing kernel of H^2_∞ at λ. Then the reproducing kernel K_λ lies in $\mathcal{M}_\lambda^\perp$, and thus $\mathcal{M}_\lambda$ is not dense.

Now assume that $\lambda \notin \mathbb{D}_2^\infty$ and there are two cases to distinguish: either $\lambda \notin \mathbb{D}^\infty$ or $\lambda \notin l^2$.

Case 1. $\lambda \notin \mathbb{D}^\infty$. Then there exists a positive integer N such that $|\lambda_N| \geqslant 1$. Since $\lambda_N - z_N$ is an outer function in $H^2(\mathbb{D})$ (in z_N-variable), there are polynomials p_k in z_N such that $(\lambda_N - z_N)p_k(z_N)$ tends to 1 in the norm of $H^2(\mathbb{D})$. Then it follows that $\mathcal{P}_\infty \subseteq [\mathcal{M}_\lambda]$, and thus $\mathcal{M}_\lambda$ is dense in H^2_∞.

Case 2. $\lambda \notin l^2$. Let $\mathbf{c}_{00}$ denote the linear space consisting of all finitely-supported complex sequences (c_n). Then λ induces an unbounded linear functional F on $(\mathbf{c}_{00}, \|\cdot\|_{l^2})$, defined by

$$F((c_1, c_2, \ldots)) = \sum_{i=1}^{\infty} c_i \lambda_i.$$

Since F is not continuous at zero, there is a sequence $\{\mathbf{a}^{(k)}\}$ in $\mathbf{c}_{00}$ such that for $k \geqslant 1$, $F(\mathbf{a}^{(k)}) = 1$ and

$$\|\mathbf{a}^{(k)}\|_{l^2} \to 0 \quad (k \to \infty).$$

Put

$$q_k = \sum_{i=1}^{\infty} a_i^{(k)}(\lambda_i - z_i) = F(\mathbf{a}^{(k)}) - \sum_{i=1}^{\infty} a_i^{(k)} z_i, \quad \text{(finite sum)}$$

and then $\{q_k\}$ converges to 1 in the norm of H^2_∞. Since $q_k \in \mathcal{M}_\lambda$ for each k, $1 \in [\mathcal{M}_\lambda]$, forcing $\mathcal{P}_\infty \subseteq [\mathcal{M}_\lambda]$. Therefore $\mathcal{M}_\lambda$ is dense in H^2_∞. ∎

The following is the H^2_∞ version of Ahern and Clark's characterization for finite codimensional submodules of $H^2(\mathbb{D}^n)$.

THEOREM 3.2. *Suppose M is a submodule of finite codimension k. Then $M \cap \mathcal{P}_\infty$ is a closed ideal of $\mathcal{P}_\infty$ and*

(i) *$M \cap \mathcal{P}_\infty$ is dense in M;*
(ii) *$\dim \mathcal{P}_\infty / M \cap \mathcal{P}_\infty = k$;*
(iii) *$Z(M \cap \mathcal{P}_\infty)$ is a finite subset of $\mathbb{D}_2^\infty$.*

Proof. Our method of proof follows the same techniques used in Proposition 2.4 of [11].

Since $(\mathcal{P}_\infty + M)/M$ is dense in H^2_∞/M and $\dim H^2_\infty/M < \infty$, it follows that $(\mathcal{P}_\infty + M)/M$ is closed, and $(\mathcal{P}_\infty + M)/M = H^2_\infty/M$. Then

$$\dim\, \mathcal{P}_\infty/M \cap \mathcal{P}_\infty = \dim\, (\mathcal{P}_\infty + M)/M = \dim\, H^2_\infty/M = k.$$

Thus there exists a k-dimensional subspace N of $\mathcal{P}_\infty$ such that

$$M \cap \mathcal{P}_\infty + N = \mathcal{P}_\infty,$$

and hence $M + N \supseteq [M \cap \mathcal{P}_\infty] + N \supseteq [\mathcal{P}_\infty] = H^2_\infty$, forcing

$$M + N = [M \cap \mathcal{P}_\infty] + N = H^2_\infty. \tag{3.1}$$

Since $\dim H^2_\infty/M = \dim N = k$, $M \cap N = \{0\}$, which by (3.1) gives

$$M = [M \cap \mathcal{P}_\infty].$$

Thus both (i) and (ii) are proved. By (i), $M \cap \mathcal{P}_\infty$ is closed.

It remains to prove (iii). In fact, for each point λ in $Z(M \cap \mathcal{P}_\infty)$,

$$M \cap \mathcal{P}_\infty \subseteq \mathcal{M}_\lambda \subseteq \mathcal{P}_\infty.$$

By (ii) $M \cap \mathcal{P}_\infty$ has finite codimension in $\mathcal{P}_\infty$, and thus there exists a finite dimensional subspace N_λ of $\mathcal{P}_\infty$ such that $\mathcal{M}_\lambda = M \cap \mathcal{P}_\infty + N_\lambda$. For each polynomial $q \in (M + N_\lambda) \cap \mathcal{P}_\infty$, write $q = q_1 + q_2$ with $q_1 \in M$ and $q_2 \in N_\lambda$. Then $q_1 = q - q_2 \in \mathcal{P}_\infty$, and thus $q \in M \cap \mathcal{P}_\infty + N_\lambda$. This shows that

$$M \cap \mathcal{P}_\infty + N_\lambda \supseteq (M + N_\lambda) \cap \mathcal{P}_\infty,$$

which immediately implies

$$\mathcal{M}_\lambda = M \cap \mathcal{P}_\infty + N_\lambda = (M + N_\lambda) \cap \mathcal{P}_\infty.$$

Then by (i)

$$\mathcal{M}_\lambda = (M + N_\lambda) \cap \mathcal{P}_\infty = ([M \cap \mathcal{P}_\infty] + N_\lambda) \cap \mathcal{P}_\infty \supseteq [\mathcal{M}_\lambda] \cap \mathcal{P}_\infty,$$

which gives $[\mathcal{M}_\lambda] \neq H^2_\infty$. Thus Lemma 3.1 implies that $\lambda \in \mathbb{D}_2^\infty$. Therefore, $Z(M \cap \mathcal{P}_\infty) \subseteq \mathbb{D}_2^\infty$.

To show $Z(M \cap \mathcal{P}_\infty)$ is finite, note that for $\lambda \in Z(M \cap \mathcal{P}_\infty)$,

$$K_\lambda \in (M \cap \mathcal{P}_\infty)^\perp = M^\perp$$

where K_λ is the reproducing kernel of H^2_∞ at λ. Since

$$\dim M^\perp = \operatorname{codim} M < \infty,$$

it follows from the linear independence of reproducing kernel vectors that the zero variety $Z(M \cap \mathcal{P}_\infty)$ contains only finitely many points. ∎

Concerning the opposite direction of Ahern and Clark's theorem in infinite variables, there is a great difference from the case in finitely many variables. This is illustrated by the next example.

EXAMPLE 3.3. Let $\mathcal{I}$ be the ideal generated by the following polynomials:

$$\{z_1^2, z_2^2\}; \quad \{z_1 - z_{2n-1} : n \geqslant 2\}; \quad \{z_2 - 2nz_{2n} : n \geqslant 2\}.$$

Writing $\mathbf{0} = (0, 0, \ldots)$, then $Z(\mathcal{I}) = \{\mathbf{0}\} \subseteq \mathbb{D}_2^\infty$, and one can show that

$$\dim \mathcal{P}_\infty / \mathcal{I} = 4 > \dim H^2_\infty / [\mathcal{I}]. \tag{3.2}$$

Hence $\mathcal{I}$ is not closed.

Below we present the proof of (3.2). It is straightforward to show

$$\mathcal{P}_\infty = \mathcal{I} \dotplus \mathbb{C}1 \dotplus \mathbb{C}z_1 \dotplus \mathbb{C}z_2 \dotplus \mathbb{C}z_1 z_2, \tag{3.3}$$

which gives $\dim \mathcal{P}_\infty / \mathcal{I} = 4$. Since $\{z_1 - z_{2n-1}\}$ converges weakly to z_1 in the Hilbert space H^2_∞, z_1 belongs to $[\mathcal{I}]$. It follows that $\mathcal{I} \subsetneq [\mathcal{I}] \cap \mathcal{P}_\infty$; that is, $\mathcal{I}$ is not closed. Also, by (3.3), $\dim H^2_\infty / [\mathcal{I}] < \infty$, and hence the map $\sigma : \mathcal{P}_\infty / [\mathcal{I}] \cap \mathcal{P}_\infty \to H^2_\infty / [\mathcal{I}]$ is an isomorphism by mapping $\widetilde{p}$ to $\widehat{p}$. This implies that

$$\dim H^2_\infty / [\mathcal{I}] = \dim \mathcal{P}_\infty / [\mathcal{I}] \cap \mathcal{P}_\infty < \dim \mathcal{P}_\infty / \mathcal{I} = 4.$$

However, from Lemma 3.1 and Theorem 3.2, the map $\mathcal{M}_\lambda \mapsto [\mathcal{M}_\lambda]$ establishes a one-to-one correspondence from maximal ideals $\mathcal{M}_\lambda$ ($\lambda \in \mathbb{D}_2^\infty$) of $\mathcal{P}_\infty$ onto submodules of H^2_∞ with codimension 1.

Before going on, we say more about Theorem 3.2. From Theorem 3.2 we see that being different from the case of finitely many variables, each proper submodule of H^2_∞ generated by finitely many polynomials is necessarily of infinite codimension. Now applying this to the semigroup (P_n) of power dilation operators on $H^2_0(\mathbb{D})$ mentioned in Introduction, shows that each finite codimensional invariant subspace of (P_n) is necessarily generated by infinitely many polynomials since $\mathbf{B}(\mathbb{C}_0[z]) = \mathcal{P}_\infty$, where $\mathbb{C}_0[z] = \{p \in \mathbb{C}[z] : p(0) = 0\}$. This means that each invariant subspace of (P_n) generated by finitely many polynomials either is $H^2_0(\mathbb{D})$, or is of infinite codimension.

Below we apply Theorem 3.2 to a Hilbert space of Dirichlet series. A Dirichlet series is a formal series $f(s) = \sum\limits_{n=1}^{\infty} a_n n^{-s}$ involving a complex variable s. In the norm of

$$\|f\| = \Big(\sum_n |a_n|^2\Big)^{1/2} < \infty,$$

such Dirichlet series form, in a natural way, a Hilbert space of analytic functions on the half-plane $\mathcal{R}s > 1/2$, denoted by $\mathcal{H}$. This space is a reproducing kernel function space on the half-plane $\mathcal{R}s > 1/2$ with the kernel $K_w(z) = \zeta(z + \overline{w})$ [15], where $\zeta(z)$ is the Riemann zeta function. Note that the space $\mathcal{H}$ has a canonical orthonormal basis $\{e_n = n^{-s} : n = 1, 2, \ldots\}$. For each natural number m, define an isometric operator M_m on $\mathcal{H}$ by $M_m e_k = e_{mk}$ ($k = 1, 2, \ldots$). Then $M_m f(s) = m^{-s} f(s)$ for $f \in \mathcal{H}$ and $m = 1, 2, \ldots$. This says that invariant subspaces of (M_m) are just those for multipliers of $\mathcal{H}$. For a complex number s with $\mathcal{R}s > 1/2$, the space $\mathcal{M}_s = \{f \in \mathcal{H} : f(s) = 0\}$ is an invariant subspace of codimension 1. Not every 1-codimensional invariant subspace has such a form. An example is the invariant subspace generated by $\{n^{-s} : n = 2, 3, \ldots\}$. By Theorem 3.2 and the Bohr transform, for $\phi \in \mathcal{H}$, if $\mathbf{B}\phi$ dose not have a zero point in $\mathbb{D}_2^\infty$, then ϕ has necessarily no zero point in the half-plane $\mathcal{R}s > 1/2$. For $\mathcal{R}\tau > 1/2$, write $\lambda = (p_1^{-\tau}, p_2^{-\tau}, \ldots)$, where $p_1, p_2, \ldots$ are consecutive prime numbers. Then $\lambda \in \mathbb{D}_2^\infty$. Setting $\phi_\tau(s) = \sum\limits_{n=1}^{\infty} n^{-\tau} n^{-s}$, one has $\phi_\tau(s) \in \mathcal{H}$, and

$$\mathbf{B}\phi_\tau(z) = \sum_n n^{-\tau} z^{\alpha(n)} = \sum_n \lambda^{\alpha(n)} z^{\alpha(n)} = \prod_n \frac{1}{1 - p_n^{-\tau} z_n} = K_{\overline{\lambda}}(z),$$

where $K_{\overline{\lambda}}(z)$ is the reproducing kernel of H^2_∞ at $\overline{\lambda}$. Since $K_{\overline{\lambda}}(z)$ has no zero point in $\mathbb{D}_2^\infty$, $\phi_\tau(s)$ has no zero point in the half-plane $\mathcal{R}s > 1/2$ for each $\mathcal{R}\tau > 1/2$. This gives the well known fact that the Riemann zeta function $\zeta(z) = \sum\limits_{n=1}^{\infty} n^{-s}$ has no zero point in the half-plane $\mathcal{R}s > 1$. A stronger conclusion is that $\phi_\tau(z)$ is a cyclic vector in $\mathcal{H}$, see [15], [18].

By developing the techniques of characteristic spaces, the codimension formula related to zero varieties, and some algebraic reduction theorems were achieved in [9], [13], [14].

In what follows we will try to generalize the codimension formula in finitely many variables to the case of H^2_∞. Firstly let us recall the characteristic space method used in [9], [13], [14]. Let $q = \sum_{\alpha\in\mathbb{Z}^n_+} c_\alpha z^\alpha$ be a complex polynomial involving finitely many variables $z_1, \ldots, z_n$, where all $c_\alpha = 0$ except for finitely many α. Denote by $q(D)$ the linear partial differential operator

$$q(D) = \sum_{\alpha\in\mathbb{Z}^n_+} c_\alpha \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}.$$

For a proper ideal $\mathcal{I}$ of $\mathcal{P}_\infty$ and a point $\lambda \in Z(\mathcal{I})$, the set

$$\mathcal{I}_\lambda = \{q \in \mathcal{P}_\infty : q(D)p|_\lambda = 0 \text{ for every } p \in \mathcal{I}\}$$

is called the characteristic space of $\mathcal{I}$ at the zero point λ. The multiplicity of $\mathcal{I}$ at the zero point λ is defined to be $\dim \mathcal{I}_\lambda$. A careful verification shows that for any polynomials q and p,

$$q(D)(z_j p)|_\lambda = \lambda_j q(D)p|_\lambda + \frac{\partial q}{\partial z_j}(D)p|_\lambda, \quad j = 1, 2, \ldots. \tag{3.4}$$

Therefore $\mathcal{I}_\lambda$ is invariant under the action of the basic partial differential operators $\{\partial/\partial z_1, \partial/\partial z_2, \ldots\}$.

The envelope, $\mathcal{I}^{\mathrm{e}}_\lambda$ of $\mathcal{I}$ at λ is defined as

$$\mathcal{I}^{\mathrm{e}}_\lambda = \{p \in \mathcal{P}_\infty : q(D)p|_\lambda = 0,\ \forall q \in \mathcal{I}_\lambda\}.$$

The preceding equalities imply that $\mathcal{I}^{\mathrm{e}}_\lambda$ is an ideal of $\mathcal{P}_\infty$ containing $\mathcal{I}$.

Similarly, for a submodule M of H^2_∞, we can define the characteristic space M_λ of M at a zero point $\lambda \in Z(M)$. Then M_λ is invariant under the basic partial differential operators $\{\partial/\partial z_1, \partial/\partial z_2, \ldots\}$. The multiplicity of M at a zero point λ is defined to be $\dim M_\lambda$.

THEOREM 3.4. *If M is a submodule of H^2_∞ of finite codimension k, then*

$$\sum_{\lambda\in Z(M)} \dim\, M_\lambda \leqslant k. \tag{3.5}$$

To prove this theorem we need the following lemma.

LEMMA 3.5. *If M is a submodule of H^2_∞ of finite codimension k, then for each $\lambda \in Z(M)$, the multiplicity of M at the zero point λ is finite, that is,* $\dim M_\lambda \leqslant k$.

Proof. For each polynomial q in M_λ, define a linear functional γ_q on H^2_∞ by setting $\gamma_q(f) = q(D)f|_\lambda$. It will be proved that γ_q is continuous. In fact, there is a positive integer N such that $q \in \mathcal{P}_N$. Define a linear map E_N from H^2_∞ to $H^2(\mathbb{D}^N)$ by putting $E_N f(z_1, z_2, \ldots) = f(z_1, \ldots, z_N, \lambda_{N+1}, \lambda_{N+2}, \ldots)$.

We claim that E_N is continuous. For this, put

$$A = \{\alpha \in \mathbb{Z}_+^\infty : \alpha_n = 0, n \geqslant N+1\}, \quad B = \{\alpha \in \mathbb{Z}_+^\infty : \alpha_1 = \cdots = \alpha_N = 0\},$$

and

$$\widetilde{\lambda} = (0, \ldots, 0, \lambda_{N+1}, \lambda_{N+2}, \ldots).$$

Since $\lambda \in \mathbb{D}_2^\infty$, $\sum\limits_{n \geqslant N+1} |\lambda_n|^2 < \infty$, then

$$\sum_{\beta \in B} |\widetilde{\lambda}^\beta|^2 = \prod_{n \geqslant N+1} \frac{1}{1 - |\lambda_n|^2} < \infty.$$

For each $f \in H^2_\infty$, write $f(z) = \sum\limits_{\alpha \in \mathbb{Z}_+^\infty} c_\alpha z^\alpha$. Thus

$$E_N f(z_1, \ldots, z_N) = \sum_{\alpha \in A, \beta \in B} c_{\alpha+\beta} \widetilde{\lambda}^\beta z_1^{\alpha_1} \cdots z_N^{\alpha_N}.$$

Then

$$\begin{aligned}\|E_N f\|^2 &= \sum_{\alpha \in A} \Big| \sum_{\beta \in B} c_{\alpha+\beta} \widetilde{\lambda}^\beta \Big|^2 \leqslant \sum_{\alpha \in A} \Big(\sum_{\beta \in B} |c_{\alpha+\beta}|^2 \Big) \Big(\sum_{\beta \in B} |\widetilde{\lambda}^\beta|^2 \Big) \\ &= \Big(\sum_{\beta \in B} |\widetilde{\lambda}^\beta|^2 \Big) \cdot \Big(\sum_{\alpha \in A, \beta \in B} |c_{\alpha+\beta}|^2 \Big) = \Big(\sum_{\beta \in B} |\widetilde{\lambda}^\beta|^2 \Big) \cdot \|f\|^2.\end{aligned}$$

The proof of the claim is finished.

Note that $\gamma_q(f) = \gamma_q(E_N f)$ for each $f \in H^2_\infty$. Since γ_q is continuous on $H^2(\mathbb{D}^N)$ and E_N is continuous, γ_q is continuous on H^2_∞. By the Riesz representation theorem there exists a unique function f_q in H^2_∞ such that $\gamma_q(f) = \langle f, f_q \rangle$ for $f \in H^2_\infty$. Then $q \mapsto f_q$ defines an injective conjugate linear map from M_λ to $M^\perp$. Therefore, $\dim M_\lambda \leqslant k$. The proof is finished. ∎

Proof of Theorem 3.4. Write $\mathcal{J} = M \cap \mathcal{P}_\infty$; then by Theorem 3.2 we see that $Z(\mathcal{J}) = Z(M)$ is a finite subset of $\mathbb{D}_2^\infty$. Let this subset be $\{\lambda^{(1)}, \ldots, \lambda^{(m)}\}$. Take a positive integer d such that $\mu_i \neq \mu_j$ for $i \neq j$, where $\mu_i = (\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_d^{(i)})$ for $i = 1, 2, \ldots, m$. By Theorem 3.2 the codimension of $\mathcal{J}$ in $\mathcal{P}_\infty$ also is k, and hence there exists a linear subspace $\mathcal{R}$ of $\mathcal{P}_\infty$ with dimension k such that

$$\mathcal{P}_\infty = \mathcal{J} \dot{+} \mathcal{R}.$$

By Lemma 3.5 for each zero point $\lambda^{(i)}$ of M, the characteristic space $M_{\lambda^{(i)}}$ is a polynomial space of finite dimension. This insures that there exists a positive integer l such that all polynomials in $\mathcal{R}$ and $M_{\lambda^{(i)}}$ $(i = 1, \ldots, m)$ only depend on variables $z_1, z_2, \ldots, z_l$. Take a positive integer $N \geqslant \max\{d, l\}$, then it holds

$$\mathcal{P}_N = \mathcal{P}_\infty \cap \mathcal{P}_N = \mathcal{J} \cap \mathcal{P}_N \dot{+} \mathcal{R}.$$

Let W denote the closure of $\mathcal{J} \cap \mathcal{P}_N$ in $H^2(\mathbb{D}^N)$. Then it is a submodule of $H^2(\mathbb{D}^N)$, and $\mu_i \in Z(W)$ for $i = 1, 2, \ldots, m$. Since $H^2(\mathbb{D}^N) = W + \mathcal{R}$, the submodule W has codimension at most k in $H^2(\mathbb{D}^N)$. It is obvious that $M_{\lambda^{(i)}} \subseteq W_{\mu_i}$

for $i = 1, 2, \ldots, m$. Thus by Theorem 3.1 of [13] we see that

$$\sum_{\lambda \in Z(M)} \dim M_\lambda = \sum_{i=1}^{m} \dim M_{\lambda^{(i)}} \leqslant \sum_{i=1}^{m} \dim W_{\mu_i} \leqslant \dim(H^2(\mathbb{D}^N)/W) \leqslant k.$$

The proof is complete. ∎

In Theorem 3.4 equality does not always hold in (3.5) in general, but this is not the case in finitely many variables [13]. To see this, let us revisit Example 3.3. Let $\mathcal{I}$ be the ideal given in Example 3.3 and put $M = [\mathcal{I}]$. Then $Z(M) = Z(\mathcal{I}) = \{\mathbf{0}\}$, and $\dim H^2_\infty/M < 4$. By a verification similar to that in the subsequent Example 3.6, the characteristic space $M_{\mathbf{0}}$ of M at the point $\mathbf{0}$ is $\mathbb{C}$, and hence $\dim M_{\mathbf{0}} = 1$. Below we will show $\dim H^2_\infty/M \geqslant 2$. Obviously $1 \in M^\perp$. Let $f(z) = z_2 + \sum_{n \geqslant 2} z_{2n}/2n$. We claim that $f \perp M$. In fact, for $n \geqslant 2$ and $\alpha \in \mathbb{Z}_+^\infty$,

$$\langle f, z_1^2 z^\alpha \rangle = \langle f, z_2^2 z^\alpha \rangle = \langle f, (z_1 - z_{2n-1}) z^\alpha \rangle = 0;$$

for $n \geqslant 2$ and $|\alpha| \geqslant 1$, $\langle f, (z_2 - 2nz_{2n}) z^\alpha \rangle = 0$, and

$$\langle f, z_2 - 2nz_{2n} \rangle = \Big\langle z_2 + \sum_{n \geqslant 2} \frac{z_{2n}}{2n}, z_2 - 2nz_{2n} \Big\rangle = 0.$$

Hence $\{1, f\} \subseteq M^\perp$ and it follows that $\dim H^2_\infty/M \geqslant 2$.

The next example shows that in the case of infinitely many variables, the relation between multiplicities of zeros and codimensions of submodules is unclear.

EXAMPLE 3.6. Let $\mathcal{I}$ be the ideal generated by $\{z_n^2 : n \geqslant 2\}$ and

$$\{z_n - kz_{n^k} : n \geqslant 1, k \geqslant 2\}.$$

Taking $n = 1, k = 2$, then $z_1 \in \mathcal{I}$, and hence $Z(\mathcal{I}) = \{\mathbf{0}\}$. We will show that $\dim \mathcal{I}_{\mathbf{0}} = 1$, but $\dim H^2_\infty/[\mathcal{I}] = \infty$.

First, we prove that $\dim \mathcal{I}_{\mathbf{0}} = 1$. For this, assume q is a polynomial in $\mathcal{I}_{\mathbf{0}}$. For each polynomial p, by the equality (3.4) we have

$$\frac{\partial q}{\partial z_1}(D)p|_{\mathbf{0}} = q(D)(z_1 p)|_{\mathbf{0}} = 0.$$

Then $\partial q/\partial z_1 = 0$. Similarly, for $n \geqslant 2$ and $k \geqslant 2$,

$$\Big(\frac{\partial q}{\partial z_n} - k\frac{\partial q}{\partial z_{n^k}}\Big)(D)p|_{\mathbf{0}} = q(D)((z_n - kz_{n^k})p)|_{\mathbf{0}} = 0,$$

and hence

$$\frac{\partial q}{\partial z_n} = k\frac{\partial q}{\partial z_{n^k}}.$$

Since there is a positive integer l such that q only depends on $z_1, \dots, z_{2^l}$, then $n^{l+1} > n^l \geqslant 2^l$ for $n \geqslant 2$, and thus

$$\frac{\partial q}{\partial z_n} = (l+1)\frac{\partial q}{\partial z_{n^{l+1}}} = 0.$$

In summary, $\partial q/\partial z_n = 0$ for $n \geqslant 1$, and hence q is a constant. Therefore, $\mathcal{I}_{\mathbf{0}} = \mathbb{C}$ and $\dim \mathcal{I}_{\mathbf{0}} = 1$.

It remains to prove that $\dim H^2_\infty/[\mathcal{I}] = \infty$. For this, it is sufficient to show that $f_p(z) = \sum\limits_{m\geqslant 1} z_{p^m}/m \in [\mathcal{I}]^\perp$ for each prime number p; and this reduces to show the following:

(1) $\langle f_p, z_n^2 z^\alpha \rangle = 0$ for each $n \geqslant 2$ and $\alpha \in \mathbb{Z}_+^\infty$;
(2) $\langle f_p, (z_n - k z_{n^k}) z^\alpha \rangle = 0$ for each $n \geqslant 1$, $k \geqslant 2$ and $|\alpha| \geqslant 1$;
(3) $\langle f_p, z_n - k z_{n^k} \rangle = 0$ for each $n \geqslant 1$ and $k \geqslant 2$.

In fact, (1) and (2) follow from the fact that $\langle f_p, z^\beta \rangle = 0$ for each $\beta \in \mathbb{Z}_+^\infty$ with $|\beta| \geqslant 2$. For (3), if $n = p^l$ for some positive integer l, then

$$\langle f_p, z_n - k z_{n^k} \rangle = \Big\langle \sum_{m\geqslant 1} \frac{z_{p^m}}{m}, z_{p^l} - k z_{p^{lk}} \Big\rangle = \frac{1}{l} - \frac{k}{lk} = 0;$$

otherwise, n is not a power of p, and then $\langle f_p, z_n \rangle = \langle f_p, z_{n^k} \rangle = 0$, forcing $\langle f_p, z_n - k z_{n^k} \rangle = 0$. This completes the proof.

REMARK 3.7. It is shown in Theorem 2.1 of [14] that for each proper ideal $\mathcal{I}$ of $\mathcal{P}_n$,

$$\mathcal{I} = \bigcap_{\lambda \in Z(\mathcal{I})} \mathcal{I}_\lambda^{\mathrm{e}},$$

where $\mathcal{I}_\lambda^{\mathrm{e}} = \{p \in \mathcal{P}_n : q(D)p|_\lambda = 0 \text{ for each } q \in \mathcal{I}_\lambda\}$. Examples 3.3 and 3.6 imply that this conclusion fails in the case of infinitely many variables. In both examples, $Z(\mathcal{I}) = \{\mathbf{0}\}$, $\mathcal{I}_{\mathbf{0}} = \mathbb{C}1$ and $\mathcal{I}_{\mathbf{0}}^{\mathrm{e}} = \{p \in \mathcal{P}_\infty : p|_{\mathbf{0}} = 0\} = \mathcal{M}_{\mathbf{0}}$. However, $\dim H^2_\infty/[\mathcal{I}] > 1 = \dim H^2_\infty/[\mathcal{I}_{\mathbf{0}}^{\mathrm{e}}]$, which leads to $\mathcal{I} \neq \mathcal{I}_{\mathbf{0}}^{\mathrm{e}}$.

4. CLOSEDNESS AND DENSITY OF IDEALS OF $\mathcal{P}_\infty$

This section first studies those ideals in $\mathcal{P}_\infty$ with finitely many zeros, and determines when they are closed in $\mathcal{P}_\infty$ or dense in H^2_∞. By a closed ideal $\mathcal{I}$ we mean $[\mathcal{I}] \cap \mathcal{P}_\infty = \mathcal{I}$.

Let $\mathcal{I}$ be an ideal of finitely many zeros $\lambda^{(1)}, \dots, \lambda^{(k)}$. By Proposition 2.4, $\mathcal{I}$ admits a primary decomposition; that is,

$$\mathcal{I} = \bigcap_{i=1}^{k} \mathfrak{p}^{(i)},$$

where the radical ideal of $\mathfrak{p}^{(i)}$ is the maximal ideal $\mathcal{M}_{\lambda^{(i)}}$. In this case, $\mathcal{I}$ is dense in H^2_∞ if and only if each ideal $\mathfrak{p}^{(i)}$ is dense in H^2_∞. Note that each ideal $\mathfrak{p}^{(i)}$ contains exactly one zero point $\lambda^{(i)}$. Let $\mathbb{D}^\infty$ denote the product of countably many unit disks; i.e., $\mathbb{D}^\infty = \{(c_1, c_2, \ldots) : |c_n| < 1 \text{ for every } n\}$. If $\lambda^{(i)} \notin \mathbb{D}^\infty$, then $|\lambda^{(i)}_n| \geqslant 1$ for some positive integer n. Since $r(\mathfrak{p}^{(i)}) = \mathcal{M}_{\lambda^{(i)}}$, there exists a positive integer k such that

$$(z_n - \lambda^{(i)}_n)^k \in \mathfrak{p}^{(i)},$$

which implies that $\mathfrak{p}^{(i)}$ is dense in H^2_∞. In conclusion, if $\mathcal{I}$ is an ideal of finitely many zeros and $Z(\mathcal{I}) \cap \mathbb{D}^\infty = \emptyset$, then $\mathcal{I}$ is dense in H^2_∞.

It is worthy to mention that Lemma 3.1 characterizes the density of maximal ideals: the maximal ideal $\mathcal{M}_\lambda$ is dense in H^2_∞ if and only if $\lambda \notin \mathbb{D}^\infty_2$. But in general, this is not true for a primary ideal with one zero point.

The following will first elaborate on the closedness and density of a class of ideals for which each is only of one zero point inside $\mathbb{D}^\infty$. To be precise, suppose $|a_n| < 1$ for each $n \geqslant 1$, and $\{k_n\}$ is a sequence of positive integers. Let $\mathcal{J}$ denote the ideal generated by

$$\{(z_n - a_n)^{k_n} : n \geqslant 1\}.$$

Then $\mathcal{J}$ is primary since $r(\mathcal{J}) = \mathcal{M}_{\mathbf{a}}$, where $\mathbf{a} = (a_1, a_2, \ldots)$. The closedness and density of the ideal $\mathcal{J}$ are completely characterized by the following theorem.

THEOREM 4.1. *The ideal $\mathcal{J}$ is closed if and only if*

$$\sum_{n \geqslant 1} |a_n|^{2k_n} < +\infty.$$

Moreover, in the case $\sum\limits_{n \geqslant 1} |a_n|^{2k_n} = \infty$, $\mathcal{J}$ is dense in H^2_∞.

To prove Theorem 4.1, we need some preparations. Write

$$\mathcal{J}(n) = (z_n - a_n)^{k_n} \mathbb{C}[z_n],$$

and $\mathcal{J}^\perp(n)$ is the orthogonal complement of $\mathcal{J}(n)$ in the Hardy space $H^2(\mathbb{D})$, in variable z_n. Put

$$\mathcal{H}'_n = \overline{\operatorname{span}}\{z^\alpha : \alpha_1 = \cdots = \alpha_n = 0\};$$

that is, $\mathcal{H}'_n$ consists of functions in H^2_∞ only depending on the variables z_{n+1}, $z_{n+2}, \ldots$. Set $\mathcal{P}'_n = \mathcal{H}'_n \cap \mathcal{P}_\infty$, the polynomial ring in the variables $z_{n+1}, z_{n+2}, \ldots$. Denote by $\mathcal{J}_n$ the ideal of $\mathcal{P}_n$ generated by

$$\{(z_i - a_i)^{k_i} : 1 \leqslant i \leqslant n\},$$

and $\mathcal{J}'_n$ the ideal of $\mathcal{P}'_n$ generated by $\{(z_i - a_i)^{k_i} : i \geqslant n+1\}$. It is clear that

$$\mathcal{J}_n = \mathcal{J}_{n-1}[z_n] + \mathcal{J}(n)[z_1, \ldots, z_{n-1}] \tag{4.1}$$

for $n \geqslant 2$, and

$$\mathcal{J} = \langle \mathcal{J}_n \rangle + \langle \mathcal{J}'_n \rangle, \tag{4.2}$$

where $\langle \mathcal{J}_n \rangle$, $\langle \mathcal{J}_n' \rangle$ denote the ideals of $\mathcal{P}_\infty$ generated by $\mathcal{J}_n$ and $\mathcal{J}_n'$, respectively. Therefore,

$$\langle \mathcal{J}_n \rangle = \operatorname{span}\{pq : p \in \mathcal{J}_n, q \in \mathcal{P}_n'\}, \quad \text{and} \tag{4.3}$$

$$\langle \mathcal{J}_n' \rangle = \operatorname{span}\{pq : p \in \mathcal{P}_n, q \in \mathcal{J}_n'\}. \tag{4.4}$$

Define $\mathcal{H}_n = H^2(\mathbb{D}^n)$. Then by (4.3) and (4.4)

$$[\langle \mathcal{J}_n \rangle] = [\mathcal{J}_n] \otimes \mathcal{H}_n', \ [\langle \mathcal{J}_n' \rangle] = \mathcal{H}_n \otimes [\mathcal{J}_n'], \tag{4.5}$$

where $[\mathcal{J}_n]$ and $[\mathcal{J}_n']$ are the closures of $\mathcal{J}_n$ and $\mathcal{J}_n'$ in $\mathcal{H}_n$ and $\mathcal{H}_n'$, respectively. Similarly, for $n \geqslant 2$,

$$[\mathcal{J}_{n-1}[z_n]] = [\mathcal{J}_{n-1}] \otimes H^2(\mathbb{D}), \quad [\mathcal{J}(n)[z_1, \dots, z_{n-1}]] = \mathcal{H}_{n-1} \otimes [\mathcal{J}(n)]. \tag{4.6}$$

Denote

$$\mathcal{K}_n = \mathcal{H}_n \ominus [\mathcal{J}_n] \quad \text{and} \quad \mathcal{K}_n' = \mathcal{H}_n' \ominus [\mathcal{J}_n'].$$

We need the following lemma.

LEMMA 4.2. *For each positive integer n, we have $\mathcal{K}_n = \mathcal{J}^\perp(1) \otimes \cdots \otimes \mathcal{J}^\perp(n)$, and $\mathcal{J}^\perp = \mathcal{K}_n \otimes \mathcal{K}_n'$.*

Proof. By (4.2) and (4.5) it follows that

$$\begin{aligned}\mathcal{J}^\perp &= \langle \mathcal{J}_n \rangle^\perp \cap \langle \mathcal{J}_n' \rangle^\perp = [\langle \mathcal{J}_n \rangle]^\perp \cap [\langle \mathcal{J}_n' \rangle]^\perp = ([\mathcal{J}_n] \otimes \mathcal{H}_n')^\perp \cap (\mathcal{H}_n \otimes [\mathcal{J}_n'])^\perp \\ &= (\mathcal{K}_n \otimes \mathcal{H}_n') \cap (\mathcal{H}_n \otimes \mathcal{K}_n') = \mathcal{K}_n \otimes \mathcal{K}_n'. \end{aligned} \tag{4.7}$$

Similarly, for $n \geqslant 2$, by (4.8) and (4.6)

$$\begin{aligned}\mathcal{K}_n &= (\mathcal{H}_n \ominus ([\mathcal{J}_{n-1}] \otimes H^2(\mathbb{D}))) \cap (\mathcal{H}_n \ominus (\mathcal{H}_{n-1} \otimes \mathcal{J}(n))) \\ &= (\mathcal{K}_{n-1} \otimes H^2(\mathbb{D})) \cap (\mathcal{H}_{n-1} \otimes \mathcal{J}^\perp(n)) = \mathcal{K}_{n-1} \otimes \mathcal{J}^\perp(n),\end{aligned}$$

and so by induction,

$$\mathcal{K}_n = \mathcal{J}^\perp(1) \otimes \cdots \otimes \mathcal{J}^\perp(n). \quad \blacksquare$$

Now we are ready to present the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\phi_n(z) = ((a_n - z_n)/(1 - \bar{a}_n z_n))^{k_n}$, and let K_n^0 denote the reproducing kernel of $\mathcal{J}^\perp(n)$ at 0 in $H^2(\mathbb{D})$. Then by a simple computation,

$$K_n^0(z_n) = (I - T_{\phi_n} T_{\phi_n}^*) K_0 = 1 - \phi_n(z_n)\overline{\phi}(0) = 1 - \bar{a}_n^{k_n} \phi_n(z_n),$$

where T_{ϕ_n} is the Toeplitz operator on $H^2(\mathbb{D})$ with symbol ϕ_n, and K_0 is the reproducing kernel of $H^2(\mathbb{D})$ at 0. Therefore, we have

$$\|K_n^0\|^2 = K_n^0(0) = 1 - |a_n|^{2k_n}.$$

Recall that $\mathcal{J}$ is the ideal generated by $\{(z_n - a_n)^{k_n} : n \geqslant 1\}$. Assume that $\sum\limits_{n\geqslant 1} |a_n|^{2k_n} < \infty$. First we prove that $\mathcal{J}^\perp \neq \{0\}$. Let

$$f_n = \frac{K_n^0}{\|K_n^0\|^2}, \quad \text{and}$$

$$h_n(z) = \prod_{i=1}^{n} f_i(z_i). \tag{4.8}$$

Then for each n

$$h_{n+1}(z_1, \ldots, z_n, 0) = h_n(z).$$

Write

$$c_\alpha(n) = \langle h_n, z^\alpha \rangle,$$

and for each $\alpha \in \mathbb{Z}_+^\infty$, $\{c_\alpha(n)\}$ is an eventually constant sequence with the limit c_α. Since $\sum\limits_{n\geqslant 1} |a_n|^{2k_n} < \infty$, the infinite product $\prod\limits_{n\geqslant 1} (1 - |a_n|^{2k_n})$ converges to a positive number C. Then by (4.8)

$$\|h_n\| = \prod_{i=1}^{n} \|f_i\| = \frac{1}{\prod\limits_{i=1}^{n} \sqrt{1 - |a_i|^{2k_i}}} \leqslant \frac{1}{\sqrt{C}}.$$

Therefore,

$$\|h_n\|^2 = \sum_{\alpha\in\mathbb{Z}_+^\infty} |c_\alpha(n)|^2 \leqslant \frac{1}{C}.$$

Letting $n \to \infty$ yields that

$$\sum_{\alpha\in\mathbb{Z}_+^\infty} |c_\alpha|^2 \leqslant \frac{1}{C}.$$

Also, $(c_\alpha(n))$ converges to (c_α) in l^2; that is, $\{h_n\}$ converges to a nonzero function $h(z) = \sum\limits_\alpha c_\alpha z^\alpha$ in H_∞^2. Since $f_i(z_i) \perp \mathcal{J}(i)$ for each i $(1 \leqslant i \leqslant n)$, Lemma 4.2 implies that $h_n \perp \mathcal{J}_n$. Then by (4.8)

$$h_{n+k} \perp \langle \mathcal{J}_n \rangle, \quad k = 0, 1, \ldots.$$

Since $\{h_n\}$ converges to h, $h \perp \langle \mathcal{J}_n \rangle$ for each n, which immediately leads to $h \perp \mathcal{J}$. Therefore, $\mathcal{J}^\perp \neq 0$.

Now we are ready to prove that $\mathcal{J}$ is closed in $\mathcal{P}_\infty$. For each polynomial $p \in [\mathcal{J}] \cap \mathcal{P}_\infty$, there exists a positive integer N such that $p \in \mathcal{P}_N$. By Lemma 4.2

$$\mathcal{J}^\perp = \mathcal{K}_N \otimes \mathcal{K}_N'.$$

This gives $\mathcal{K}_N' \neq 0$, and then there are nonzero functions ψ in $\mathcal{K}_N'$ and q in $\mathcal{P}_N'$ such that $\langle q, \psi \rangle \neq 0$. For each $\varphi \in \mathcal{K}_N$, since $\varphi \otimes \psi \in \mathcal{J}^\perp$ and $pq \in [\mathcal{J}] \cap \mathcal{P}_\infty$, it follows that $\langle pq, \varphi \otimes \psi \rangle = 0$. That is,

$$\langle p, \varphi \rangle \langle q, \psi \rangle = 0,$$

forcing $\langle p, \varphi \rangle = 0$. By arbitrariness of φ, $p \perp \mathcal{K}_N$, and hence p belongs to $[\mathcal{J}_N] \cap \mathcal{P}_N$. By the Ahern–Clark theorem [1] (Theorem 1.2), $[\mathcal{J}_N] \cap \mathcal{P}_N = \mathcal{J}_N$, and thus $p \in \mathcal{J}$. Then $\mathcal{J} = [\mathcal{J}] \cap \mathcal{P}_\infty$; that is, $\mathcal{J}$ is closed.

It remains to prove that if

$$\sum_{n \geqslant 1} |a_n|^{2k_n} = \infty,$$

then $\mathcal{J}$ is dense in H^2_∞. For this, assume, to the contrary, that $\mathcal{J}^\perp \neq \{0\}$. Since $\mathcal{J}$ is an ideal, the submodule $[\mathcal{J}]$ of H^2_∞ does not contain 1. This implies that there exists a function $g \in \mathcal{J}^\perp$ with $g(\mathbf{0}) \neq 0$. To get a contradiction it suffices to derive $g(\mathbf{0}) = 0$. For this, write $g_{(n)}(z) = g(z_1, \ldots, z_n, 0, 0, \ldots)$. Since $g \in \mathcal{J}^\perp$, this induces $g_{(n)} \in \mathcal{J}_n^\perp$. By Lemma 4.2, it holds

$$g(\mathbf{0}) = g_{(n)}(\mathbf{0}) = \Big\langle g_{(n)}, \prod_{i=1}^{n} K_i^0(z_i) \Big\rangle,$$

which leads to

$$|g(\mathbf{0})| \leqslant \|g_{(n)}\| \prod_{i=1}^{n} \|K_i^0\| \leqslant \|g\| \prod_{i=1}^{n} \sqrt{1 - |a_i|^{2k_i}}.$$

Then $|g(\mathbf{0})|^2 \leqslant \|g\|^2 \prod\limits_{n \geqslant 1} (1 - |a_n|^{2k_n}) = 0$, where the last identity follows from $\sum\limits_{n \geqslant 1} |a_n|^{2k_n} = \infty$. Thus $g(\mathbf{0}) = 0$, a contradiction.

The proof of Theorem 4.1 is complete. ∎

If $\mathcal{I}$ is an ideal of the polynomial ring $\mathcal{P}_n$ with one single zero point outside $\mathbb{D}^n$, then the ideal $\mathcal{I}$ contains a power of a maximal ideal that is dense in $H^2(\mathbb{D}^n)$ ([2], Proposition 7.14), and therefore $\mathcal{I}$ is dense. However, Theorem 4.1 implies that there exists an ideal $\mathcal{J}$ of the ring $\mathcal{P}_\infty$ such that $Z(\mathcal{J})$ contains exactly a single point outside the domain $\mathbb{D}_2^\infty$, but $\mathcal{J}$ is not dense in H^2_∞. To see this, we have the following example.

EXAMPLE 4.3. Set $a_n = 1/\sqrt{n+1}$, $n = 1, 2, \ldots$. Then $\mathbf{a} = (a_n) \notin \mathbb{D}_2^\infty$. Let $\mathcal{J}$ be the ideal generated by $\{(z_n - a_n)^2 : n \geqslant 1\}$. By Theorem 4.1, $\mathcal{J}$ is closed in $\mathcal{P}_\infty$, and not dense in H^2_∞.

In fact, Theorem 4.1 could be generalized as follows through a modified proof.

THEOREM 4.4. *Suppose that $\{p_n\}_{n=1}^\infty$ is a sequence of non-constant monic polynomials in one complex variable. Let $\mathcal{I}$ denote the ideal generated by $\{p_n(z_n) : n \geqslant 1\}$. Then the ideal $\mathcal{I}$ is closed in $\mathcal{P}_\infty$ if and only if $Z(p_n) \subseteq \mathbb{D}$ for each positive integer n and*

$$\sum_{n \geqslant 1} |p_n(0)|^2 < +\infty.$$

As in the proof of Theorem 4.1, we first give a characterization of the density of the ideal $\mathcal{I}$. More generally, we come to the following conclusion.

PROPOSITION 4.5. *Suppose that* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of non-zero functions in the Hardy space* $H^2(\mathbb{D})$. *Then* $\{f_n(z_n) : n \geqslant 1\}$ *is a generating set of the Hardy module* H^2_∞ *if and only if either there is a positive integer* N *such that* f_N *is outer in* $H^2(\mathbb{D})$ *or*

$$\sum_{n\geqslant 1} |\eta_n(0)|^2 = +\infty,$$

where η_n *is the inner factor of* f_n.

Before giving the proof of Proposition 4.5, let us recall some classical results on quotient modules of $H^2(\mathbb{D})$. It follows from Beurling's theorem that every quotient module of $H^2(\mathbb{D})$ is of the form $H^2(\mathbb{D}) \ominus \eta H^2(\mathbb{D})$ for some inner function η. Rewrite $\mathcal{Q}_\eta$ for this quotient module. It is well known that the orthogonal projection P_η from $H^2(\mathbb{D})$ onto $\mathcal{Q}_\eta$ coincides with the operator $I - T_\eta T_\eta^*$, where T_η is the Toeplitz operator on $H^2(\mathbb{D})$ with symbol η. Thus, one can compute the distance from the constant function 1 to $\eta H^2(\mathbb{D})$ as follows:

$$\|P_\eta 1\|^2 = \|1 - T_\eta T_\eta^* 1\|^2 = \|1 - \overline{\eta(0)}\eta\|^2 = \|\eta - \eta(0)\|^2 = 1 - |\eta(0)|^2. \tag{4.9}$$

Now we need to introduce a bit more notations. Let M denote the submodule generated by the set $\{f_n(z_n) : n \geqslant 1\}$. Denote by M_n the submodule of $\mathcal{H}_n$ generated by $\{f_i(z_i) : 1 \leqslant i \leqslant n\}$, and M_n' the submodule of $\mathcal{H}_n'$ generated by $\{f_i(z_i) : i \geqslant n+1\}$, where $\mathcal{H}_n = H^2(\mathbb{D}^n)$ and $\mathcal{H}_n' = \overline{\text{span}}\{z^\alpha : \alpha_1 = \cdots = \alpha_n = 0\}$. Write N_n and N_n' for the quotient modules of M_n and M_n', respectively; that is, $N_n = \mathcal{H}_n \ominus M_n$ and $N_n' = \mathcal{H}_n' \ominus M_n'$. The following conclusion is actually a generalization of Lemma 4.2.

LEMMA 4.6. *For each positive integer* n, *we have* $N_n = \mathcal{Q}_{\eta_1} \otimes \cdots \otimes \mathcal{Q}_{\eta_n}$ *and* $M^\perp = N_n \otimes N_n'$.

The proof is similar to that of Lemma 4.2.

REMARK 4.7. The notion of Jordan block plays an important role in operator theory. In [20], [21], [22], [23], Qin, Yang and Sarkar studied Jordan blocks of the multi-variable Hardy module. From the point of view in [21] and Lemma 4.6, the quotient module $M^\perp$ can be considered as a Jordan block in the infinite-variables setting.

Now we are ready to prove Proposition 4.5 and Theorem 4.4.

Proof of Proposition 4.5. For a set $E \subseteq H^2_\infty$, denote by $\text{dist}(1, E)$ the distance from 1 to E. It is clear that $\text{dist}(1, M_n) \to \text{dist}(1, M)$ $(n \to \infty)$. Then $M = H^2_\infty$ if and only if $\text{dist}(1, M_n) \to 0$ $(n \to \infty)$. By Lemma 4.6, we have

$$P_{N_n} = P_{\eta_1} \otimes \cdots \otimes P_{\eta_n},$$

where P_{N_n} is the orthogonal projection from H^2_∞ onto N_n. Therefore, by (4.9)

$$\operatorname{dist}(1, M_n)^2 = \|P_{N_n}1\|^2 = \|P_{\eta_1}1 \otimes \cdots \otimes P_{\eta_n}1\|^2 = \prod_{i=1}^{n} \|P_{\eta_i}1\|^2 = \prod_{i=1}^{n}(1 - |\eta_i(0)|^2).$$

This implies that $M = H^2_\infty$ if and only if either $|\eta_N(0)| = 1$ for some positive integer N or

$$\sum_{n \geqslant 1} |\eta_n(0)|^2 = +\infty,$$

which completes the proof. ∎

Proof of Theorem 4.4. For each positive integer n denote by B_n the inner factor of p_n. Note that B_n is a finite Blaschke product.

Suppose that $\mathcal{I}$ is closed in $\mathcal{P}_\infty$. Then $\mathcal{I}$ is not dense in H^2_∞, and by Proposition 4.5,

$$\sum_{n \geqslant 1} |B_n(0)|^2 < +\infty.$$

Now we claim that $Z(p_n) \subseteq \mathbb{D}$ for each positive integer n. To reach a contradiction, assume conversely that there exist a positive integer N and a number $a \in \mathbb{C} \setminus \mathbb{D}$ such that $p_N(a) = 0$. Then $p_N(z_N) = (z_N - a)q(z_N)$ for some polynomial q in one complex variable. Since $z_N - a$ is cyclic in H^2_∞, it follows that $q(z_N) \in [\mathcal{I}] \cap \mathcal{P}_\infty = \mathcal{I}$. Then there exist a positive integer d and polynomials $\{q_n\}_{n=1}^{d}$, such that

$$q(z_N) = \sum_{n=1}^{d} q_n(z) p_n(z_n).$$

For $n \neq N$, let z_n be a zero point of p_n in the identity above. Therefore, there is a polynomial r in one complex variable satisfying $q(z_N) = r(z_N)p_N(z_N)$. It follows that $1 = (z_N - a)r(z_N)$, which leads to a contradiction. Thus the claim is proved.

Since p_n is monic, we have $|p_n(0)| = |B_n(0)|$ for each positive integer n. Therefore,

$$\sum_{n \geqslant 1} |p_n(0)|^2 = \sum_{n \geqslant 1} |B_n(0)|^2 < +\infty.$$

Now suppose that for each positive integer n all zero points of p_n are contained in $\mathbb{D}$ and $\sum_{n \geqslant 1} |p_n(0)|^2 < +\infty$, which immediately gives that

$$\sum_{n \geqslant 1} |B_n(0)|^2 < +\infty. \tag{4.10}$$

Take an arbitrary polynomial $p \in [\mathcal{I}] \cap \mathcal{P}$, there is a positive integer N such that $p \in \mathcal{P}_N$. By (4.10) and Proposition 4.5, $\mathcal{I}^\perp \neq 0$. Then one can apply Lemma 4.6 and the argument in the proof of Theorem 4.1 to show that $p \in [\mathcal{I}_N] \cap \mathcal{P}_N$, where $\mathcal{I}_N$ is the ideal of $\mathcal{P}_N$ generated by $\{p_i(z_i) : 1 \leqslant i \leqslant N\}$. Since $Z(\mathcal{I}_N) \subseteq \mathbb{D}^N$, the Ahern–Clark theorem implies that $[\mathcal{I}_N] \cap \mathcal{P}_N = \mathcal{I}_N$. Thus $p \in \mathcal{I}_N \subseteq \mathcal{I}$ as desired. ∎

To conclude this section, we record the following example.

EXAMPLE 4.8. For a complex number a, denote by $\mathcal{I}$ the ideal generated by $\{z_n^n - a^n : n \geqslant 1\}$. Note that $Z(\mathcal{I}) \cap \mathbb{D}_2^\infty = \emptyset$ whenever $a \neq 0$. By Theorem 4.4 and Proposition 4.5, $\mathcal{I}$ is closed in $\mathcal{P}_\infty$ if and only if $|a| < 1$; $\mathcal{I}$ is dense in H^2_∞ if and only if $|a| \geqslant 1$. When $a \neq 0$, $Z(\mathcal{I})$ is an uncountable set. In this case, it is easy to see that $\mathcal{I}$ has no finite primary decomposition.

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HUI DAN, SCHOOL OF MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, SHANGHAI, 200433, CHINA
E-mail address: danhuimath@gmail.com

KUNYU GUO, SCHOOL OF MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, SHANGHAI, 200433, CHINA
E-mail address: kyguo@fudan.edu.cn

HANSONG HUANG, DEPARTMENT OF MATHEMATICS, EAST CHINA UNIVERSITY OF SCIENCE AND TECHNOLOGY, SHANGHAI, 200237, CHINA
E-mail address: hshuang@ecust.edu.cn

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