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CLASSIFICATION OF DRURY–ARVESON-TYPE HILBERT MODULES ASSOCIATED WITH CERTAIN DIRECTED GRAPHS

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ABSTRACT. Given a directed Cartesian product $\mathscr{T}$ of locally finite, leafless, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ of finite joint branching index, one may associate with $\mathscr{T}$ the Drury–Arveson-type $\mathbb{C}[z_1, \ldots, z_d]$-Hilbert module $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ of vector-valued holomorphic functions on the unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$, where $a > 0$. The main result of this paper classifies all directed Cartesian products $\mathscr{T}$ for which the Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ are isomorphic in case a is an integer. Indeed, a careful analysis of these Hilbert modules allows us to prove that the cardinality of generations of $\mathscr{T}_1, \ldots, \mathscr{T}_d$ are complete invariants for $\mathscr{H}_{\mathfrak{c}_a}(\cdot)$ if $ad \neq 1$.

KEYWORDS: *Hilbert module, reproducing kernel, representing measure.*

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1. A CLASSIFICATION PROBLEM

In [9], we introduced and studied the notion of multishifts on the directed Cartesian product of finitely many leafless, rooted directed trees. This was indeed an attempt to unify the theory of weighted shifts on rooted directed trees [17] and that of classical unilateral multishifts [18]. Besides a finer analysis of various joint spectra and wandering subspace property of these multishifts, this work provided a scheme to associate a one parameter family of reproducing kernel Hilbert spaces $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ $(a > 0)$ with every directed Cartesian product $\mathscr{T}$ of finite joint branching index, see Corollary 2.12 below (cf. Proposition 4.4 of [1] and Definition 4.1 of [21]). These spaces consist of vector-valued holomorphic functions defined on the unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$, and can be thought of as *tree analogs* of the reproducing kernel Hilbert spaces $\mathscr{H}_a$ associated with the positive definite kernels

$$(1.1) \qquad \kappa_a(z, w) := \frac{1}{(1 - \langle z, w\rangle)^a}, \quad z, w \in \mathbb{B}^d$$

(refer to [5] and [14]; refer also to [27] for a comprehensive account of the theory of Hilbert spaces of holomorphic functions on the unit ball). Indeed, the reproducing kernels $\kappa_{\mathscr{H}_{\mathfrak{c}_a}}(z,w)$ associated with $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ are certain positive operator linear combinations of $\kappa_a(z,w)$ and multivariable hypergeometric functions (see Theorem 5.2.6 of [9]). In particular, the Hilbert space $\mathscr{H}_a$ is *contractively contained* in $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ (see Theorem 5.1 of [22]; cf. Proposition 4.5(1) of [1]). It is interesting to note that $\kappa_{\mathscr{H}_{\mathfrak{c}_a}}(z,w)$ can be obtained by integrating certain perturbations of $\kappa_a(z,w)$ with respect to a finite family of spectral measures (see Remark 2.11). Further, the Hilbert space $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ carries a natural *Hilbert module* structure over the polynomial ring $\mathbb{C}[z_1,\ldots,z_d]$ with module action

$$(p,h) \in \mathbb{C}[z_1,\ldots,z_d] \times \mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}) \longmapsto p(\mathscr{M}_z)h \in \mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}), \tag{1.2}$$

where $\mathbb{C}[z_1,\ldots,z_d]$ denotes the ring of polynomials in the complex variables $z_1,\ldots,z_d$ and $\mathscr{M}_z$ is the d-tuple of multiplication operators $\mathscr{M}_{z_1},\ldots,\mathscr{M}_{z_d}$ acting on $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ (refer to Section 2 of [23] for the general theory of Hilbert modules over the algebra of polynomials). We refer to $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ as the *Drury–Arveson-type Hilbert module* associated with $\mathscr{T}$.

A thorough study of the Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ had been carried out in Chapter 5, Section 2 of [9]. In particular, the essential normality of $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ is shown to be closely related to the notion of finite joint branching index of $\mathscr{T}$ (see Proposition 5.2.9 and Example 5.2.20 of [9]). In the present work, we continue our study of the Drury–Arveson-type Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$. The investigations herein are motivated by the following classification problem for the Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ associated with the directed Cartesian product $\mathscr{T}$ (see Theorem 4.5 of [3], Theorem 2.4 of [8] for variants of this problem; see also Theorem 4.6 of [3]).

PROBLEM 1.1. *For $j=1,2$, let $\mathscr{T}^{(j)} = (V^{(j)}, \mathcal{E}^{(j)})$ denote the directed Cartesian product of locally finite, leafless, rooted directed trees $\mathscr{T}_1^{(j)},\ldots,\mathscr{T}_d^{(j)}$ of finite joint branching index. Under what conditions on $\mathscr{T}^{(1)}$ and $\mathscr{T}^{(2)}$, the Drury–Arveson-type Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(1)})$ and $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(2)})$ are isomorphic ?*

Recall that the Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(1)})$ and $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(2)})$ are *isomorphic* if there exists a unitary map $U : \mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(1)}) \to \mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(2)})$ such that

$$U\mathscr{M}_{z_k}^{(1)} = \mathscr{M}_{z_k}^{(2)}U, \quad k=1,\ldots,d,$$

where $\mathscr{M}_{z_k}^{(j)}$ denotes the operator of multiplication by the coordinate function z_k on $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(j)})$ for $j=1,2$. We refer to U as a *Hilbert module isomorphism* between $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(1)})$ and $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(2)})$.

It turns out that for graph-isomorphic directed Cartesian products, the associated Drury–Arveson-type Hilbert modules are always isomorphic (see Remark 2.10). However, given any positive integer k, one can produce k number of non-isomorphic directed Cartesian products for which the associated Drury–Arveson-type Hilbert modules are isomorphic (see Corollary 1.7(ii)). Thus graph-isomorphism of directed Cartesian products is sufficient but not necessary to ensure the isomorphism of the associated Drury–Arveson-type Hilbert modules. This is in contrast with Theorem 2.11 of [19], where countable directed graphs completely determine the associated tensor (quiver) algebras (up to Banach space isomorphism) (cf. Theorem 3.7 of [26]).

The main result of the present paper answers when two Drury–Arveson-type Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(j)})$ $(j = 1, 2)$ are isomorphic in case a is a positive integer (see Theorem 1.4 and Remark 1.5). In particular, it provides complete unitary invariants for the Drury–Arveson-type Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ in terms of some discrete data associated with $\mathscr{T}$. Before we state this result, we need to reproduce several notions from [9] and [17] (the reader is advised to recall all the relevant definitions pertaining to the directed trees from [17]).

1.1. MULTISHIFTS ON DIRECTED CARTESIAN PRODUCT OF DIRECTED TREES. We first set some standard notations. For a positive integer d and a set X, X^d stands for the d-fold Cartesian product of X, while $\operatorname{card}(X)$ stands for the cardinality of X. The symbol $\mathbb{N}$ denotes the set of nonnegative integers, and $\mathbb{C}$ denotes the field of complex numbers. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we use $\alpha! := \prod_{j=1}^{d} \alpha_j!$ and $|\alpha| := \sum_{j=1}^{d} \alpha_j$. The modulus of a complex number z is denoted by $|z|$. The complex conjugate of $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ is given by $\bar{z} := (\bar{z}_1, \ldots, \bar{z}_d)$, while the Euclidean norm $(|z_1|^2 + \cdots + |z_d|^2)^{1/2}$ of z is denoted by $\|z\|_2$. The open ball in $\mathbb{C}^d$ centered at the origin and of radius $r > 0$ is denoted by $\mathbb{B}^d_r$, while the sphere centered at the origin and of radius $r > 0$ is denoted by $\partial\mathbb{B}^d_r$. For simplicity, the unit ball $\mathbb{B}^d_1$ and the unit sphere $\partial\mathbb{B}^d_1$ are denoted respectively by $\mathbb{B}^d$ and $\partial\mathbb{B}^d$. Throughout this paper, we follow the standard conventions that the sum over the empty set is 0, while the product over the empty set is always 1.

For $j = 1, \ldots, d$, let $\mathscr{T}_j = (V_j, \mathcal{E}_j)$ be a leafless, rooted directed tree with root root_j. The *directed Cartesian product* of $\mathscr{T}_1, \ldots, \mathscr{T}_d$ is the directed graph $\mathscr{T} = (V, \mathcal{E})$ given by

$$\begin{aligned} V &:= V_1 \times \cdots \times V_d, \\ \mathcal{E} &:= \{(v, w) \in V \times V : \text{ there is a positive integer } k \in \{1, \ldots, d\} \\ &\qquad\qquad \text{such that } v_j = w_j \text{ for } j \neq k \text{ and the edge } (v_k, w_k) \in \mathcal{E}_k\}, \end{aligned}$$

where $v \in V$ is always understood as $v = (v_1, \ldots, v_d)$ with $v_j \in V_j$, $j = 1, \ldots, d$. The d-fold directed Cartesian product of a directed tree $\mathscr{T}$ is denoted by $\mathscr{T}^d$.

We briefly recall some relevant notions from Chapter 2 of [9] for the sake of completeness. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of directed trees

$\mathscr{T}_1, \ldots, \mathscr{T}_d$. For $j = 1, \ldots, d$ and $v \in V$, we set

$$\mathsf{Chi}_j(v) := \{w \in V : w_j \in \mathsf{Chi}(v_j) \text{ and } w_k = v_k \text{ for } k \neq j\}.$$

We denote $\mathsf{Chi}(v) = \bigcup_{j=1}^{d} \mathsf{Chi}_j(v)$. Further, for $W \subseteq V$ and $k \in \mathbb{N}$, we define

$$\mathsf{Chi}_j(W) := \bigcup_{w \in W} \mathsf{Chi}_j(w), \quad \mathsf{Chi}_j^{\langle k \rangle}(W) := \underbrace{\mathsf{Chi}_j \cdots \mathsf{Chi}_j}_{k \text{ times}}(W),$$

where we understand that $\mathsf{Chi}_j^{\langle 0 \rangle}(W) = W$. Further, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $W \subseteq V$, we define

$$\mathsf{Chi}^{\ll \alpha \gg}(W) := \mathsf{Chi}_1^{\langle \alpha_1 \rangle} \cdots \mathsf{Chi}_d^{\langle \alpha_d \rangle}(W).$$

If $W = \{v\}$ for some $v \in V$, then we use the simpler notation $\mathsf{Chi}^{\ll \alpha \gg}(v)$ for $\mathsf{Chi}^{\ll \alpha \gg}(\{v\})$. For $j = 1, \ldots, d$ and $v \in V$, we set

$$\mathsf{par}_j(v) := \begin{cases} \{w \in V : w_j = \mathsf{par}(v_j) \text{ and } w_k = v_k \text{ for } k \neq j\} & \text{if } v_j \neq \mathsf{root}_j, \\ \emptyset & \text{otherwise.} \end{cases}$$

Moreover, for $W \subseteq V$ and $k \in \mathbb{N}$, we define

$$\mathsf{par}_j(W) := \bigcup_{w \in W} \mathsf{par}_j(w), \quad \mathsf{par}_j^{\langle k \rangle}(W) := \underbrace{\mathsf{par}_j \cdots \mathsf{par}_j}_{k \text{ times}}(W),$$

where we understand that $\mathsf{par}_j^{\langle 0 \rangle}(W) = W$. We denote $\mathsf{Par}(v) = \bigsqcup_{j=1}^{d} \mathsf{par}_j(v)$. Furthrmore, for $u \in V$ and $j = 1, \ldots, d$, we set

$$\mathsf{sib}_j(u) := \begin{cases} \mathsf{Chi}_j(\mathsf{par}_j(u)) & \text{if } u_j \neq \mathsf{root}_j, \\ \emptyset & \text{otherwise.} \end{cases}$$

The *depth* of a vertex $v \in V$ is the unique multiindex $\mathsf{d}_v \in \mathbb{N}^d$ such that

$$v \in \mathsf{Chi}^{\ll \mathsf{d}_v \gg}(\mathsf{root}),$$

where root denotes the root $(\mathsf{root}_1, \ldots, \mathsf{root}_d)$ of $\mathscr{T}$. The depth of a vertex always exists (see Lemma 2.1.10(vi) of [9]). For $k \in \mathbb{N}$, the set

$$\mathcal{G}_k := \{v \in V : |\mathsf{d}_v| = k\}$$

is referred to as the k^{th} *generation* of $\mathscr{T}$. A vertex $v \in V$ is called a *branching vertex* of $\mathscr{T}$ if $\mathrm{card}(\mathsf{Chi}(v_j)) \geqslant 2$ for all $j = 1, \ldots, d$. The *branching index* $k_{\mathscr{T}}$ of $\mathscr{T}$ is the multiindex $(k_{\mathscr{T}_1}, \ldots, k_{\mathscr{T}_d}) \in \mathbb{N}^d$ given by

$$k_{\mathscr{T}_j} := \begin{cases} 1 + \sup\{\mathsf{d}_w : w \in V_{\prec}^{(j)}\} & \text{if } V_{\prec}^{(j)} \text{ is non-empty,} \\ 0 & \text{otherwise,} \end{cases}$$

where $V_{\prec}^{(j)}$ is the set of branching vertices of $\mathscr{T}_j$, $j = 1, \dots, d$. It is recorded in Proposition 2.1.19 of [9] that

$$\mathsf{Chi}^{\ll k_{\mathscr{T}} \gg}(V_{\prec}) \cap V_{\prec} = \emptyset,$$

where $V_{\prec}$ denotes the set of branching vertices of $\mathscr{T}$.

Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \dots, \mathscr{T}_d$ and let $V^\circ := V \setminus \{\mathsf{root}\}$. Consider the complex Hilbert space $l^2(V)$ of square summable complex functions on V equipped with the standard inner product. Note that $l^2(V)$ admits the orthonormal basis $\{e_v : v \in V\}$, where $e_v : V \to \mathbb{C}$ denotes the indicator function of the set $\{v\}$, $v \in V$. Given a system $\boldsymbol{\lambda} = \{\lambda_j(v) : v \in V^\circ,\ j = 1, \dots, d\}$ of positive numbers, we define the *multishift* $S_{\boldsymbol{\lambda}}$ *on* $\mathscr{T}$ with weights $\boldsymbol{\lambda}$ as the d-tuple of linear (possibly unbounded) operators $S_1, \dots, S_d$ in $l^2(V)$ given by

$$\mathscr{D}(S_j) := \{f \in l^2(V) \colon \Lambda_{\mathscr{T}}^{(j)} f \in l^2(V)\},$$
$$S_j f := \Lambda_{\mathscr{T}}^{(j)} f, \quad f \in \mathscr{D}(S_j),$$

where $\Lambda_{\mathscr{T}}^{(j)}$ is the mapping defined on complex functions f on V by

$$(\Lambda_{\mathscr{T}}^{(j)} f)(v) := \begin{cases} \lambda_j(v) \cdot f(\mathsf{par}_j(v)) & \text{if } v_j \in V_j^\circ, \\ 0 & \text{otherwise.} \end{cases}$$

It is shown in Lemma 3.1.5 of [9] that $S_j \in B(l^2(V))$ if and only if

$$\sup_{v \in V} \sum_{w \in \mathsf{Chi}_j(v)} \lambda_j(w)^2 < \infty, \tag{1.3}$$

where $B(\mathcal{H})$ denotes the space of bounded linear operators on the Hilbert space $\mathcal{H}$. Further, an examination of the proof of Proposition 3.1.7 in [9] reveals that for $1 \leqslant i, j \leqslant d$, $S_i S_j = S_j S_i$ if and only if

$$\lambda_j(u)\lambda_i(\mathsf{par}_j(u)) = \lambda_i(u)\lambda_j(\mathsf{par}_i(u)), \quad u \in \mathsf{Chi}_j\mathsf{Chi}_i(v),\ v \in V. \tag{1.4}$$

We say that $S_{\boldsymbol{\lambda}}$ is a *commuting multishift on* $\mathscr{T}$ if $\boldsymbol{\lambda}$ satisfies (1.3) and (1.4) for all $i, j = 1, \dots, d$.

We assume that all the directed trees under consideration are countably infinite and leafless, that is, the cardinality of set of vertices is $\aleph_0$ and for every vertex u, $\mathrm{card}(\mathsf{Chi}(u)) \geqslant 1$.

For future reference, we reproduce from Proposition 3.1.7 of [9] some general properties of commuting multishifts.

LEMMA 1.2. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \dots, \mathscr{T}_d$ and let $S_{\boldsymbol{\lambda}}$ be a commuting multishift on $\mathscr{T}$. Then, for any $\alpha \in \mathbb{N}^d$, the following statements hold:*

(i) *$S_{\boldsymbol{\lambda}}^{*\alpha} S_{\boldsymbol{\lambda}}^{\alpha}$ is a diagonal operator (with respect to the orthonormal basis $\{e_v\}_{v \in V}$) with diagonal entries $\|S_{\boldsymbol{\lambda}}^{\alpha} e_v\|^2$, $v \in V$;*

(ii) *for distinct vertices* $v, w \in V$, $\langle S^{\alpha}_{\boldsymbol\lambda} e_v, S^{\alpha}_{\boldsymbol\lambda} e_w\rangle = 0$.

Let $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$ with weight system $\boldsymbol\lambda$. Assume that $S_{\boldsymbol\lambda}$ is *joint left invertible*, that is, $\sum_{j=1}^{d} S_j^* S_j$ is invertible. Then the *spherical Cauchy dual* $S^{\mathfrak{s}}_{\boldsymbol\lambda} = (S^{\mathfrak{s}}_1, \ldots, S^{\mathfrak{s}}_d)$ of $S_{\boldsymbol\lambda}$ is given by

$$S^{\mathfrak{s}}_j e_v = \Big(\sum_{i=1}^{d} \|S_i e_v\|^2\Big)^{-1} \sum_{w \in \mathsf{Chi}_j(v)} \lambda_j(w) e_w, \quad v \in V,\ j = 1, \ldots, d.$$

Note that $S^{\mathfrak{s}}_{\boldsymbol\lambda}$ is the multishift on $\mathscr{T}$ with weights

$$\lambda_j(w)\Big(\sum_{i=1}^{d} \|S_i e_v\|^2\Big)^{-1}, \quad w \in \mathsf{Chi}_j(v),\ v \in V,\ j = 1, \ldots, d. \tag{1.5}$$

In general, $S^{\mathfrak{s}}_{\boldsymbol\lambda}$ is not commuting (see Proposition 5.2.10 of [9]). However, if $S^{\mathfrak{s}}_{\boldsymbol\lambda}$ is commuting, then it is a joint left invertible commuting multishift such that $(S^{\mathfrak{s}}_{\boldsymbol\lambda})^{\mathfrak{s}} = S_{\boldsymbol\lambda}$.

1.2. JOINT COKERNEL OF MULTISHIFTS. The main result of this paper relies heavily on the description of the joint cokernel $\ker \mathscr{M}_z^*$ of the multiplication tuple $\mathscr{M}_z$ acting on the Drury–Arveson-type Hilbert space $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$. The first step in this direction is to realize $\ker \mathscr{M}_z^*$ as the solution space of certain systems of linear equations arising from the eigenvalue problem for the adjoint of a commuting multishift. For this realization, we find it necessary to collect required graph-theoretic jargon as introduced in Chapter 4 of [9].

For a set A, let $\mathscr{P}(A)$ denote the set of all subsets of A. In the case when $A = \{1, \ldots, d\}$, we simply write $\mathscr{P}$ in place of $\mathscr{P}(A)$. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Consider the set-valued function $\Phi : \mathscr{P} \rightarrow \mathscr{P}(V)$ given by

$$\begin{aligned} &\Phi(F) = \Phi_F, \quad F \in \mathscr{P}, \\ &\Phi_F := \{v \in V : v_j \in V_j^{\circ} \text{ if } j \in F, \text{ and } v_j = \mathsf{root}_j \text{ if } j \notin F\}, \end{aligned} \tag{1.6}$$

where $V_j^{\circ} := V_j \setminus \{\mathsf{root}_j\}$, $j = 1, \ldots, d$. Note that

(•) if $F \neq G$, then $\Phi_F \cap \Phi_G = \emptyset$;
(•) if $v \in V$, then $v \in \Phi_F$ for $F := \{j \in \{1, \ldots, d\} : v_j \neq \mathsf{root}_j\}$.

Thus it follows that

$$V = \bigsqcup_{F \in \mathscr{P}} \Phi_F \quad \text{(disjoint sum)}. \tag{1.7}$$

For $F \in \mathscr{P}$ and $u \in \Phi_F$, define

$$\mathsf{sib}_F(u) := \begin{cases} \mathsf{sib}_{i_1}\mathsf{sib}_{i_2}\cdots\mathsf{sib}_{i_k}(u) & \text{if } F = \{i_1, \ldots, i_k\}, \\ \{u\} & \text{if } F = \emptyset. \end{cases}$$

Define an equivalence relation $\sim$ on Φ_F by

$$u \sim v \text{ if and only if } u \in \mathsf{sib}_F(v),$$

and note that for any $u \in \Phi_F$, the equivalence class containing u is precisely $\mathsf{sib}_F(u)$. An application of the axiom of choice allows us to form a set Ω_F by picking up exactly one element from each of the equivalence classes $\mathsf{sib}_F(u)$, $u \in \Phi_F$. We refer to Ω_F as an *indexing set corresponding to F*. Thus we have the disjoint union

$$\Phi_F = \bigsqcup_{u \in \Omega_F} \mathsf{sib}_F(u). \tag{1.8}$$

This combined with (1.7) yields the following decomposition of $l^2(V)$:

$$l^2(V) = \bigoplus_{F \in \mathscr{P}} \bigoplus_{u \in \Omega_F} l^2(\mathsf{sib}_F(u)). \tag{1.9}$$

For $F \in \mathscr{P}$ and $v = (v_1, \ldots, v_d) \in V$, let $v_F \in V$ denote the d-tuple with j^{th} coordinate, $1 \leqslant j \leqslant d$, given by

$$(v_F)_j = \begin{cases} v_j & \text{if } j \in F, \\ \mathsf{root}_j & \text{if } j \notin F. \end{cases}$$

Further, for $i = 1, \ldots, d$ such that $i \notin F$ and $u_i \in V_i$, we define $v_F|_{u_i} \in V$ to be the d-tuple $(w_1, \ldots, w_d)$, where

$$w_j = \begin{cases} u_i & \text{if } j = i, \\ (v_F)_j & \text{otherwise.} \end{cases}$$

For $F, G \in \mathscr{P}$ such that $G \subseteq F$ and $u \in \Phi_F$, define

$$\mathsf{sib}_{F,G}(u) := \{v_G : v \in \mathsf{sib}_F(u)\}.$$

In view of (1.9), it can be deduced from Lemma 4.1.6 of [9] that the joint kernel $E := \bigcap_{j=1}^{d} \ker S_j^*$ of $S_{\boldsymbol{\lambda}}^*$ is given by

$$E = [e_{\mathsf{root}}] \oplus \bigoplus_{F \in \mathscr{P}, F \neq \emptyset} \bigoplus_{u \in \Omega_F} \mathcal{L}_{u,F}, \tag{1.10}$$

where $\mathcal{L}_{u,F} \subseteq l^2(\mathsf{sib}_F(u))$ is the solution space of the following system of equations

$$\sum_{w \in \mathsf{sib}_j(v_G|_{u_j})} f(w)\lambda_j(w) = 0, \quad j \in F,\ v_G \in \mathsf{sib}_{F,G}(u) \text{ and } G = F \setminus \{j\} \tag{1.11}$$

(see the discussion following Lemma 4.1.6 of [9] for more details). The number of variables $M_{u,F}$ and number of equations $N_{u,F}$ in the above system are given by

$$M_{u,F} = \mathrm{card}(\mathsf{sib}_F(u)) = \prod_{j \in F} \mathrm{card}(\mathsf{sib}(u_j)), \quad N_{u,F} = \sum_{j \in F} \prod_{i \in F, i \neq j} \mathrm{card}(\mathsf{sib}_i(u)). \tag{1.12}$$

In particular, $\mathcal{L}_{u,F}$ is finite dimensional whenever the directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ are locally finite. Indeed, $M_{u,F}$ and $N_{u,F}$ are finite in this case.

We present the following useful lemma for future reference.

LEMMA 1.3. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let S_λ be a commuting multishift on $\mathscr{T}$. Then the joint kernel E of S_λ^* is given by*

$$E = \bigoplus_{F \in \mathscr{P}} \bigoplus_{u \in \Omega_F} \mathcal{L}_{u,F}, \tag{1.13}$$

where $\mathcal{L}_{u,F} \subseteq l^2(\mathsf{sib}_F(u))$ is the solution space of the system (1.11). *Moreover, if $\mathscr{T}$ is of finite joint branching index, then*

(i) *for any $F \in \mathscr{P}$, $\mathcal{L}_{u,F} \neq \{0\}$ for at most finitely many $u \in \Omega_F$;*

(ii) *E is finite dimensional.*

Proof. Note that $\Omega_\emptyset = \{\mathsf{root}\}$. Thus the system (1.11) is vacuous, and hence $\mathcal{L}_{\mathsf{root},\emptyset} = [e_{\mathsf{root}}]$. The desired expression for E is now obvious from (1.10). The part (ii) is immediate from Corollary 3.1.14 of [9], while (i) is clear in view of (1.13) and (ii). ∎

1.3. STATEMENT OF THE MAIN RESULT. We recall from Theorem 5.2.6 of [9] that $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ is a reproducing kernel Hilbert space of E-valued holomorphic functions defined on the open unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$. The reproducing kernel $\kappa_{\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})} : \mathbb{B}^d \times \mathbb{B}^d \to B(E)$ associated with $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ is given by

$$\kappa_{\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})}(z, w) = \sum_{F \in \mathscr{P}} \sum_{u \in \Omega_F} \Big(\sum_{\alpha \in \mathbb{N}^d} \frac{\mathsf{d}_u!}{(\mathsf{d}_u + \alpha)!} \prod_{j=0}^{|\alpha|-1} (|\mathsf{d}_u| + a + j)\, z^\alpha \overline{w}^\alpha \Big) P_{\mathcal{L}_{u,F}},$$

where $z, w \in \mathbb{B}^d$ and $P_{\mathcal{L}_{u,F}}$ is the orthogonal projection on $\mathcal{L}_{u,F}$ (see (1.13)).

We are now ready to state the main result of this paper.

THEOREM 1.4. *Let a, d be positive integers such that $ad \neq 1$, and fix $j = 1, 2$. Let $\mathscr{T}^{(j)} = (V^{(j)}, \mathcal{E}^{(j)})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1^{(j)}, \ldots, \mathscr{T}_d^{(j)}$ of finite joint branching index. Let $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(j)})$ be the Drury–Arveson-type Hilbert module associated with $\mathscr{T}^{(j)}$. Let $E^{(j)}$ be the subspace of constant functions in $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(j)})$ and let $\mathcal{L}_{u,F}^{(j)}$ be as appearing in the decomposition* (1.13) *of $E^{(j)}$. Then the following statements are equivalent:*

(i) *the Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(1)})$ and $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(2)})$ are isomorphic;*

(ii) *for any $\alpha \in \mathbb{N}^d$ and $F \in \mathscr{P}$,*

$$\sum_{u \in \Omega_F^{(1)},\, \mathsf{d}_u = \alpha} \dim \mathcal{L}_{u,F}^{(1)} = \sum_{v \in \Omega_F^{(2)},\, \mathsf{d}_v = \alpha} \dim \mathcal{L}_{v,F}^{(2)};$$

(iii) *for any* $n \in \mathbb{N}$ *and* $l = 1, \ldots, d,$

$$\sum_{u \in \Omega^{(1)}_{\{l\}},\, \mathsf{d}_u = n\epsilon_l} (\mathrm{card}(\mathsf{sib}_l(u)) - 1) = \sum_{u \in \Omega^{(2)}_{\{l\}},\, \mathsf{d}_u = n\epsilon_l} (\mathrm{card}(\mathsf{sib}_l(u)) - 1),$$

where ϵ_l *is the d-tuple with* 1 *in the* l^{th} *place and zeros elsewhere;*

(iv) *for any* $n \in \mathbb{N}$ *and* $l = 1, \ldots, d,$

$$\mathrm{card}(\mathcal{G}_n(\mathscr{T}_l^{(1)})) = \mathrm{card}(\mathcal{G}_n(\mathscr{T}_l^{(2)})),$$

where $\mathcal{G}_n(\mathscr{T}_l^{(j)})$ *is the* n^{th} *generation of* $\mathscr{T}_l^{(j)}$.

REMARK 1.5. The above result does not hold true in case $ad = 1$. This may be attributed to the von Neumann–Wold decomposition for isometries ([12], Chapter I; see the discussion following Problem 2.3 of [8]). In case $d = 1$, (iv) is equivalent to the following:

$$(1.14)\qquad \sum_{v \in V^{(1)}_{\prec} \cap \mathcal{G}_n(\mathscr{T}^{(1)})} (\mathrm{card}(\mathsf{Chi}(v)) - 1) = \sum_{v \in V^{(2)}_{\prec} \cap \mathcal{G}_n(\mathscr{T}^{(2)})} (\mathrm{card}(\mathsf{Chi}(v)) - 1).$$

In particular, the invariant appearing in (iii) of Theorem 1.4 can be seen as a multivariable counterpart of k^{th} generation branching degree as defined in equation (4.5) of [3]. Further, it is evident from the equivalence of (i) and (iv) above that non-graph-isomorphic directed Cartesian products can yield isomorphic Drury–Arveson-type Hilbert modules. Finally, note that the operator theoretic statements (i) and (ii) are equivalent to purely graph theoretic statements (iii) and (iv).

We discuss here some immediate consequences of Theorem 1.4. Recall that two directed graphs are *isomorphic* if there exists a bijection between their sets of vertices which preserves (directed) edges.

COROLLARY 1.6. *Let* a, d *be positive integers and let* $\mathscr{T} = (V, \mathcal{E})$ *be the directed Cartesian product of locally finite rooted directed trees* $\mathscr{T}_1, \ldots, \mathscr{T}_d$ *of finite joint branching index. Then the Drury–Arveson-type Hilbert module* $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ *associated with* $\mathscr{T}$ *is isomorphic to the classical Drury–Arveson-type Hilbert module* $\mathscr{H}_a$ *if and only if for any* $j = 1, \ldots, d$, *the directed tree* $\mathscr{T}_j$ *is graph isomorphic to the rooted directed tree without any branching vertex.*

Proof. The sufficiency part is immediate from Remark 3.1.1 of [9], while the necessary part follows from the equivalence of (i) and (iv) of Theorem 1.4, and the fact that a rooted directed tree without any branching vertex is unique up to graph isomorphism. ∎

COROLLARY 1.7. *Given positive integers a and d, we have the following statements:*

(i) *There exist infinitely many mutually non-isomorphic Drury–Arveson-type Hilbert modules.*

(ii) *Given any positive integer k, there exist k number of mutually non-isomorphic directed Cartesian products $\mathscr{T} = (V, \mathcal{E})$ of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ such that the associated Drury–Arveson-type Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ are isomorphic.*

Proof. We need the following example of rooted directed tree discussed in Chapter 6 of [17]. For a positive integer n_0, consider the directed tree $\mathscr{T}_{n_0,0} = (V, \mathcal{E})$ as follows:

$$V = \mathbb{N}, \quad \mathcal{E} = \{(0, j) : j = 1, \ldots, n_0\} \cup \bigcup_{j=1}^{n_0} \{(j + (l-1)n_0, j + ln_0) : l \geqslant 1\}.$$

(i) Consider the directed Cartesian product

$$\mathscr{T}^{(k)} := \mathscr{T}_{k,0} \times \mathscr{T}_{1,0}^{d-1}, \quad k \geqslant 1.$$

It is now immediate from Theorem 1.4 that the Drury–Arveson-type Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(k)})$ associated with the directed Cartesian product $\mathscr{T}^{(k)}$, $k \geqslant 1$ are mutually non-isomorphic.

(ii) Fix a positive integer k. For $j = 1, \ldots, k$, consider the rooted directed tree $\mathscr{T}_{1j}$ with $\mathsf{Chi}(\mathsf{root}) = \{u, v\}$, $\mathrm{card}(\mathsf{Chi}(u)) = 2k - j$, $\mathrm{card}(\mathsf{Chi}(v)) = j$, and $\mathrm{card}(\mathsf{Chi}(w)) = 1$ for all remaining vertices w in $\mathscr{T}_{1j}$. Consider the directed Cartesian product

$$\mathscr{T}^{(j)} := \mathscr{T}_{1j} \times \mathscr{T}_{1,0}^{d-1}, \quad j = 1, \ldots, k.$$

Then $\mathscr{T}^{(j)}$, $1 \leqslant j \leqslant k$ are mutually non-isomorphic. Now apply Theorem 1.4 to obtain the desired conclusion in (ii). ∎

Our proof of Theorem 1.4 occupies a substantial part of this paper. It is fairly long and quite involved as compared to the case of $d = 1$ (see Theorem 5.1 of [8]). It is worth mentioning that in case $d = 1$, the conclusion of Theorem 1.4 holds without the assumption of finite branching index. An essential reason for this is the fact that any left-invertible analytic operator admits an analytic model in the sense of [24]. On the other hand, in dimension $d \geqslant 2$, there is no successful counterpart of Shimorin's construction of an analytic model for joint left-invertible tuples (cf. Theorem 4.2.4, Remark 4.2.5 and Appendix of [9]). The proof of the main theorem is comprised of two parts, namely, Sections 2 and 3. Here is a brief overview of these sections.

In Section 2, we introduce and study a one parameter family $\mathcal{S}_{\mathscr{T}}$ of spherically balanced multishifts $S_{\lambda_{\mathfrak{c}}}$. This is carried out in three subsections:

(1) The first subsection is devoted to an elaborate description of joint cokernel E of multishifts in the family $\mathcal{S}_{\mathscr{T}}$. It turns out that the building blocks $\mathcal{L}_{u,F}$ appearing in the decomposition (1.13) of E can be identified with tensor product of certain hyperplanes (see Theorem 2.3). These hyperplanes further can be looked upon as the kernel of row matrices with all entries equal to 1 and of size dependent on coordinate siblings of u. This description readily provides a neat formula for the dimension of E (see Corollary 2.4).

(2) In the second subsection, we show that the multishifts in $\mathcal{S}_{\mathscr{T}}$ can be modeled as multiplication d-tuples on reproducing kernel Hilbert spaces $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$ of vector-valued holomorphic functions defined on a ball in $\mathbb{C}^d$ (see Theorem 2.9). We also provide a compact formula for the reproducing kernel associated with $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$ (see (2.21)). In particular, these results apply to Drury–Arveson-type multishifts $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ and their spherical Cauchy dual tuples $S^{\mathfrak{s}}_{\boldsymbol\lambda_{\mathfrak{c}_a}}$. The sequences $\mathfrak{c}$ associated with $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ and $S^{\mathfrak{s}}_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ are given respectively by

$$\mathfrak{c}(t) := \mathfrak{c}_a(t) = \frac{t+d}{t+a}, \quad \mathfrak{c}(t) := \frac{1}{\mathfrak{c}_a(t)}, \quad t \in \mathbb{N}, \tag{1.15}$$

where a is a positive real number. We emphasize that if we relax the assumption that $\mathscr{T}$ is of finite joint branching index, then the above model theorem fails unless the dimension $d = 1$ (see Remark 2.11).

(3) In the last subsection, we introduce and study the notion of an operator-valued representing measure for the Hilbert module $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$. Existence of a representing measure for $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$ is shown to be equivalent to the assertion that $\{\prod_{j=0}^{n-1} \mathfrak{c}(j)\}_{n\in\mathbb{N}}$ is a Hausdorff moment sequence (see Theorem 2.15). As an application, we show that Drury–Arveson-type Hilbert module $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ admits a representing measure if a is an integer bigger than or equal to d. We also show that $\mathscr{H}_{\mathfrak{c}_a^{\mathfrak{s}}}(\mathscr{T})$ (model space of $S^{\mathfrak{s}}_{\boldsymbol\lambda_{\mathfrak{c}_a}}$) admits a representing measure if a is a positive integer less than d. Further, we explicitly compute the representing measures in both these situations (see Corollaries 2.17 and 2.18).

In Section 3, we prove the main theorem. This section begins with the observation that the classification of Drury–Arveson-type Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ is equivalent to the unitary equivalence of operator-valued representing measures of $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ (respectively $\mathscr{H}_{\mathfrak{c}_a^{\mathfrak{s}}}(\mathscr{T})$) if $a \geqslant d$ (respectively if $a < d$) (see Lemma 3.1). We then establish another key observation that any isomorphism between two Drury–Arveson-type Hilbert modules preserves the decomposition (1.13) of E over each generation (see Proposition 3.2). Finally, we put all the pieces together to obtain a proof of Theorem 1.4.

A strictly higher dimensional fact in graph theory (*constant on parents is constant on generations*) closely related to the notion of spherically balanced multishift is added as an appendix (see Theorem A.1).

2. A FAMILY OF SPHERICALLY BALANCED MULTISHIFTS

Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Given a sequence $\mathfrak{c} : \mathbb{N} \to (0, \infty)$, we can associate a system $\boldsymbol\lambda_{\mathfrak{c}} = \{\lambda_j(w) : w \in V^\circ,\ j = 1, \ldots, d\}$ to $\mathscr{T}$ as follows:

$$\lambda_j(w) = \sqrt{\frac{\mathfrak{c}(|\mathsf{d}_v|)}{\mathsf{card}(\mathsf{Chi}_j(v))}}\sqrt{\frac{\mathsf{d}_{v_j}+1}{|\mathsf{d}_v|+d}}, \quad w \in \mathsf{Chi}_j(v),\ v \in V,\ j = 1, \ldots, d, \tag{2.1}$$

where d_v denotes the depth of the vertex v in $\mathscr{T}$. Note that

$$\sup_{v\in V}\sum_{w\in \mathsf{Chi}_j(v)} \lambda_j(w)^2 = \sup_{v\in V} \mathfrak{c}(|\mathsf{d}_v|)\frac{\mathsf{d}_{v_j}+1}{|\mathsf{d}_v|+d}.$$

It follows that the multishift $S_{\boldsymbol\lambda_{\mathfrak{c}}}$ with weights $\boldsymbol\lambda_{\mathfrak{c}}$ is bounded if and only if the sequence $\mathfrak{c}$ is bounded (see (1.3)). In this case, as shown in Proposition 5.2.3 of [9], the multishift $S_{\boldsymbol\lambda_{\mathfrak{c}}}$ turns out to be commuting and spherically balanced. Recall that a commuting multishift $S_{\boldsymbol\lambda} = (S_1, \ldots, S_d)$ is *spherically balanced* if the function $\mathfrak{C} : V \to (0, \infty)$ given by

$$\mathfrak{C}(v) := \sum_{j=1}^{d} \|S_j e_v\|^2, \quad v \in V$$

is constant on every generation $\mathcal{G}_t$, $t \in \mathbb{N}$. In case $S_{\boldsymbol\lambda} = S_{\boldsymbol\lambda_{\mathfrak{c}}}$,

$$\mathfrak{C}(v) = \mathfrak{c}(|\mathsf{d}_v|), \quad v \in V. \tag{2.2}$$

In case the directed trees $\mathscr{T}_j$ are without branching vertices, $S_{\boldsymbol\lambda_{\mathfrak{c}}}$ is spherical (or homogeneous with respect to the group of unitary $d \times d$ matrices) in the sense of Definition 1.1 in [11]. This may be concluded from Theorem 2.1 of [11]. On the other hand, for a spherically balanced multishift $S_{\boldsymbol\lambda}$, the spherical Cauchy dual tuple $S^{\mathfrak{s}}_{\boldsymbol\lambda}$ is always commuting. In fact, by Proposition 5.2.10 of [9], $S^{\mathfrak{s}}_{\boldsymbol\lambda}$ is commuting if and only if

$$\mathfrak{C} \text{ is constant on } \mathsf{Par}(v) := \bigcup_{j=1}^{d} \mathsf{par}_j(v) \quad \text{for all } v \in V^{\circ}. \tag{2.3}$$

In dimension bigger than 1, there is a curious fact that the apparently weaker condition (2.3) implies that $\mathfrak{C}$ is constant on every generation $\mathcal{G}_t$, $t \in \mathbb{N}$. It follows that if $d \geqslant 2$, then $S_{\boldsymbol\lambda}$ is a spherically balanced d-tuple if and only if its spherical Cauchy dual $S^{\mathfrak{s}}_{\boldsymbol\lambda}$ is commuting. Since the above facts play no essential role in the main result of this paper, we relegate their proof to an appendix. Needless to say, these facts are strictly higher dimensional.

The following family of multishifts plays a central role in the present investigations:

$$\mathcal{S}_{\mathscr{T}} := \{S_{\boldsymbol\lambda_{\mathfrak{c}}} : \inf \mathfrak{c} > 0,\ \sup \mathfrak{c} < \infty\}. \tag{2.4}$$

Our proof of Theorem 1.4 is based on a thorough study of this family. This includes a dimension formula for joint cokernel, an analytic model and existence of operator-valued representing measures for multishifts in this family.

2.1. A DIMENSION FORMULA. In this subsection, we obtain a neat formula for the dimension of joint cokernel of members $S_{\boldsymbol\lambda_{\mathfrak{c}}}$ belonging to the family $\mathcal{S}_{\mathscr{T}}$. It is worth noting that this formula is independent of $\mathfrak{c}$ due to the specific form of the weight system $\boldsymbol\lambda_{\mathfrak{c}}$ of multishifts from $\mathcal{S}_{\mathscr{T}}$ (see Lemma 2.1 below). First a definition (recall all required notations from Subsection 1.2).

Fix a nonempty $F \in \mathscr{P}$ and let $u \in \Omega_F$. For $j \in F$, define the linear functional $X_j : l^2(\mathsf{sib}(u_j)) \to \mathbb{C}$ by

$$X_j(f) = \sum_{\eta \in \mathsf{sib}(u_j)} f(\eta), \quad f \in l^2(\mathsf{sib}(u_j)). \tag{2.5}$$

The description of the joint cokernel for a member of $\mathcal{S}_{\mathscr{T}}$ is intimately related to the kernel of the linear functionals X_j, $j \in F$.

LEMMA 2.1. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol{\lambda}_{\mathfrak{c}}}$ be a multishift belonging to the family $\mathcal{S}_{\mathscr{T}}$. Let $\mathcal{L}_{u,F}$ be as appearing in* (1.13). *Then, for $F \in \mathscr{P}$ and $u \in \Omega_F$, the following are equivalent:*

(i) *f belongs to $\mathcal{L}_{u,F}$;*

(ii) $\sum_{w \in \mathsf{sib}_j(v_G|u_j)} f(w) = 0$ *for any $j \in F$, $v_G \in \mathsf{sib}_{F,G}(u)$ with $G = F \setminus \{j\}$;*

(iii) $\sum_{\eta \in \mathsf{sib}(u_j)} f(v_G|_\eta) e_\eta \in \ker X_j$ *for any $j \in F$, $v_G \in \mathsf{sib}_{F,G}(u)$ with $G = F \setminus \{j\}$,*

where X_j is as given in (2.5).

Proof. By (2.1), for each $j = 1, \ldots, d$, $\lambda_j(\cdot)$ is constant on $\mathsf{sib}_F(u)$, and hence the equivalence of (i) and (ii) follows from (1.11). The equivalence of (ii) and (iii) is immediate from the definition (2.5) of X_j. ∎

REMARK 2.2. It follows from the implication (i) ⇒ (ii) above that $\mathcal{L}_{u,F}$ (and hence by (1.13) the joint kernel E of $S^*_{\boldsymbol{\lambda}_{\mathfrak{c}}}$) is independent of the choice of $\mathfrak{c}$.

The following result identifies the building blocks $\mathcal{L}_{u,F}$ appearing in the orthogonal decomposition of the joint cokernel of $S_{\boldsymbol{\lambda}_{\mathfrak{c}}}$ with a tensor product of kernels of X_j, $j \in F$.

THEOREM 2.3. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_{\boldsymbol{\lambda}_{\mathfrak{c}}}$ be a member of $\mathcal{S}_{\mathscr{T}}$. Let $\mathcal{L}_{u,F}$ be as appearing in* (1.13) *with $F = \{i_1, \ldots, i_k\}$ for some positive integer $k \in \{1, \ldots, d\}$. Then $\mathcal{L}_{u,F}$ is isomorphic to the finite dimensional space* $\ker X_{i_1} \otimes \cdots \otimes \ker X_{i_k}$ *where X_j, $j \in F$, is as defined in* (2.5). *In particular,*

$$\dim \mathcal{L}_{u,F} = \prod_{j \in F} (\mathrm{card}(\mathsf{sib}(u_j)) - 1). \tag{2.6}$$

Proof. We begin with the fact that $l^2(\mathsf{sib}_F(u))$ can be identified as the tensor product of $l^2(\mathsf{sib}(u_j))$, $j \in F$. Since our proof utilizes the precise form of the isomorphism between these spaces, we provide elementary details essential in this identification.

Define $\phi : l^2(\mathsf{sib}(u_{i_1})) \oplus \cdots \oplus l^2(\mathsf{sib}(u_{i_k})) \to l^2(\mathsf{sib}_F(u))$ by

$$\phi((f_{i_1}, \ldots, f_{i_k}))(v) = \prod_{j \in F} f_j(v_j), \quad v = (v_1, \ldots, v_d) \in \mathsf{sib}_F(u). \tag{2.7}$$

It is easy to see that ϕ is multilinear. By the universal property of tensor product of vector spaces ([16], Theorem 4.14), there exists a unique linear map Φ such that the following diagram commutes:

$$\begin{array}{ccc} l^2(\mathsf{sib}(u_{i_1})) \oplus \cdots \oplus l^2(\mathsf{sib}(u_{i_k})) & \xrightarrow{\ \otimes\ } & l^2(\mathsf{sib}(u_{i_1})) \otimes \cdots \otimes l^2(\mathsf{sib}(u_{i_k})) \\ & \searrow^{\phi} & \downarrow{\Phi} \\ & & l^2(\mathsf{sib}_F(u)) \end{array}$$

By (2.7) and $\Phi \circ \otimes = \phi$, the action of Φ on elementary tensors is given by

$$\Phi(f_{i_1} \otimes \cdots \otimes f_{i_k})(v_1, \ldots, v_d) = \prod_{j \in F} f_j(v_j), \quad v \in \mathsf{sib}_F(u). \tag{2.8}$$

The map Φ turns out to be an isomorphism. Since we are not aware of an appropriate reference, we include necessary details. We first verify that Φ is injective. Let $f = \sum_{j=1}^{N} f_{j1} \otimes \cdots \otimes f_{jk} \in \ker \Phi$. Suppose to the contrary that $f \neq 0$. By Lemma 1.1 of [20],

$$\{f_{ji} : j = 1, \ldots, N\} \text{ is linearly independent for } i = 1, \ldots, k. \tag{2.9}$$

Since $\Phi(f) = 0$, it follows that

$$\sum_{j=1}^{N} f_{j1}(v_{i_1}) \cdots f_{jk}(v_{i_k}) = 0, \quad v \in \mathsf{sib}_F(u).$$

Fixing all coordinates of $v \in \mathsf{sib}_F(u)$ except i_k, and using (2.9), we conclude that

$$f_{j1}(v_{i_1}) \cdots f_{jk-1}(v_{i_{k-1}}) = 0, \quad v \in \mathsf{sib}_F(u),\ j = 1, \ldots, N.$$

Since $f_{jk-1} \neq 0$, by fixing all coordinates of $v \in \mathsf{sib}_F(u)$ except i_{k-1}, we conclude that

$$f_{j1}(v_{i_1}) \cdots f_{jk-2}(v_{i_{k-2}}) = 0, \quad v \in \mathsf{sib}_F(u),\ j = 1, \ldots, N.$$

Continuing like this, we arrive at the conclusion that f_{j1} is identically 0 for $j = 1, \ldots, N$, which contradicts (2.9). Hence we must have $f = 0$, that is, Φ is injective. Further, since $\mathrm{card}(\mathsf{sib}_F(u)) = \prod_{j \in F} \mathrm{card}(\mathsf{sib}(u_j))$ (see (1.12)), we obtain

$$\dim(l^2(\mathsf{sib}(u_{i_1})) \otimes \cdots \otimes l^2(\mathsf{sib}(u_{i_k}))) = \dim l^2(\mathsf{sib}_F(u)).$$

It follows that Φ is an isomorphism. However, for the rest of the proof, we also need to know the action of Φ^{-1}. To see that, let $f \in l^2(\mathsf{sib}_F(u))$. Then

$$f = \sum_{w \in \mathsf{sib}_F(u)} f(w) e_w = \sum_{w \in \mathsf{sib}_F(u)} f(w) \prod_{i \in F} \chi_{w_i},$$

where, for $i \in F$ and $v \in \mathsf{sib}_F(u)$,

$$\chi_{w_i}(v) = \begin{cases} 1 & \text{if } v_i = w_i, \\ 0 & \text{otherwise.} \end{cases}$$

It is now easy to see using (2.8) that

$$\Phi^{-1}(f) = \sum_{w \in \mathsf{sib}_F(u)} f(w)(e_{w_{i_1}} \otimes \cdots \otimes e_{w_{i_k}}). \tag{2.10}$$

We now check that Φ maps $\ker X_{i_1} \otimes \cdots \otimes \ker X_{i_k}$ into $\mathcal{L}_{u,F}$. To see this, let $f_j \in l^2(\mathsf{sib}(u_{i_j}))$ be such that

$$X_{i_j} f_j = 0, \quad j = 1, \ldots, k. \tag{2.11}$$

In view of Lemma 2.1, it suffices to check that for $j = 1, \ldots, k$,

$$\sum_{w \in \mathsf{sib}_{i_j}(v_G|_{u_{i_j}})} \Phi(f_1 \otimes \cdots \otimes f_k)(w) = 0, \quad v_G \in \mathsf{sib}_{F,G}(u),\ G = F \setminus \{i_j\}. \tag{2.12}$$

However, for $v_G \in \mathsf{sib}_{F,G}(u),\ G = F \setminus \{i_j\},\ j = 1, \ldots, k$,

$$\begin{aligned} \sum_{w \in \mathsf{sib}_{i_j}(v_G|_{u_{i_j}})} \Phi(f_1 \otimes \cdots \otimes f_k)(w) &\overset{(2.8)}{=} \sum_{w \in \mathsf{sib}_{i_j}(v_G|_{u_{i_j}})} \prod_{l=1}^{k} f_l(w_{i_l}) \\ &= \sum_{\eta \in \mathsf{sib}(u_{i_j})} \Big(\prod_{l \neq j,\, l=1}^{k} f_l(v_{i_l}) \Big) f_j(\eta) \\ &\overset{(2.5)}{=} \Big(\prod_{l \neq j,\, l=1}^{k} f_l(v_{i_l}) \Big) X_{i_j} f_j, \end{aligned}$$

which is 0 in view of (2.11). This yields (2.12) for every $j = 1, \ldots, k$, and hence

$$\Phi(\ker X_{i_1} \otimes \cdots \otimes \ker X_{i_k}) \subseteq \mathcal{L}_{u,F}.$$

To see that this inclusion is an equality, let $f \in \mathcal{L}_{u,F}$. By Lemma 4.1.5(i) of [9],

$$\mathsf{sib}_F(u) = \bigsqcup_{v_G \in \mathsf{sib}_{F,G}(u)} \mathsf{sib}_j(v_G|_{u_j}), \quad G = F \setminus \{j\},\ j \in F. \tag{2.13}$$

It follows that for $G = F \setminus \{i_j\}$, $j = 1, \ldots, k$,

$$\begin{aligned} \Phi^{-1}(f) &\overset{(2.10)}{=} \sum_{w \in \mathsf{sib}_F(u)} f(w)(e_{w_{i_1}} \otimes \cdots \otimes e_{w_{i_k}}) \\ &\overset{(2.13)}{=} \sum_{v_G \in \mathsf{sib}_{F,G}(u)} \ \sum_{w \in \mathsf{sib}_{i_j}(v_G|_{u_{i_j}})} f(w)(e_{w_{i_1}} \otimes \cdots \otimes e_{w_{i_k}}). \end{aligned}$$

It follows that $\Phi^{-1}(f)$ is equal to

$$\sum_{v_G \in \mathsf{sib}_{F,G}(u)} e_{w_{i_1}} \otimes \cdots \otimes e_{w_{i_{j-1}}} \otimes \sum_{w_{i_j} \in \mathsf{sib}(u_{i_j})} f(v_G|_{w_{i_j}}) e_{w_{i_j}} \otimes e_{w_{i_{j+1}}} \otimes \cdots \otimes e_{w_{i_k}}.$$

However, since $f \in \mathcal{L}_{u,F}$, by Lemma 2.1,

$$\sum_{w_{i_j} \in \mathsf{sib}(u_{i_j})} f(v_G|_{w_{i_j}}) e_{w_{i_j}} \in \ker X_{i_j}.$$

It is now clear that $\Phi^{-1}(f) \in \ker X_{i_1} \otimes \cdots \otimes \ker X_{i_k}$. Since the dimension of tensor product of vector spaces is a product of dimensions of respective vector spaces ([16], Theorem 4.14) the remaining part is immediate. ∎

A careful examination of the proof of Theorem 2.3 shows that the formula for the joint cokernel holds for any multishift with constant weight system taking value 1 (commonly known as *adjacency operator* in dimension $d = 1$; refer to [17]). The following is immediate from (1.10), (1.13) and (2.6) (see also Lemma 1.3).

COROLLARY 2.4. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ of finite joint branching index. Let $S_{\lambda_\mathfrak{c}}$ be a member of $\mathcal{S}_\mathscr{T}$ and let E denote the joint kernel of $S^*_{\lambda_\mathfrak{c}}$. Then the dimension of E is given by*

$$\dim E = \sum_{F \in \mathscr{P}} \sum_{u \in \Omega_F} \prod_{i \in F} (\mathrm{card}(\mathsf{sib}(u_i)) - 1) = 1 + \sum_{F \neq \emptyset,\, F \in \mathscr{P}} \sum_{u \in \Omega_F} \prod_{i \in F} (\mathrm{card}(\mathsf{sib}(u_i)) - 1).$$

REMARK 2.5. In case $d = 1$, the above formula for E simplifies to

$$\dim E = 1 + \sum_{u \in \Omega_{\{1\}}} (\mathrm{card}(\mathsf{sib}(u)) - 1),$$

which holds for any choice of positive weights λ (see Proposition 3.5.1(ii) of [17]). This formula resembles the expression for the (undirected) graph invariant $\Upsilon(\mathscr{T})$ introduced in p. 3 of [4], which counts precisely the number of the so-called partial conjugations of the right angled Artin group $A_\mathscr{T}$ defined by deep nodes.

2.2. AN ANALYTIC MODEL.

In this section, we obtain an analytic model for multishifts belonging to the family $\mathcal{S}_\mathscr{T}$ (see (2.4)). The treatment here relies on a technique developed in the proof of Theorem 5.2.6 in [9]. We begin with an important aspect of the family $\mathcal{S}_\mathscr{T}$, namely that it is closed under the operation of taking spherical Cauchy dual.

LEMMA 2.6. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. If $S_{\lambda_\mathfrak{c}} \in \mathcal{S}_\mathscr{T}$, then $S^{\mathfrak{s}}_{\lambda_\mathfrak{c}}$ is well-defined and $S^{\mathfrak{s}}_{\lambda_\mathfrak{c}}$ belongs to $\mathcal{S}_\mathscr{T}$.*

Proof. Assume that $S_{\lambda_\mathfrak{c}} = (S_1, \ldots, S_d) \in \mathcal{S}_\mathscr{T}$. Note that by (2.2),

$$\inf_{v \in V} \sum_{j=1}^{d} \|S_j e_v\|^2 = \inf \mathfrak{c} > 0, \quad \sup_{v \in V} \sum_{j=1}^{d} \|S_j e_v\|^2 = \sup \mathfrak{c} < \infty.$$

It follows that $S_{\boldsymbol\lambda_{\mathfrak c}}$ is joint left-invertible, and hence $S^{\mathfrak s}_{\boldsymbol\lambda_{\mathfrak c}}$ is well-defined. On the other hand, by (1.5) and (2.2), the weights of $S^{\mathfrak s}_{\boldsymbol\lambda_{\mathfrak c}}$ are given by

$$\lambda_j(w) = \frac{1}{\sqrt{\mathfrak c(|\mathsf d_v|)}}\sqrt{\frac{1}{\mathrm{card}(\mathsf{Chi}_j(v))}}\sqrt{\frac{\mathsf d_{v_j}+1}{|\mathsf d_v|+d}}, \quad w \in \mathsf{Chi}_j(v),\ v \in V,\ j = 1,\dots,d.$$

It is now clear that $S^{\mathfrak s}_{\boldsymbol\lambda_{\mathfrak c}} \in \mathcal S_{\mathscr T}$. ∎

We skip the proof of the following simple yet useful fact, which may be obtained by a routine inductive argument (cf. Proof of Corollary 5.2.12 in [9]).

LEMMA 2.7. *Let $\mathscr T = (V,\mathcal E)$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr T_1,\dots,\mathscr T_d$ and let $S_{\boldsymbol\lambda}$ be a commuting multishift on $\mathscr T$ with weight system $\boldsymbol\lambda = \{\lambda_j(v) : v \in V^\circ,\ j = 1,\dots,d\}$. For a bounded sequence $\mathfrak w$ of positive numbers, let $\boldsymbol\lambda_{\mathfrak w}$ denote the system*

$$\mathfrak w(|\mathsf d_v|)\lambda_j(w), \quad w \in \mathsf{Chi}_j(v),\ v \in V,\ j = 1,\dots,d. \tag{2.14}$$

Then the multishift $S_{\boldsymbol\lambda_{\mathfrak w}}$ on $\mathscr T$ with weight system as given in (2.14) *is commuting. Moreover, for any $v \in V$ and $\beta \in \mathbb N^d$, we obtain:*

(i) $S^{\beta}_{\boldsymbol\lambda_{\mathfrak w}} e_v = \Big(\prod_{p=0}^{|\beta|-1} \mathfrak w(|\mathsf d_v|+p)\Big) S^{\beta}_{\boldsymbol\lambda} e_v$;

(ii) $\|S^{\beta}_{\boldsymbol\lambda_{\mathfrak w}} e_v\|^2 = \Big(\prod_{p=0}^{|\beta|-1} \mathfrak w(|\mathsf d_v|+p)^2\Big) \|S^{\beta}_{\boldsymbol\lambda} e_v\|^2$.

Here is a key observation in obtaining an analytic model for members of $\mathcal S_{\mathscr T}$.

LEMMA 2.8. *Let $\mathscr T = (V,\mathcal E)$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr T_1,\dots,\mathscr T_d$ of finite joint branching index. Let $S_{\boldsymbol\lambda_{\mathfrak c}}$ be in the family $\mathcal S_{\mathscr T}$ and let E denote the joint kernel of $S^*_{\boldsymbol\lambda_{\mathfrak c}}$. Then the following statements are true:*

(i) *E is invariant under $S^{*\alpha}_{\boldsymbol\lambda_{\mathfrak c}} S^{\alpha}_{\boldsymbol\lambda_{\mathfrak c}}$ and $S^{*\alpha}_{\boldsymbol\lambda_{\mathfrak c}} S^{\alpha}_{\boldsymbol\lambda_{\mathfrak c}}|_E$ is boundedly invertible for all $\alpha \in \mathbb N^d$;*

(ii) *the multisequence $\{S^{\alpha}_{\boldsymbol\lambda_{\mathfrak c}} E\}_{\alpha\in\mathbb N^d}$ of subspaces of $l^2(V)$ is mutually orthogonal.*

Proof. The proof relies on the case in which $\mathfrak c = \mathfrak c_a$, $a > 0$ (see (1.15)) as established in Lemma 5.2.7 of [9]. Let $\alpha \in \mathbb N^d$. We apply Lemma 2.7(ii) to the system $\boldsymbol\lambda_{\mathfrak c_a}$ given by

$$\lambda_j(w) = \frac{1}{\sqrt{\mathrm{card}(\mathsf{Chi}_j(v))}}\sqrt{\frac{\mathsf d_{v_j}+1}{|\mathsf d_v|+a}}, \quad w \in \mathsf{Chi}_j(v),\ v \in V,\ j = 1,\dots,d, \tag{2.15}$$

with

$$\mathfrak w(t) := \sqrt{\mathfrak c(t)}\sqrt{\frac{t+a}{t+d}}, \quad t \in \mathbb N, \tag{2.16}$$

to conclude that

$$\|S^{\alpha}_{\boldsymbol\lambda_{\mathfrak c}} e_v\|^2 = K(|\alpha|, |\mathsf d_v|)\|S^{\alpha}_{\boldsymbol\lambda_{\mathfrak c_a}} e_v\|^2, \quad v \in V, \tag{2.17}$$

where

$$K(s,t) = \prod_{p=0}^{s-1} \mathfrak{w}(t+p)^2 \overset{(2.16)}{=} \prod_{p=0}^{s-1} \mathfrak{c}(t+p) \prod_{p=0}^{s-1} \frac{t+a+p}{t+d+p}, \quad s,t \in \mathbb{N}. \tag{2.18}$$

For $F \in \mathscr{P}$ and $u \in \Omega_F$, let $f \in \mathcal{L}_{u,F}$. It is now easy to see from Lemma 1.2(i) and (2.17) that

$$S^{*\alpha}_{\lambda_\mathfrak{c}} S^{\alpha}_{\lambda_\mathfrak{c}} f = K(|\alpha|, |\mathsf{d}_u|) S^{*\alpha}_{\lambda_{\mathfrak{c}_a}} S^{\alpha}_{\lambda_{\mathfrak{c}_a}} f. \tag{2.19}$$

Since $\mathcal{L}_{u,F}$ is invariant under $S^{*\alpha}_{\lambda_{\mathfrak{c}_a}} S^{\alpha}_{\lambda_{\mathfrak{c}_a}}$ ([9], Proof of Lemma 5.2.7), by Remark 2.2, we must have $S^{*\alpha}_{\lambda_\mathfrak{c}} S^{\alpha}_{\lambda_\mathfrak{c}} f \in \mathcal{L}_{u,F} \subseteq E$. Also, since $S^{*\alpha}_{\lambda_{\mathfrak{c}_a}} S^{\alpha}_{\lambda_{\mathfrak{c}_a}}|_E$ is boundedly invertible for all $\alpha \in \mathbb{N}^d$, the remaining part in (i) is now immediate from (2.19), (2.18), Lemma 1.3(i) and the assumption that $\inf \mathfrak{c} > 0$.

To see (ii), let $\alpha \in \mathbb{N}^d$ and

$$f = \sum_{F \in \mathscr{P}} \sum_{u \in \Omega_F} f_{u,F} \in E, \quad f_{u,F} \in \mathcal{L}_{u,F}.$$

Since $\mathcal{L}_{u,F} \subseteq l^2(\mathsf{sib}_F(u))$, we have

$$\mathcal{L}_{u,F} \subseteq \bigvee \{e_v : v \in \mathcal{G}_{|\mathsf{d}_u|}\}. \tag{2.20}$$

This combined with Lemma 2.7(i) and (2.18) implies that

$$S^{\alpha}_{\lambda_\mathfrak{c}} f_{u,F} = \sqrt{K(|\alpha|, |\mathsf{d}_u|)}\, S^{\alpha}_{\lambda_{\mathfrak{c}_a}} f_{u,F}.$$

Thus we obtain

$$S^{\alpha}_{\lambda_\mathfrak{c}} f = \sum_{F \in \mathscr{P}} \sum_{u \in \Omega_F} S^{\alpha}_{\lambda_\mathfrak{c}} f_{u,F} = \sum_{F \in \mathscr{P}} \sum_{u \in \Omega_F} \sqrt{K(|\alpha|, |\mathsf{d}_u|)}\, S^{\alpha}_{\lambda_{\mathfrak{c}_a}} f_{u,F}.$$

The desired conclusion in (ii) now follows from the fact that $\{S^{\alpha}_{\lambda_{\mathfrak{c}_a}} E\}_{\alpha \in \mathbb{N}^d}$ is mutually orthogonal (see Lemma 5.2.7 of [9]). ∎

We note that the conclusion of (ii) in the above lemma holds for any (spherically) balanced, injective weighted shift on a rooted directed tree (see Lemma 15 and Theorem 16 of [6]). We believe that this result fails in higher dimensions.

We now present the promised analytic model for multishifts belonging to the family $\mathcal{S}_\mathscr{T}$ (cf. Theorem 5.2.6 of [9] and Theorem 2.2 of [10]).

THEOREM 2.9. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ of finite joint branching index. Let $S_{\lambda_\mathfrak{c}}$ be a multishift belonging to $\mathcal{S}_\mathscr{T}$. Let E denote the joint kernel of $S^*_{\lambda_\mathfrak{c}}$ and let $\mathcal{L}_{u,F}$ be as appearing in* (1.13). *Then $S_{\lambda_\mathfrak{c}}$ is unitarily equivalent to the multiplication d-tuple $\mathscr{M}_z = (\mathscr{M}_{z_1}, \ldots, \mathscr{M}_{z_d})$ on a reproducing kernel Hilbert space $\mathscr{H}_\mathfrak{c}(\mathscr{T})$ of E-valued holomorphic functions defined on the open ball $\mathbb{B}^d_r$ in $\mathbb{C}^d$ for some positive number r. Further, the reproducing kernel*

$\kappa_{\mathscr{H}_{\mathfrak{c}}(\mathscr{T})} : \mathbb{B}^d_r \times \mathbb{B}^d_r \to B(E)$ *associated with* $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$ *is given by*

$$\kappa_{\mathscr{H}_{\mathfrak{c}}(\mathscr{T})}(z,w) = \sum_{F\in\mathscr{P}}\sum_{u\in\Omega_F}\Big(\sum_{\alpha\in\mathbb{N}^d}\frac{\mathsf{d}_u!}{(\mathsf{d}_u+\alpha)!}\prod_{j=0}^{|\alpha|-1}\frac{|\mathsf{d}_u|+d+j}{\mathfrak{c}(|\mathsf{d}_u|+j)}\,z^\alpha\overline{w}^\alpha\Big)P_{\mathcal{L}_{u,F}}, \tag{2.21}$$

where $z, w \in \mathbb{B}^d_r$ *and* $P_{\mathcal{L}_{u,F}}$ *is the orthogonal projection on* $\mathcal{L}_{u,F}$.

REMARK 2.10. Note that the reproducing kernel Hilbert space $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$ is a module over the polynomial ring $\mathbb{C}[z_1,\ldots,z_d]$ (see (1.2)). Fix $j = 1,2$. Let $\mathscr{T}^{(j)} = (V^{(j)},\mathcal{E}^{(j)})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}^{(j)}_1,\ldots,\mathscr{T}^{(j)}_d$ of finite joint branching index. Let $S^{(j)}_{\boldsymbol{\lambda}_\mathfrak{c}}$ be a member of $\mathcal{S}_{\mathscr{T}^{(j)}}$. It may be concluded from Remark 3.1.1 of [9] that if $\mathscr{T}^{(1)}$ and $\mathscr{T}^{(2)}$ are graph isomorphic, then the multishifts $S^{(1)}_{\boldsymbol{\lambda}_\mathfrak{c}}$ and $S^{(2)}_{\boldsymbol{\lambda}_\mathfrak{c}}$ are unitarily equivalent. It follows that the Hilbert modules $\mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(1)})$ and $\mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(2)})$ are isomorphic in this case.

Proof. We adapt the argument of Theorem 5.2.6 in [9] to the present situation. The verification of the first part is along the lines of Step I of the proof of Theorem 5.2.6 in [9]. Indeed, the space $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$ can be explicitly written as

$$\mathscr{H}_{\mathfrak{c}}(\mathscr{T}) = \Big\{F(z) = \sum_{\alpha\in\mathbb{N}^d} f_\alpha z^\alpha : f_\alpha \in E\ (\alpha\in\mathbb{N}^d),\ \sum_{\alpha\in\mathbb{N}^d}\|S^\alpha_{\boldsymbol{\lambda}_\mathfrak{c}} f_\alpha\|^2 < \infty\Big\}$$

with inner product

$$\langle F(z),\, G(z)\rangle_{\mathscr{H}_{\mathfrak{c}}(\mathscr{T})} = \sum_{\alpha\in\mathbb{N}^d}\langle S^\alpha_{\boldsymbol{\lambda}_\mathfrak{c}} f_\alpha,\, S^\alpha_{\boldsymbol{\lambda}_\mathfrak{c}} g_\alpha\rangle_{l^2(V)}, \quad F,G\in\mathscr{H}_{\mathfrak{c}}(\mathscr{T}).$$

We leave the details to the interested reader. We also skip the routine verification of the fact that $\kappa_{\mathscr{H}_{\mathfrak{c}}(\mathscr{T})}$ is a reproducing kernel for $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$:

$$\langle F,\, \kappa_{\mathscr{H}_{\mathfrak{c}}}(\cdot,w)g\rangle = \langle F(w),\, g\rangle_E, \quad F\in\mathscr{H}_{\mathfrak{c}}(\mathscr{T}), g\in E, w\in\mathbb{B}^d_r.$$

Let us check that the series on the right hand side of (2.21) converges for any z, w in some open ball centred at the origin in $\mathbb{C}^d$. Note that the reproducing kernel $\kappa_{\mathscr{H}_{\mathfrak{c}}}$ admits the orthogonal decomposition:

$$\kappa_{\mathscr{H}_{\mathfrak{c}}(\mathscr{T})}(z,w) = \bigoplus_{u\in\Omega_F,\, F\in\mathscr{P}}\kappa_{u,F}(z,w)P_{\mathcal{L}_{u,F}},$$

where, for $F\in\mathscr{P}$ and $u\in\Omega_F$,

$$\kappa_{u,F}(z,w) = \sum_{\alpha\in\mathbb{N}^d}\frac{\mathsf{d}_u!}{(\mathsf{d}_u+\alpha)!}\prod_{j=0}^{|\alpha|-1}\frac{|\mathsf{d}_u|+d+j}{\mathfrak{c}(|\mathsf{d}_u|+j)}\,z^\alpha\overline{w}^\alpha.$$

Further, the domain of convergence of $\kappa_{u,F}(\cdot,w)$ contains the open ball of radius $\inf\mathfrak{c}$ for fixed w in the unit ball in $\mathbb{C}^d$. Indeed, $\inf\mathfrak{c}$ is positive since $S_{\boldsymbol{\lambda}_\mathfrak{c}}$ belongs to

$\mathcal{S}_{\mathscr{T}}$, and hence for any $F \in \mathscr{P}$ and $u \in \Omega_F$,

$$|\kappa_{u,F}(z,w)| \leqslant \sum_{\alpha \in \mathbb{N}^d} \frac{\mathsf{d}_u!}{(\mathsf{d}_u + \alpha)!} \prod_{j=0}^{|\alpha|-1} (|\mathsf{d}_u| + d + j) \frac{1}{(\inf \mathfrak{c})^{|\alpha|}} |z^\alpha||w^\alpha|$$
$$\overset{(\star)}{\leqslant} \frac{\mathsf{d}_u!}{(|\mathsf{d}_u| + d - 1)!} \frac{1}{(1 - (\inf \mathfrak{c})^{-1}\langle z, w\rangle)^d} |z^\beta||w^\gamma|,$$

where the inequality $(\star)$ can be deduced from the multinomial formula (see proof of Lemma 4.4 in [15] for details), and for $j = 1, \ldots, d$,

$$\beta_j = \begin{cases} -\mathsf{d}_{u_j} & \text{if } z_j \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \gamma_j = \begin{cases} -\mathsf{d}_{u_j} & \text{if } w_j \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Also, since $\mathscr{T}$ is of finite joint branching index, by Lemma 1.3(i), for any $F \in \mathscr{P}$, $\mathcal{L}_{u,F} \neq \{0\}$ for at most finitely many $u \in \Omega_F$. It follows that for fixed $w \in \mathbb{B}^d$, the domain of convergence of $\kappa_{\mathscr{H}_\mathfrak{c}(\mathscr{T})}(\cdot, w)$ contains the open ball of radius $\inf \mathfrak{c}$. If $r := \min\{\inf \mathfrak{c}, 1\}$, then the absolute convergence of $\kappa_{\mathscr{H}_\mathfrak{c}(\mathscr{T})}(z,w)$ for $z, w \in \mathbb{B}^d_r$ is now immediate from

$$\kappa_{\mathscr{H}_\mathfrak{c}(\mathscr{T})}(z,w) = \overline{\kappa_{\mathscr{H}_\mathfrak{c}(\mathscr{T})}(w,z)}, \quad z, w \in \mathbb{B}^d_r.$$

To see (2.21), one can argue as in Step II of the proof of Theorem 5.2.6 in [9] to obtain

$$\kappa_{\mathscr{H}_\mathfrak{c}(\mathscr{T})}(z,w) = \sum_{\alpha \in \mathbb{N}^d} D_\alpha z^\alpha \overline{w}^\alpha, \quad z, w \in \mathbb{B}^d_r, \tag{2.22}$$

where D_α is the inverse of $S^{*\alpha}_{\lambda_\mathfrak{c}} S^\alpha_{\lambda_\mathfrak{c}}|_E$ as ensured by Lemma 2.8(i). On the other hand, as recorded in Step III of the proof of Theorem 5.2.6 in [9],

$$\|S^\alpha_{\lambda_{\mathfrak{c}_a}} e_u\|^{-2} = \frac{\mathsf{d}_u!}{(\mathsf{d}_u + \alpha)!} (|\mathsf{d}_u| + a)(|\mathsf{d}_u| + a + 1) \cdots (|\mathsf{d}_u| + a + |\alpha| - 1), \quad u \in V. \tag{2.23}$$

This combined with (2.17) and (2.18) yields that

$$\|S^\alpha_{\lambda_\mathfrak{c}} e_u\|^{-2} = \frac{\mathsf{d}_u!}{(\mathsf{d}_u + \alpha)!} \prod_{j=0}^{|\alpha|-1} \frac{|\mathsf{d}_u| + d + j}{\mathfrak{c}(|\mathsf{d}_u| + j)}, \quad u \in V. \tag{2.24}$$

The desired expression in (2.21) is now immediate from (2.20), Lemma 1.2(i) and (2.22). ∎

REMARK 2.11. The reproducing kernel $\kappa_{\mathscr{H}_\mathfrak{c}(\mathscr{T})}(z,w)$ can be obtained by integrating a family of scalar-valued reproducing kernels (cf. (1.1)) with respect to a finite family of spectral measures. Indeed,

$$\kappa_{\mathscr{H}_\mathfrak{c}(\mathscr{T})}(z,w) = \sum_{F \in \mathscr{P}} \int_{\Omega_F} \Big(\sum_{\alpha \in \mathbb{N}^d} \frac{\mathsf{d}_u!}{(\mathsf{d}_u + \alpha)!} \prod_{j=0}^{|\alpha|-1} \frac{|\mathsf{d}_u| + d + j}{\mathfrak{c}(|\mathsf{d}_u| + j)} z^\alpha \overline{w}^\alpha \Big) dP_F(u),$$

where $z, w \in \mathbb{B}^d_r$ and $P_F(\cdot)$, $F \in \mathscr{P}$ is the spectral measure given by

$$P_F(\sigma) = \sum_{u \in \sigma} P_{\mathcal{L}_{u,F}}, \quad \sigma \in \mathscr{P}(\Omega_F).$$

Further, note that Theorem 2.9 fails in dimension $d \geqslant 2$ if we relax the assumption that $\mathscr{T}$ is of finite joint branching index. Indeed, if $d = 2$, $\mathscr{T}_1$ is the binary tree (see Section 4.3 of [17]) and $\mathscr{T}_2$ is the rooted directed tree $\mathscr{T}_{1,0}$ without any branching vertex, then $\mathcal{L}_{u,\{1\}} \neq \{0\}$ for infinitely many $u \in \Omega_{\{1\}}$ (cf. Lemma 1.3(i)), and hence it may be concluded from Lemma 1.2 and (2.23) that $S^{*\alpha}_{\lambda_{\mathfrak{c}_a}} S^{\alpha}_{\lambda_{\mathfrak{c}_a}}|_E$ is not boundedly invertible for $\alpha = (1, 1)$. It is worth noting that this phenomenon is not possible in dimension $d = 1$ (see Proposition 3.1 of [8]).

From now onwards, the pair $(\mathscr{M}_z, \mathscr{H}_{\mathfrak{c}}(\mathscr{T}))$, as obtained in Theorem 2.9, will be referred to as the *analytic model* of the multishift $S_{\lambda_{\mathfrak{c}}}$ on $\mathscr{T}$. In case $\mathfrak{c} = \mathfrak{c}_a$, $a > 0$ (see (1.15)), the multishift $S_{\lambda_{\mathfrak{c}_a}}$ will be referred to as *Drury–Arveson-type multishift on* $\mathscr{T}$ (see (2.15)). In case each directed tree $\mathscr{T}_j$ is isomorphic to $\mathbb{N}$, $S_{\lambda_{\mathfrak{c}_1}}$ is unitarily equivalent to the Drury–Arveson d-shift, $S_{\lambda_{\mathfrak{c}_d}}$ is unitarily equivalent to the Szegö d-shift, while $S_{\lambda_{\mathfrak{c}_{d+1}}}$ is unitarily equivalent to the Bergman d-shift (refer to [15] for elementary properties of classical Drury–Arveson-type multishifts). The analytic model for $S_{\lambda_{\mathfrak{c}_a}}$ can be described as follows.

COROLLARY 2.12. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ of finite joint branching index and let $S_{\lambda_{\mathfrak{c}_a}}$ be the Drury–Arveson-type multishift on $\mathscr{T}$. Let E denote the joint kernel of $S^*_{\lambda_{\mathfrak{c}_a}}$ and let $\mathcal{L}_{u,F}$ be as appearing in* (1.13). *Then $S_{\lambda_{\mathfrak{c}_a}}$ is unitarily equivalent to the multiplication d-tuple $\mathscr{M}_z = (\mathscr{M}_{z_1}, \ldots, \mathscr{M}_{z_d})$ on a reproducing kernel Hilbert space $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ of E-valued holomorphic functions defined on the open unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$. Further, the reproducing kernel $\kappa_{\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})} : \mathbb{B}^d \times \mathbb{B}^d \to B(E)$ associated with $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ is given by*

$$\kappa_{\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})}(z, w) = \sum_{F \in \mathscr{P}} \sum_{u \in \Omega_F} \Big(\sum_{\alpha \in \mathbb{N}^d} \frac{\mathsf{d}_u!}{(\mathsf{d}_u + \alpha)!} \prod_{j=0}^{|\alpha|-1} (|\mathsf{d}_u| + a + j)\, z^{\alpha} \overline{w}^{\alpha} \Big) P_{\mathcal{L}_{u,F}},$$

where $z, w \in \mathbb{B}^d$ and $P_{\mathcal{L}_{u,F}}$ is the orthogonal projection on $\mathcal{L}_{u,F}$.

We discuss here one more instance in which Theorem 2.9 is applicable. Let $S_{\lambda_{\mathfrak{c}}} \in \mathcal{S}_{\mathscr{T}}$. By Lemma 2.6, the spherical Cauchy dual $S^{\mathfrak{s}}_{\lambda_{\mathfrak{c}}}$ of $S_{\lambda_{\mathfrak{c}}}$ belongs to $\mathcal{S}_{\mathscr{T}}$. By Theorem 2.9, $S^{\mathfrak{s}}_{\lambda_{\mathfrak{c}}}$ admits an analytic model, say, $(\mathscr{M}_z, \mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}}(\mathscr{T}))$. Sometimes, we refer to the Hilbert module $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}}(\mathscr{T})$ as the *Cauchy dual of the Hilbert module* $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$. It follows that the spherical Cauchy dual d-tuple $S^{\mathfrak{s}}_{\lambda_{\mathfrak{c}_a}}$ of the Drury–Arveson-type multishift $S_{\lambda_{\mathfrak{c}_a}}$ on $\mathscr{T}$ is unitarily equivalent to the multiplication d-tuple $(\mathscr{M}_{z_1}, \ldots, \mathscr{M}_{z_d})$ on the reproducing kernel Hilbert space $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T})$ of E-valued holomorphic functions defined on the open unit ball $\mathbb{B}^d$ in $\mathbb{C}^d$. Further, the reproducing kernel $\kappa_{\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T})} : \mathbb{B}^d \times \mathbb{B}^d \to B(E)$ associated with $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T})$ is given by

$$\kappa_{\mathscr{H}_{\mathfrak{c}_a}^{\mathfrak{s}}(\mathscr{T})}(z,w) = \sum_{F\in\mathscr{P}}\sum_{u\in\Omega_F}\Big(\sum_{\alpha\in\mathbb{N}^d}\frac{\mathsf{d}_u!}{(\mathsf{d}_u+\alpha)!}\prod_{j=0}^{|\alpha|-1}\frac{(|\mathsf{d}_u|+d+j)^2}{(|\mathsf{d}_u|+a+j)}z^\alpha\overline{w}^\alpha\Big)P_{\mathcal{L}_{u,F}}$$

for $z, w \in \mathbb{B}^d$. The last formula is immediate from

$$\|(S^{\mathfrak{s}}_{\lambda_{\mathfrak{c}_a}})^\alpha f\|^{-2} = \frac{\mathsf{d}_u!}{(\mathsf{d}_u+\alpha)!}\prod_{j=0}^{|\alpha|-1}\frac{(|\mathsf{d}_u|+d+j)^2}{(|\mathsf{d}_u|+a+j)}\|f\|^2, \quad f\in\mathcal{L}_{u,F},\ u\in V,\ F\in\mathscr{P}, \tag{2.25}$$

which, in turn, can be derived from (2.24) and the fact that

$$S^{\mathfrak{s}}_{\lambda_{\mathfrak{c}_a}} \text{ is of the form } S_{\lambda_{\mathfrak{c}}} \text{ with } \mathfrak{c}(t) = \frac{t+a}{t+d}, t\in\mathbb{N}. \tag{2.26}$$

REMARK 2.13. The joint kernel of $S^*_{\lambda_{\mathfrak{c}_a}}$ is the same as that of $S^{\mathfrak{s}*}_{\lambda_{\mathfrak{c}_a}}$. Thus, in the model spaces $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ and $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T})$ of $S_{\lambda_{\mathfrak{c}_a}}$ and $S^{\mathfrak{s}}_{\lambda_{\mathfrak{c}_a}}$ respectively, the subspaces of constant functions are the same. Indeed, they are equal to the joint kernel of $S^*_{\lambda_{\mathfrak{c}_a}}$ (see Remark 2.2).

2.3. OPERATOR-VALUED REPRESENTING MEASURES. In this subsection, we formally introduce the notion of an operator-valued representing measure for the Hilbert module $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$. This is reminiscent of the well-studied notion of the Berger measure appearing in the study of subnormal operators in one and several variables (refer to [12], [13] and [18]). The main result here provides a necessary and sufficient condition to ensure its existence and uniqueness. We conclude this section by computing explicitly the representing measures for Drury–Arveson-type Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$, $a \geqslant d$ and their Cauchy dual Hilbert modules $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T})$, $a < d$.

DEFINITION 2.14. Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ of finite joint branching index. Let $S_{\lambda_{\mathfrak{c}}}$ be a multishift belonging to $\mathcal{S}_{\mathscr{T}}$ and let $(\mathscr{M}_z, \mathscr{H}_{\mathfrak{c}}(\mathscr{T}))$ be the analytic model of the multishift $S_{\lambda_{\mathfrak{c}}}$ on $\mathscr{T}$. Let E denote the joint kernel of $S^*_{\lambda_{\mathfrak{c}}}$ as described in (1.13). We say that $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$ *admits a representing measure* if there exists a $B(E)$-valued product measure $\rho_{\mathscr{T}} \times \nu_{\mathscr{T}} = (\rho_u \times \nu_u)_{u\in\Omega_F, F\in\mathscr{P}}$ supported on $[0,b]\times\partial\mathbb{B}^d$, $b > 0$ such that the following hold:

(i) (Integral representation) For any $f \in E$ and $\alpha \in \mathbb{N}^d$,

$$\|z^\alpha f\|^2_{\mathscr{H}_{\mathfrak{c}}(\mathscr{T})} = \int_{[0,b]}\int_{\partial\mathbb{B}^d} s^{|\alpha|}|z^\alpha|^2\langle \mathrm{d}\rho_{\mathscr{T}}(s)\times\mathrm{d}\nu_{\mathscr{T}}(z)f, f\rangle;$$

(ii) (Diagonal measure) ρ_u and ν_u are scalar-valued measures such that for any $g_{u,F} \in \mathcal{L}_{u,F}$,

$$\mathrm{d}\rho_{\mathscr{T}}(s)\times\mathrm{d}\nu_{\mathscr{T}}(z)\Big(\sum_{F\in\mathscr{P}}\sum_{u\in\Omega_F} g_{u,F}\Big) = \sum_{F\in\mathscr{P}}\sum_{u\in\Omega_F}\mathrm{d}\rho_u(s)\mathrm{d}\nu_u(z)g_{u,F}; \tag{2.27}$$

(iii) (Normalization) ρ_{root} and ν_{root} are probability measures.

The existence of a representing measure is connected to the Hausdorff moment problem (refer to Chapter 4 of [25] for the definition and basic theory of Hausdorff moment sequences).

THEOREM 2.15. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ of finite joint branching index. Let $S_{\boldsymbol\lambda_{\mathfrak{c}}}$ be a multishift belonging to $\mathcal{S}_{\mathscr{T}}$ and let $(\mathscr{M}_z, \mathscr{H}_{\mathfrak{c}}(\mathscr{T}))$ be the analytic model of the multishift $S_{\boldsymbol\lambda_{\mathfrak{c}}}$ on $\mathscr{T}$. Then the following statements are equivalent:*

(i) *$\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$ admits a representing measure $\rho_{\mathscr{T}} \times \nu_{\mathscr{T}} = (\rho_u \times \nu_u)_{u \in \Omega_F, F \in \mathscr{P}}$ supported on $[0, \sup \mathfrak{c}] \times \partial\mathbb{B}^d$;*

(ii) *the sequence $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ given below is a Hausdorff moment sequence:*

$$\mathfrak{a}_n = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{j=0}^{n-1} \mathfrak{c}(j) & \text{if } n \geqslant 1. \end{cases} \tag{2.28}$$

If any of the above statements holds, then for $u \in \Omega_F$ and $F \in \mathscr{P}$, the positive scalar-valued measures ρ_u and ν_u are given by

$$d\rho_u(s) = \frac{s^{|\mathsf{d}_u|}}{\mathfrak{a}_{|\mathsf{d}_u|}} d\rho_{\mathsf{root}}(s), \tag{2.29}$$

$$d\nu_u(z) = \frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d, \sigma)}} d\sigma(z), \tag{2.30}$$

where ρ_{root} is the representing measure of $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ supported on $[0, \sup \mathfrak{c}]$ and σ is the normalized surface area measure on $\partial\mathbb{B}^d$.

REMARK 2.16. The measure ρ_{root}, as appearing in (2.29), turns out to be the Berger measure of the weighted shift S_θ on $\mathscr{T}^{\otimes}_{\mathsf{root}}$ associated with $S_{\boldsymbol\lambda_{\mathfrak{c}}}$, where $\mathscr{T}^{\otimes}_{\mathsf{root}}$ is the connected component of the tensor product $\mathscr{T}^{\otimes}$ of $\mathscr{T}_1, \ldots, \mathscr{T}_d$ that contains root (see Definitions 2.2.1 and 5.2.13 of [9] for definitions of $\mathscr{T}^{\otimes}$ and S_θ). This may be deduced from (5.26) of [9]. The representing measure of $\mathscr{H}_{\mathfrak{c}}(\mathscr{T})$ can be considered as complex and operator-valued version of the Berger measure arising in the subnormality of commuting tuples (refer to [13], [18]).

Proof. It follows from Lemma 1.2, (2.20) and (2.24) that for $f \in \mathcal{L}_{u,F}$ and $\alpha \in \mathbb{N}^d$,

$$\begin{aligned} \|S^{\alpha}_{\boldsymbol\lambda_{\mathfrak{c}}} f\|^2_{l^2(V)} &= \frac{(\alpha + \mathsf{d}_u)!}{\mathsf{d}_u!} \prod_{j=0}^{|\alpha|-1} \frac{\mathfrak{c}(|\mathsf{d}_u| + j)}{|\mathsf{d}_u| + d + j} \|f\|^2_{l^2(V)} \\ &\overset{(2.28)}{=} \frac{\mathfrak{a}_{|\alpha|+|\mathsf{d}_u|}}{\mathfrak{a}_{|\mathsf{d}_u|}} Q(\mathsf{d}_u, \alpha)\, \|f\|^2_{l^2(V)}, \end{aligned} \tag{2.31}$$

where

$$Q(\beta,\alpha) := \frac{(\alpha+\beta)!}{\beta!}\prod_{j=0}^{|\alpha|-1}\frac{1}{|\beta|+d+j}, \quad \alpha,\beta\in\mathbb{N}^d.$$

By Lemma 1.11 of [27], for $\alpha,\beta\in\mathbb{N}^d$, we obtain

$$\begin{aligned}\int_{\partial\mathbb{B}^d}\frac{|z^{\alpha+\beta}|^2}{\|z^\beta\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}\sigma(z) &= \frac{\|z^{\alpha+\beta}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}{\|z^\beta\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\\ &= \frac{(\alpha+\beta)!(d-1)!}{(|\beta|+|\alpha|+d-1)!}\frac{(|\beta|+d-1)!}{\beta!(d-1)!} = Q(\beta,\alpha). \qquad (2.32)\end{aligned}$$

It is now immediate from (2.31) that

$$\|S^\alpha_{\lambda_\mathfrak{c}}f\|^2_{l^2(V)} = \frac{\mathfrak{a}_{|\alpha|+|\mathsf{d}_u|}}{\mathfrak{a}_{|\mathsf{d}_u|}}\Big(\int_{\partial\mathbb{B}^d}\frac{|z^{\alpha+\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}\sigma(z)\Big)\|f\|^2_{l^2(V)}, \quad f\in\mathcal{L}_{u,F}. \qquad (2.33)$$

(ii) $\Rightarrow$ (i) Assume that (ii) holds. Thus there exists a probability measure $\mu_\mathfrak{c}$ supported on a finite interval $[0,b]$ such that

$$\mathfrak{a}_n = \int_{[0,b]} s^n\mathrm{d}\mu_\mathfrak{c}(s), \quad n\in\mathbb{N}. \qquad (2.34)$$

By Lemma 2 of [7] and (2.28), $b=\sup\mathfrak{c}$. It is easy to see from (2.33) and (2.34) that for $f\in\mathcal{L}_{u,F}$,

$$\|z^\alpha f\|^2_{\mathscr{H}_\mathfrak{c}(\mathscr{T})} = \|S^\alpha_{\lambda_\mathfrak{c}}f\|^2_{l^2(V)} = \Big(\int_{[0,b]}\int_{\partial\mathbb{B}^d}s^{|\alpha|}|z^\alpha|^2\mathrm{d}\rho_u(s)\mathrm{d}\nu_u(z)\Big)\|f\|^2_{l^2(V)}, \qquad (2.35)$$

where $\mathrm{d}\rho_u$ and $\mathrm{d}\nu_u$ are as defined in (2.29) (with ρ_{root} replaced by $\mu_\mathfrak{c}$) and (2.30) respectively. To see the integral representation of $\|z^\alpha f\|^2_{\mathscr{H}_\mathfrak{c}(\mathscr{T})}$ for arbitrary $f\in E$, note that by (1.13), any $f\in E$ is of the form

$$f=\sum_{F\in\mathscr{P}}\sum_{u\in\Omega_F}g_{u,F}, \quad g_{u,F}\in\mathcal{L}_{u,F}.$$

By (2.27),

$$\begin{aligned}&\int_{[0,b]}\int_{\partial\mathbb{B}^d}s^{|\alpha|}|z^\alpha|^2\langle\mathrm{d}\rho_\mathscr{T}\times\mathrm{d}\nu_\mathscr{T}f,f\rangle\\ &\quad=\sum_{F\in\mathscr{P}}\sum_{u\in\Omega_F}\Big(\int_{[0,b]}\int_{\partial\mathbb{B}^d}s^{|\alpha|}|z^\alpha|^2\mathrm{d}\rho_u(s)\mathrm{d}\nu_u(z)\Big)\|g_{u,F}\|^2\\ &\quad\overset{(2.35)}{=}\sum_{F\in\mathscr{P}}\sum_{u\in\Omega_F}\|z^\alpha g_{u,F}\|^2_{\mathscr{H}_\mathfrak{c}(\mathscr{T})} = \|z^\alpha f\|^2_{\mathscr{H}_\mathfrak{c}(\mathscr{T})},\end{aligned}$$

where we used the orthogonality of $\{z^\alpha g_{u,F} : F\in\mathscr{P}, u\in\Omega_F\}$ in the last equality (see Lemma 1.2). This completes the proof of (ii) $\Rightarrow$ (i).

(i) ⇒ (ii) Assume that (i) holds. It may be concluded from (5.24) and (5.26) of [9] that

$$\sum_{\alpha\in\mathbb{N}^d,\ |\alpha|=n}\frac{n!}{\alpha!}\|S^{\alpha}_{\boldsymbol\lambda_{\mathfrak c}}e_v\|^2_{l^2(V)}=\prod_{p=0}^{n-1}\mathfrak{c}(|\mathsf{d}_v|+p),\quad n\in\mathbb{N},\ v\in V. \tag{2.36}$$

Letting $f=e_{\mathsf{root}}$ in the integral representation (see Definition 2.14(i)), we obtain for any $n\in\mathbb{N}$,

$$\begin{aligned}\mathfrak{a}_n=\prod_{p=0}^{n-1}\mathfrak{c}(p)&\overset{(2.36)}{=}\sum_{\alpha\in\mathbb{N}^d,\,|\alpha|=n}\frac{n!}{\alpha!}\|z^{\alpha}e_{\mathsf{root}}\|^2_{\mathscr{H}_{\mathfrak c}(\mathscr{T})}\\&=\Big(\int_{[0,b]}s^n\mathrm{d}\rho_{\mathsf{root}}(s)\Big)\int_{\partial\mathbb{B}^d}\sum_{\alpha\in\mathbb{N}^d,\,|\alpha|=n}\frac{n!}{\alpha!}|z^{\alpha}|^2\mathrm{d}\nu_{\mathsf{root}}(z)=\int_{[0,b]}s^n\mathrm{d}\rho_{\mathsf{root}}(s),\end{aligned}$$

where we used the assumption that ν_{root} is a probability measure along with the multinomial theorem in the last equality. This completes the verification of (i) ⇒ (ii).

To see the uniqueness part, note that by (2.28) and (2.36), the sequence $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ is uniquely determined by the action of $S_{\boldsymbol\lambda_{\mathfrak c}}$ on e_{root}. By the determinacy of the Hausdorff moment problem ([25], Theorem 4.17.1), the probability measure ρ_{root} is unique. It now follows from (2.29) and (2.30) that the representing measure $\rho_{\mathscr{T}}\times\nu_{\mathscr{T}}$ of $\mathscr{H}_{\mathfrak c}(\mathscr{T})$ is unique. ∎

Let us see two particular instances in which representing measures can be determined explicitly.

COROLLARY 2.17. *Let $\mathscr{T}=(V,\mathcal{E})$ denote the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1,\ldots,\mathscr{T}_d$ of finite joint branching index and let $(\mathscr{M}_z,\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}))$ be the analytic model of the Drury–Arveson-type multishift $S_{\boldsymbol\lambda_{\mathfrak{c}_a}}$ on $\mathscr{T}$. If a is a positive integer such that $a\geqslant d$, then $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ admits the representing measure $\rho_{\mathscr{T}}\times\nu_{\mathscr{T}}=(\rho_u\times\nu_u)_{u\in\Omega_F,F\in\mathscr{P}}$. In this case, for $u\in\Omega_F$ and $F\in\mathscr{P}$, the positive scalar-valued measures ρ_u and ν_u are given by*

$$\mathrm{d}\rho_u(s)=\begin{cases}w_{|\mathsf{d}_u|}(s)\mathrm{d}m(s) & \text{if } a>d,\\ \mathrm{d}\delta_1(s) & \text{if } a=d,\end{cases}\qquad \mathrm{d}\nu_u(z)=\frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}\sigma(z), \tag{2.37}$$

where m is the Lebesgue measure on $[0,1]$, δ_1 is the Borel probability measure supported at $\{1\}$, σ is the normalized surface area measure on $\partial\mathbb{B}^d$, and

$$w_l(s)=(l+d)\cdots(l+a-1)\sum_{i=d}^{a-1}\frac{s^{i+l-1}}{\prod_{d\leqslant j\neq i\leqslant a-1}(j-i)},\quad s\in[0,1],\ l\in\mathbb{N}. \tag{2.38}$$

Proof. Suppose that a is a positive integer such that $a \geqslant d$. By (1.15), $\mathfrak{c}_a(t) = (t+d)/(t+a)$, $t \in \mathbb{N}$, and hence by (2.28),

$$\mathfrak{a}_n = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{j=0}^{n-1} \frac{j+d}{j+a} & \text{if } n \geqslant 1. \end{cases} \tag{2.39}$$

Note that in case $a = d$, $\mathfrak{a}$ is the constant sequence with value 1. Clearly, it is a Hausdorff moment sequence with representing measure δ_1. Suppose that $a > d$. Then

$$\mathfrak{a}_n = \prod_{j=d}^{a-1} \frac{j}{n+j}, \quad n \in \mathbb{N}.$$

It follows from Corollary 3.8 of [2] that $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ is a Hausdorff moment sequence with representing measure ρ_{root} being the weighted Lebesgue measure with the weight function $w_0 : [0,1] \to [0,\infty)$ as given by (2.38). Thus

$$d\rho_{\mathsf{root}}(s) = \begin{cases} d\delta_1 & \text{if } a = d, \\ w_0(s)dm(s) & \text{if } a > d. \end{cases}$$

It follows that

$$d\rho_u(s) \overset{(2.29)}{=} \frac{s^{|\mathsf{d}_u|}}{\mathfrak{a}_{|\mathsf{d}_u|}} d\rho_{\mathsf{root}}(s) = \begin{cases} \frac{s^{|\mathsf{d}_u|}}{\mathfrak{a}_{|\mathsf{d}_u|}} d\delta_1 \overset{(2.39)}{=} d\delta_1 & \text{if } a = d, \\ \frac{s^{|\mathsf{d}_u|}}{\mathfrak{a}_{|\mathsf{d}_u|}} w_0(s)dm(s) \overset{(2.38)}{=} w_{|\mathsf{d}_u|}(s)dm(s) & \text{if } a > d. \end{cases}$$

The expression for $d\nu_u$ in (2.37) follows from (2.30). ∎

COROLLARY 2.18. *Let $\mathscr{T} = (V, \mathcal{E})$ denote the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ of finite joint branching index and let $(\mathscr{M}_z, \mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T}))$ be the analytic model of the spherical Cauchy dual $S^{\mathfrak{s}}_{\lambda_{\mathfrak{c}_a}}$ of the Drury–Arveson-type multishift $S_{\lambda_{\mathfrak{c}_a}}$ on $\mathscr{T}$. If a is a positive integer such that $a < d$, then $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T})$ admits the representing measure $\rho_{\mathscr{T}} \times \nu_{\mathscr{T}} = (\rho_u \times \nu_u)_{u\in\Omega_F, F\in\mathscr{P}}$. In this case, for $u \in \Omega_F$ and $F \in \mathscr{P}$, the positive scalar-valued measures ρ_u and ν_u are given by*

$$d\rho_u(s) = \omega_{|\mathsf{d}_u|}(s)dm(s), \quad d\nu_u(z) = \frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}} d\sigma(z), \tag{2.40}$$

where m is the Lebesgue measure on $[0,1]$, σ is the normalized surface area measure on $\partial\mathbb{B}^d$, and

$$\omega_l(s) = (l+a)\cdots(l+d-1)\sum_{i=a}^{d-1} \frac{s^{i+l-1}}{\prod_{a\leqslant j\neq i\leqslant d-1}(j-i)}, \quad s \in [0,1],\ l \in \mathbb{N}. \tag{2.41}$$

Proof. Suppose that a is a positive integer such that $a < d$. By (2.26) and (2.28),

$$\mathfrak{a}_n = \begin{cases} 1 & \text{if } n = 0, \\ \prod\limits_{j=0}^{n-1} \frac{j+a}{j+d} = \prod\limits_{j=a}^{d-1} \frac{j}{n+j} & \text{if } n \geqslant 1. \end{cases}$$

It follows from Corollary 3.8 of [2] that $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ is a Hausdorff moment sequence with representing measure being the weighted Lebesgue measure with weight function $\omega_0 : [0,1] \to [0,\infty)$ as given by (2.41). The desired expressions in (2.40) can now be derived as in the proof of Corollary 2.17. ∎

3. PROOF OF THE MAIN RESULT

In this section, we present a proof of Theorem 1.4. We begin with a simple observation which reduces Problem 1.1 to the problem of unitary equivalence of representing measures arising from the Drury–Arveson-type Hilbert modules (as ensured by Corollaries 2.17 and 2.18).

LEMMA 3.1. *Fix $j = 1,2$. Let $\mathscr{T}^{(j)} = (V^{(j)}, \mathcal{E}^{(j)})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1^{(j)}, \ldots, \mathscr{T}_d^{(j)}$ of finite joint branching index. Let $S_{\boldsymbol\lambda_\mathfrak{c}}^{(j)}$ be a multishift belonging to $\mathcal{S}_{\mathscr{T}^{(j)}}$ and let $(\mathscr{M}_z^{(j)}, \mathscr{H}_\mathfrak{c}(\mathscr{T}^{(j)}))$ be the analytic model of the multishift $S_{\boldsymbol\lambda_\mathfrak{c}}^{(j)}$ on $\mathscr{T}^{(j)}$. Let $E^{(j)}$ be the subspace of constant functions in $\mathscr{H}_\mathfrak{c}(\mathscr{T}^{(j)})$. If $\mathscr{H}_\mathfrak{c}(\mathscr{T}^{(j)})$ admits the representing measure $\rho_{\mathscr{T}^{(j)}} \times \nu_{\mathscr{T}^{(j)}} = (\rho_u^{(j)} \times \nu_u^{(j)})_{u\in\Omega_F, F\in\mathscr{P}}$ supported on $[0,1] \times \partial\mathbb{B}^d$, then the Hilbert modules $\mathscr{H}_\mathfrak{c}(\mathscr{T}^{(1)})$ and $\mathscr{H}_\mathfrak{c}(\mathscr{T}^{(2)})$ are isomorphic if and only if there exists a unitary transformation $U : E^{(1)} \to E^{(2)}$ such that for every Borel subset A of $[0,1] \times \partial\mathbb{B}^d$,*

$$U\rho_{\mathscr{T}^{(1)}} \times \nu_{\mathscr{T}^{(1)}}(A)U^* = \rho_{\mathscr{T}^{(2)}} \times \nu_{\mathscr{T}^{(2)}}(A). \tag{3.1}$$

Proof. Suppose that $\mathscr{H}_\mathfrak{c}(\mathscr{T}^{(j)})$ admits the representing measure $\rho_{\mathscr{T}^{(j)}} \times \nu_{\mathscr{T}^{(j)}}$ supported on $[0,1] \times \partial\mathbb{B}^d$, $j = 1,2$. Let $U : \mathscr{H}_\mathfrak{c}(\mathscr{T}^{(1)}) \to \mathscr{H}_\mathfrak{c}(\mathscr{T}^{(2)})$ be a unitary map such that

$$U\mathscr{M}_{z_j}^{(1)}U^* = \mathscr{M}_{z_j}^{(2)}, \quad j = 1, \ldots, d. \tag{3.2}$$

It follows that

$$U \text{ maps } E^{(1)} = \bigcap_{j=1}^{d} \ker \mathscr{M}_{z_j}^{(1)^*} \text{ unitarily onto } E^{(2)} = \bigcap_{j=1}^{d} \ker \mathscr{M}_{z_j}^{(2)^*}. \tag{3.3}$$

Thus, if $f \in E^{(1)}$, then $Uf \in E^{(2)}$, and hence for every $\alpha \in \mathbb{N}^d$,

$$\begin{aligned}
&\int_0^1 \int_{\partial\mathbb{B}^d} s^{|\alpha|}|z^\alpha|^2 \langle \mathrm{d}\rho_{\mathscr{T}^{(1)}}(s) \times \mathrm{d}\nu_{\mathscr{T}^{(1)}}(z) f, f\rangle \\
&\qquad = \|z^\alpha f\|^2_{\mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(1)})} = \|U(\mathscr{M}^{(1)}_{z,a})^\alpha f\|^2_{\mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(2)})} \\
&\qquad = \|(\mathscr{M}^{(2)}_{z,a})^\alpha U f\|^2_{\mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(2)})} = \|z^\alpha U f\|^2_{\mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(2)})} \\
&\qquad = \int_0^1 \int_{\partial\mathbb{B}^d} s^{|\alpha|}|z^\alpha|^2 \langle \mathrm{d}\rho_{\mathscr{T}^{(2)}}(s) \times \mathrm{d}\nu_{\mathscr{T}^{(2)}}(z) Uf, Uf\rangle \\
&\qquad = \int_0^1 \int_{\partial\mathbb{B}^d} s^{|\alpha|}|z^\alpha|^2 \langle U^*\mathrm{d}\rho_{\mathscr{T}^{(2)}}(s) \times \mathrm{d}\nu_{\mathscr{T}^{(2)}}(z) Uf, f\rangle.
\end{aligned}$$

By uniqueness of the representing measure (see Theorem 2.15), we obtain the necessary part.

To see the converse, assume that (3.1) holds for a unitary transformation $U : E^{(1)} \to E^{(2)}$. Define $\widetilde{U} : \mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(1)}) \to \mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(2)})$ by

$$(\widetilde{U} f)(z) = U(f(z)), \quad f \in \mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(1)}),\ z \in \mathbb{B}^d. \tag{3.4}$$

It is easy to see using (3.1) that $\widetilde{U}$ is a unitary map. Also, it is a routine matter to verify that

$$\widetilde{U}\mathscr{M}^{(1)}_{z_j} = \mathscr{M}^{(2)}_{z_j}\widetilde{U}, \quad j = 1, \ldots, d.$$

This completes the proof. ∎

The following rather technical result says that any Hilbert module isomorphism between two Drury–Arveson-type Hilbert modules preserves the orthogonal decomposition (1.13) of joint cokernels of associated multiplication tuples over each generation.

PROPOSITION 3.2. *Let a, d be positive integers such that $ad \neq 1$, and fix $j = 1, 2$. Let $\mathscr{T}^{(j)} = (V^{(j)}, \mathcal{E}^{(j)})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}^{(j)}_1, \ldots, \mathscr{T}^{(j)}_d$ of finite joint branching index. Let $\mathsf{root}^{(j)}$ denote the root of $\mathscr{T}^{(j)}$ and let $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(j)})$ be the Drury–Arveson-type Hilbert module associated with $\mathscr{T}^{(j)}$. Let $E^{(j)}$ be the subspace of constant functions in $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(j)})$ and let $\mathcal{L}^{(j)}_{u,F}$ be as appearing in the decomposition* (1.13) *of $E^{(j)}$. Suppose there exists a Hilbert module isomorphism $U : \mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(1)}) \to \mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(2)})$. Then, for any $\alpha \in \mathbb{N}^d$ and $F \in \mathscr{P}$,*

$$U \text{ maps } \bigoplus_{u \in \Omega^{(1)}_F,\ \mathsf{d}_u = \alpha} \mathcal{L}^{(1)}_{u,F} \text{ onto } \bigoplus_{v \in \Omega^{(2)}_F,\ \mathsf{d}_v = \alpha} \mathcal{L}^{(2)}_{v,F},$$

where we used the convention that orthogonal direct sum over empty collection is $\{0\}$. In particular, U maps $e_{\mathsf{root}^{(1)}}$ to a unimodular scalar multiple of $e_{\mathsf{root}^{(2)}}$.

Proof. Note that two joint left-invertible tuples are unitarily equivalent if and only if their spherical Cauchy dual d-tuples are unitarily equivalent. It follows that the Hilbert modules $\mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(1)})$ and $\mathscr{H}_{\mathfrak{c}}(\mathscr{T}^{(2)})$ are isomorphic if and only if their Cauchy dual Hilbert modules $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}}(\mathscr{T}^{(1)})$ and $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}}(\mathscr{T}^{(2)})$ are isomorphic. Since $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ $(a \geqslant d)$ and $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T})$ $(a < d)$ admit representing measures (Corollaries 2.17 and 2.18), in view of Remark 2.2, it suffices to treat the case in which $a \geqslant d$.

Suppose that $a \geqslant d$. Then, by Corollary 2.17, $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(j)})$ admits the representing measure $\rho_{_{\mathscr{T}^{(j)}}} \times \nu_{_{\mathscr{T}^{(j)}}} = (\rho^{(j)}_u \times \nu^{(j)}_u)_{u \in \Omega^{(j)}_F, F \in \mathscr{P}}$ as given by (2.37). Let $f \in \mathcal{L}^{(1)}_{u,F}$ for some $u \in \Omega^{(1)}_F$ and $F \in \mathscr{P}$. Then, by (3.3), U maps $E^{(1)}$ into $E^{(2)}$, and hence

$$Uf = \sum_{F \in \mathscr{P}} \sum_{v \in \Omega^{(2)}_F} g_{v,F}, \quad g_{v,F} \in \mathcal{L}^{(2)}_{v,F}. \tag{3.5}$$

Further, by Lemma 3.1, for any Borel subset $\Delta_1 \times \Delta_2$ of $[0,1] \times \partial\mathbb{B}^d$,

$$\int_{\Delta_1}\int_{\Delta_2} \mathrm{d}\rho_{_{\mathscr{T}^{(2)}}}(s) \times \mathrm{d}\nu_{_{\mathscr{T}^{(2)}}}(z) Uf = \int_{\Delta_1}\int_{\Delta_2} U \mathrm{d}\rho_{_{\mathscr{T}^{(1)}}}(s) \times \mathrm{d}\nu_{_{\mathscr{T}^{(1)}}}(z) f. \tag{3.6}$$

We verify that for almost every $z \in \partial\mathbb{B}^d$,

$$\frac{|z^{\mathsf{d}_v}|^2}{\|z^{\mathsf{d}_v}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}} = \frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}} \quad \text{if } v \in \Omega^{(2)}_F,\ g_{v,F} \neq 0. \tag{3.7}$$

We divide the verification into two cases.

Case 1. When $a = d$.

By the definition of the representing measure (see (2.27)) and Corollary 2.17,

$$\int_{\Delta_1}\int_{\Delta_2} U \mathrm{d}\rho_{_{\mathscr{T}^{(1)}}}(s) \times \mathrm{d}\nu_{_{\mathscr{T}^{(1)}}}(z) f = \int_{\Delta_1}\int_{\Delta_2} \frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}} \mathrm{d}\delta_1(s)\mathrm{d}\sigma(z)\, Uf. \tag{3.8}$$

Arguing similarly and using (3.5), we obtain

$$\begin{aligned}\int_{\Delta_1}\int_{\Delta_2} & \mathrm{d}\rho_{_{\mathscr{T}^{(2)}}}(s) \times \mathrm{d}\nu_{_{\mathscr{T}^{(2)}}}(z) Uf \\ &= \int_{\Delta_1}\int_{\Delta_2} \Big(\sum_{F \in \mathscr{P}} \sum_{v \in \Omega^{(2)}_F} \frac{|z^{\mathsf{d}_v}|^2}{\|z^{\mathsf{d}_v}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}} g_{v,F} \Big) \mathrm{d}\delta_1(s)\mathrm{d}\sigma(z).\end{aligned} \tag{3.9}$$

Hence, by (3.9), (3.6) and (3.8), we obtain

$$\int_{\Delta_1}\int_{\Delta_2}\Big(\sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)}}\frac{|z^{\mathsf{d}_v}|^2}{\|z^{\mathsf{d}_v}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}g_{v,F}\Big)\mathrm{d}\delta_1(s)\mathrm{d}\sigma(z)$$
$$=\int_{\Delta_1}\int_{\Delta_2}\frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}\delta_1(s)\mathrm{d}\sigma(z)\ Uf$$
$$\overset{(3.5)}{=}\int_{\Delta_1}\int_{\Delta_2}\frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}\delta_1(s)\mathrm{d}\sigma(z)\ \Big(\sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)}}g_{v,F}\Big). \tag{3.10}$$

Letting $\Delta_1=[0,1]$, we get

$$\int_{\Delta_2}\Big(\sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)}}\frac{|z^{\mathsf{d}_v}|^2}{\|z^{\mathsf{d}_v}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}g_{v,F}\Big)\mathrm{d}\sigma(z)$$
$$=\int_{\Delta_2}\frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}\sigma(z)\ \Big(\sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)}}g_{v,F}\Big)$$

for every Borel subset Δ_2 of $\partial\mathbb{B}^d$. Comparing the coefficients of nonzero $g_{v,F}$ on both sides, we obtain (3.7).

Case 2. When $a>d$.

By the definition of the representing measure (see (2.27)) and Corollary 2.17,

$$\int_{\Delta_1}\int_{\Delta_2}U\mathrm{d}\rho_{\mathscr{T}^{(1)}}(s)\times\mathrm{d}\nu_{\mathscr{T}^{(1)}}(z)f$$
$$=\int_{\Delta_1}\int_{\Delta_2}w_{|\mathsf{d}_u|}(s)\frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}m(s)\mathrm{d}\sigma(z)\ Uf, \tag{3.11}$$

(see (2.38) for the definition of the weight function $w_l(\cdot)$). Also, by (3.5), we obtain

$$\int_{\Delta_1}\int_{\Delta_2}\mathrm{d}\rho_{\mathscr{T}^{(2)}}(s)\times\mathrm{d}\nu_{\mathscr{T}^{(2)}}(z)Uf$$
$$=\int_{\Delta_1}\int_{\Delta_2}\Big(\sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)}}w_{|\mathsf{d}_v|}(s)\frac{|z^{\mathsf{d}_v}|^2}{\|z^{\mathsf{d}_v}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}g_{v,F}\Big)\mathrm{d}m(s)\mathrm{d}\sigma(z). \tag{3.12}$$

Hence, by (3.12), (3.6) and (3.11), we obtain

$$\int_{\Delta_1}\int_{\Delta_2}\Big(\sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)}} w_{|\mathsf{d}_v|}(s)\frac{|z^{\mathsf{d}_v}|^2}{\|z^{\mathsf{d}_v}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}g_{v,F}\Big)\mathrm{d}m(s)\mathrm{d}\sigma(z)$$

$$= \int_{\Delta_1}\int_{\Delta_2} w_{|\mathsf{d}_u|}(s)\frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}m(s)\mathrm{d}\sigma(z)\; Uf$$

$$\overset{(3.5)}{=} \int_{\Delta_1}\int_{\Delta_2} w_{|\mathsf{d}_u|}(s)\frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}m(s)\mathrm{d}\sigma(z)\;\Big(\sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)}} g_{v,F}\Big). \tag{3.13}$$

Letting $\Delta_2 = \partial\mathbb{B}^d$, we get

$$\int_{\Delta_1}\Big(\sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)}} w_{|\mathsf{d}_v|}(s)g_{v,F}\Big)\mathrm{d}m(s) = \int_{\Delta_1} w_{|\mathsf{d}_u|}(s)\mathrm{d}m(s)\;\Big(\sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)}} g_{v,F}\Big)$$

for every Borel subset Δ_1 of $[0,1]$. Comparing the coefficients of nonzero $g_{v,F}$ on both sides, we obtain for almost every $s\in[0,1]$,

$$w_{|\mathsf{d}_v|}(s) = w_{|\mathsf{d}_u|}(s) \quad \text{if } v\in\Omega_F^{(2)},\ g_{v,F}\neq 0.$$

By (2.38), $w_k \neq w_l$ as integrable functions for non-negative integers $k\neq l$. Thus

$$g_{v,F}\neq 0 \text{ implies that } |\mathsf{d}_v| = |\mathsf{d}_u| \text{ for all } v\in\Omega_F^{(2)}. \tag{3.14}$$

Thus (3.13) becomes

$$\sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)},\,|\mathsf{d}_v|=|\mathsf{d}_u|}\Big(\int_{\Delta_1}\int_{\Delta_2} w_{|\mathsf{d}_u|}(s)\frac{|z^{\mathsf{d}_v}|^2}{\|z^{\mathsf{d}_v}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}m(s)\mathrm{d}\sigma(z)\Big)g_{v,F}$$

$$= \sum_{F\in\mathscr{P}}\sum_{v\in\Omega_F^{(2)},\,|\mathsf{d}_v|=|\mathsf{d}_u|}\Big(\int_{\Delta_1}\int_{\Delta_2} w_{|\mathsf{d}_u|}(s)\frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}m(s)\mathrm{d}\sigma(z)\Big)g_{v,F}.$$

Letting $\Delta_1 = [0,1]$ and comparing the coefficients of nonzero $g_{v,F}$, for every Borel subset Δ_2 of $\partial\mathbb{B}^d$, we get

$$\int_{\Delta_2}\frac{|z^{\mathsf{d}_v}|^2}{\|z^{\mathsf{d}_v}\|_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}\sigma(z) = \int_{\Delta_2}\frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|_{L^2(\partial\mathbb{B}^d,\sigma)}}\mathrm{d}\sigma(z) \quad \text{if } v\in\Omega_F^{(2)},\ g_{v,F}\neq 0,$$

where we used the fact that $\int_{[0,1]} w_l(s)\mathrm{d}m(s) \neq 0$, $l\in\mathbb{N}$. Thus (3.7) holds in this case as well.

We next claim that

$$\mathsf{d}_v = \mathsf{d}_u \quad \text{if } v\in\Omega_F^{(2)},\ g_{v,F}\neq 0. \tag{3.15}$$

In case $a > d$ and $d = 1$, the claim is trivial in view of (3.14). Assume that $d \geqslant 2$. In view of continuity of the monomials and the fact that (3.7) holds on a dense set, the equality in (3.7) holds for all $z \in \partial\mathbb{B}^d$. Consider $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ given by $\alpha_j = \min\{\mathsf{d}_{u_j}, \mathsf{d}_{v_j}\}$, $j = 1, \ldots, d$. Thus for every $z \in \partial\mathbb{B}^d$,

$$\prod_{j=1}^{d} |z_j^{\mathsf{d}_{v_j}-\alpha_j}|^2 = \frac{\|z^{\mathsf{d}_v}\|_{L^2(\partial\mathbb{B}^d,\sigma)}}{\|z^{\mathsf{d}_u}\|_{L^2(\partial\mathbb{B}^d,\sigma)}} \prod_{j=1}^{d} |z_j^{\mathsf{d}_{u_j}-\alpha_j}|^2, \quad v \in \Omega_F^{(2)},\ g_{v,F} \neq 0. \tag{3.16}$$

Suppose to the contrary that $\mathsf{d}_u \neq \mathsf{d}_v$ for some $v \in \Omega_F^{(2)}$. Without loss of generality, we may assume that $\mathsf{d}_{u_1} \neq \mathsf{d}_{v_1}$. Let $w = (0, w_2, \ldots, w_d) \in \partial\mathbb{B}^d$ be such that $w_j \neq 0$ for $j = 2, \ldots, d$. Then evaluating (3.16) at w, we get one side of (3.16) equal to zero, while the other side remains nonzero. This contradicts (3.16), and hence $\mathsf{d}_u = \mathsf{d}_v$. Thus the claim stands verified. It is now immediate from (3.5) and (3.15) that

$$U(\mathcal{L}_{u,F}^{(1)}) \subseteq \bigoplus_{H \in \mathscr{P}} \bigoplus_{v \in \Omega_H^{(2)},\, \mathsf{d}_v = \mathsf{d}_u} \mathcal{L}_{v,H}^{(2)}.$$

Note that by (1.6) and (1.8), for any $H \in \mathscr{P}$ and $v \in \Omega_H^{(j)}$ $(j = 1, 2)$, $k \in H$ if and only if $\mathsf{d}_{v_k} \neq 0$, and therefore

$$U(\mathcal{L}_{u,F}^{(1)}) \subseteq \bigoplus_{v \in \Omega_F^{(2)},\, \mathsf{d}_v = \mathsf{d}_u} \mathcal{L}_{v,F}^{(2)}.$$

This also yields

$$U\Big(\bigoplus_{u \in \Omega_F^{(1)},\, \mathsf{d}_u = \alpha} \mathcal{L}_{u,F}^{(1)}\Big) \subseteq \bigoplus_{v \in \Omega_F^{(2)},\, \mathsf{d}_v = \alpha} \mathcal{L}_{v,F}^{(2)}$$

for every $\alpha \in \mathbb{N}^d$ such that $\alpha_j \neq 0$ if and only if $j \in F$. Applying this fact to U^{-1}, we obtain the desired conclusion in the first part. The remaining part follows by applying the first part to $\alpha = 0$ and $F = \emptyset$ (see the proof of Lemma 1.3). ∎

REMARK 3.3. In case $a = 1 = d$, the conclusion in (3.7) always holds, while (3.15) does not follow from (3.7) unlike the case $ad \neq 1$.

In the proof of the main result, we also need a couple of facts related to the indexing set Ω_F, $F \in \mathscr{P}$. The following is immediate from the definition of $\Omega_{\{l\}}$, $l = 1, \ldots, d$:

$$\mathcal{G}_{n-1}(\mathscr{T}_l) = \bigsqcup_{u \in \Omega_{\{l\}},\, \mathsf{d}_u = n\epsilon_l} \{\mathsf{par}(u_l)\}, \quad \mathcal{G}_n(\mathscr{T}_l) = \bigsqcup_{u \in \Omega_{\{l\}},\, \mathsf{d}_u = n\epsilon_l} \mathsf{sib}(u_l), \quad n \geqslant 1. \tag{3.17}$$

In the proof of Theorem 1.4, we also need a *canonical choice* for the indexing set Ω_F. Indeed, the choices of $\Omega_{\{j\}}$, $j \in F$ yield the following natural choice for Ω_F:

$$\widetilde{\Omega}_F := \bigcap_{j \in F} \{(u_1, \ldots, u_d) \in V : u_l = \mathsf{root}_l \text{ if } l \notin F,\ u_j = \widetilde{u}_j \text{ for some } \widetilde{u} \in \Omega_{\{j\}}\}. \tag{3.18}$$

To see that the above collection satisfies requirements of an indexing set Ω_F (as ensured by the axiom of choice), note the following:

(•) If $u, v \in \widetilde{\Omega}_F$ such that $u \neq v$, then $\mathsf{sib}_F(u) \cap \mathsf{sib}_F(v) = \emptyset$.

($\bullet$) $\Phi_F = \bigsqcup_{v \in \widetilde{\Omega}_F} \mathsf{sib}_F(v)$. To see this, let $u \in \Phi_F$, and fix $j \in F$. Note that the d-tuple $u^{(j)} := (\mathsf{root}_1, \ldots, \mathsf{root}_{j-1}, u_j, \mathsf{root}_{j+1}, \ldots, \mathsf{root}_d) \in \Phi_{\{j\}}$. However,

$$\Phi_{\{j\}} = \bigsqcup_{v \in \Omega_{\{j\}}} \mathsf{sib}_{\{j\}}(v).$$

Thus $u^{(j)} \in \mathsf{sib}_{\{j\}}(v^{(j)})$ for some $v^{(j)} \in \Omega_{\{j\}}$. Let v denote the d-tuple with j^{th} entry given by

$$v_j = \begin{cases} j^{\text{th}} \text{ coordinate of } v^{(j)} & \text{if } j \in F, \\ \mathsf{root}_j & \text{if } j \notin F. \end{cases}$$

Then $u \in \mathsf{sib}_F(v)$ and $v \in \widetilde{\Omega}_F$.

The utility of the canonical choice of the indexing set is illustrated in the following lemma.

LEMMA 3.4. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of locally finite rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Then, for any $\alpha \in \mathbb{N}^d$ and a nonempty $F \in \mathscr{P}$,*

$$\prod_{l \in F} \sum_{u \in \Omega_{\{l\}}, \, \mathsf{d}_u = \alpha_l \epsilon_l} (\mathrm{card}(\mathsf{sib}(u_l)) - 1) = \sum_{u \in \Omega_F, \, \mathsf{d}_u = \alpha} \prod_{l \in F} (\mathrm{card}(\mathsf{sib}(u_l)) - 1). \tag{3.19}$$

Proof. Let $F = \{i_1, \ldots, i_k\}$, and note that

$$\begin{aligned} &\prod_{l \in F} \sum_{u \in \Omega_{\{l\}}, \, \mathsf{d}_u = \alpha_l \epsilon_l} (\mathrm{card}(\mathsf{sib}(u_l)) - 1) \\ &= \sum_{u^{(1)} \in \Omega_{\{i_1\}}, \, \mathsf{d}_{u^{(1)}} = \alpha_{i_1} \epsilon_{i_1}} \cdots \sum_{u^{(k)} \in \Omega_{\{i_k\}}, \, \mathsf{d}_{u^{(k)}} = \alpha_{i_k} \epsilon_{i_k}} (\mathrm{card}(\mathsf{sib}(u^{(1)}_{i_1})) - 1) \cdots (\mathrm{card}(\mathsf{sib}(u^{(k)}_{i_k})) - 1) \\ &= \sum_{u \in \widetilde{\Omega}_F, \, \mathsf{d}_u = \alpha} \prod_{l \in F} (\mathrm{card}(\mathsf{sib}(u_l)) - 1), \end{aligned}$$

where $\widetilde{\Omega}_F$ is the canonical choice as given in (3.18). The desired formula now follows from the fact that the summation on right hand side of (3.19) is independent of the choice of Ω_F. ∎

We now complete the proof of the main result of this paper.

Proof of Theorem 1.4. The implication (i) $\Rightarrow$ (ii) follows from Proposition 3.2. To see the implication (ii) $\Rightarrow$ (i), suppose that (ii) holds. Thus, for every $\alpha \in \mathbb{N}^d$ and every $F \in \mathscr{P}$, there exists a unitary

$$U_{F,\alpha} : \bigoplus_{u \in \Omega^{(1)}_F, \, \mathsf{d}_u = \alpha} \mathcal{L}^{(1)}_{u,F} \to \bigoplus_{v \in \Omega^{(2)}_F, \, \mathsf{d}_v = \alpha} \mathcal{L}^{(2)}_{v,F}.$$

Define $U : E^{(1)} \to E^{(2)}$ by setting $U = \bigoplus_{F \in \mathscr{P}, \, \alpha \in \mathbb{N}^d} U_{F,\alpha}$. Let $\rho_{\mathscr{T}^{(j)}} \times \mathrm{d}\nu_{\mathscr{T}^{(j)}}$ be the representing measure of $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(j)})$ (respectively $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T}^{(j)})$) in case $a \geqslant d$

(respectively $a < d$), $j = 1, 2$. Fix $F \in \mathscr{P}$ and $\alpha \in \mathbb{N}^d$. Note that by (2.27), for any Borel subset A of $[0,1] \times \partial\mathbb{B}^d$,

$$\begin{aligned}
& U\mathrm{d}\rho_{\mathscr{T}^{(1)}} \times \mathrm{d}\nu_{\mathscr{T}^{(1)}}(A)\Big(\sum_{u \in \Omega_F^{(1)},\, \mathsf{d}_u = \alpha} g_{u,F}\Big) \\
&= \int_A \Big(\sum_{u \in \Omega_F^{(1)},\, \mathsf{d}_u = \alpha} w_{|\mathsf{d}_u|}(s) \frac{|z^{\mathsf{d}_u}|^2}{\|z^{\mathsf{d}_u}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}} U_{F,\alpha}(g_{u,F})\Big)\mathrm{d}m(s)\mathrm{d}\sigma(z) \\
&= \int_A w_{|\alpha|}(s) \frac{|z^{\alpha}|^2}{\|z^{\alpha}\|^2_{L^2(\partial\mathbb{B}^d,\sigma)}} \Big(\sum_{u \in \Omega_F^{(1)},\, \mathsf{d}_u = \alpha} U_{F,\alpha}(g_{u,F})\Big)\mathrm{d}m(s)\mathrm{d}\sigma(z) \\
&= \mathrm{d}\rho_{\mathscr{T}^{(2)}} \times \mathrm{d}\nu_{\mathscr{T}^{(2)}}(A)\, U\Big(\sum_{u \in \Omega_F^{(1)},\, \mathsf{d}_u = \alpha} g_{u,F}\Big).
\end{aligned}$$

Hence, by Lemma 3.1, the Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(1)})$ and $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(2)})$ (respectively $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T}^{(1)})$ and $\mathscr{H}^{\mathfrak{s}}_{\mathfrak{c}_a}(\mathscr{T}^{(2)})$) are isomorphic in case $a \geqslant d$ (respectively $a < d$). By the first paragraph of the proof of Proposition 3.2, in both these cases, $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(1)})$ and $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(2)})$ are isomorphic.

We now see the equivalence of (iii) and (iv). Note that for every integer $n \geqslant 1$,

$$\begin{aligned}
\sum_{u \in \Omega_{\{l\}}^{(j)},\, \mathsf{d}_u = n\epsilon_l} (\mathrm{card}(\mathsf{sib}_l(u)) - 1) &= \sum_{u \in \Omega_{\{l\}}^{(j)},\, \mathsf{d}_u = n\epsilon_l} (\mathrm{card}(\mathsf{sib}(u_l)) - \mathrm{card}(\mathsf{par}(u_l))) \\
&\overset{(3.17)}{=} \mathrm{card}(\mathcal{G}_n(\mathscr{T}_l^{(j)})) - \mathrm{card}(\mathcal{G}_{n-1}(\mathscr{T}_l^{(j)})).
\end{aligned}$$

Since each $\mathscr{T}_l^{(j)}$ is rooted, $\mathrm{card}(\mathcal{G}_0(\mathscr{T}_l^{(j)})) = 1$. A routine telescopic sum argument now establishes the equivalence of (iii) and (iv).

To see (iii) $\Rightarrow$ (ii), note first that (ii) holds trivially in case $F = \emptyset$. Let $\alpha \in \mathbb{N}^d$ and $F \in \mathscr{P}$ be nonempty. For $j = 1, 2$,

$$\begin{aligned}
\prod_{l \in F} \sum_{u \in \Omega_{\{l\}}^{(j)},\, \mathsf{d}_u = \alpha_l\epsilon_l} (\mathrm{card}(\mathsf{sib}_l(u)) - 1) &= \prod_{l \in F} \sum_{u \in \Omega_{\{l\}}^{(j)},\, \mathsf{d}_u = \alpha_l\epsilon_l} (\mathrm{card}(\mathsf{sib}(u_l)) - 1) \\
&\overset{(3.19)}{=} \sum_{u \in \Omega_F^{(j)},\, \mathsf{d}_u = \alpha} \prod_{l \in F} (\mathrm{card}(\mathsf{sib}(u_l)) - 1).
\end{aligned}$$

However, by (2.6),

$$\sum_{u \in \Omega_F^{(j)},\, \mathsf{d}_u = \alpha} \dim \mathcal{L}_{u,F}^{(j)} = \sum_{u \in \Omega_F^{(j)},\, \mathsf{d}_u = \alpha} \prod_{l \in F} (\mathrm{card}(\mathsf{sib}(u_l)) - 1), \tag{3.20}$$

which establishes (ii). To complete the proof of the theorem, it now suffices to see that (ii) $\Rightarrow$ (iii). This follows immediately by taking $F = \{l\}$ in (3.20). ∎

We conclude this paper with some possibilities for further investigations. It is clear that the existence of the operator-valued representing measures for Drury–Arveson-type modules or its spherical Cauchy dual modules is one of the crucial ingredients for the proof of Theorem 1.4. In case the parameter a is non-integral, we do not know whether or not the Hilbert modules $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T})$ or $\mathscr{H}_{\mathfrak{c}_a}^{\mathfrak{s}}(\mathscr{T})$ admit representing measures. Further, the following classification problem arises naturally in the realm of graph-theoretic operator theory.

PROBLEM 3.5. *For $j = 1, 2$, let $\mathscr{T}^{(j)} = (V^{(j)}, \mathcal{E}^{(j)})$ denote the directed Cartesian product of locally finite, leafless, rooted directed trees $\mathscr{T}_1^{(j)}, \ldots, \mathscr{T}_d^{(j)}$ of finite joint branching index, and consider the (graded) submodules $\mathscr{N}^{(j)}$ of the Drury–Arveson-type Hilbert module $\mathscr{H}_{\mathfrak{c}_a}(\mathscr{T}^{(j)})$ generated by (homogeneous) polynomials $p_1, \ldots, p_l \in \mathbb{C}[z_1, \ldots, z_d]$. Under what conditions on $\mathscr{T}^{(1)}$ and $\mathscr{T}^{(2)}$, the submodules $\mathscr{N}^{(1)}$ and $\mathscr{N}^{(2)}$ are isomorphic ?*

Appendix A. CONSTANT ON PARENTS IS CONSTANT ON GENERATIONS

The main result of this appendix is a rigidity theorem showing that in higher dimensions ($d \geqslant 2$) the conditions (2.2) and (2.3) are equivalent. Here is the precise statement.

THEOREM A.1. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of leafless, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$ and let $S_\lambda = (S_1, \ldots, S_d)$ be a commuting multishift on $\mathscr{T}$. Consider the function $\mathfrak{C} : V \to (0, \infty)$ given by*

$$\mathfrak{C}(v) := \sum_{j=1}^{d} \|S_j e_v\|^2, \quad v \in V.$$

If $d \geqslant 2$, then the following conditions are equivalent:

(i) *$\mathfrak{C}$ is constant on every generation $\mathcal{G}_t$, $t \in \mathbb{N}$;*

(ii) *$\mathfrak{C}$ is constant on $\mathsf{Par}(v)$ for every $v \in V^\circ$.*

Recall the following notations from [9]:

$$\mathsf{Chi}(v) := \bigcup_{j=1}^{d} \mathsf{Chi}_j(v), \quad \mathsf{Par}(v) := \bigcup_{j=1}^{d} \mathsf{par}_j(v), \quad v \in V.$$

In the proof of the above theorem, we need two general facts.

LEMMA A.2. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of leafless, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$. Set*

$$V_\beta := \{v \in V : \mathsf{d}_v = \beta\}, \quad \beta \in \mathbb{N}^d. \tag{A.1}$$

Let f be a complex valued function on V such that f is constant on each of the sets $\mathsf{Chi}(v)$, $v \in V$. If $d \geqslant 2$, then the following statements are equivalent:

(i) *f is constant on each of the sets $V_{t\epsilon_1}$, $t \in \mathbb{N}$;*
(ii) *f is constant on each of the sets V_β and $f(V_\beta) = f(V_{|\beta|\epsilon_1})$, $\beta \in \mathbb{N}^d$;*
(iii) *f is constant on each of the generations $\mathcal{G}_t$, $t \in \mathbb{N}$.*

Proof. Assume that $d \geqslant 2$. Clearly, the implications (iii) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (i) hold. To see that (i) $\Rightarrow$ (ii), let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$. By (i), f is constant on $V_{|\beta|\epsilon_1}$. Consider the sequence $\{\Gamma_l\}_{l=1}^{d-1}$ given by

$$\Gamma_1 := \{\gamma_1^{(j)} = (|\beta| - j, j, 0, \ldots, 0) \in \mathbb{N}^d : 1 \leqslant j \leqslant \beta_2\},$$
$$\Gamma_k := \{\gamma_k^{(j)} = (|\beta| - \beta_2 - \cdots - \beta_k - j, \beta_2, \ldots, \beta_k, j, \underbrace{0, \ldots, 0}_{\substack{d-k-1 \\ \text{entries}}}) \in \mathbb{N}^d : \ 1 \leqslant j \leqslant \beta_{k+1}\} \quad 2 \leqslant k \leqslant d-1.$$

Let $v \in V_{\gamma_1^{(1)}}$. Then $v \in \mathsf{Chi}_2(\mathsf{par}_2(v))$ and $\mathsf{Chi}_1(\mathsf{par}_2(v)) \subseteq V_{|\beta|\epsilon_1}$. Thus the set $\mathsf{Chi}(\mathsf{par}_2(v))$ intersects with $V_{|\beta|\epsilon_1}$. Since f is constant on $\mathsf{Chi}(\mathsf{par}_2(v))$, it follows that $f(v)$ is equal to the constant value of f on $V_{|\beta|\epsilon_1}$. Since v was chosen arbitrarily, we get that f is constant on $V_{\gamma_1^{(1)}}$ and $f(V_{\gamma_1^{(1)}}) = f(V_{|\beta|\epsilon_1})$.

We claim that for any $k = 1, \ldots, d-1$, if f is constant on $V_{\gamma_k^{(j)}}$, then it is also constant on $V_{\gamma_k^{(j+1)}}$ (if $j < \beta_{k+1}$) or $V_{\gamma_{k+1}^{(1)}}$ (if $j = \beta_{k+1}$) with the same constant value. We divide this verification into two cases.

Case 1. When $j < \beta_{k+1}$.
Note that

$$\gamma_k^{(j)} = (|\beta| - \beta_2 - \cdots - \beta_k - j, \beta_2, \ldots, \beta_k, j, 0, \ldots, 0),$$
$$\gamma_k^{(j+1)} = (|\beta| - \beta_2 - \cdots - \beta_k - j - 1, \beta_2, \ldots, \beta_k, j+1, 0, \ldots, 0).$$

Suppose f is constant on $V_{\gamma_k^{(j)}}$. Let $v \in V_{\gamma_k^{(j+1)}}$. Then $v \in \mathsf{Chi}_{k+1}(\mathsf{par}_{k+1}(v))$ and $\mathsf{Chi}_1(\mathsf{par}_{k+1}(v)) \subseteq V_{\gamma_k^{(j)}}$. Thus $\mathsf{Chi}(\mathsf{par}_{k+1}(v))$ intersects with $V_{\gamma_k^{(j)}}$. Since f is constant on $\mathsf{Chi}(\mathsf{par}_{k+1}(v))$, it follows that $f(v)$ is equal to the constant value of f on $V_{\gamma_k^{(j)}}$. Since v was chosen arbitrarily, we get that f is constant on $V_{\gamma^{(j+1)}}$ and $f(V_{\gamma_k^{(j+1)}}) = f(V_{\gamma_k^{(j)}})$.

Case 2. When $j = \beta_{k+1}$.
Note that

$$\gamma_k^{(\beta_{k+1})} = (|\beta| - \beta_2 - \cdots - \beta_{k+1}, \beta_2, \ldots, \beta_{k+1}, 0, \ldots, 0),$$
$$\gamma_{k+1}^{(1)} = (|\beta| - \beta_2 - \cdots - \beta_{k+1} - 1, \beta_2, \ldots, \beta_{k+1}, 1, 0, \ldots, 0).$$

Suppose f is constant on $V_{\gamma_k^{(\beta_{k+1})}}$. Let $v \in V_{\gamma_{k+1}^{(1)}}$. Then $v \in \mathsf{Chi}_{k+2}(\mathsf{par}_{k+2}(v))$ and $\mathsf{Chi}_1(\mathsf{par}_{k+2}(v)) \subseteq V_{\gamma_k^{(\beta_{k+1})}}$. Thus $\mathsf{Chi}(\mathsf{par}_{k+2}(v))$ intersects with $V_{\gamma_k^{(\beta_{k+1})}}$. Since f is constant on $\mathsf{Chi}(\mathsf{par}_{k+2}(v))$, it follows that $f(v)$ is equal to the constant value of f

on $V_{\gamma_k^{(\beta_{k+1})}}$. Since v was chosen arbitrarily, we get that f is constant on $V_{\gamma_{k+1}^{(1)}}$ and $f(V_{\gamma_{k+1}^{(1)}}) = f(V_{\gamma_k^{(\beta_{k+1})}})$. Thus the claim stands verified. Since $\gamma_{d-1}^{(\beta_d)} = \beta$, we obtain

$$f(V_{|\beta|\epsilon_1}) = f(V_{\gamma_1^{(1)}}) = \cdots = f(V_{\gamma_{d-1}^{(\beta_d)}}) = f(V_\beta).$$

This yields (ii).

Now we show that (ii) $\Rightarrow$ (iii). Let u, v be any two vertices in $\mathcal{G}_t$, $t \in \mathbb{N}$. Then $|\mathsf{d}_u| = |\mathsf{d}_v| = t$ and by (ii), $f(V_{\mathsf{d}_v}) = f(V_{\mathsf{d}_u}) = f(V_{t\epsilon_1})$. This shows that $f(u) = f(v)$, which proves (iii). ∎

LEMMA A.3. *Let $\mathscr{T} = (V, \mathcal{E})$ be the directed Cartesian product of leafless, rooted directed trees $\mathscr{T}_1, \ldots, \mathscr{T}_d$, $d \geqslant 2$ and let f be a complex valued function on V. Then the following statements are equivalent:*

(i) *f is constant on each of the sets $\mathsf{Chi}(v)$, $v \in V$;*

(ii) *f is constant on each of the sets $\mathsf{Par}(v)$, $v \in V^\circ$.*

Proof. To see that (i) $\Rightarrow$ (ii), let f be constant on each of the sets $\mathsf{Chi}(v)$, $v \in V$. Let $v \in V^\circ$ and $u, w \in \mathsf{Par}(v)$. Then $u = \mathsf{par}_i(v)$ and $w = \mathsf{par}_j(v)$ for some $1 \leqslant i, j \leqslant d$. Note that

$$u \in \mathsf{Chi}_j(\mathsf{par}_i\mathsf{par}_j(v)) \quad \text{and} \quad w \in \mathsf{Chi}_i(\mathsf{par}_i\mathsf{par}_j(v)).$$

Thus $u, w \in \mathsf{Chi}(\mathsf{par}_i\mathsf{par}_j(v))$. Hence, by the hypothesis, $f(u) = f(w)$, proving (ii).

To see that (ii) $\Rightarrow$ (i), let f be constant on each of the sets $\mathsf{Par}(v)$, $v \in V^\circ$. Let $v \in V$ and $u, w \in \mathsf{Chi}(v)$. Then $u \in \mathsf{Chi}_i(v)$ and $w \in \mathsf{Chi}_j(v)$ for some $1 \leqslant i, j \leqslant d$.

Case 1. When $i \neq j$.

Without loss of generality, assume that $i < j$. In this case, consider the vertex

$$\eta = (v_1, \ldots, u_i, \ldots, w_j, \ldots, v_d).$$

Note that $u_i \in \mathsf{Chi}(v_i)$, $w_j \in \mathsf{Chi}(v_j)$, $u = \mathsf{par}_j(\eta)$ and $w = \mathsf{par}_i(\eta)$. Thus $u, w \in \mathsf{Par}(\eta)$. Hence, by (ii), $f(u) = f(w)$.

Case 2. When $i = j$.

For any positive integer $k \in \{1, \ldots, d\}$ such that $k \neq i$, consider the vertices

$$\theta = (v_1, \ldots, \eta_k, \ldots, u_i, \ldots, v_d) \quad \text{and} \quad \xi = (v_1, \ldots, \eta_k, \ldots, w_i, \ldots, v_d),$$

where $\eta_k \in \mathsf{Chi}(v_k)$. Note that $u_i, w_i \in \mathsf{Chi}(v_i)$, and $\mathsf{par}_i(\theta) = \mathsf{par}_i(\xi)$. Hence $\mathsf{Par}(\theta) \cap \mathsf{Par}(\xi) \neq \emptyset$. Further, $u = \mathsf{par}_k(\theta) \in \mathsf{Par}(\theta)$ and $w = \mathsf{par}_k(\xi) \in \mathsf{Par}(\xi)$. Since f is constant on $\mathsf{Par}(\theta)$ as well as on $\mathsf{Par}(\xi)$, and $\mathsf{Par}(\theta)$, $\mathsf{Par}(\xi)$ have a common vertex, it follows that $f(u) = f(w)$.

This proves (i). ∎

Proof of Theorem A.1. Assume that $d \geqslant 2$. In view of Lemmas A.2 and A.3 (applied to $f = \mathfrak{C}$), it suffices to show that if

(A.2) $\qquad \mathfrak{C}$ is constant on each of the sets $\mathsf{Chi}(v)$, $\quad v \in V$, then it is constant on $V_{t\epsilon_1}$ for every $t \in \mathbb{N}$, where V_β is as defined in (A.1). To this end, let $t \in \mathbb{N}$ and let u, v be any two vertices in $V_{t\epsilon_1}$. We need to show that $\mathfrak{C}(u) = \mathfrak{C}(v)$. Observe that as $\mathsf{d}_u = t\epsilon_1 = \mathsf{d}_v$, we must have $u = (u_1, \mathsf{root}_2, \ldots, \mathsf{root}_d)$ and $v = (v_1, \mathsf{root}_2, \ldots, \mathsf{root}_d)$ for some vertices u_1, v_1 of depth t in $\mathscr{T}_1$. Let k denote the unique least non-negative integer such that

$$\mathsf{par}^{\langle k \rangle}(u_1) = \mathsf{par}^{\langle k \rangle}(v_1). \tag{A.3}$$

If $k = 0$, then $u = v$, and hence $\mathfrak{C}(u) = \mathfrak{C}(v)$ holds trivially. So assume that $k \geqslant 1$. Consider the sequence $\{v_2^{(l)}\}_{l \in \mathbb{N}}$ of vertices in V_2 with the following conditions:

$$\mathsf{par}(v_2^{(1)}) = \mathsf{root}_2 = v_2^{(0)}, \quad \mathsf{par}(v_2^{(l)}) = v_2^{(l-1)}, \quad l \geqslant 2.$$

Now consider the sequence $\{v^{(l)}\}_{l=1}^k$ of vertices in $\mathscr{T}$ given as follows:

$$v^{(l)} = (\mathsf{par}^{\langle l \rangle}(v_1), v_2^{(l-1)}, \mathsf{root}_3, \ldots, \mathsf{root}_d), \quad l = 1, \ldots, k.$$

Further, consider the sequence $\{\theta^{(l)}\}_{l=1}^k$ of vertices in $\mathscr{T}$ given as follows:

$$\theta^{(l)} = (\mathsf{par}^{\langle l \rangle}(v_1), v_2^{(l)}, \mathsf{root}_3, \ldots, \mathsf{root}_d), \quad l = 1, \ldots, k.$$

Notice that $v \in \mathsf{Chi}_1(v^{(1)})$ and $\theta^{(1)} \in \mathsf{Chi}_2(v^{(1)})$. Thus $v, \theta^{(1)} \in \mathsf{Chi}(v^{(1)})$, and hence by (A.2), $\mathfrak{C}(v) = \mathfrak{C}(\theta^{(1)})$. Further, observe that $\theta^{(1)} \in \mathsf{Chi}_1(v^{(2)})$ and $\theta^{(2)} \in \mathsf{Chi}_2(v^{(2)})$. Once again, by (A.2), $\mathfrak{C}(\theta^{(1)}) = \mathfrak{C}(\theta^{(2)})$. A finite inductive argument together with (A.2) shows $\mathfrak{C}(\theta^{(l-1)}) = \mathfrak{C}(\theta^{(l)})$ for $l = 2, \ldots, k-1$. Thus we obtain

$$\mathfrak{C}(v) = \mathfrak{C}(\theta^{(1)}) = \cdots = \mathfrak{C}(\theta^{(k)}). \tag{A.4}$$

Note that by (A.3),

$$v^{(k)} = (\mathsf{par}^{\langle k \rangle}(v_1), v_2^{(k-1)}, \mathsf{root}_3, \ldots, \mathsf{root}_d) = (\mathsf{par}^{\langle k \rangle}(u_1), v_2^{(k-1)}, \mathsf{root}_3, \ldots, \mathsf{root}_d).$$

Now consider the sequence $\{w^{(l)}\}_{l=1}^{k-1}$ of vertices in $\mathscr{T}$ given as follows:

$$w^{(l)} = (\mathsf{par}^{\langle k-l \rangle}(u_1), v_2^{(k-l-1)}, \mathsf{root}_3, \ldots, \mathsf{root}_d), \quad l = 1, \ldots, k-1.$$

Further, consider the sequence $\{\eta^{(l)}\}_{l=1}^k$ of vertices in $\mathscr{T}$ given as follows:

$$\eta^{(l)} = (\mathsf{par}^{\langle k-l \rangle}(u_1), v_2^{(k-l)}, \mathsf{root}_3, \ldots, \mathsf{root}_d), \quad l = 1, \ldots, k.$$

Observe that $\eta^{(1)} \in \mathsf{Chi}_1(v^{(k)})$ and $\theta^{(k)} \in \mathsf{Chi}_2(v^{(k)})$. Thus $\eta^{(1)}, \theta^{(k)} \in \mathsf{Chi}(v^{(k)})$, and hence by (A.2), $\mathfrak{C}(\theta^{(k)}) = \mathfrak{C}(\eta^{(1)})$. Further, observe that $\eta^{(1)} \in \mathsf{Chi}_2(w^{(1)})$ and $\eta^{(2)} \in \mathsf{Chi}_1(w^{(1)})$. Arguing as above, we have $\mathfrak{C}(\eta^{(1)}) = \mathfrak{C}(\eta^{(2)})$. A finite inductive argument now shows that

$$\mathfrak{C}(\theta^{(k)}) = \mathfrak{C}(\eta^{(1)}) = \cdots = \mathfrak{C}(\eta^{(k)}) = \mathfrak{C}(u). \tag{A.5}$$

Combining (A.4) and (A.5), we get $\mathfrak{C}(v) = \mathfrak{C}(u)$. ∎

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REFERENCES

[1] D. ALPAY, D. VOLOK, Point evaluation and Hardy space on a homogeneous tree, *Integral Equations Operator Theory* **53**(2005), 1–22.

[2] A. ANAND, S. CHAVAN, A moment problem and joint q-isometry tuples, *Complex Anal. Oper. Theory* **11**(2017), 785–810.

[3] A. ANAND, S. CHAVAN, Z. JABŁOŃSKI, J. STOCHEL, Complete systems of unitary invariants for some classes of 2-isometries, *Banach J. Math. Anal.*, to appear.

[4] J. ARAMAYONA, J. FERNÁNDEZ, P. FERNÁNDEZ, C. MARTÍNEZ-PÉREZ, Trees, homology, and automorphism groups of RAAGs, *J. Algebraic Combinatorics*, to appear.

[5] W. ARVESON, Subalgebras of C^*-algebras. III. Multivariable operator theory, *Acta Math.* **181**(1998), 159–228.

[6] P. BUDZYŃSKI, P. DYMEK, A. PLANETA, M. PTAK, Weighted shifts on directed trees. Their multiplier algebras, reflexivity and decompositions, *Studia Math.* **244**(2019), 285–308.

[7] P. BUDZYŃSKI, Z. JABŁOŃSKI, I.B. JUNG, J. STOCHEL, Unbounded subnormal composition operators in L^2-spaces, *J. Funct. Anal.* **269**(2015), 2110–2164.

[8] S. CHAVAN, D. PRADHAN, S. TRIVEDI, Dirichlet spaces associated with locally finite rooted directed trees, *Integral Equations Operator Theory* **89**(2017), 209–232.

[9] S. CHAVAN, D. PRADHAN, S. TRIVEDI, Multishifts on directed Cartesian product of rooted directed trees, *Diss. Math.* **527**(2017), 1–102.

[10] S. CHAVAN, S. TRIVEDI, An analytic model for left-invertible weighted shifts on directed trees, *J. London Math. Soc.* **94**(2016), 253–279.

[11] S. CHAVAN, D. YAKUBOVICH, Spherical tuples of Hilbert space operators, *Indiana Univ. Math. J.* **64**(2015), 577–612.

[12] J. CONWAY, *The Theory of Subnormal Operators*, Math. Surveys Monogr., vol. 36, Amer. Math. Soc. Providence, RI 1991.

[13] R. CURTO, J. YOON, Disintegration-of-measure techniques for commuting multivariable weighted shifts, *Proc. London Math. Soc.* **92**(2006), 381–402.

[14] S. DRURY, A generalization of von Neumann's inequality to the complex ball, *Proc. Amer. Math. Soc.* **68**(1978), 300–304.

[15] J. GLEASON, S. RICHTER, m-Isometric commuting tuples of operators on a Hilbert space, *Integral Equations Operator Theory* **56**(2006), 181–196.

[16] B. HALL, *Lie Groups, Lie Algebras, and Representations. An Elementary Introduction*, Second edition, Grad. Texts in Math., vo. 222, Springer, Cham 2015.

[17] Z. JABŁOŃSKI, I.B. JUNG, J. STOCHEL, Weighted shifts on directed trees, *Mem. Amer. Math. Soc.* **216**(2012), no. 1017.

[18] N.P. JEWELL, A.R. LUBIN, Commuting weighted shifts and analytic function theory in several variables, *J. Operator Theory* **1**(1979), 207–223.

[19] E. KATSOULIS, D. KRIBS, Isomorphisms of algebras associated with directed graphs, *Math. Ann.* **330**(2004), 709–728.

[20] W. LIGHT, E. CHENEY, *Approximation Theory in Tensor Product Spaces*, Lecture Notes in Math., vol. 1169, Springer-Verlag, Berlin 1985.

[21] Y. NERETIN, Groups of hierarchomorphisms of trees and related Hilbert spaces, *J. Funct. Anal.* **200**(2003), 505–535.

[22] V. PAULSEN, M. RAGHUPATHI, *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces*, Cambridge Stud. Adv. Math., vol. 152, Cambridge Univ. Press, Cambridge 2016.

[23] J. SARKAR, Applications of Hilbert module approach to multivariable operator theory, in *Handbook of Operator Theory*, Springer, Basel 2015, pp. 1035–1091.

[24] S. SHIMORIN, Wold-type decompositions and wandering subspaces for operators close to isometries, *J. Reine Angew. Math.* **531**(2001), 147–189.

[25] B. SIMON, *Real Analysis. A Comprehensive Course in Analysis*, Part 1. Amer. Math. Soc., Providence, RI 2015.

[26] B. SOLEL, You can see the arrows in a quiver operator algebra, *J. Austral. Math. Soc.* **77**(2004), 111–122.

[27] K. ZHU, *Spaces of Holomorphic Functions in the Unit Ball*, Grad. Texts in Math., vol. 226, Springer-Verlag, New York 2005.

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