HYPERRIGID SUBSETS OF CUNTZ–KRIEGER ALGEBRAS AND THE PROPERTY OF RIGIDITY AT ZERO

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ABSTRACT. A subset $\mathcal{G}$ generating a C^*-algebra A is said to be *hyperrigid* if for every faithful nondegenerate $*$-representation $A \subseteq B(H)$ and a sequence $\phi_n : B(H) \to B(H)$ of unital completely positive maps, we have that

$$\lim_{n \to \infty} \phi_n(g) = g \quad \text{for all } g \in \mathcal{G} \implies \lim_{n \to \infty} \phi_n(a) = a \quad \text{for all } a \in A.$$

We show that in the Cuntz–Krieger algebra of a row-finite directed graph with no isolated vertices, the set of all edge partial-isometries is hyperrigid. We also examine, both in general and in the context of graphs, a related property named *rigidity at* 0 that sheds light on the phenomenon of hyperrigidity.

KEYWORDS: *Cuntz–Krieger algebra, directed graph, hyperrigidity, C^*-envelope.*

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1. INTRODUCTION

A *directed graph* $G = (V, E, s, r)$ consists of a set V of vertices, a set E of edges, and two maps $s, r : E \to V$, called the *source* and the *range*; if $v = s(e)$ and $w = r(e)$ we say that v emits e and w receives it. In this paper, we consider only *countable* directed graphs, meaning that both the sets V and E are countable. A directed graph is said to be *row-finite* if every vertex receives at most finitely many edges, i.e., $r^{-1}(v)$ is a finite subset of E, for all $v \in V$.

A *Cuntz–Krieger G-family* $(\mathcal{V}, \mathcal{E})$ of a directed graph G consists of a set of mutually orthogonal projections $\mathcal{V} := \{P_v : v \in V\}$ and a set of partial isometries $\mathcal{E} := \{S_e : e \in E\}$ which satisfy the relations:

(I) $S_e^* S_e = P_{s(e)}$, for every $e \in E$;

(TCK) $\sum\limits_{e \in F} S_e S_e^* \leqslant P_v$ for every finite subset $F \subset r^{-1}(v)$; and

(CK) $\sum\limits_{r(e)=v} S_e S_e^* = P_v$, for every $v \in V$, with $0 < |r^{-1}(v)| < \infty$.

There exists a universal C^*-algebra $\mathcal{O}(G)$ generated by a Cuntz–Krieger G-family, which is called the *Cuntz–Krieger algebra* of the graph G. The original definition of this C^*-algebra is due to Cuntz and Krieger [7]; for a comprehensive background on Cuntz–Krieger algebras associated to directed graphs we refer the reader to Raeburn's book [21].

When G is row-finite, it is known ([10], [12], [14]) that inside the Cuntz–Krieger algebra $\mathcal{O}(G)$, the Cuntz–Krieger family is not just a generating set, but in fact, a *hyperrigid* generating set.

DEFINITION 1.1. Let A be a C^*-algebra and $\mathcal{G}$ be a generating subset of A. We say that $\mathcal{G}$ is *hyperrigid* in A if for every faithful nondegenerate $*$-representation $A \subseteq B(H)$ and a sequence of unital completely positive maps $\phi_n : B(H) \to B(H)$ we have that

$$\lim_{n\to\infty} \phi_n(g) = g \quad \text{for all } g \in \mathcal{G} \implies \lim_{n\to\infty} \phi_n(a) = a \quad \text{for all } a \in A$$

where all convergences are in norm.

The notion of hyperrigidity was extensively studied during the last decade in various contexts; see for example [4], [8], [16], [18] and Section 7 of [15]. There is one question that naturally arises whenever a set $\mathcal{G}$ is hyperrigid in a C^*-algebra A: what is the smallest subset of $\mathcal{G}$ that is already hyperrigid in A?

In [4], Arveson showed that $\{1, x, x^2\}$ is hyperrigid in $C([0,1])$ while the smaller generating set $\{1, x\}$ is not. In [18], Kennedy and Shalit considered the Cuntz algebra $\mathcal{O}_I$ associated to a homogeneous ideal $I \triangleleft \mathbb{C}[z_1, \dots, z_d]$. Without getting into technical details, let us just say that the C^*-algebra $\mathcal{O}_I$ is naturally generated by a set of d generators $\mathcal{G} = \{Z_1, \dots, Z_d\}$. While it can be easily shown that the larger generating set $\mathcal{G} \cup \mathcal{G}^*\mathcal{G}$ is always hyperrigid in $\mathcal{O}_I$ for every homogeneous ideal I, the authors showed in Theorem 4.12 of [18] that hyperrigidity of $\mathcal{G}$ itself is equivalent to the well known essential normality conjecture ([2], Problem 2) of Arveson.

In [10], Dor-On and the author showed that G being row-finite does not only imply the hyperrigidity of the Cuntz–Krieger G-family inside $\mathcal{O}(G)$, but is in fact equivalent to the latter (in the paper, the authors considered the norm-closed algebra generated by the Cuntz–Krieger family — namely, the tensor-algebra $\mathcal{T}_+(G)$ — but the proof works the same if one replaces $\mathcal{T}_+(G)$ with the Cuntz–Krieger family itself; see the proofs of Theorems 3.5 and 3.9 in [10]).

It is therefore natural to ask the following question regarding a row-finite graph G: in case $\mathcal{O}(G)$ is already generated by a subset $\mathcal{G}$ of the Cuntz–Krieger family, is this subset hyperrigid as well? For simplicity, if the graph is assumed to contain no isolated vertices, then the subset $\mathcal{E} := \{S_e : e \in E\}$ is a (minimal) subset of the Cuntz–Krieger family generating $\mathcal{O}(G)$; must $\mathcal{E}$ be hyperrigid in this case?

In this paper, we show that answer is *yes*: $\mathcal{E}$ is hyperrigid in $\mathcal{O}(G)$ if and only if G is a row-finite. In the proof, we use a generalization to a not-necessarily-unital

C^*-algebra A of Arveson's characterization for the hyperrigidity of a generating subset $\mathcal{G}$: $\mathcal{G}$ is hyperrigid in A if and only if for every unital $*$-representation $\pi : A \to B(H)$ there exists a unique unital completely positive extension of $\pi|_{\mathcal{G}}$ to A, namely π itself. We also prove that in a proper sense hyperrigidity is preserved under inductive limits, and use this to show the above result.

The fact that for a row-finite graph G, whenever $\mathcal{E}$ generates $\mathcal{O}(G)$ it must be hyperrigid, also implies that for row-finite graphs the C^*-envelope of $A(\mathcal{E}) := \overline{\text{alg}}\{S_e : e \in E\}$ — the operator algebra generated by $\mathcal{E}$ — is $\mathcal{O}(G)$. This is known when the algebra $A(\mathcal{E})$ is replaced by the tensor algebra $\mathcal{T}_+(G) = \overline{\text{alg}}\{P_v, S_e : v \in V, e \in E\}$; see Theorem 2.5 of [17].

In addition, we define and examine a closely related notion: the property of rigidity at 0. A generating subset $\mathcal{G}$ of a C^*-algebra A is said to be *rigid at* 0 if for every sequence of contractive positive maps $\varphi_n : A \to \mathbb{C}$ satisfying $\lim_{n\to\infty} \varphi_n(g) = 0$ for every $g \in \mathcal{G}$, we have that $\lim_{n\to\infty} \varphi_n(a) = 0$ for every $a \in A$.

Joining hyperrigidity the property of rigidity at 0 yields a somewhat stronger notion of hyperrigidity. For example, for a nonunital C^*-algebra A, these two properties (when combined together) are equivalent to a version of Definition 1.1 in which the faithful $*$-representation $A \subseteq B(H)$ is not assumed to be nondegenerate. In this case, where A is nonunital, the two properties are also equivalent to the hyperrigidity of $\mathcal{G} \cup \{1\}$ inside the minimal unitization A^1.

We show that while the Cuntz–Krieger family of any directed graph is always rigid at 0, the smaller generating set $\mathcal{E}$ is never rigid at 0. More precisely, we show that for a row-finite graph, a subset $\mathcal{G}$ of the Cuntz–Krieger family is rigid at 0 if and only if it contains every vertex projection.

One immediate consequence of the latter characterization is that whenever a row-finite graph G has infinitely many vertices (or, equivalently, whenever $\mathcal{O}(G)$ is nonunital), $\mathcal{G} \cup \{1\}$ is hyperrigid in the unitized C^*-algebra $\mathcal{O}(G)^1$ if and only if $\mathcal{G}$ contains every vertex projection.

2. HYPERRIGIDITY

We now describe a few properties of hyperrigidity.

PROPOSITION 2.1. *Let A be a C^*-algebra and $\mathcal{G}$ a generating subset of A. Then the following conditions are equivalent:*

   (i) *$\mathcal{G}$ is hyperrigid in A;*

  (ii) *$\mathcal{G} \cup \mathcal{G}^*$ is hyperrigid in A;*

 (iii) *$\text{span}(\mathcal{G})$ is hyperrigid in A;*

If A is unital, then (i)–(iii) *are also equivalent to*

  (iv) *$\mathcal{G} \cup \{1\}$ is hyperrigid in A.*

The proof follows directly from the definition of hyperrigidity.

REMARK 2.2. Suppose A is a nonunital C^*-algebra generated by a subset $\mathcal{G}$. Let A^1 denote the minimal unitization of A. If $\mathcal{G} \cup \{1\}$ is hyperrigid in A^1, then $\mathcal{G}$ is clearly hyperrigid in A. The converse, however, fails; see Proposition 3.8.

The notion of the *unique extension property*, originally defined by Arveson, has developed in various settings over the last decades. The common definition, Definition 2.1 of [3], is for a unital completely positive map defined on an operator system S which generates a unital C^*-algebra A. The notion has a parallel version for a unital completely contractive map defined on a unital operator algebra, and in Definition 2.1 of [10] the latter was generalized to the nonunital case.

In this paper, we will need the following version of the unique extension property which concerns a restriction (to a generating set) of a $*$-representation: the C^*-algebra is not assumed to be unital, and the generating set is not assumed to be an operator system or an operator algebra.

DEFINITION 2.3. Let A be a C^*-algebra, $\mathcal{G}$ a generating subset of A and $\pi : A \to B(H)$ a $*$-representation. We say that $\pi|_{\mathcal{G}}$ has the *unique extension property* if for every completely contractive completely positive map $\rho : A \to B(H)$ we have that

$$\phi(g) = \pi(g) \quad \text{for all } g \in \mathcal{G} \implies \phi(a) = \pi(a) \quad \text{for all } a \in A.$$

If A is a unital C^*-algebra, S is an operator system generating A, and $\pi : A \to B(H)$ is a unital $*$-representation, then $\pi|_S$ has the unique extension property if and only if there exists a unique *unital completely positive* extension of $\pi|_S$ to A, namely π itself. This shows that if the generating set is an operator system, then our definition for the unique extension property agrees with the one in Definition 2.1 of [3].

Now let A be a nonunital C^*-algebra and $\rho : A \to B(H)$ a completely contractive completely positive map. Let A^1 be the minimal unitization of A. The *unitization* of ρ is the unital completely positive map $\rho^1 : A^1 \to B(H)$ defined by

$$\rho^1(a + \lambda \cdot 1) = \rho(a) + \lambda \cdot I_H \quad \text{for all } a \in A, \lambda \in \mathbb{C};$$

see Proposition 2.2.1 of [6] for a proof that ρ^1 is indeed a well defined unital completely positive map. If ρ is a $*$-representation, then ρ^1 is a $*$-representation as well.

The proof of the following proposition follows immediately from the definition of the unique extension property and the discussion above, and is therefore omitted.

PROPOSITION 2.4. *Let A be a nonunital C^*-algebra, $\mathcal{G}$ a generating subset of A and $\pi : A \to B(H)$ a $*$-representation. Then the following conditions are equivalent:*
  (i) *$\pi|_{\mathcal{G}}$ has the unique extension property;*
  (ii) *$\pi^1|_{\mathcal{G} \cup \{1\}}$ has the unique extension property.*

In [4], Arveson showed that the unique extension property (in the sense of operator systems and unital C^*-algebras) is preserved under direct sums. The proof of the following version is similar and is therefore omitted.

PROPOSITION 2.5. *Let A be a C^*-algebra generated by a subset $\mathcal{G}$. Let I be a set, and for every $i \in I$ let $\pi_i : A \to B(H_i)$ be a $*$-representation such that $\pi_i|_{\mathcal{G}}$ has the unique extension property. Set $\pi := \bigoplus_{i \in I} \pi_i$. Then $\pi|_{\mathcal{G}}$ has the unique extension property.*

The next proposition shows that having the unique extension property for every restriction of a $*$-representation is equivalent to having it only for restrictions of *nondegenerate* $*$-representations and for the trivial $*$-representation.

PROPOSITION 2.6. *Let A be a C^*-algebra generated by a subset $\mathcal{G}$. Then the following conditions are equivalent:*
   (i) *for every $*$-representation $\pi : A \to B(H)$, $\pi|_{\mathcal{G}}$ has the unique extension property;*
   (ii) *for every nondegenerate $*$-representation $\pi : A \to B(H)$, $\pi|_{\mathcal{G}}$ has the unique extension property, and for the zero map $0 : A \to \mathbb{C}$, $0|_{\mathcal{G}}$ has the unique extension property.*
If A is unital and $1 \in \mathcal{G}$, then (i) and (ii) are also equivalent to
   (iii) *for every unital $*$-representation $\pi : A \to B(H)$, $\pi|_{\mathcal{G}}$ has the unique extension property.*
If A is nonunital, then (i) and (ii) are also equivalent to
   (iii)[1] *for every unital $*$-representation $\pi : A^1 \to B(H)$, $\pi|_{\mathcal{G} \cup \{1\}}$ has the unique extension property.*

Proof. (i) $\Rightarrow$ (ii) is clear. For the opposite direction, if $\pi : A \to B(H)$ is any $*$-representation, let $\pi_{\mathrm{nd}} : A \to B(K)$ denote its nondegenerate part, so that $\pi = \pi_{\mathrm{nd}} \oplus 0^{\oplus \lambda}$ for some cardinality λ. By Proposition 2.5 π has the unique extension property.

If A is unital and $1 \in \mathcal{G}$, then for the zero map $0 : A \to \mathbb{C}$, $0|_{\mathcal{G}}$ must always have the unique extension property, and nondegenerate $*$-representations are unital, so that (ii) $\Leftrightarrow$ (iii).

If A is nonunital, then the equivalence (i) $\Leftrightarrow$ (iii)[1] follows from Proposition 2.4. ∎

Let A be a C^*-algebra generated by a subset $\mathcal{G}$ and let $\pi : A \to B(H)$ be a (perhaps degenerate) $*$-representation. Let $\phi_n : A \to B(H)$ be a sequence of completely contractive completely positive maps satisfying

$$\lim_{n \to \infty} \|\phi_n(g) - \pi(g)\| = 0 \quad \text{for every } g \in \mathcal{G}.$$

A crucial ingredient in the proof of Theorem 2.1 in [4] was to define a faithful unital $*$-representation $\iota : B(H) \to \ell^\infty(B(H))/c_0(B(H))$ by

$$\iota(x) = (x, x, \dots) + c_0(B(H)),$$

to choose some faithful unital $*$-representation $\kappa : \ell^\infty(B(H))/c_0(B(H)) \to B(K)$, and to show that if $\kappa \circ \iota \circ \pi|_{\mathcal{G}}$ has the unique extension property, then

$$\lim_{n\to\infty} \|\phi_n(a) - \pi(a)\| = 0 \quad \text{for every } a \in A.$$

The original proof of this implication is only for the case where A is unital, $\mathcal{G}$ is an operator system, and π is unital, but remains the same in our more generalized setting.

It is worth to note that if π is nondegenerate, then $\kappa \circ \iota \circ \pi$ is nondegenerate, and if π is the zero map, then $\kappa \circ \iota \circ \pi$ is the zero map. This observation gives rise to the following approximation lemma.

LEMMA 2.7. *Let A be a C^*-algebra generated by a subset $\mathcal{G}$ and let $\pi : A \to B(H)$ be $*$-representation. Let $\phi_n : A \to B(H)$ be a sequence of completely contractive completely positive maps satisfying*

$$\lim_{n\to\infty} \|\phi_n(g) - \pi(g)\| = 0 \quad \text{for every } g \in \mathcal{G}.$$

If either

(i) *every $*$-representation of A has the unique extension property when restricted to $\mathcal{G}$; or*

(ii) *every nondegenerate $*$-representation of A has the unique extension property when restricted to $\mathcal{G}$, and π is nondegenerate; or*

(iii) *the zero representation of A has the unique extension property when restricted to $\mathcal{G}$, and π is the zero representation.*
Then

$$\lim_{n\to\infty} \|\phi_n(a) - \pi(a)\| = 0 \quad \text{for every } a \in A.$$

Proposition 2.1 implies that hyperrigidity does not depend on whether $\mathcal{G}$ contains the unit of A (if exists) or not. There is therefore no loss in assuming that $\mathcal{G}$ contains 1 whenever A is unital. The following theorem is a generalization of a theorem of Arveson ([4], Theorem 2.1) and Lemma 2.7(ii) is a key ingredient in its proof. The proof essentially consists of minor modifications of the proof of the original theorem and is therefore omitted as well.

THEOREM 2.8. *Let A be a separable C^*-algebra and $\mathcal{G}$ be a generating subset of A. If A is unital, assume in addition that $1 \in \mathcal{G}$. Then $\mathcal{G}$ is hyperrigid in A if and only if for every nondegenerate $*$-representation $\pi : A \to B(K)$ on a separable Hilbert space K, $\pi|_{\mathcal{G}}$ has the unique extension property.*

3. RIGIDITY AT ZERO

We now define a closely related notion concerning a certain rigidity property of a generating subset $\mathcal{G}$ of a C^*-algebra A.

DEFINITION 3.1. Let A be a C^*-algebra and $\mathcal{G}$ be a generating subset of A. We say that $\mathcal{G}$ is *rigid at* 0 in A if for every sequence of contractive positive maps $\varphi_n : A \to \mathbb{C}$, we have

$$\lim_{n \to \infty} \varphi_n(g) = 0 \quad \text{for all } g \in \mathcal{G} \implies \lim_{n \to \infty} \varphi_n(a) = 0 \quad \text{for all } a \in A.$$

REMARK 3.2. If A is a unital C^*-algebra and $\mathcal{G}$ is a generating subset of A containing the unit, then $\mathcal{G}$ must be rigid at 0 in A. To see this, let $\varphi_n : A \to \mathbb{C}$ be a sequence of contractive positive maps satisfying $\lim_{n \to \infty} \varphi_n(g) = 0$ for all $g \in \mathcal{G}$. Then for every $a \in A$

$$\limsup_{n \to \infty} \|\varphi_n(a)\| \leqslant \limsup_{n \to \infty} \|\varphi_n(1)\| \|a\| = 0.$$

This is not the case, however, for nonunital C^*-algebras or unital C^*-algebras with a generating set that does not contain the unit; see Examples 3.4 and 5.5.

In the following theorem, we give some equivalent conditions to the property of rigidity at 0.

THEOREM 3.3. *Let A be a separable C^*-algebra generated by a subset $\mathcal{G}$. Then the following conditions are equivalent:*

(i) *$\mathcal{G}$ is rigid at 0;*

(ii) *for every faithful nondegenerate $*$-representation $A \subseteq B(H)$ and for every sequence of unital completely positive maps $\phi_n : B(H) \to B(H)$, we have*

$$\lim_{n \to \infty} \phi_n(g) = 0 \quad \text{for all } g \in \mathcal{G} \implies \lim_{n \to \infty} \phi_n(a) = 0 \quad \text{for all } a \in A;$$

(iii) *for the zero map $0 : A \to \mathbb{C}$, $0|_{\mathcal{G}}$ has the unique extension property;*

(iv) *for every separable Hilbert space K and $0 : A \to B(K)$, $0|_{\mathcal{G}}$ has the unique extension property;*

(v) *there are no states on A vanishing on $\mathcal{G}$.*

Proof. (v) $\Rightarrow$ (iii) Clearly, if $0|_{\mathcal{G}}$ has the unique extension property, then there are no states on A vanishing on $\mathcal{G}$, and conversely, if $0|_{\mathcal{G}}$ has a nontrivial positive contractive extension φ, then $(1/\|\varphi\|)\varphi$ is a state on A vanishing on $\mathcal{G}$.

(i) $\Rightarrow$ (iii) is clear.

(ii) $\Rightarrow$ (v) Assume that (v) does not hold and let $A \subseteq B(H)$ be a faithful nondegenerate $*$-representation. By Arveson's extension theorem we obtain a state φ on $B(H)$ vanishing on $\mathcal{G}$ but not on A. Define $\phi : B(H) \to B(H)$ by $\phi(x) := \varphi(x) I_H$, then ϕ is a unital completely positive map vanishing on $\mathcal{G}$ but not on A. Setting $\phi_n = \phi$ for every $n \in \mathbb{N}$ yields a contradiction to (ii).

The assertion (iii) $\Rightarrow$ (iv) follows by Proposition 2.5, while the assertions (iv) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) both follow by Lemma 2.7(iii). ∎

In Corollary 3.4 of [4], Arveson shows that for every $n \in \mathbb{N}$ the set of n isometries $\mathcal{G}_n = \{S_1, \ldots, S_n\}$ generating the Cuntz algebra $\mathcal{O}_n$ is hyperrigid. In Continuation of Example 2.7 of [20], Muhly and Solel show that the infinite set of

isometries $\mathcal{G}_\infty = \{S_1, S_2, \dots\}$ generating the Cuntz algebra $\mathcal{O}_\infty$ is not hyperrigid (more precisely, they showed that the norm-closed algebra generated by $\mathcal{G}_\infty$ is not hyperrigid). In the next example we show that for any n, finite or not, $\mathcal{G}_n$ is not rigid at 0 in $\mathcal{O}_n$.

EXAMPLE 3.4. Let $n \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{O}_n$ be the Cuntz algebra, and let $\mathcal{G}_n = \{S_1, S_2, \dots, S_n\}$ be the set of n isometries generating it. We will show that there exists a state on $\mathcal{O}_n$ vanishing on $\mathcal{G}$.

For $n = 1$ note that $\mathcal{O}_1 \cong C(\mathbb{T})$, and the state

$$\rho(f) = \int f(z) \mathrm{d}\mu(z),$$

where μ is the normalized Lebesgue measure on the unit circle, vanishes on z; a precomposition with the above isomorphism yields a state on $\mathcal{O}_1$ vanishing on the unitary S_1.

Assume now that $n \geqslant 2$. Let $\mathcal{O}_n \subseteq B(H)$ be some faithful nondegenerate $*$-representation, and let $\rho : \mathcal{O}_n \to B(H)$ be the completely contractive completely positive map defined by

$$\theta(a) = S_2^* S_1^* a S_1 S_2, \quad \text{for all } a \in \mathcal{O}_n.$$

Then $\rho|_{\mathcal{G}_n} = 0$.

REMARK 3.5. When n is finite, the study of Davidson and Pitts on Cuntz algebra atomic representations [9] gives rise to an alternative proof of $\mathcal{G}_n$ being not rigid at 0 in $\mathcal{O}_n$. Indeed, suppose that $\pi : \mathcal{O}_n \to B(H)$ is any atomic $*$-representation, namely, that there exists an orthonormal basis $\{\xi_j\}$ of H, n endomorphisms $\sigma_i : \mathbb{N} \to \mathbb{N}$, and scalars $\lambda_{i,j} \in \mathbb{T}$ such that $\pi(S_i)\xi_j = \lambda_{i,j}\xi_{\sigma_i(j)}$. Let ξ be a wandering vector for the set of noncommutative words in $\pi(\mathcal{G}_n)$ (see Corollary 3.6 of [9]). Then the state defined by

$$\varphi(a) = \langle \pi(a)\xi, \xi \rangle, \quad \text{for all } a \in \mathcal{O}_n$$

vanishes on $\mathcal{G}_n$.

PROPOSITION 3.6. *Let A be a C*-algebra and $\mathcal{G}$ a generating subset of A. If* $\mathrm{span}(\mathcal{G})$ *contains an approximate identity for A, then it must be rigid at 0 in A.*

Proof. Let $\{e_n\} \subseteq \mathrm{span}(\mathcal{G})$ be an approximate identity for A. If $\varphi : A \to \mathbb{C}$ is a contractive positive linear functional vanishing on $\mathcal{G}$, then for every $0 \leqslant a \in A$

$$\varphi(a) = \varphi(\lim e_n^{1/2} a e_n^{1/2}) \leqslant \|a\| \lim \varphi(e_n) = 0.$$

Thus $\varphi = 0$, so by Theorem 3.3 $\mathcal{G}$ is rigid at 0 in A. ∎

EXAMPLE 3.7. Let $\mathcal{G}$ be a directed graph, and let $\mathcal{G} := \{P_v, S_e : v \in V, e \in E\}$ be the universal Cuntz–Krieger family generating $\mathcal{O}(G)$. Then $\mathrm{span}(\mathcal{G})$ obviously contains an approximate identity for $\mathcal{O}(G)$, namely, the finite sums of the form $\sum_{v \in F} P_v$ where F runs over all finite subsets of V. Thus, $\mathcal{G}$ is rigid at 0 in $\mathcal{O}(G)$. Note

that when G is row-finite, this already follows from Theorem 3.9 of [10] (together with Theorem 3.3).

When A is a nonunital C^*-algebra generated by a subset G, it is natural to ask not only whether G is hyperrigid in A, but also whether $G \cup \{1\}$ is hyperrigid in A^1; see Section 6 of [4]. The answer depends on whether G is rigid at 0 or not.

PROPOSITION 3.8. *Let A be a nonunital C^*-algebra generated by a subset G, and let A^1 denote its minimal unitization. Then $G \cup \{1\}$ is hyperrigid in A^1 if and only if G is rigid at* 0 *and hyperrigid in A.*

Proof. If $G \cup \{1\}$ is hyperrigid in A^1, then obviously G is hyperrigid in A. As for rigidity at 0, by Theorem 2.8 we have that for every unital $*$-representation $\pi : A^1 \to B(H)$, $\pi|_G$ has the unique extension property. By Proposition 2.6, this implies that for every (perhaps degenerate) $*$-representation $\pi : A \to B(H)$ — and, in particular, for the zero representation — $\pi|_G$ has the unique extension property. By Theorem 3.3, G is rigid at 0 in A.

Conversely, assume that G is rigid at 0 and hyperrigid in A. As G is hyperrigid, Theorem 2.8 implies that for any nondegenerate $*$-representation $\pi : A \to B(H)$, $\pi|_G$ has the unique extension property. As G is rigid at 0, Theorem 3.3 implies that for the zero representation $0 : A \to \mathbb{C}$, $0|_G$ has the unique extension property. By Proposition 2.6, for every unital $*$-representation $\pi : A^1 \to B(H)$, $\pi|_{G \cup \{1\}}$ has the unique extension property. Thus, Theorem 2.8 implies that $G \cup \{1\}$ is hyperrigid in A^1. ∎

In the following theorem we give some equivalent conditions for a set of generators G of a C^*-algebra A being both rigid at 0 and hyperrigid in A; we assume, without the loss of generality, that when A is unital G conatins the unit.

THEOREM 3.9. *Let A be a C^*-algebra generated by a subset G. Then the following conditions are equivalent:*

(i) *G is rigid at* 0 *and hyperrigid in A;*

(ii) *for every $*$-representation $\pi : A \to B(H)$ on a separable Hilbert space H, $\pi|_G$ has the unique extension property;*

(iii) *for every faithful $*$-representation $A \subseteq B(H)$ and every sequence of unital completely positive maps $\phi_n : B(H) \to B(H)$, we have that*

$$\lim_{n \to \infty} \phi_n(g) = g \quad \text{for all } g \in G \implies \lim_{n \to \infty} \phi_n(a) = a \quad \text{for all } a \in A$$

where all convergences are in norm.
If A is unital , then (i)–(iii) *are also equivalent to*

(iv) *G is hyperrigid in A.*
If A is nonunital, then (i)–(iii) *are also equivalent to*

(v)[1] *$G \cup \{1\}$ is hyperrigid in A^1.*

Proof. If A is unital and $1 \in \mathcal{G}$, then $\mathcal{G}$ is always rigid at 0 (see Remark 3.2), so in this case we have (i) $\Leftrightarrow$ (iv). If A is nonunital, then by Proposition 3.8 we have (i) $\Leftrightarrow$ (iv)[1].

(i) $\Rightarrow$ (ii) Assume that $\mathcal{G}$ is both rigid at 0 and hyperrigid in A. As $\mathcal{G}$ is rigid at 0 by Theorem 3.3 we have that for the zero representation $0 : A \to \mathbb{C}$, $0|_{\mathcal{G}}$ has the unique extension property. As $\mathcal{G}$ is hyperrigid, by Theorem 2.8 we have that for any nondegenerate $*$-representation $\pi : A \to B(H)$, $\pi|_{\mathcal{G}}$ has the unique extension property. By the equivalence (i) $\Leftrightarrow$ (ii) of Proposition 2.6, we are done.

(ii) $\Rightarrow$ (iii) follows by Lemma 2.7(i).

(iii) $\Rightarrow$ (i) We clearly have that condition (iii) implies hyperrigidity, so we only need to show it implies rigidity at 0. We will do so by obtaining the equivalent condition (ii) from Theorem 3.3. Let $\iota : A \hookrightarrow B(H)$ be a faithful nondegenerate $*$-representation and let $\phi_n : B(H) \to B(H)$ be a sequence of unital completely positive maps satisfying $\lim_{n \to \infty} \phi_n(g) = 0$ for every $g \in \mathcal{G}$. Consider the faithful (degenerate) $*$-representation $\iota \oplus 0 : A \hookrightarrow B(H \oplus H)$ and identify A with $\iota \oplus 0(A)$. For every $n \in \mathbb{N}$, let ψ_n denote the extension to $B(H \oplus H)$ of $\mathrm{id}_{B(H)} \oplus \phi_n$. Then $\lim_{n \to \infty} \psi_n(g) = g$ for every $g \in \mathcal{G}$, so $\lim_{n \to \infty} \psi_n(a) = a$ for every $a \in A$. Thus, $\lim_{n \to \infty} \phi_n(a) = 0$ for every $a \in A$. ∎

4. HYPERRIGIDITY OF THE EDGE SET

Let $G = (V, E, s, r)$ be a directed graph. Recall that the Cuntz–Krieger algebra $\mathcal{O}(G)$ is the universal C^*-algebra generated by a Cuntz–Krieger G-family $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{P_v : v \in V\}$ and $\mathcal{E} := \{S_e : e \in E\}$ satisfying the relations (I), (TCK) and (CK) described in the Introduction. We sometimes call $\mathcal{V}$ and $\mathcal{E}$ the vertex set and the edge set of $\mathcal{O}(G)$, respectively. As was mentioned in the Introduction (see Theorems 3.5 and 3.9 of [10]), it is known that $\mathcal{V} \cup \mathcal{E}$ is hyperrigid in $\mathcal{O}(G)$ if and only if G is row-finite, so if G is non-row-finite, $\mathcal{E}$ can never be hyperrigid in $\mathcal{O}(G)$.

If G is row-finite, then $\mathcal{E}$ generates $\mathcal{O}(G)$ if and only if G contains no isolated vertices (namely, there are no vertices that emit and receive no edges). In this section, we show that in this case $\mathcal{E}$ is hyperrigid in $\mathcal{O}(G)$. We start with a finite graph and then continue, by taking inductive limits, to any row-finite graph.

4.1. FINITE GRAPHS. A directed graph $G = (V, E, s, r)$ is called *finite* if both V and E are finite. To show that for finite graphs with no isolated vertices $\mathcal{E}$ is hyperrigid in $\mathcal{O}(G)$, we will use the machinery of maximal dilations of unital completely positive maps on operator systems.

Let S be an operator system. A unital completely positive map $\phi : S \to B(H)$ is said to be *maximal* if whenever $\psi : S \to B(K)$ is a unital completely positive map dilating ϕ — that is, $K \supseteq H$ and $\phi = P_H \psi(\cdot)|_H$ — then $\psi = \phi \oplus \rho$ for

some unital completely positive map ρ. It is known that any completely positive map $\phi : S \to B(H)$ dilates to a maximal map; see Theorem 1.3 of [3] (this was originally proved in [11], but in terms of operator algebras rather than operator systems).

The notions of maximality and the unique extension property are strongly related. Suppose A is a unital C^*-algebra generated by an operator system S, then a unital completely positive map $\phi : S \to B(H)$ is maximal if and only if it extends uniquely to a unital completely positive map on A and additionally this extension is $*$-representation. In particular, if $\pi : A \to B(H)$ is a unital $*$-representation, then $\pi|_S$ is maximal if and only if it has the unique extension property; see Proposition 2.2 of [3] (the original proof appeared in [19], but again in terms of operator algebras rather than operator systems).

THEOREM 4.1. *Let $G = (V, E, s, r)$ be a finite graph with no isolated vertices. Then $\mathcal{E} := \{S_e : e \in E\}$ is hyperrigid in $\mathcal{O}(G)$.*

Proof. Let $\pi : \mathcal{O}(G) \to B(H)$ be a unital $*$-representation. By Proposition 2.1 and Theorem 2.8 we need to show that $\pi|_{\mathcal{E}+\mathcal{E}^*+\mathbb{C}1}$ has the unique extension property, or equivalently, that it is maximal. Let $\widetilde{\rho}$ be a maximal dilation of $\pi|_{\mathcal{E}+\mathcal{E}^*+\mathbb{C}1}$ to a Hilbert space $K \supseteq H$, and let $\rho : \mathcal{O}(G) \to B(K)$ be its extension to a $*$-representation. Denote

$$\rho(S_e) = \begin{bmatrix} \pi(S_e) & X_e \\ Y_e & Z_e \end{bmatrix} \quad \text{for all } e \in E.$$

Let W be a subset of V containing all sources and no sinks. For every $v \in W$ choose a representative $e_v \in s^{-1}(v)$. Then

$$1 = \sum_{v \in W} P_v + \sum_{v \in V \setminus W} P_v = \sum_{v \in W} S_{e_v}^* S_{e_v} + \sum_{v \in V \setminus W} \sum_{r(e)=v} S_e S_e^*.$$

Thus,

$$1 = \sum_{v \in W} \rho(S_{e_v})^* \rho(S_{e_v}) + \sum_{v \in V \setminus W} \sum_{r(e)=v} \rho(S_e) \rho(S_e)^*$$

$$= \sum_{v \in W} \begin{bmatrix} \pi(S_{e_v})^* \pi(S_{e_v}) + Y_{e_v}^* Y_{e_v} & * \\ * & * \end{bmatrix} + \sum_{v \in V \setminus W} \sum_{r(e)=v} \begin{bmatrix} \pi(S_e) \pi(S_e)^* + X_e X_e^* & * \\ * & * \end{bmatrix},$$

so $\sum\limits_{v \in W} Y_{e_v}^* Y_{e_v} + \sum\limits_{v \in V \setminus W} \sum\limits_{r(e)=v} X_e X_e^* = 0$. We therefore conclude that $X_e = 0$ for all $e \in r^{-1}(V \setminus W)$, and $Y_{e_v} = 0$ for all $v \in W$. Since the representatives $\{e_v\}_{v \in W}$ were chosen arbitrarily, we have $Y_e = 0$ for all $e \in s^{-1}(W)$.

Finally, note that choosing W to be the set of all sources in the graph, we obtain that $r^{-1}(V \setminus W) = E$, so $X_e = 0$ for *all* $e \in E$; similarly choosing $V \setminus W$ to be the set of all sinks in the graph, we obtain that $s^{-1}(W) = E$, so $Y_e = 0$ for *all* $e \in E$. Thus, $\pi|_{\mathcal{E}+\mathcal{E}^*+\mathbb{C}1}$ is maximal. ∎

4.2. ROW-FINITE GRAPHS. We will now show that for a row-finite directed graph G, whenever $\mathcal{E}$ generates $\mathcal{O}(G)$, it must be hyperrigid. For this, we first recall that a row-finite graph G is an inductive limit (or equivalently, a *direct union*) of a sequence of finite subgraphs, and consequently that $\mathcal{O}(G)$ is the inductive limit of the corresponding finite graph Cuntz–Krieger algebras. These results are considered as folklore, but we provide the details for completeness.

Let $G_1 = (V_1, E_1, s_1, r_1)$ and $G_2 = (V_2, E_2, s_2, r_2)$ be two directed graphs. Then G_1 is said to be a *subgraph* of G_2 if $V_1 \subseteq V_2$, $E_1 \subseteq E_2$, $s_1 = s_2|_{E_1}$ and $r_1 = r_2|_{E_1}$. If furthermore whenever $v \in V_1$ is not a source or an infinite receiver in G_1 we have $r_1^{-1}(v) = r_2^{-1}(v)$, then G_1 is said to be a *CK subgraph* of G_2.

Let $\{G_\alpha = (V_\alpha, E_\alpha, s_\alpha, r_\alpha)\}_{\alpha \in A}$ be a family of CK subgraphs of $G = (V, E, s, r)$ *directed under CK inclusion*, i.e., $(A, \leqslant)$ is a directed set of indices, and whenever $\alpha \leqslant \beta$ we have that G_α is a CK subgraph of G_β. Then $G = (V, E, r, s)$ is said to be the *direct union* of the family $\{G_\alpha\}_{\alpha \in A}$ if $V = \bigcup_\alpha V_\alpha$, $E = \bigcup_\alpha E_\alpha$, and for every $\alpha \in A, e \in E_\alpha$ we have $r(e) := r_\alpha(e), s(e) := s_\alpha(e)$.

PROPOSITION 4.2. *Let $G = (V, E, s, r)$ be a directed graph. Then G is row-finite if and only if it is a direct union of finite CK subgraphs. If G is a row-finite graph containing no isolated vertices, then we can choose the finite CK subgraph to contain no isolated vertices as well.*

Proof. Suppose G is row-finite. For every X finite subset of V, let E_X consists of all edges with $r(e) \in X$ (by the row-finiteness assumption E_X is finite) and let $V_X := X \cup s(E_X)$ (so V_X is finite as well). Let $s_X := s|_{E_X}$, $r_X := r|_{E_X}$, and $G_X := (V_X, E_X, s_X, r_X)$. Clearly, G_X is a subgraph of G. Since each $v \in V_X \setminus X$ is a source in G_X, and each $v \in X$ has $r_X^{-1}(v) = r^{-1}(v)$, we have that G_X is a CK subgraph of G. A similar argument shows that if $X \subseteq Y$, then G_X is a CK subgraph of G_Y. Thus, G is the inductive union of the family $\{G_X\}$ indexed over all finite subsets $X \subset V$, which is directed under CK inclusion.

If G is not row-finite, there is an infinite receiver $v \in V$, and any finite CK subgraph of G cannot contain all the edges $r^{-1}(v)$. Therefore, v is a source for *every* finite CK subgraph of G containing it. Indeed, if $G_1 = (V_1, E_1, s_1, r_1)$ is a finite CK subgraph of G containing v, and the latter is not a source for G_1, then $r_1^{-1}(v) = r^{-1}(v)$, which is impossible as $r^{-1}(v)$ is infinite. Thus, v must be a source in any union of finite CK subgraphs of G, so G is not the union of finite CK subgraphs.

Suppose now that G contains no isolated vertices, and let $S \subseteq V$ be the subset of all sources in G. Since G contains no isolated vertices, we can choose for each $v \in S$ a representative $e_v \in s^{-1}(v)$. For every X finite subset of V, let $X' = X \cup \{r(e_v) : v \in S \cap X\}$ (so X' is finite as well). Let $G_{X'} = (V_{X'}, E_{X'}, s_{X'}, r_{X'})$ be the CK subgraph of G described in the first paragraph of the proof (constructed with respect to X'). Then clearly, G is still the inductive union of the family $\{G_{X'}\}$ where X is a finite subset of V. Assume that $V_{X'}$ contain an isolated vertex v of

$G_{X'}$. As v is isolated it is not in $s(E_{X'})$, so $v \in X'$. But $E_{X'} = r^{-1}(X')$, so that v must be a source (of G). But then $v \in S \cap X' = S \cap X$, so that $r(e_v) \in X'$. Thus, $e_v \in E_{X'}$ and $s(e_v) = v$, which is a contradiction. ∎

PROPOSITION 4.3. *If $G_1 = (V_1, E_1, s_1, r_1)$ is a CK subgraph of a directed graph $G_2 = (V_2, E_2, s_2, r_2)$, then there is an embedding of C*-algebras $\mathcal{O}(G_1) \hookrightarrow \mathcal{O}(G_2)$ mapping generators to generators.*

Proof. As G_1 is a subgraph of G_2 we have $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. We show that the Cuntz–Krieger family $\{P_v, S_e : v \in V_1, e \in E_1\} \subseteq \mathcal{O}(G_2)$ of G_2 is also a Cuntz–Krieger family for G_1. Condition (I) is clearly satisfied. As for (CK), let $v \in V_1$. If v is either a source in G_1 or an infinite receiver, then there is nothing to check. Otherwise, since G_1 is a *CK subgraph* in G_2, we have $r_1^{-1}(v) = r_2^{-1}(v)$, so $\sum_{e \in r_1^{-1}(v)} S_e S_e^* = \sum_{e \in r_2^{-1}(v)} S_e S_e^* = P_v$, and (CK) is satisfied as well. Thus, by the universality of $\mathcal{O}(G_1)$, there exists a $*$-homomorphism $\varphi : \mathcal{O}(G_1) \to \mathcal{O}(G_2)$, and by the gauge-invariant uniqueness theorem, φ must be injective. ∎

We now obtain the following corollary.

COROLLARY 4.4. *If G is a row-finite graph, then there is a directed family $\{G_\alpha : \alpha \in A\}$ of finite CK subgraphs of G such that the union of $\mathcal{O}(G_\alpha)$ is dense in $\mathcal{O}(G)$. If G is a row-finite graph containing no isolated vertices, then we can choose the finite CK subgraph G_α to contain no isolated vertices as well.*

We now show that hyperrigidity is preserved by taking inductive limits with injective connecting maps.

PROPOSITION 4.5. *Let A be a C*-algebra and $\mathcal{G}$ a generating subset of A. Suppose that there is a collection $\{A_\alpha\}$ of C*-subalgebras of A directed under inclusion and with a dense union, and let $\mathcal{G}_\alpha := \mathcal{G} \cap A_\alpha$. If for every α we have that $\mathcal{G}_\alpha$ is hyperrigid in A_α, then $\mathcal{G}$ is hyperrigid in A.*

Proof. Let $\pi : A \to B(H)$ be a nondegenerate $*$-representation. For every α, let $\pi_\alpha : A_\alpha \to B(H_\alpha)$ be the nondegenerate part of $\pi|_{A_\alpha} : A_\alpha \to B(H)$, that is, $H_\alpha = \overline{\pi(A_\alpha)H}$ and $\pi_\alpha = P_{H_\alpha} \pi|_{A_\alpha}(\cdot)|_{H_\alpha}$. By assumption, $\pi_\alpha|_{\mathcal{G}_\alpha}$ has the unique extension property.

Let $\varphi : A \to B(H)$ be a completely contractive completely positive extension of $\pi|_{\mathcal{G}}$, and consider $\varphi|_{A_\alpha} : A_\alpha \to B(H)$. Then $x \mapsto P_{H_\alpha} \varphi|_{A_\alpha}(x)|_{H_\alpha}$ is a completely contractive completely positive map $A_\alpha \to B(H_\alpha)$, and for every $g \in \mathcal{G}_\alpha$ we have

$$P_{H_\alpha} \varphi|_{A_\alpha}(g)|_{H_\alpha} = P_{H_\alpha} \pi(g)|_{H_\alpha} = \pi_\alpha(g).$$

Thus, $\pi(a) = P_{H_\alpha} \varphi(a)|_{H_\alpha}$ for every $a \in A_\alpha$. Now for every $a \in A_\alpha$ and $\xi \in H_\beta$ choose $\gamma \geqslant \alpha, \beta$. Then,

$$\pi(a)\xi = P_{H_\gamma} \pi(a)|_{H_\gamma}\xi = \pi_\gamma(a)\xi = P_{H_\gamma} \varphi(a)P_{H_\gamma}\xi = P_{H_\gamma} \varphi(a)\xi.$$

Since this holds for any $\gamma \geqslant \alpha, \beta$ and since $\bigcup H_\gamma$ is dense in H, we have that $\pi(a)\xi = \varphi(a)\xi$. As this is true for every $\xi \in H_\beta$ and $a \in A_\alpha$ and for every α and β we conclude that $\pi = \varphi$. ∎

Proposition 4.5 together with Theorem 4.1 and Corollary 4.4 implies that for row-finite graphs, whenever $\mathcal{E}$ generates $\mathcal{O}(G)$, then it must be hyperrigid.

THEOREM 4.6. *Let G be a row-finite graph with no isolated vertices. Then $\mathcal{E}$ is hyperrigid in $\mathcal{O}(G)$.*

Note that while for row-finite graphs an equivalent condition for $\mathcal{E}$ to generate $\mathcal{O}(G)$ is that the graph contains no isolated vertices, in the not necessarily row-finite case one needs to add to the latter condition the requirement that every infinite receiver is also an emitter.

COROLLARY 4.7. *Let G be a directed graph with no isolated vertices and in which every infinite receiver is an emitter. Then $\mathcal{E}$ is hyperrigid in $\mathcal{O}(G)$ if and only if $\mathcal{G}$ is row-finite.*

Proof. If G is not row-finite, then by (the proof of) Theorem 3.9 of [10], even the larger set $\{P_v, S_e : v \in V, e \in E\}$ is not hyperrigid in $\mathcal{O}(G)$, so that $\mathcal{E}$ itself obviously cannot be hyperrigid. If G is row-finite, then Theorem 4.6 implies that $\mathcal{E}$ is hyperrigid in $\mathcal{O}(G)$. ∎

Let A be an operator algebra. A C^*-*cover* of A is a pair (B, ι) consisting of a C^*-algebra B and a completely isometric homomorphism $\iota : A \to B$ such that $\iota(A)$ generates B. Arveson defined the notion of the C^*-envelope. Let A be a unital operator algebra. A C^*-cover $(C_e^*(A), \kappa)$ is called a C^*-*envelope* of A if for every other C^*-cover (B, ι) there exists a (necessarily unique and onto) $*$-homomorphism $\pi : B \to C_e^*(A)$ such that $\pi \circ \iota = \kappa$. In this case, $(C_e^*(A), \kappa)$ must be unique up to $*$-isomorphism. Hamana proved that every unital operator algebra admits a C^*-envelope ([13], Theorem 4.4).

If an operator algebra A is nonunital, then a theorem of Meyer ([5], Corollary 2.1.15) shows that it admits a unique minimal unitization. More precisely, if (ι, B) is a C^*-cover for the operator algebra A, and $B \subseteq B(H)$ is some faithful nondegenerate $*$-representation of B, then the operator algebraic structure of $A^1 \cong \iota(A) + \mathbb{C}I_H$ is independent of the C^*-cover B and of the faithful nondegenerate $*$-representation $B \subseteq B(H)$.

In this case, the C^*-envelope of A is defined as follows: let $(C_e^*(A^1), \kappa)$ be the C^*-envelope of A^1, then the C^*-envelope of A is $(C^*(\kappa(A)), \kappa)$ where $C^*(\kappa(A))$ is the C^*-algebra generated by $\kappa(A)$ in $C_e^*(A^1)$. It satisfies a universal property generalizing the one for C^*-envelopes of unital operator algebras: for any other C^*-cover (B, ι) of A there exists a (necessarily unique and onto) $*$-homomorphism $\pi : B \to C_e^*(A)$ such that $\pi \circ \iota = \kappa$.

The following proposition and corollary are known for unital operator algebras; see Theorem 4.1 of [11] and Section 2.2 of [1].

PROPOSITION 4.8. *Let B be a C^*-algebra generated by an operator algebra A. Let $\iota : B \to B(H)$ be some nondegenerate faithful $*$-representation. If $\iota|_A$ has the unique extension property, then B is the C^*-envelope of A.*

Proof. If A is unital, then this follows from the proof of Theorem 4.1 in [11]. If B is nonunital, then by Proposition 2.4, for the unital faithful $*$-representation $\iota^1 : B^1 \to B(H)$, $\iota^1|_{A^1}$ has the unique extension property. Thus, by the unital case $C_e^*(A^1) = B^1$, so that $C_e^*(A)$ is the C^*-algebra generated by A in B^1, namely, B.

Finally, if B is unital and A is not, then as $\iota|_A$ has the unique extension property, $\iota|_{A^1}$ must have the unique extension property as well. Thus, $C_e^*(A^1) = B$, so that $C_e^*(A)$ is the C^*-algebra generated by A in B, namely, B itself. ∎

COROLLARY 4.9. *Let B be a C^*-algebra generated by an operator algebra A. If A is hyperrigid in B, then $C_e^*(A) = B$.*

Proof. If B is nonunital or A is unital, then this follows from Proposition 4.8 together with Theorem 2.8. Otherwise, B is unital and A is not. If A is hyperrigid in B, then A^1 is hyperrigid in B. Thus, by the unital case, $C_e^*(A^1) = B$, so that $C_e^*(A)$ is the C^*-algebra generated by A in B, namely, B itself. ∎

Katsoulis and Kribs showed in Theorem 2.5 of [17] that $\mathcal{O}(G)$ is the C^*-envelope of the norm-closed algebra generated by the Cuntz–Krieger family — namely, the tensor-algebra $\mathcal{T}_+(G) = \overline{\mathrm{alg}}\{P_v, S_e : v \in V, e \in E\}$ (in fact, the tensor-algebra is usually defined as the norm-closed algebra generated by the universal *Toeplitz*–Cuntz–Krieger, but as $\mathcal{O}(G)$ is a C^*-cover of the latter algebra, the two definitions are equivalent). Our hyperrigidity result shows that for a row-finite graph G the smaller operator algebra $A(\mathcal{E}) := \overline{\mathrm{alg}}\{S_e : e \in E\}$ is hyperrigid in $\mathcal{O}(G)$, and therefore, the latter must be the C^*-envelope of $A(\mathcal{E})$.

COROLLARY 4.10. *Let G be a row-finite graph containing no isolated vertices, and let $A(\mathcal{E}) := \overline{\mathrm{alg}}\{S_e : e \in E\}$ be the operator algebra generated by $\mathcal{E}$ inside $\mathcal{O}(G)$. Then, $C_e^*(A(\mathcal{E})) = \mathcal{O}(G)$.*

5. RIGIDITY AT ZERO OF THE EDGE SET

Let $G = (V, E, s, r)$ be a directed graph. As was mentioned in the Introduction and in Example 3.7, the Cuntz–Krieger G-family $\mathcal{V} \cup \mathcal{E}$ is hyperrigid if and only if G is row-finite, and is rigid at 0 in any case. In the last section we showed that whenever $\mathcal{E}$ generates $\mathcal{O}(G)$, then it is hyperrigid in $\mathcal{O}(G)$ if and only if G is row-finite. In this section we show that $\mathcal{E}$ is never rigid at 0 in $\mathcal{O}(G)$. More precisely, we show that a generating subset $\mathcal{G} \subseteq \mathcal{V} \cup \mathcal{E}$ is rigid at 0 if and only if $\mathcal{V} \subseteq \mathcal{G}$.

THEOREM 5.1. *Let $G = (V, E, s, r)$ be a directed graph and let $\mathcal{G}$ be a subset of the Cuntz–Krieger family $\{P_v, S_e : v \in V, e \in E\}$. If there exists a vertex $v \in V$ such that $P_v \notin \mathcal{G}$, then $\mathcal{G}$ is not rigid at 0 in $\mathcal{O}(G)$.*

Proof. Let v be a vertex in V such that $P_v \notin \mathcal{G}$ and let $\mathcal{O}(G) \subseteq B(H)$ be a nondegenerate faithful $*$-representation. By Theorem 3.3 it suffices to show that for the zero map $0 : \mathcal{O}(G) \to B(H)$, $0|_{\mathcal{G}}$ does not have the unique extension property.

If there are no edges $e \in E$ with $r(e) = s(e) = v$ — namely, if there are no loops at v — then the nontrivial completely contractive completely positive map $\rho : \mathcal{O}(G) \to B(H)$ defined by $\rho(a) = P_v a P_v$ vanishes on $\mathcal{G}$, and we are done. On the other hand, if there are two (or more) loops, e and f, at v, then the nontrivial completely contractive completely positive map $\rho : \mathcal{O}(G) \to B(H)$ defined by $\rho(a) = S_f^* S_e^* P_v a P_v S_e S_f$ vanishes on $\mathcal{G}$. Thus, we may assume that there is exactly one loop at v, which we denote by e.

If there exists an edge $f \in E \setminus \{e\}$ with $r(f) = v$, then the nontrivial completely contractive completely positive map $\rho : \mathcal{O}(G) \to B(H)$ defined by $\rho(a) = S_f^* P_v a P_v S_f$ vanishes on $\mathcal{G}$.

Thus, the only case we need to consider is when there is only one edge going into v, namely, e. In this case $S_e^* S_e = P_v = S_e S_e^*$, so there exists a $*$-homomorphism $\pi : C(\mathbb{T}) \to C^*(S_e)$ mapping z to S_e. As $\mathcal{O}(G)$ admits a natural gauge action and π must be equivariant with respect to this gauge action, the gauge invariant uniqueness theorem implies that π is faithful. Thus, by Example 3.4, there exists a state on $C^*(S_e)$ vanishing on $\{S_e\}$. By Arveson's extension theorem it extends to a state φ on $P_v \mathcal{O}(G) P_v$. The nontrivial completely contractive completely positive map $\rho : \mathcal{O}(G) \to B(H)$ defined by $\rho(a) := \varphi(P_v a P_v) I_H$ must vanish on $\mathcal{G}$. ∎

REMARK 5.2. As the referee pointed out, one can prove Theorem 5.1 alternatively using an integration over the gauge action. More precisely, if $P_v \notin \mathcal{G}$ and γ denotes the natural gauge action, then an integration of $P_v \gamma_z(\cdot) P_v$ over the unit circle gives rise to a nontrivial completely contractive completely positive extension of $0|_{\mathcal{G}}$.

Recall that $\mathcal{O}(G)$ is unital if and only if G contains only finitely many vertices. In case $\mathcal{O}(G)$ is nonunital, then due to Proposition 3.8, whenever $\mathcal{G}$ is not rigid at 0, $\mathcal{G} \cup \{1\}$ is not hyperrigid in $\mathcal{O}(G)^1$.

COROLLARY 5.3. *Let $G = (V, E, s, r)$ be a directed graph with infinitely many vertices and let $\mathcal{G}$ be any subset of the Cuntz–Krieger family. If there exists a vertex $v \in V$ such that $P_v \notin \mathcal{G}$, then $\mathcal{G} \cup \{1\}$ is not hyperrigid in $\mathcal{O}(G)^1$.*

COROLLARY 5.4. *Let G be a directed graph with no isolated vertices and in which every infinite receiver is an emitter (or equivalently assume that $\mathcal{O}(G)$ is generated by $\mathcal{E}$). Then $\mathcal{E}$ is not rigid at 0 in $\mathcal{O}(G)$.*

EXAMPLE 5.5. Let $n \in \mathbb{N} \cup \{\infty\}$. Define a directed graph by setting $V = \{i : 1 \leqslant i < n+1\}$ and $E = \{e_i : 2 \leqslant i < n+1\}$ where for each $i \in V$ we have $s(e_i) = i$ and $r(e_i) = i - 1$. Set

$$E_{i,j} := S_{e_{i+1}} S_{e_{i+2}} \cdots S_{e_j}, \quad \text{for every } i < j;$$
$$E_{i,j} := E_{j,i}^*, \quad \text{for every } i > j; \quad \text{and}$$
$$E_{i,i} := E_{i,1} E_{1,i} = P_i.$$

A simple computation shows that the $E_{i,j}$'s are matrix units, so that $\mathcal{O}(G) = C^*(\mathcal{E})$ can be identified with the C^*-algebra $\mathcal{K}(H)$ of compact operators on a separable Hilbert space H (in case n is finite, then we obtain $M_n(\mathbb{C})$). By Corollaries 4.7 and 5.4, $\mathcal{E} = \{E_{i-1,i} : 2 \leqslant i < n+1\}$ is hyperrigid but not rigid at 0 in $\mathcal{O}(G)$. In addition, note that if $n = \infty$, by Proposition 3.8, $\mathcal{E} \cup \{1\}$ is not hyperrigid in $K(H)^1$.

To conclude this section we give the following observation for the case where the edge set $\mathcal{E}$ generates $\mathcal{O}(G)$. As we showed, in this case, $\mathcal{E}$ is hyperrigid if and only if G is row-finite, and $\mathcal{E}$ is never rigid at 0. The set $\mathcal{E} \cup \mathcal{E}^*\mathcal{E}$ is also hyperrigid if and only if G is row-finite and is always rigid at 0. To complete the picture, we show that for any graph G, row-finite or not, $\mathcal{E} \cup \mathcal{E}^*\mathcal{E} \cup \mathcal{E}\mathcal{E}^*$ is hyperrigid (and rigid at 0). More precisely, we have the following theorem.

THEOREM 5.6. *Let $G = (V, E, s, r)$ be a directed graph. Then the set*

$$\{P_v, S_e, S_e S_e^* : v \in V, e \in E\}$$

is hyperrigid and rigid at 0 in $\mathcal{O}(G)$.

Proof. First assume that V is infinite. By Propositions 3.8 and 2.1 we must show that the operator system $S = \text{span}\{1, P_v, S_e, S_e S_e^* : v \in V, e \in E\}$ is hyperrigid in $\mathcal{O}(G)^1$. To this end, we use Theorem 2.8. Let $\pi : \mathcal{O}(G)^1 \to B(H)$ be a unital $*$-representation. We will show that $\pi|_S$ has the unique extension property, or equivalently, that it is maximal. Let $\tilde{\rho} : S \to B(K)$ be a maximal dilation of $\pi|_S$, and let $\rho : \mathcal{O}(G)^1 \to B(K)$ be its extension to a $*$-representation. Denote

$$\rho(S_e) = \begin{bmatrix} \pi(S_e) & X_e \\ Y_e & Z_e \end{bmatrix} \quad \text{for all } e \in E, \quad \rho(P_v) = \begin{bmatrix} \pi(P_v) & X_v \\ Y_v & Z_v \end{bmatrix} \quad \text{for all } v \in V, \quad \text{and}$$

$$\rho(S_e S_e^*) = \begin{bmatrix} \pi(S_e S_e^*) & X_e' \\ Y_e' & Z_e' \end{bmatrix} \quad \text{for all } e \in E.$$

Let $P : K \to H$ denote the orthogonal projection of K onto H, then for every $v \in V$

$$P\rho(P_v)^*(1-P)\rho(P_v)P = P\rho(P_v)P - P\rho(P_v)P\rho(P_v)P = \pi(P_v) - \pi(P_v)\pi(P_v) = 0,$$

and the C^*-identity implies $Y_v = (1-P)\rho(P_v)P = 0$ for every $v \in V$. As $\rho(P_v)$ is self-adjoint, we have $X_v = 0$ for every $v \in V$ as well. A similar argument shows that $X_e' = Y_e' = 0$ for every $e \in E$. Now, for all $e \in E$, we have that $S_e^* S_e = P_{s(e)}$ so

that

$$\begin{bmatrix} \pi(P_{s(e)}) & 0 \\ 0 & * \end{bmatrix} = \rho(S_e^* S_e) = \rho(S_e)^* \rho(S_e) = \begin{bmatrix} \pi(S_e)^* \pi(S_e) + Y_e^* Y_e & * \\ * & * \end{bmatrix},$$

which implies $Y_e = 0$ for all $e \in E$.

Finally, for all $e \in E$ we have

$$\begin{bmatrix} \pi(S_e S_e^*) & 0 \\ 0 & * \end{bmatrix} = \rho(S_e S_e^*) = \rho(S_e)\rho(S_e)^* = \begin{bmatrix} \pi(S_e)\pi(S_e)^* + X_e X_e^* & * \\ * & * \end{bmatrix},$$

and we have that $X_e = 0$ for all $e \in E$. Thus π is maximal.

If V is finite, then a similar argument shows that the operator system $S = \mathrm{span}\{P_v, S_e, S_e S_e^* : v \in V, e \in E\}$ is hyperrigid and rigid at 0 in $\mathcal{O}(G)$. ∎

To summarize our results, we state the following theorem.

THEOREM 5.7. *Let G be a directed graph with no isolated vertices and in which every infinite receiver is an emitter (or equivalently assume that $\mathcal{O}(G)$ is generated by $\mathcal{E}$). Then inside $\mathcal{O}(G)$*

(i) *$\mathcal{E} \cup \mathcal{E}^*\mathcal{E} \cup \mathcal{E}\mathcal{E}^*$ is rigid at 0 and hyperrigid;*

(ii) *$\mathcal{E} \cup \mathcal{E}^*\mathcal{E}$ is rigid at 0, and is hyperrigid if and only if G is row-finite; and*

(iii) *$\mathcal{E}$ itself is not rigid at 0, and is hyperrigid if and only if G is row-finite.*

If, furthermore, G has infinitely many vertices, then inside $\mathcal{O}(G)^1$

(i[1]) *$\mathcal{E} \cup \mathcal{E}^*\mathcal{E} \cup \mathcal{E}\mathcal{E}^* \cup \{1\}$ is hyperrigid;*

(ii[1]) *$\mathcal{E} \cup \mathcal{E}^*\mathcal{E} \cup \{1\}$ is hyperrigid if and only if G is row-finite; and*

(iii[1]) *$\mathcal{E} \cup \{1\}$ is not hyperrigid.*

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