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# TAYLOR SPECTRA AND COMMON INVARIANT SUBSPACES THROUGH THE DUGGAL AND GENERALIZED ALUTHGE TRANSFORMS FOR COMMUTING $n$-TUPLES OF OPERATORS

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ABSTRACT. In the first part of this paper, we introduce two notions of multivariable Duggal transforms (toral and spherical), and study their basic properties including hyponormality and norm-continuity. In the second part, we study how the Taylor spectrum and Taylor essential spectrum of 2-variable weighted shifts behave under the toral and spherical Duggal transforms including generalized Aluthge transforms. In the last part, we investigate nontrivial common invariant subspaces between the toral (respectively spherical) Duggal transform and the original $n$-tuple of bounded operators with dense ranges. We also study the sets of common invariant subspaces among them.

KEYWORDS: *Duggal transforms, generalized Aluthge transforms, common invariant subspaces, Taylor spectra, commuting n-tuples, 2-variable weighted shifts.*

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## 1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. Recall that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *subnormal* if $T = N|_{\mathcal{H}}$, where $N$ is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *hyponormal* if $T^*T \geqslant TT^*$. We say that an $n$-tuple $\mathbf{T} := (T_1, \dots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & \cdots & [T_n^*, T_1] \\ \vdots & \ddots & \vdots \\ [T_1^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive (semi-definite) on the direct sum of $n$ copies of $\mathcal{H}$ (cf. [2], [4], [10]), where $[T_i^*, T_i] := T_i^*T_i - T_iT_i^*$. For $i = 1, 2$ and $T_i \in \mathcal{B}(\mathcal{H})$, we say that a pair $\mathbf{T} = (T_1, T_2)$ of operators on $\mathcal{H}$ is (jointly) *normal* if $\mathbf{T}$ is commuting and each $T_i$

is normal, and $\mathbf{T}$ is (jointly) *subnormal* if $\mathbf{T}$ is the restriction of a normal pair to a common invariant subspace. For $k \geqslant 1$, $\mathbf{T} = (T_1, T_2)$ is said to be *k-hyponormal* ([9]) if $\mathbf{T}(k) := (T_1, T_2, \ldots, T_1^k, T_2 T_1^{k-1}, \ldots, T_2^k)$ is (jointly) hyponormal. Clearly, for all $k \geqslant 1$, normal $\Rightarrow$ subnormal $\Rightarrow$ $k$-hyponormal.

For $T \in \mathcal{B}(\mathcal{H})$, the *polar decomposition* of $T$ is $T = U|T|$, the *Aluthge transform* $\widetilde{T}$ of $T$ is $\widetilde{T} := |T|^{1/2}U|T|^{1/2}$, the *generalized Aluthge transform* $\widetilde{T}^{\varepsilon}$ of $T$ is $\widetilde{T}^{\varepsilon} := |T|^{\varepsilon}U|T|^{1-\varepsilon}$, where $0 < \varepsilon < 1$, and the *Duggal transform* $\widetilde{T}^D$ of $T$ is $\widetilde{T}^D := |T|U$ ([1], [3], [15], [18], [20], [24]). In this paper, we set out to extend the Duggal and generalized Aluthge transforms to commuting $n$-tuples of bounded operators.

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting $n$-tuple of operators and consider $T = \begin{bmatrix} T_1 \\ \vdots \\ T_n \end{bmatrix}$ as an operator from $\mathcal{H}$ into $\bigoplus_{i=1}^{n} \mathcal{H}_i$, where $\mathcal{H}_i = \mathcal{H}$ for each $i = 1, \ldots, n$. Since $T$ is an operator from $\mathcal{H}$ into $\bigoplus_{i=1}^{n} \mathcal{H}_i$, $T$ has a standard singular-operator polar decomposition $T = VP$, that is,

$$\begin{bmatrix} T_1 \\ \vdots \\ T_n \end{bmatrix} = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} P,$$

where $V = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$ is a partial isometry from $\mathcal{H}$ to $\bigoplus_{i=1}^{n} \mathcal{H}_i$ and

$$P = (T^*T)^{1/2} = (T_1^*T_1 + \cdots + T_n^*T_n)^{1/2}$$

is a positive operator on $\mathcal{H}$. Then, $V^*V = V_1^*V_1 + \cdots + V_n^*V_n$ is the (orthogonal) projection onto the initial space of the partial isometry $V$ which in turn is

$$(\ker T)^{\perp} = \left( \bigcap_{i=1}^{n} \ker T_i \right)^{\perp}.$$

We can now define a polar decomposition of $\mathbf{T}$:

$$\mathbf{T} = (V_1 P, \ldots, V_n P)$$

(cf. [13], [14], [19]). Then, naturally we can get the *spherical* Duggal transform $\widehat{\mathbf{T}}^{\mathrm{D}}$ for $\mathbf{T}$ as the following:

$$\widehat{\mathbf{T}}^{\mathrm{D}} := (\widehat{T}_1^{\mathrm{D}}, \ldots, \widehat{T}_n^{\mathrm{D}}) = (PV_1, \ldots, PV_n). \tag{1.1}$$

Even though $\widehat{T}_i^{\mathrm{D}} = PV_i$ is not the Duggal transform of $T_i$ (for $i = 1, 2, \ldots, n$), the restriction of $V^*V = V_1^*V_1 + \cdots + V_n^*V_n$ to the range of $P$ is the identity operator. For a commuting $n$-tuple of operators $\mathbf{T} = (T_1, \ldots, T_n)$, we can define another polar decomposition of $\mathbf{T}$, that is,

$$\mathbf{T} = (T_1, \ldots, T_n) = (U_1|T_1|, \ldots, U_n|T_n|).$$

Thus, there is another Duggal transform (called *toral* Duggal transform) of $\mathbf{T}$ defined by

$$\widetilde{\mathbf{T}}^{\mathrm{D}} := (\widetilde{T}_1^{\mathrm{D}}, \ldots, \widetilde{T}_n^{\mathrm{D}}) = (|T_1|U_1, \ldots, |T_n|U_n).$$

We next define the generalized Aluthge transforms of $\mathbf{T} = (T_1, \ldots, T_n)$. The *generalized toral Aluthge transform* of $\mathbf{T}$ is defined by

$$\widetilde{\mathbf{T}}^{\varepsilon} := (\widetilde{T}_1^{\varepsilon}, \ldots, \widetilde{T}_n^{\varepsilon}) = (|T_1|^{\varepsilon}U_1|T|^{1-\varepsilon}, \ldots, |T_n|^{\varepsilon}U_n|T|^{1-\varepsilon}) \quad (0 < \varepsilon < 1). \tag{1.2}$$

The *generalized spherical Aluthge transform* of $\mathbf{T}$ is defined by

$$\widehat{\mathbf{T}}^{\varepsilon} := (\widehat{T}_1^{\varepsilon}, \ldots, \widehat{T}_n^{\varepsilon}) = (P^{\varepsilon}V_1P^{1-\varepsilon}, \ldots, P^{\varepsilon}V_nP^{1-\varepsilon}) \quad (0 < \varepsilon < 1). \tag{1.3}$$

For $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let

$$W_\alpha = \text{shift}(\alpha_0, \alpha_1, \ldots) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$$

be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geqslant 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The moments of $W_\alpha$ are given as

$$\gamma_k = \gamma_k(W_\alpha) := \begin{cases} 1 & \text{if } k = 0, \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0. \end{cases}$$

Similarly, consider double-indexed positive bounded sequences $\alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_+^2$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. We define the 2-variable weighted shift $W_{(\alpha,\beta)} = (T_1, T_2)$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\epsilon_1} \quad \text{and} \quad T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\epsilon_2},$$

where $\boldsymbol{\epsilon}_1 := (1, 0)$ and $\boldsymbol{\epsilon}_2 := (0, 1)$. Clearly,

$$T_1T_2 = T_2T_1 \Leftrightarrow \beta_{\mathbf{k}+\epsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\epsilon_2}\beta_{\mathbf{k}} \quad (\text{all } \mathbf{k} \in \mathbb{Z}_+^2). \tag{1.4}$$

In an entirely similar way one can define multivariable weighted shifts. Trivially, a pair of unilateral weighted shifts $W_\alpha$ and $W_\beta$ gives rise to a 2-variable weighted shift $W_{(\alpha,\beta)}$, if we let $\alpha_{(k_1,k_2)} := \alpha_{k_1}$ and $\beta_{(k_1,k_2)} := \beta_{k_2}$ (all $(k_1, k_2) \in \mathbb{Z}_+^2$). In this case, $W_{(\alpha,\beta)}$ is subnormal (respectively hyponormal) if and only if $T_1$ and $T_2$ are; in fact, under the canonical identification of $\ell^2(\mathbb{Z}_+^2)$ with $\ell^2(\mathbb{Z}_+) \otimes \ell^2(\mathbb{Z}_+)$, we have $T_1 \cong I \otimes W_\alpha$ and $T_2 \cong W_\beta \otimes I$, and so $W_{(\alpha,\beta)}$ is doubly commuting (see the definition given below).

Given $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}_+^2$, the moment of $W_{(\alpha,\beta)}$ of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}}(W_{(\alpha,\beta)}) := \begin{cases} 1 & \text{if } k_1 = 0 \text{ and } k_2 = 0, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geqslant 1 \text{ and } k_2 = 0, \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geqslant 1, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geqslant 1 \text{ and } k_2 \geqslant 1. \end{cases} \tag{1.5}$$

Due to the commutativity condition (1.4), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to $(k_1, k_2)$.

## 2. MAIN RESULTS

### 2.1. BASIC PROPERTIES FOR THE SPHERICAL AND TORAL DUGGAL TRANSFORMS.

Compared to the commuting properties of the spherical and toral Aluthge transforms ([13], [19]), the spherical and toral Duggal transforms for commuting $n$-tuples have better commuting properties. First, we have the following proposition.

PROPOSITION 2.1. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be a commuting n-tuple of operators. Then, the spherical Duggal transform* $\widehat{\mathbf{T}}^{\mathrm{D}}$ *is also commuting.*

*Proof.* Since $\mathbf{T} = (V_1 P, \dots, V_n P)$ is commuting, for $i, j \in \{1, 2, \dots, n\}$

$$[T_i, T_j] = V_i P V_j P - V_j P V_i P = (V_i P V_j - V_j P V_i) P = 0.$$

Thus, we have

$$V_i P V_j = V_j P V_i \quad \text{on} \overline{(\mathrm{Ran} P)}.$$

On the other hand, $\ker \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} = \ker P$ and so $\ker P \subset \ker V_i \cap \ker V_j$. Hence,

$$V_i P V_j = V_j P V_i \quad \text{on} \ker P.$$

Since $\mathcal{H} = \ker P \bigoplus \overline{(\mathrm{Ran} P)}$, we have

$$V_i P V_j = V_j P V_i. \tag{2.1}$$

Therefore, it follows from (2.1) that

$$[\widehat{T}_i^{\mathrm{D}}, \widehat{T}_j^{\mathrm{D}}] = P V_i P V_j - P V_j P V_i = P(V_i P V_j - V_j P V_i) = 0. \quad \blacksquare$$

For the toral Duggal transform $\widetilde{\mathbf{T}}^{\mathrm{D}}$ of a commuting $n$-tuple $\mathbf{T}$, $\widetilde{\mathbf{T}}^{\mathrm{D}}$ is not commuting in general (cf. Proposition 2.3(iv)). However, we can make it commuting in many cases. First, we recall that a commuting $n$-tuple $\mathbf{T} = (T_1, \dots, T_n)$ is said to *doubly commute* if $T_i T_j = T_j T_i$ and $T_i T_j^* = T_j^* T_i$ for all $i, j = 1, 2, \dots, n$ and $i \neq j$. Then, we have the following lemma.

LEMMA 2.2 ([19]). *Let* $\mathbf{T} = (U_1 |T_1|, \dots, U_n |T_n|)$ *be a doubly commuting n-tuple of injective operators. Then, we have for* $i, j = 1, 2, \dots, n$*, and* $i \neq j$

(i) $|T_i||T_j| = |T_j||T_i|$,
(ii) $U_i U_j = U_j U_i$, *and*
(iii) $|T_i|^{1/2} U_j = U_j |T_i|^{1/2}$.

Further, recall that a commuting $n$-tuple

$$\mathbf{T} = (T_1, \dots, T_n) = (U_1 |T_1|, \dots, U_n |T_n|)$$

is said to *isometrically commute* if $T_i U_j = T_j U_i$ for all $i, j = 1, \dots, n$ [19].

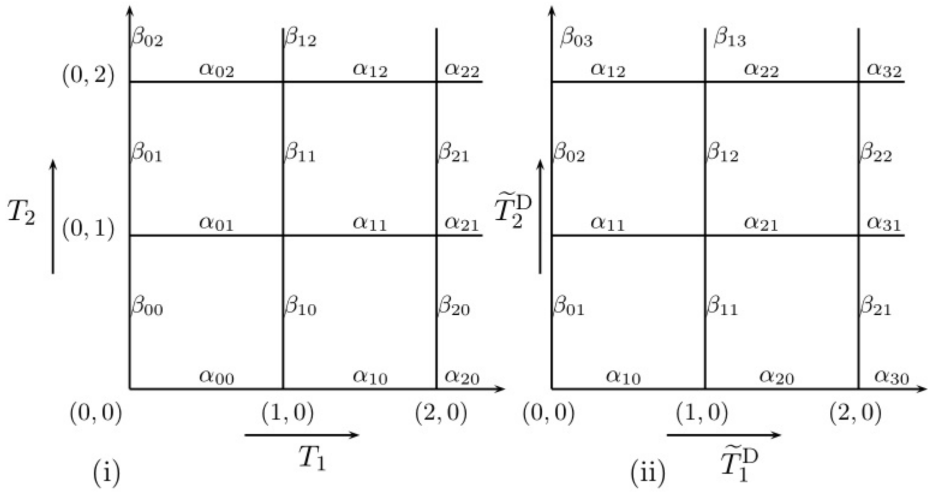


FIGURE 1. Weight diagram of a commutative 2-variable weighted shift $W_{(\alpha,\beta)} = (T_1, T_2)$ and weight diagram of its toral Duggal transform $\widetilde{W}_{(\alpha,\beta)} = (\widetilde{T}_1^{\mathrm{D}}, \widetilde{T}_2^{\mathrm{D}})$, respectively.

It is known ([13]) that for the toral Aluthge transform $\widetilde{W}_{(\alpha,\beta)}$ of $W_{(\alpha,\beta)}$ and $(k_1,k_2) \in \mathbb{Z}_+^2$,

$$\widetilde{W}_{(\alpha,\beta)} \text{ is commuting} \Leftrightarrow \alpha_{(k_1,k_2+1)}\alpha_{(k_1+1,k_2+1)} = \alpha_{(k_1+1,k_2)}\alpha_{(k_1,k_2+2)}.$$

The following results show when the toral Duggal transform $\widetilde{\mathbf{T}}^{\mathrm{D}}$ of $\mathbf{T}$ is commuting.

PROPOSITION 2.3. *Consider* $\mathbf{T} = (T_1,\dots,T_n) = (U_1|T_1|,\dots,U_n|T_n|)$.

(i) *If* $\{U_i\}_{i=1}^n$ *are self-adjoint, then* $\widetilde{\mathbf{T}}^{\mathrm{D}}$ *is commuting.*

(ii) *If* $\mathbf{T}$ *is doubly-commuting, then* $\widetilde{\mathbf{T}}^{\mathrm{D}}$ *is commuting.*

(iii) *If* $\mathbf{T}$ *is an isometrically commuting n-tuple of injective operators, then* $\widetilde{\mathbf{T}}^{\mathrm{D}}$ *is commuting.*

(iv) *If* $\mathbf{T} = W_{(\alpha,\beta)}$, *then* $\widetilde{\mathbf{T}}^{\mathrm{D}}$ *is commuting if and only if for all* $k_1,k_2 \geqslant 0$,

$$\alpha_{(k_1,k_2+1)}\alpha_{(k_1+1,k_2+1)} = \alpha_{(k_1+1,k_2)}\alpha_{(k_1,k_2+2)}.$$

*Proof.* (i) For $i,j = 1,2,\dots,n$, we let $T_i = U_i|T_i|$ and $T_j = U_j|T_j|$, where $U_i^* = U_i$ and $U_j^* = U_j$. Then $T_i^* = (U_i|T_i|)^* = |T_1|U_1^* = |T_i|U_i = \widetilde{T}_i^{\mathrm{D}}$ and $T_j^* = \widetilde{T}_j^{\mathrm{D}}$. Since $T_iT_j = T_jT_i$, we have that $T_i^*T_j^* = T_j^*T_i^*$ and so $\widetilde{T}_i^{\mathrm{D}}\widetilde{T}_j^{\mathrm{D}} = \widetilde{T}_j^{\mathrm{D}}\widetilde{T}_i^{\mathrm{D}}$.

(ii) By Lemma 2.2(iii), we have that for $i,j = 1,2,\dots,n$

$$[\widetilde{T}_i^{\mathrm{D}}, \widetilde{T}_j^{\mathrm{D}}] = |T_i||T_j|U_iU_j - |T_j||T_i|U_jU_i = |T_i||T_j|U_iU_j - |T_i||T_j|U_iU_j = 0.$$

(iii) By the definition of $\widetilde{\mathbf{T}}^{\mathrm{D}}$ and following the same way used in Proposition 2.16 of [19], we have

$$|T_i| = |T_j| \quad \text{for all } i, j = 1, \ldots, n. \tag{2.2}$$

Hence, we get that

$$[\widetilde{T}_i^{\mathrm{D}}, \widetilde{T}_j^{\mathrm{D}}] = |T_i|(U_i|T_j|U_j - U_j|T_i|U_i) = |T_i|(U_i|T_i|U_j - U_j|T_j|U_i) = 0.$$

(iv) Let $W_\alpha = \mathrm{shift}(\alpha_0, \alpha_1, \ldots)$. Then its polar decomposition is $W_\alpha = U_+ D$, where $U_+$ is the unilateral shift and $D = \mathrm{diag}(\alpha_0, \alpha_1, \ldots)$. Hence, the Duggal transform $\widetilde{W}_\alpha^{\mathrm{D}}$ of $W_\alpha$ is $\widetilde{W}_\alpha^{\mathrm{D}} = DU_+ = \mathrm{shift}(\alpha_1, \alpha_2, \ldots)$. Consider a 2-variable weighted shift $W_{(\alpha,\beta)} = (T_1, T_2)$ whose diagram is given as in Figure 1(i). Then, for $(k_1, k_2) \in \mathbb{Z}_+^2$ we have

$$\widetilde{T}_1^{\mathrm{D}} \widetilde{T}_2^{\mathrm{D}} e_{(k_1,k_2)} = \widetilde{T}_1^{\mathrm{D}}(\beta_{(k_1,k_2+1)} e_{(k_1,k_2+1)}) = \alpha_{(k_1+1,k_2+1)} \beta_{(k_1,k_2+1)} e_{(k_1+1,k_2+1)}. \tag{2.3}$$

On the other hand,

$$\widetilde{T}_2^{\mathrm{D}} \widetilde{T}_1^{\mathrm{D}} e_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)} \beta_{(k_1+1,k_2+1)} e_{(k_1+1,k_2+1)}. \tag{2.4}$$

From (1.4), (2.3), and (2.4) it follows that $\widetilde{T}_1^{\mathrm{D}} \widetilde{T}_2^{\mathrm{D}} = \widetilde{T}_2^{\mathrm{D}} \widetilde{T}_1^{\mathrm{D}}$ if and only if

$$\alpha_{(k_1,k_2+1)} \alpha_{(k_1+1,k_2+1)} = \alpha_{(k_1+1,k_2)} \alpha_{(k_1,k_2+2)}, \tag{2.5}$$

as desired. ∎

For an arbitrary 2-variable weighted shift $W_{(\alpha,\beta)}$, we let $\mathcal{M}_i$ (respectively $\mathcal{N}_j$) be the subspace of $\ell^2(\mathbb{Z}_+^2)$ spanned by the canonical orthonormal basis associated to indices $\mathbf{k} = (k_1, k_2)$ with $k_1 \geqslant 0$ and $k_2 \geqslant i$ (respectively $k_1 \geqslant j$ and $k_2 \geqslant 0$). We will often write $\mathcal{M}_1$ simply as $\mathcal{M}$ and $\mathcal{N}_1$ as $\mathcal{N}$. The *core* $c(W_{(\alpha,\beta)})$ of $W_{(\alpha,\beta)}$ is the restriction of $W_{(\alpha,\beta)}$ to the invariant subspace $\mathcal{M}_1 \cap \mathcal{N}_1$. $W_{(\alpha,\beta)}|_{\mathcal{M}}$ means the restriction of $W_{(\alpha,\beta)}$ to the invariant subspace $\mathcal{M}$. A 2-variable weighted shift $W_{(\alpha,\beta)} = (T_1, T_2)$ is said to be of *tensor form* if it is of the form $(I \otimes W_\alpha, W_\beta \otimes I)$, where $W_\alpha$ and $W_\beta$ are unilateral weighted shifts. The class of all 2-variable weighted shifts $W_{(\alpha,\beta)}$ whose core is of tensor form will be denoted by

$$\mathcal{TC} := \{W_{(\alpha,\beta)} : c(W_{(\alpha,\beta)}) \text{ is of tensor form}\}. \tag{2.6}$$

Given $W_{(\alpha,\beta)}$ and for $i, j \geqslant 0$, we let

$$W_{\alpha^{(j)}} = \mathrm{shift}(a_{0j}, a_{1j}, \ldots) \quad \text{and} \quad W_{\beta^{(i)}} = \mathrm{shift}(\beta_{i0}, \beta_{i1}, \ldots)$$

denote the associated $j$-th horizontal and $i$-th vertical slice of $W_{(\alpha,\beta)}$, respectively.

EXAMPLE 2.4. Consider a 2-variable weighted shift $W_{(\alpha,\beta)} = (T_1, T_2)$ whose diagram is given as in Figure 1(i), where $c(W_{(\alpha,\beta)}) = (I \otimes U_+, U_+ \otimes I)$, $W_{\alpha^{(0)}} = \mathrm{shift}(x_0, x_1, \ldots)$, $W_{\beta^{(0)}} = \mathrm{shift}(y_0, y_1, \ldots)$, and $U_+$ is the unilateral shift. Then, we have that

$$\widetilde{W}_{(\alpha,\beta)}^{\mathrm{D}} \text{ is commuting if and only if } x_1 = y_1 \text{ and } x_n = y_n = 1 \text{ for all } n \geqslant 1.$$

The proof follows from Proposition 2.3(iv) and

$$c(W_{(\alpha,\beta)}) = (I \otimes U_+, U_+ \otimes I).$$

The following example shows that the subnormality of the toral Duggal transform does not guarantee the hyponormality of the original one.

EXAMPLE 2.5. Consider a 2-variable weighted shift $W_{(\alpha,\beta)} = (T_1, T_2)$ whose diagram is given as in Figure 1(i), where $\alpha_{(0,0)} = 2$, $\beta_{(k_1,0)} = \frac{1}{2}$ ($k_1 \geqslant 1$), and otherwise 1. Then, $\widetilde{W}^{\mathrm{D}}_{(\alpha,\beta)}$ is commuting and subnormal, so it is $k$-hyponormal for all $k \geqslant 1$. However, we observe that $W_{(\alpha,\beta)}$ is not hyponormal, because $W_{\alpha^{(0)}} = \mathrm{shift}(2,1,1,\ldots)$ is not hyponormal.

It is known that for $T \in \mathcal{B}(\mathcal{H})$, the Aluthge transform map $T \to \widetilde{T}$ is $(\|\cdot\|, \|\cdot\|)$-continuous on $\mathcal{B}(\mathcal{H})$ [15]. We next study the norm-continuity property of the Duggal transform.

PROPOSITION 2.6. *For a single operator $T \in \mathcal{B}(\mathcal{H})$, the Duggal transform map $T \to \widetilde{T}^{\mathrm{D}}$ is $(\|\cdot\|, \|\cdot\|)$-contractive and continuous on $\mathcal{B}(\mathcal{H})$.*

*Proof.* Let $T = U|T|$ be the polar decomposition of $T$. Since $U$ is a partial isometry, $U$ is a contraction and so $U^*$ is also a contraction. Hence $|T|^2 \geqslant |T|UU^*|T|$, i.e., $T^*T = |T|^2 \geqslant \widetilde{T}^{\mathrm{D}}(\widetilde{T}^{\mathrm{D}})^*$. Therefore, $\|T\| \geqslant \|(\widetilde{T}^{\mathrm{D}})^*\| = \|\widetilde{T}^{\mathrm{D}}\|$, that is, the Duggal transform map is $(\|\cdot\|, \|\cdot\|)$-contractive.

For any $\varepsilon > 0$, let $p(t)$ be a real polynomial such that

$$\max_{t \in [0, \|T\|+1]} |p(t^2) - t| < \varepsilon.$$

Then, by the continuous functional calculus, we have that $\|p(|T|^2) - |T|\| < \varepsilon$, i.e., $\|p(T^*T) - |T|\| < \varepsilon$. Let $\delta > 0$ such that $\|T - S\| < \delta$ implies that $\|p(T^*T) - p(S^*S)\| < \varepsilon$. Then

$$\begin{aligned}
\|\widetilde{T}^{\mathrm{D}} - \widetilde{S}^{\mathrm{D}}\| &= \|\widetilde{T}^{\mathrm{D}} - p(T^*T)U + p(T^*T)U - p(S^*S)U + p(S^*S)U - \widetilde{S}^{\mathrm{D}}\| \\
&\leqslant \|\widetilde{T}^{\mathrm{D}} - p(T^*T)U\| + \|p(T^*T)U - p(S^*S)U\| + \|p(S^*S)U - \widetilde{S}^{\mathrm{D}}\| \\
&\leqslant \||T| - p(T^*T)\| + \|p(T^*T) - p(S^*S)\| + \|p(S^*S) - |S|\| \leqslant 3\varepsilon.
\end{aligned}$$

Thus, the Duggal transform map is $(\|\cdot\|, \|\cdot\|)$-continuous on $B(\mathcal{H})$. ∎

REMARK 2.7. We note that Proposition 2.6 can be extended to the multivariable case with the operator norm

$$\|\mathbf{T}\| := \max\{\|T_1\|, \ldots, \|T_n\|\} \quad \text{for } \mathbf{T} = (T_1, \ldots, T_n). \tag{2.7}$$

Thus, the toral Duggal transform map $\mathbf{T} \to \widetilde{\mathbf{T}}^{\mathrm{D}}$ is $(\|\cdot\|, \|\cdot\|)$-continuous on $\mathcal{B}(\mathcal{H})$.

We next investigate the commutativity of the spherical Duggal transform $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)} = (\widehat{T}^{\mathrm{D}}_1, \widehat{T}^{\mathrm{D}}_2)$ for $W_{(\alpha,\beta)} = (T_1, T_2)$. By the direct calculation, the spherical

Duggal transform $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}$ is a pair of weighted shifts with the following weights:

$$\begin{aligned} \widehat{\alpha}^{\mathrm{D}}_{(k_1,k_2)} &:= \alpha_{(k_1,k_2)}\sqrt{\frac{\alpha^2_{(k_1+1,k_2)}+\beta^2_{(k_1+1,k_2)}}{\alpha^2_{(k_1,k_2)}+\beta^2_{(k_1,k_2)}}} \quad \text{and} \\ \widehat{\beta}^{\mathrm{D}}_{(k_1,k_2)} &:= \beta_{(k_1,k_2)}\sqrt{\frac{\alpha^2_{(k_1,k_2+1)}+\beta^2_{(k_1,k_2+1)}}{\alpha^2_{(k_1,k_2)}+\beta^2_{(k_1,k_2)}}}. \end{aligned} \tag{2.8}$$

Therefore, by (1.4) and (2.8), we have

$$\widehat{T}^{\mathrm{D}}_1\widehat{T}^{\mathrm{D}}_2 = \widehat{T}^{\mathrm{D}}_2\widehat{T}^{\mathrm{D}}_1.$$

We summarize the above results as follows.

PROPOSITION 2.8. *Consider* $W_{(\alpha,\beta)} = (T_1, T_2)$*. Then,* $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)} = (\widehat{T}^{\mathrm{D}}_1, \widehat{T}^{\mathrm{D}}_2)$ *is commuting.*

Next, we consider the following proposition.

PROPOSITION 2.9. *Let* $W_{(\alpha,\beta)} = (T_1, T_2)$ *be a* 2*-variable weighted shift of hyponormal operators. Then so is* $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}$.

*Proof.* Clearly, $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}$ is a 2-variable weighted shift. Thus, we only show that the hyponormality of $\widehat{T}^{\mathrm{D}}_i$ $(i = 1, 2)$. Fix a lattice point $(k_1, k_2)$. Since the hyponormality of an operator is invariant under multiplication by a nonzero scalar, we may assume that $\alpha_{(k_1,k_2)} = 1$ and $\beta_{(k_1,k_2)} = 1$. For simplicity, let $\alpha_{(k_1,k_2+1)} = a$, $\alpha_{(k_1+1,k_2)} = b$, $\alpha_{(k_1+2,k_2)} = c$ and $\beta_{(k_1+2,k_2)} = d$. Now, without loss of generality, we will show that $\widehat{\alpha}^{\mathrm{D}}_{(k_1,k_2)} \leqslant \widehat{\alpha}^{\mathrm{D}}_{(k_1+1,k_2)}$. Since $T_1$ is hyponormal, we have $a \leqslant \frac{bd}{a}$, i.e., $a^2 \leqslant bd$, and $b \leqslant c$. Thus, we obtain that

$$\begin{aligned} (\widehat{\alpha}^{\mathrm{D}}_{(k_1,k_2)})^2 &= \frac{(a^2+b^2)^2}{2(a^2+b^2)} \leqslant \frac{a^4+2a^2b^2+b^4}{2(a^2+b^2)} \leqslant \frac{a^4+b^4}{a^2+b^2} \\ &\leqslant \frac{b^2d^2+b^2c^2}{a^2+b^2} = b^2\frac{c^2+d^2}{a^2+b^2} = (\widehat{\alpha}^{\mathrm{D}}_{(k_1+1,k_2)})^2. \quad \blacksquare \end{aligned}$$

REMARK 2.10. (i) Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be a quasi-affinity if it has a trivial kernel and dense range. If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-affinity hyponormal operator, then $\widehat{T}$ is also hyponormal. Indeed, let $T = U|T|$ be the polar decomposition. Then, $U$ is unitary and $|T|$ is a quasi-affinity. Since $T$ is hyponormal and $U$ is unitary, we have that

$$T^*T \geqslant TT^* \Rightarrow |T|^2 \geqslant U|T|^2U^* \Rightarrow U^*|T|^2U \geqslant |T|^2.$$

On the other hand, $1 \geqslant UU^*$ and so

$$|T|^2 \geqslant |T|UU^*|T|.$$

Hence, we get

$$U^*|T|^2U \geqslant |T|UU^*|T| \Leftrightarrow \widehat{T}^*\widehat{T} \geqslant \widehat{T}\widehat{T}^*.$$

(ii) Since a unilateral weighted shift is not a quasi-affinity, we still need to prove Proposition 2.9 for the general case.

EXAMPLE 2.11. For $0 < x, y < 1$, let $W_{(\alpha,\beta)}$ be a 2-variable weighted shift as in Figure 1(i), where $\alpha_{(0,0)} = x$, $\beta_{(0,0)} = y$, $\alpha_{(0,k_2)} = \frac{\sqrt{3}}{2}$ $(k_2 \geqslant 1)$, $\beta_{(k_1,0)} = \frac{\sqrt{3}y}{2x}$ $(k_1 \geqslant 1)$, and otherwise 1. Then, we have:

(i) $W_{(\alpha,\beta)}$ is hyponormal $\Leftrightarrow y \leqslant x\sqrt{\frac{16(1-x^2)}{9-8x^2}}$;

(ii) $W_{(\alpha,\beta)}$ is subnormal $\Leftrightarrow y \leqslant 2\sqrt{1-x^2}$;

(iii) $\widetilde{W}^{\mathrm{D}}_{(\alpha,\beta)}$ is always subnormal;

(iv) $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}$is always subnormal.

*Proof.* (i) By the definition of $W_{(\alpha,\beta)}$, it is obvious that

$$W_{(\alpha,\beta)}|_{\mathcal{M}} \cong (I \otimes S_{\sqrt{3}/2}, U_+ \otimes I) \quad \text{and} \quad W_{(\alpha,\beta)}|_{\mathcal{N}} \cong (I \otimes U_+, S_{\sqrt{3}y/2x} \otimes I),$$

where $S_{\sqrt{3}/2} = \text{shift}(\frac{\sqrt{3}}{2}, 1, 1, \ldots)$, $S_{\sqrt{3}y/2x} = \text{shift}(\frac{\sqrt{3}y}{2x}, 1, 1, \ldots)$, and $U_+$ is the unilateral shift. $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal with Berger measure

$$\mu_{\mathcal{M}} = \Big[\frac{1}{4}\delta_0 + \frac{3}{4}\delta_1\Big] \times \delta_1$$

and $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal with Berger measure $\mu_{\mathcal{N}} = \delta_1 \times \big[(1 - \frac{3y^2}{4x^2})\delta_0 + \frac{3y^2}{4x^2}\delta_1\big]$. By the result in Theorem 2.4 of [9], to show the joint hyponormality of $W_{(\alpha,\beta)}$ it is enough to check that

$$M_{(0,0)}(1)(W_{(\alpha,\beta)}) \geqslant 0.$$

Since $x < 1$, the positivity of $M_{(0,0)}(1)(W_{(\alpha,\beta)})$ is equivalent to

$$\det(M_{(0,0)}(1)(W_{(\alpha,\beta)})) \geqslant 0,$$

i.e.,

$$(1-x^2)(1-y^2) \geqslant \Big(\frac{3y}{4x} - yx\Big)^2,$$

which in turn is equivalent to

$$y \leqslant x\sqrt{\frac{16(1-x^2)}{9-8x^2}}.$$

(ii) Since $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are both subnormal, by the result in Proposition 3.10 of [11], we have that

$$\begin{aligned} W_{(\alpha,\beta)} \text{ is subnormal} &\Leftrightarrow \beta_{00}^2\Big\|\frac{1}{t}\Big\|_{L^1(\mu_{\mathcal{M}})}(\mu_{\mathcal{M}})^X_{\text{ext}} \leqslant \nu \\ &\Leftrightarrow y \leqslant 2\sqrt{1-x^2} \quad (\text{because}\sqrt{3}y \leqslant 2x). \end{aligned}$$

(iii) Since the Duggal transform $\widetilde{W}_{\alpha}^{\mathrm{D}}$ of $W_{\alpha} = \mathrm{shift}(\alpha_0, \alpha_1, \ldots)$ is

$$\widetilde{W}_{\alpha}^{\mathrm{D}} = \mathrm{shift}(\alpha_1, \alpha_2, \ldots),$$

we can observe that $\widetilde{W}_{(\alpha,\beta)}^{\mathrm{D}} = (I \otimes U_+, U_+ \otimes I)$. Hence $\widetilde{W}_{(\alpha,\beta)}^{\mathrm{D}}$ is subnormal.

(iv) By (2.8), we first observe that

$$c(\widehat{W}_{(\alpha,\beta)}^{\mathrm{D}}) = (I \otimes U_+, U_+ \otimes I),$$

$$\widehat{\alpha}_{(0,0)}^{\mathrm{D}} = \frac{\sqrt{x^2 + \frac{3y^2}{4}}}{\sqrt{x^2+y^2}}, \quad \widehat{\alpha}_{(0,k_2)}^{\mathrm{D}} = \sqrt{\frac{6}{7}} \quad (\text{all } k_2 \geqslant 1),$$

$$\widehat{\beta}_{(0,0)}^{\mathrm{D}} = \frac{\sqrt{7}y}{2\sqrt{x^2+y^2}}, \quad \text{and} \quad \widehat{\beta}_{(k_1,0)}^{\mathrm{D}} = \frac{\sqrt{6}y}{2\sqrt{x^2 + \frac{3y^2}{4}}} \quad (\text{all } k_1 \geqslant 1).$$

By Proposition 2.9, we note that

$$\widehat{\beta}_{(k_1,0)}^{\mathrm{D}} \leqslant 1 \quad (\text{all } k_1 \geqslant 1).$$

Hence, if we follow the proof of (ii), we have that $\widehat{W}_{(\alpha,\beta)}^{\mathrm{D}}$ is subnormal, as desired. ∎

REMARK 2.12. Example 2.11 shows that both the spherical and toral Duggal transforms may turn the given $W_{(\alpha,\beta)}$ into a more nicely behaved 2-variable weighted shift.

2.2. TAYLOR SPECTRA. In [17], I. Jung, E. Ko and C. Pearcy proved that $T$ and $\widetilde{T}$ have the same spectrum. In [3], M. Cho, I. Jung, and W.Y. Lee also proved that $T$ and $\widetilde{T}^{\mathrm{D}}$ have the same spectrum. In this section we show that these results may be extended to the toral and spherical Duggal transform including the generalized toral and spherical Aluthge transforms under certain circumstances. For this, we first introduce some terminology needed to describe the Taylor spectrum and Taylor essential spectrum of commuting $n$-tuples $\mathbf{T} = (T_1, \ldots, T_n)$. For additional facts about this notion of a joint spectrum, the reader is referred to ([5], [6], [7]). Let $\Lambda = \Lambda_n[e]$ be the *complex exterior algebra* on $n$ generators $e_1, \ldots, e_n$ with identity $e_0 = 1$, multiplication denoted by $\wedge$ (wedge product) and complex coefficients, subject to the collapsing property $e_i \wedge e_j + e_j \wedge e_i = 0$ $(1 \leqslant i, j \leqslant 1)$. If one declares $\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} : I \in \{1, \ldots, n\}\}$ to be an orthonormal basis, the exterior algebra becomes a Hilbert space with the canonical inner product, i.e., $\langle e_I, e_J \rangle := 0$ if $I \neq J$, $\langle e_I, e_J \rangle := 1$ if $I = J$. It also admits an orthogonal decomposition $\Lambda = \bigoplus_{i=0}^{n} \Lambda^i$ with $\Lambda^i \wedge \Lambda^k \subset \Lambda^{i+k}$. Moreover, $\dim \Lambda^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$. Let $E_i : \Lambda \to \Lambda$ denote the *creation operator*, given by $\xi \longmapsto e_i \wedge \xi$ $(i = 1, \ldots, n)$. We recall that $E_i^* E_j + E_j E_i^* = \delta_{ij}$ and $E_i$ is a partial isometry (all $i, j = 1, \ldots, n$). Consider a Hilbert space $\mathcal{H}$ and set $\Lambda(\mathcal{H}) := \bigoplus_{i=0}^{n} \mathcal{H} \otimes_{\mathbb{C}} \Lambda^i$. For a commuting $n$-tuple

$\mathbf{T} = (T_1, \ldots, T_n)$ of bounded operators on $\mathcal{H}$, define

$$D_{\mathbf{T}} : \Lambda(\mathcal{H}) \to \Lambda(\mathcal{H}) \quad \text{by } D_{\mathbf{T}}(x \otimes \xi) = \sum_{i=1}^{n} T_i x \otimes e_i \wedge \xi.$$

Then $D_{\mathbf{T}} \circ D_{\mathbf{T}} = 0$, so $\mathrm{Ran} D_{\mathbf{T}} \subseteq \ker D_{\mathbf{T}}$. This naturally leads to a cochain complex, called the *Koszul complex* $K(\mathbf{T},\mathcal{H})$ associated to $\mathbf{T}$ on $\mathcal{H}$, as follows:

$$K(\mathbf{T},\mathcal{H}) : 0 \xrightarrow{0} \mathcal{H} \otimes \Lambda^0 \xrightarrow{D^0_{\mathbf{T}}} \mathcal{H} \otimes \Lambda^1 \xrightarrow{D^1_{\mathbf{T}}} \cdots \xrightarrow{D^{n-1}_{\mathbf{T}}} \mathcal{H} \otimes \Lambda^n \xrightarrow{D^n_{\mathbf{T}}=0} 0,$$

where $D^i_{\mathbf{T}}$ denotes the restriction of $D_{\mathbf{T}}$ to the subspace $\mathcal{H} \otimes \Lambda^i$. We define $\mathbf{T}$ to be *invertible* in case its associated Koszul complex $K(\mathbf{T},\mathcal{H})$ is exact. Thus, we can define the Taylor spectrum $\sigma_{\mathrm{T}}(\mathbf{T})$ of $\mathbf{T}$ as follows:

$$\sigma_{\mathrm{T}}(\mathbf{T}) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : K((T_1 - \lambda_1, \ldots, T_n - \lambda_n), \mathcal{H}) \text{ is not invertible}\}.$$

$\mathbf{T}$ is called *Fredholm* if $\mathrm{ran} D_{\mathbf{T}}$ is closed and $\dim(\ker D_{\mathbf{T}}/\mathrm{ran} D_{\mathbf{T}}) < \infty$. We also define the Taylor essential spectrum $\sigma_{\mathrm{Te}}(\mathbf{T})$ of $\mathbf{T}$ as follows:

$$\sigma_{\mathrm{Te}}(\mathbf{T}) := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : (T_1 - \lambda_1, \ldots, T_n - \lambda_n) \text{ is not Fredholm}\}.$$

J.L. Taylor showed that, if $\mathcal{H} \neq \{0\}$, then $\sigma_{\mathrm{T}}(\mathbf{T})$ is a nonempty, compact subset of the polydisc of multiradius $r(\mathbf{T}) := (r(T_1), \ldots, r(T_n))$, where $r(T_i)$ is the spectral radius of $T_i$ ($i = 1, \ldots, n$) ([22], [23]).

Now recall the following lemma.

LEMMA 2.13. (i) ([5], [7]) *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $A_i \in \mathcal{B}(\mathcal{H}_1)$, $C_i \in \mathcal{B}(\mathcal{H}_2)$ and $B_i \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $(i = 1, \ldots, n)$ be such that*

$$\begin{pmatrix} \mathbf{A} & 0 \\ \mathbf{B} & \mathbf{C} \end{pmatrix} := \left( \begin{pmatrix} A_1 & 0 \\ B_1 & C_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_n & 0 \\ B_n & C_n \end{pmatrix} \right)$$

*is commuting. Assume that $\mathbf{A}$ and $\left(\begin{smallmatrix} \mathbf{A} & 0 \\ \mathbf{B} & \mathbf{C} \end{smallmatrix}\right)$ are Taylor invertible. Then, $\mathbf{C}$ is Taylor invertible. Furthermore, if $\mathbf{A}$ and $\mathbf{C}$ are Taylor invertible, then $\left(\begin{smallmatrix} \mathbf{A} & 0 \\ \mathbf{B} & \mathbf{C} \end{smallmatrix}\right)$ is Taylor invertible.*

(ii) ([8]) *For $\mathbf{A}$ and $\mathbf{B}$ two commuting n-tuples of bounded operators on Hilbert space, we have:*

$$\sigma_{\mathrm{T}}(I \otimes \mathbf{A}, \mathbf{B} \otimes I) = \sigma_{\mathrm{T}}(\mathbf{A}) \times \sigma_{\mathrm{T}}(\mathbf{B}) \quad \textit{and}$$
$$\sigma_{\mathrm{Te}}(I \otimes \mathbf{A}, \mathbf{B} \otimes I) = \sigma_{\mathrm{Te}}(\mathbf{A}) \times \sigma_{\mathrm{T}}(\mathbf{B}) \cup \sigma_{\mathrm{T}}(\mathbf{A}) \times \sigma_{\mathrm{Te}}(\mathbf{B}).$$

The following result shows that the toral Duggal transform preserves the Taylor spectrum.

EXAMPLE 2.14. Consider the 2-variable weighted shift $W_{(\alpha,\beta)} = \mathbf{T} = (T_1, T_2)$ whose weight diagram is given as in Figure 1(i), where $\alpha_{(k_1,k_2)} = (\frac{1}{2})^{k_2}$ and $\beta_{(k_1,k_2)} = (\frac{1}{2})^{k_1+1}$ for all $k_1, k_2 \geqslant 0$. Then, $\widetilde{\mathbf{T}}^{\mathrm{D}} = (\widetilde{T}^{\mathrm{D}}_1, \widetilde{T}^{\mathrm{D}}_2)$ is commuting and we have the following:

(i) $\sigma_{\mathrm{T}}(\mathbf{T}) = \sigma_{\mathrm{T}}(\widetilde{\mathbf{T}}^{\mathrm{D}}) = (\overline{\mathbb{D}} \times \{0\}) \cup (\{0\} \times \frac{1}{2}\overline{\mathbb{D}})$;

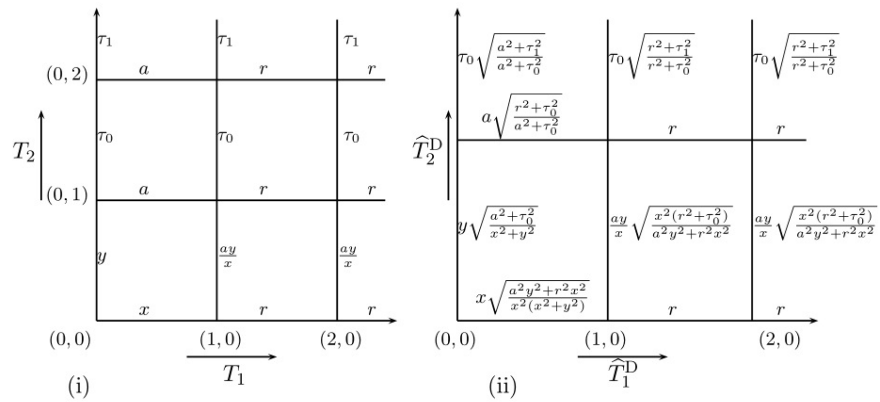

FIGURE 2. Weight diagram of the 2-variable weighted shift $W_{(\alpha,\beta)} = (T_1, T_2)$ in Example 2.17 and weight diagram of the spherical Duggal transform $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}$ of $W_{(\alpha,\beta)}$, respectively.

(ii) $\sigma_{\mathrm{Te}}(\mathbf{T}) = \sigma_{\mathrm{Te}}(\widetilde{\mathbf{T}}^{\mathrm{D}})$, where

$$\sigma_{\mathrm{Te}}(\mathbf{T}) = \Big(\Big(\{0\} \cup \Big(\textstyle\bigcup_{n=0}^{\infty} \frac{1}{2^n}\mathbb{T}\Big)\Big) \times \{0\}\Big) \cup \Big(\{0\} \times \Big(\{0\} \cup \Big(\textstyle\bigcup_{n=1}^{\infty} \frac{1}{2^n}\mathbb{T}\Big)\Big)\Big).$$

*Proof.* The commutativity of $(\widetilde{T}_1^{\mathrm{D}}, \widetilde{T}_2^{\mathrm{D}})$ is clear. Since $\ell^2(\mathbb{Z}_+^2) = \ell^2(\mathbb{Z}_+) \oplus \ell^2(\mathbb{Z}_+) \oplus \cdots$, we can write $T_1 = D \otimes U_+$ and $T_2 = U_+ \otimes \frac{1}{2}D$, where $D = \mathrm{diag}(1, \frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, \ldots)$.

(i) Let $p(z_1, z_2) := z_1 z_2$. Then by the spectral mapping theorem,

$$\begin{aligned} p(\sigma_{\mathrm{T}}(T_1, T_2)) &= \sigma_{\mathrm{T}}(p(T_1, T_2)) = \sigma_{\mathrm{T}}(T_1 T_2) = \sigma_{\mathrm{T}}\Big(\frac{1}{2}W \otimes \frac{1}{2}W\Big) \\ &= \sigma\Big(\frac{1}{2}W \otimes \frac{1}{2}W\Big) = \sigma\Big(\frac{1}{2}W\Big) \cdot \sigma\Big(\frac{1}{2}W\Big) = \{0\}, \end{aligned}$$

where $W = \mathrm{shift}(1, \frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, \ldots)$ is a compact operator with the spectrum $\{0\}$. Hence, we have that

$$\sigma_{\mathrm{T}}(T_1, T_2) \subseteq p^{-1}(\{0\}).$$

Since $T_1 = U_+ \oplus \frac{1}{2}T_1$ and $T_2 = \frac{1}{2}D \oplus T_2$, by Lemma 2.13(i), we obtain that

$$\sigma_{\mathrm{T}}(T_1, T_2) \subseteq (\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C}).$$

Now, by the result in Lemma 4.16 of [25], we have that

$$\sigma_{\mathrm{T}}(T_1, T_2) = (\pi_1(\sigma_{\mathrm{T}}(T_1, T_2)) \times \{0\}) \cup (\{0\} \times \pi_2(\sigma_{\mathrm{T}}(T_1, T_2))),$$

where $\pi_1(z_1, z_2) = z_1$ and $\pi_2(z_1, z_2) = z_2$. By the spectral mapping theorem,

$$\begin{aligned}\pi_1(\sigma_T(T_1, T_2)) &= \sigma_T(\pi_1(T_1, T_2)) = \sigma_T(T_1) = \sigma(T_1)\\ &= \sigma(U_+ \otimes D) = \sigma(U_+) \cdot \sigma(D) = \overline{\mathbb{D}} \cdot \left\{0, 1, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \dots\right\} = \overline{\mathbb{D}}.\end{aligned}$$

Similarly, we have that

$$\pi_2(\sigma_T(T_1, T_2)) = \frac{1}{2}\overline{\mathbb{D}}.$$

Hence

$$\sigma_T(T_1, T_2) = (\overline{\mathbb{D}} \times \{0\}) \cup \left(\{0\} \times \frac{1}{2}\overline{\mathbb{D}}\right). \tag{2.9}$$

(ii) By the result in Lemma 4.16 of [25], note that

$$\begin{aligned}\sigma_{Te}(T_1, T_2) &= (\pi_1(\sigma_{Te}(T_1, T_2)) \times \{0\}) \cup (\{0\} \times \pi_2(\sigma_{Te}(T_1, T_2)))\\ &= (\pi_1(\sigma_{Te}(T_1)) \times \{0\}) \cup (\{0\} \times \pi_2(\sigma_{Te}(T_2)))\\ &= (\pi_1(\sigma_e(T_1)) \times \{0\}) \cup (\{0\} \times \pi_2(\sigma_e(T_2))).\end{aligned}$$

Since $T_1 = \bigoplus_{n=0}^{\infty} \frac{1}{2^n} U_+$, we have that

$$\sigma_e(T_1) = \{0\} \cup \left(\bigcup_{n=0}^{\infty} \frac{1}{2^n}\sigma_e(U_+)\right) = \{0\} \cup \left(\bigcup_{n=0}^{\infty} \frac{1}{2^n}\mathbb{T}\right).$$

Similarly, we have that

$$\sigma_e(T_2) = \{0\} \cup \left(\bigcup_{n=1}^{\infty} \frac{1}{2^n}\mathbb{T}\right).$$

Hence, we obtain that

$$\sigma_{Te}(\mathbf{T}) = \left(\left(\{0\} \cup \left(\bigcup_{n=0}^{\infty} \frac{1}{2^n}\mathbb{T}\right)\right) \times \{0\}\right) \cup \left(\{0\} \times \left(\{0\} \cup \left(\bigcup_{n=1}^{\infty} \frac{1}{2^n}\mathbb{T}\right)\right)\right).$$

Note that $(T_1, T_2) = (\widetilde{T}_1^D, \widetilde{T}_2^D)$. Hence, we have $\sigma_T(\mathbf{T}) = \sigma_T(\widetilde{\mathbf{T}}^D)$ and $\sigma_{Te}(\mathbf{T}) = \sigma_{Te}(\widetilde{\mathbf{T}}^D)$, as desired. ∎

REMARK 2.15. It is known that if $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ is an invariant subspace for $\mathcal{H}$, then $\sigma(T|_{\mathcal{M}}) \subset \eta(\sigma(T))$, where $\eta(\sigma(T))$ is the union of $\sigma(T)$ and all bounded components of the resolvent set $\rho(T)$ ([21], Theorem 0.8). Similarly, for the $W_{(\alpha,\beta)} = \mathbf{T}$ in Theorem 2.14, we can see that $\sigma_T(\mathbf{T}|_{\mathcal{M}\cap\mathcal{N}}) \subseteq \sigma_T(\mathbf{T})$, indeed, by the spectral mapping theorem, we have that

$$\sigma_T(\mathbf{T}|_{\mathcal{M}\cap\mathcal{N}}) = \sigma_T\left(\frac{1}{2}\mathbf{T}\right) = \frac{1}{2}\sigma_T(\mathbf{T}).$$

Next recall the commuting property for the spherical Duggal transform $\widehat{W}^D_{(\alpha,\beta)}$ for $W_{(\alpha,\beta)}$ in Proposition 2.8. We now have the following proposition.

PROPOSITION 2.16. *Let $W_{(\alpha,\beta)} \in \mathcal{TC}$ with $c(W_{(\alpha,\beta)}) = (I \otimes W_\alpha, W_\beta \otimes I)$. Then, we have that for some $r, s > 0$,*

$$\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)} \in \mathcal{TC} \Leftrightarrow c(W_{(\alpha,\beta)}) = (rI \otimes U_+, W_\beta \otimes I) \quad or \quad (I \otimes W_\alpha, U_+ \otimes sI).$$

*Proof.* By (2.8), we first observe that

$$c(\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}) = \widehat{c(W_{(\alpha,\beta)})}^{\mathrm{D}}. \tag{2.10}$$

($\Rightarrow$) Since $W_{(\alpha,\beta)} \in \mathcal{TC}$ and $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)} \in \mathcal{TC}$, we observe that $\alpha_{(k_1,k_2)} = \alpha_{(k_1,k_2+1)}$, $\beta_{(k_1,k_2)} = \beta_{(k_1+1,k_2)}$, and $\widehat{T}^{\mathrm{D}}_1 e_{(k_1,k_2)} = \widehat{T}^{\mathrm{D}}_1 e_{(k_1,k_2+1)}$ for $(k_1,k_2) \geqslant (1,1)$. Thus, we have that

$$\begin{aligned}
\widehat{T}^{\mathrm{D}}_1 e_{(k_1,k_2)} &= \widehat{T}^{\mathrm{D}}_1 e_{(k_1,k_2+1)} \\
&\Leftrightarrow \frac{\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1,k_2)}}{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}} = \frac{\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1,k_2+1)}}{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2+1)}} \\
&\Leftrightarrow \frac{(\beta^2_{(k_1,k_2)} - \beta^2_{(k_1,k_2+1)})(\alpha^2_{(k_1,k_2)} - \alpha^2_{(k_1+1,k_2)})}{(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2+1)})(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)})} = 0 \\
&\Leftrightarrow \alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)} \quad \text{or} \quad \beta_{(k_1,k_2)} = \beta_{(k_1,k_2+1)}.
\end{aligned}$$

If $\beta_{(k_1,k_2)} \neq \beta_{(k_1,k_2+1)}$ for some $k_2 \geqslant 1$, then, by the commutativity of $W_{(\alpha,\beta)}$ and $W_{(\alpha,\beta)} \in \mathcal{TC}$, we have $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ for all $(k_1,k_2) \geqslant (1,1)$. Thus, $c(W_{(\alpha,\beta)}) = (rI \otimes U_+, W_\beta \otimes I)$ for some $r > 0$. If $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$ for all $(k_1,k_2) \geqslant (1,1)$, then $c(W_{(\alpha,\beta)}) = (I \otimes W_\alpha, U_+ \otimes sI)$ for some $s > 0$.

($\Leftarrow$) If $c(W_{(\alpha,\beta)}) = (rI \otimes U_+, W_\beta \otimes I)$ or $(I \otimes W_\alpha, U_+ \otimes sI)$ for some $r, s > 0$, then by (2.8), $\widehat{c(W_{(\alpha,\beta)})}^{\mathrm{D}}$ is of tensor form. Hence, by (2.10), we have $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)} \in \mathcal{TC}$, as desired. ∎

Since $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}$ is commuting, we have the following example.

EXAMPLE 2.17. Let $W_{(\alpha,\beta)} = (T_1, T_2) \in \mathcal{TC}$ be given as in Figure 2(i). Assume that $T_1$ and $T_2$ are hyponormal. Then, we have that

$$\sigma_{\mathrm{T}}(\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}) = \sigma_{\mathrm{T}}(W_{(\alpha,\beta)}) \quad \text{and} \quad \sigma_{\mathrm{Te}}(\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}) = \sigma_{\mathrm{Te}}(W_{(\alpha,\beta)}).$$

*Proof.* We have

$$W_x = \text{shift}(x, r, r, \ldots), \quad W_a = \text{shift}(a, r, r, \ldots), \quad \text{and} \quad D_y = \text{diag}\Big(y, \frac{ay}{x}, \frac{ay}{x}, \ldots\Big).$$

We first represent $T_1 = \begin{bmatrix} W_x & 0 & 0 \\ 0 & W_a & 0 \\ 0 & 0 & R_1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 0 & 0 & 0 \\ D_y & 0 & 0 \\ 0 & \tau_0 I & R_2 \end{bmatrix}$, where $R_1 = I \otimes W_a$ and

$$R_2 = \begin{bmatrix} 0 & 0 & 0 & \dots \\ \tau_1 I & 0 & 0 & \dots \\ 0 & \tau_2 I & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, by Lemma 2.13(i), we obtain that

$$\begin{aligned} \sigma_{\mathrm{T}}(T_1, T_2) &\subseteq \sigma_{\mathrm{T}}\left(\begin{bmatrix} ccW_x 0 & \\ 0 & W_a \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ D_y & 0 \end{bmatrix}\right) \cup \sigma_{\mathrm{T}}(I \otimes W_a, W_\tau \otimes I) \\ &\subseteq \sigma_{\mathrm{T}}(W_x, 0) \cup \sigma_{\mathrm{T}}(W_a, 0) \cup \sigma_{\mathrm{T}}(I \otimes W_a, W_\tau \otimes I) \\ &\subseteq (r\overline{\mathbb{D}} \times \{0\}) \cup (r\overline{\mathbb{D}} \times \{0\}) \cup (r\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}}) = r\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}}, \end{aligned} \tag{2.11}$$

where $W_\tau = \mathrm{shift}(\tau_1, \tau_2, \dots)$.

On the other hand,

$$\begin{aligned} \sigma_{\mathrm{T}}(I \otimes W_a, W_\tau \otimes I) &\subseteq \sigma_{\mathrm{T}}(T_1, T_2) \cup \sigma_{\mathrm{T}}\left(\begin{bmatrix} W_x & 0 \\ 0 & W_a \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ D_y & 0 \end{bmatrix}\right) \\ &\Rightarrow r\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}} \subseteq \sigma_{\mathrm{T}}(T_1, T_2) \cup (r\overline{\mathbb{D}} \times \{0\}) \\ &\Rightarrow (r\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}}) \backslash (r\overline{\mathbb{D}} \times \{0\}) \subseteq \sigma_{\mathrm{T}}(T_1, T_2). \end{aligned}$$

Since $\sigma_{\mathrm{T}}(T_1, T_2)$ is closed,

$$r\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}} \subseteq \sigma_{\mathrm{T}}(T_1, T_2). \tag{2.12}$$

By (2.11) and (2.12), we have

$$\sigma_{\mathrm{T}}(T_1, T_2) = r\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}}. \tag{2.13}$$

Since $W_{(\alpha,\beta)}$ is a compact perturbation of $(I \otimes W_a, W_{ay/x} \otimes I)$, where

$$W_{ay/x} = \mathrm{shift}\Big(\frac{ay}{x}, \tau_0, \tau_1, \dots\Big),$$

we have that

$$\begin{aligned} \sigma_{\mathrm{Te}}(T_1, T_2) &= \sigma_{\mathrm{Te}}(I \otimes W_a, W_{ay/x} \otimes I) \\ &= (\sigma_{\mathrm{Te}}(W_a) \times \sigma_{\mathrm{T}}(W_{ay/x})) \cup (\sigma_{\mathrm{T}}(W_a) \times \sigma_{\mathrm{Te}}(W_{ay/x})) \\ &= (r\mathbb{T} \times \|W_\tau\|\overline{\mathbb{D}}) \cup (r\overline{\mathbb{D}} \times \|W_\tau\|\mathbb{T}). \end{aligned} \tag{2.14}$$

Now, $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}$ is given as in Figure 2(ii). Since

$$\Big\|\mathrm{shift}\Big(y\frac{\sqrt{a^2+\tau_0^2}}{\sqrt{x^2+y^2}}, \tau_0\frac{\sqrt{a^2+\tau_1^2}}{\sqrt{z^2+\tau_0^2}}, \tau_1\frac{\sqrt{a^2+\tau_2^2}}{\sqrt{z^2+\tau_1^2}} \cdots\Big)\Big\| = \|W_\tau\|,$$

by the previous argument and Proposition 2.9, we have that

$$\sigma_{\mathrm{T}}(\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}) = r\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}} \quad \text{and}$$
$$\sigma_{\mathrm{Te}}(\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}) = (r\mathbb{T} \times \|W_\tau\|\overline{\mathbb{D}}) \cup (r\overline{\mathbb{D}} \times \|W_\tau\|\mathbb{T}). \quad \blacksquare$$

We next direct our attention to generalized (toral and spherical) Aluthge transforms.

PROPOSITION 2.18. (i) *If* $W_\alpha = \mathrm{shift}(\alpha_0, \alpha_1, \ldots)$ *is a weighted shift, then the generalized Aluthge transform* $\widetilde{W}^{\varepsilon}_{\alpha}$ *is*

$$\widetilde{W}^{\varepsilon}_{\alpha} = \mathrm{shift}(\alpha_0^{1-\varepsilon}\alpha_1^{\varepsilon}, \alpha_1^{1-\varepsilon}\alpha_2^{\varepsilon}, \ldots).$$

(ii) *Let* $W_{(\alpha,\beta)} = (T_1, T_2)$*. Then, we have* $\widehat{W}^{\varepsilon}_{(\alpha,\beta)} = (\widehat{T}^{\varepsilon}_1, \widehat{T}^{\varepsilon}_2)$*, where for* $(k_1, k_2) \in \mathbb{Z}^2_+$*,*

$$\widehat{T}^{\varepsilon}_1(e_{(k_1,k_2)}) = \alpha_{(k_1,k_2)} \frac{(\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1+1,k_2)})^{\varepsilon/2}}{(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)})^{\varepsilon/2}} e_{(k_1+1,k_2)} \quad \textit{and} \tag{2.15}$$

$$\widehat{T}^{\varepsilon}_2(e_{(k_1,k_2)}) = \beta_{(k_1,k_2)} \frac{(\alpha^2_{(k_1,k_2+1)} + \beta^2_{(k_1,k_2+1)})^{\varepsilon/2}}{(\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)})^{\varepsilon/2}} e_{(k_1,k_2+1)}. \tag{2.16}$$

*Proof.* (i) is clear from the definition of $\widetilde{W}^{\varepsilon}_{\alpha}$.

For (ii), note that

$$\widehat{T}^{\varepsilon}_i = P^{\varepsilon} V_i P^{1-\varepsilon} = P^{\varepsilon} T_i P^{-\varepsilon} \quad (i = 1, 2). \tag{2.17}$$

Thus, (2.15) and (2.16) are straightforward from Proposition 2.8 and (2.17). $\blacksquare$

THEOREM 2.19. *For* $W_{(\alpha,\beta)} = (T_1, T_2)$*, we have that*

$$\widetilde{W}^{\varepsilon}_{(\alpha,\beta)} = \widehat{W}^{\varepsilon}_{(\alpha,\beta)} \Leftrightarrow \alpha_{(k_1+1,k_2)} = \alpha_{(k_1,k_2+1)} \quad \textit{and}$$
$$\beta_{(k_1+1,k_2)} = \beta_{(k_1,k_2+1)} \quad \textit{(all } (k_1, k_2) \in \mathbb{Z}^2_+\textit{)}.$$

*Proof.* By Proposition 2.18, we have that

$$\widetilde{W}^{\varepsilon}_{(\alpha,\beta)} = \widehat{W}^{\varepsilon}_{(\alpha,\beta)} \quad \text{if and only if}$$
$$\frac{\alpha^{1-\varepsilon}_{(k_1,k_2)} \alpha^{\varepsilon}_{(k_1+1,k_2)}}{\alpha_{(k_1,k_2)}} = \Big( \frac{\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1+1,k_2)}}{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}} \Big)^{\varepsilon/2} \quad \text{and}$$
$$\frac{\beta^{1-\varepsilon}_{(k_1,k_2)} \beta^{\varepsilon}_{(k_1+1,k_2)}}{\alpha_{(k_1,k_2)}} = \Big( \frac{\alpha^2_{(k_1,k_2+1)} + \beta^2_{(k_1,k_2+1)}}{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}} \Big)^{\varepsilon/2}$$

for all $(k_1, k_2) \in \mathbb{Z}^2_+$. Since

$$\frac{\alpha^{1-\varepsilon}_{(k_1,k_2)} \alpha^{\varepsilon}_{(k_1+1,k_2)}}{\alpha_{(k_1,k_2)}} = \Big( \frac{\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1+1,k_2)}}{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}} \Big)^{\varepsilon/2}$$
$$\Leftrightarrow \alpha_{(k_1+1,k_2)} \beta_{(k_1,k_2)} = \alpha_{(k_1,k_2)} \beta_{(k_1+1,k_2)},$$

and, by the commutativity $\alpha_{(k_1,k_2)}\beta_{(k_1+1,k_2)} = \alpha_{(k_1,k_2+1)}\beta_{(k_1,k_2)}$, we have that

$$\frac{\alpha^{1-\varepsilon}_{(k_1,k_2)}\alpha^{\varepsilon}_{(k_1+1,k_2)}}{\alpha_{(k_1,k_2)}} = \left(\frac{\alpha^2_{(k_1+1,k_2)} + \beta^2_{(k_1+1,k_2)}}{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}}\right)^{\varepsilon/2} \Leftrightarrow \alpha_{(k_1+1,k_2)} = \alpha_{(k_1,k_2+1)}.$$

Similarly, we have that

$$\frac{\beta^{1-\varepsilon}_{(k_1,k_2)}\beta^{\varepsilon}_{(k_1+1,k_2)}}{\alpha_{(k_1,k_2)}} = \left(\frac{\alpha^2_{(k_1,k_2+1)} + \beta^2_{(k_1,k_2+1)}}{\alpha^2_{(k_1,k_2)} + \beta^2_{(k_1,k_2)}}\right)^{\varepsilon/2} \Leftrightarrow \beta_{(k_1+1,k_2)} = \beta_{(k_1,k_2+1)}.$$

Hence, the result follows. ∎

PROPOSITION 2.20. *Let $W_{(\alpha,\beta)} = (T_1, T_2)$ be a 2-variable weighted shift of hyponormal operators. Then, so is $\widehat{W}^{\varepsilon}_{(\alpha,\beta)}$.*

The proof is straightforward from the proof of Proposition 2.9.

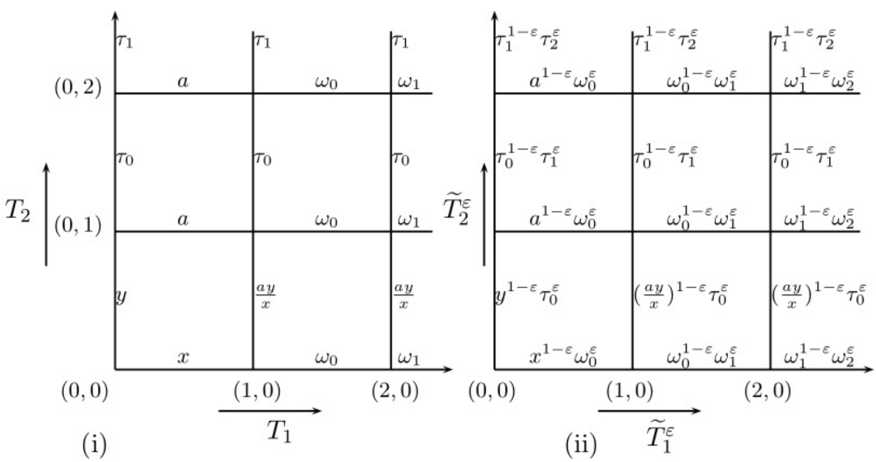

FIGURE 3. Weight diagram of the 2-variable weighted shift in Theorem 2.21 and weight diagram of its generalized toral Aluthge transform $\widetilde{W}^{\varepsilon}_{(\alpha,\beta)}$, respectively.

THEOREM 2.21. *Let $W_{(\alpha,\beta)} = (T_1, T_2)$ be given as in Figure 3(i). Assume that $T_1$ and $T_2$ are hyponormal. Then, we have:*

(i) $\sigma_{\mathrm{T}}(W_{(\alpha,\beta)}) = \|W_\omega\|\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}}$;

(ii) $\sigma_{\mathrm{Te}}(W_{(\alpha,\beta)}) = (\|W_\omega\|\mathbb{T} \times \|W_\tau\|\overline{\mathbb{D}}) \cup (\|W_\omega\|\overline{\mathbb{D}} \times \|W_\tau\|\mathbb{T})$;

(iii) $\sigma_{\mathrm{T}}(\widetilde{W}^{\varepsilon}_{(\alpha,\beta)}) = \sigma_{\mathrm{T}}(W_{(\alpha,\beta)})$ *and* $\sigma_{\mathrm{Te}}(\widetilde{W}^{\varepsilon}_{(\alpha,\beta)}) = \sigma_{\mathrm{Te}}(W_{(\alpha,\beta)})$.

*Proof.* Let

$$W_x = \mathrm{shift}(x, \omega_0, \omega_1, \ldots), \quad W_a = \mathrm{shift}(a, \omega_0, \omega_1, \ldots) \quad \text{and}$$
$$D_y = \mathrm{diag}\Big(y, \frac{ay}{x}, \frac{ay}{x}, \ldots\Big).$$

We consider

$$T_1 = \begin{bmatrix} W_x & 0 & 0 \\ 0 & W_a & 0 \\ 0 & 0 & R_1 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 & 0 & 0 \\ D_y & 0 & 0 \\ 0 & \tau_0 I & R_2 \end{bmatrix},$$

where

$$R_1 = I \otimes W_a, \quad R_2 = \begin{bmatrix} 0 & 0 & 0 & \ldots \\ \tau_1 I & 0 & 0 & \ldots \\ 0 & \tau_2 I & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = W_{\tau_1} \otimes I, \quad W_{\tau_1} = \mathrm{shift}(\tau_1, \tau_2, \ldots).$$

Hence, we get that

$$\begin{aligned} \sigma_{\mathrm{T}}(T_1, T_2) &\subseteq \sigma_{\mathrm{T}}\left( \begin{bmatrix} W_x & 0 \\ 0 & W_a \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ D_y & 0 \end{bmatrix} \right) \cup \sigma_{\mathrm{T}}(I \otimes W_a, W_{\tau_1} \otimes I) \\ &\subseteq \sigma_{\mathrm{T}}(W_x, 0) \cup \sigma_{\mathrm{T}}(W_a, 0) \cup \sigma_{\mathrm{T}}(I \otimes W_a, W_{\tau_1} \otimes I) \\ &\subseteq (\|W_\omega\|\overline{\mathbb{D}} \times \{0\}) \cup (\|W_\omega\|\overline{\mathbb{D}} \times \{0\}) \\ &\quad \cup (r\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}}) = \|W_\omega\|\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}}. \end{aligned} \tag{2.18}$$

On the other hand,

$$\begin{aligned} \sigma_{\mathrm{T}}(I \otimes W_a, W_\tau \otimes I) &\subseteq \sigma_{\mathrm{T}}(T_1, T_2) \cup \sigma_{\mathrm{T}}\left( \begin{bmatrix} W_x & 0 \\ 0 & W_a \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ D_y & 0 \end{bmatrix} \right) \\ &\Rightarrow \|W_\omega\|\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}} \subseteq \sigma_{\mathrm{T}}(T_1, T_2) \cup (\|W_\omega\|\overline{\mathbb{D}} \times \{0\}) \\ &\Rightarrow (\|W_\omega\|\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}}) \backslash (\|W_\omega\|\overline{\mathbb{D}} \times \{0\}) \subseteq \sigma_{\mathrm{T}}(T_1, T_2). \end{aligned}$$

Since $\sigma_{\mathrm{T}}(T_1, T_2)$ is closed,

$$\|W_\omega\|\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}} \subseteq \sigma_{\mathrm{T}}(T_1, T_2). \tag{2.19}$$

By (2.18) and (2.19), we have

$$\sigma_{\mathrm{T}}(T_1, T_2) = \|W_\omega\|\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}}. \tag{2.20}$$

Since $W_{(\alpha,\beta)}$ is a compact perturbation of $(I \otimes W_a, W_{ay/x} \otimes I)$, we have that

$$\begin{aligned} \sigma_{\mathrm{Te}}(T_1, T_2) &= \sigma_{\mathrm{Te}}(I \otimes W_a, W_{ay/x} \otimes I) \\ &= (\sigma_{\mathrm{Te}}(W_a) \times \sigma_{\mathrm{T}}(W_{ay/x})) \cup (\sigma_{\mathrm{T}}(W_a) \times \sigma_{\mathrm{Te}}(W_{ay/x})) \\ &= (\|W_\omega\|\mathbb{T} \times \|W_\tau\|\overline{\mathbb{D}}) \cup (\|W_\omega\|\overline{\mathbb{D}} \times \|W_\tau\|\mathbb{T}), \end{aligned} \tag{2.21}$$

where $W_{ay/x} = \text{shift}(\frac{ay}{x}, \tau_0, \tau_1, \ldots)$. Now, consider $\widetilde{W}^{\varepsilon}_{(\alpha,\beta)}$ given as in Figure 3(ii). Since $T_1$ and $T_2$ are hyponormal, we have that

$$\|\text{shift}(\omega_0^{1-\varepsilon}\omega_1^{\varepsilon}, \ldots)\| = \|W_\omega\| \quad \text{and} \quad \|\text{shift}(\tau_0^{1-\varepsilon}\tau_1^{\varepsilon}, \ldots)\| = \|W_\tau\|.$$

Hence, by the previous argument, we obtain that

$$\sigma_{\mathrm{T}}(\widetilde{W}^{\varepsilon}_{(\alpha,\beta)}) = \|W_\omega\|\overline{\mathbb{D}} \times \|W_\tau\|\overline{\mathbb{D}} \quad \text{and}$$
$$\sigma_{\mathrm{Te}}(\widetilde{W}^{\varepsilon}_{(\alpha,\beta)}) = (\|W_\omega\|\mathbb{T} \times \|W_\tau\|\overline{\mathbb{D}}) \cup (\|W_\omega\|\overline{\mathbb{D}} \times \|W_\tau\|\mathbb{T}). \quad \blacksquare$$

THEOREM 2.22. *Let* $W_{(\alpha,\beta)} = (T_1, T_2) \in \mathcal{TC}$ *be given as in Figure* 2(i). *Assume that* $T_1$ *and* $T_2$ *are hyponormal. Then, we have*

$$\sigma_{\mathrm{T}}(\widehat{W}^{\varepsilon}_{(\alpha,\beta)}) = \sigma_{\mathrm{T}}(W_{(\alpha,\beta)}) \quad \text{and} \quad \sigma_{\mathrm{Te}}(\widehat{W}^{\varepsilon}_{(\alpha,\beta)}) = \sigma_{\mathrm{Te}}(W_{(\alpha,\beta)}).$$

*Proof.* By Proposition 2.18, we easily get the weight diagram for $\widehat{W}^{\varepsilon}_{(\alpha,\beta)}$. By Proposition 2.20, since the structure of the weight diagram for $\widehat{W}^{\varepsilon}_{(\alpha,\beta)}$ is entirely similar to that of $W_{(\alpha,\beta)}$, the results follow by imitating the proof of Example 2.17. $\blacksquare$

In view of Theorems 2.21 and 2.22, it is natural to consider the following conjecture.

CONJECTURE 2.23. Let $W_{(\alpha,\beta)} = (T_1, T_2)$ be a 2-variable weighted shift with hyponormal $T_1$ and $T_2$. Assume that $\widetilde{W}^{\mathrm{D}}_{(\alpha,\beta)}$ and $\widetilde{W}^{\varepsilon}_{(\alpha,\beta)}$ are commuting. Then $W_{(\alpha,\beta)}$, $\widetilde{W}^{\mathrm{D}}_{(\alpha,\beta)}$, $\widehat{W}^{\mathrm{D}}_{(\alpha,\beta)}$, $\widetilde{W}^{\varepsilon}_{(\alpha,\beta)}$, and $\widehat{W}^{\varepsilon}_{(\alpha,\beta)}$ all have the same Taylor spectrum and the same Taylor essential spectrum.

2.3. COMMON INVARIANT SUBSPACES. In [16], I. Jung, E. Ko, and C. Pearcy proved that an operator $T \in \mathcal{B}(\mathcal{H})$ with dense range has a nontrivial invariant subspace if and only if $\widetilde{T}$ does. Our next result shows that $T$ has a nontrivial invariant subspace if and only if $\widetilde{T}^{\mathrm{D}}$ does, where $\widetilde{T}^{\mathrm{D}}$ is the Duggal transform for $T$.

THEOREM 2.24. *Let* $T = U|T|$ *in* $\mathcal{B}(\mathcal{H})$ *be an operator with dense range. Then,* $T$ *has a nontrivial invariant subspace if and only if* $\widetilde{T}^{\mathrm{D}}$ *does.*

*Proof.* (i) If $\ker T = \{0\}$, then $U$ is unitary and $|T|$ is a quasi-affinity. Since

$$U\widetilde{T}^{\mathrm{D}} = U|T|U = TU,$$

$\widetilde{T}^{\mathrm{D}}$ and $T$ are unitarily equivalent. So $\mathrm{Lat}(T) = \mathrm{Lat}(\widetilde{T}^{\mathrm{D}})$, where $\mathrm{Lat}(T)$ be the set of common invariant subspaces for $T$ and $\mathrm{Lat}(\widetilde{T}^{\mathrm{D}})$ for $\widetilde{T}^{\mathrm{D}}$.

(ii) If $\ker T \neq \{0\}$, $T$ has a nontrivial invariant subspace. Since $\ker T = \ker U$, we have that

$$\widetilde{T}^{\mathrm{D}}(\ker T) = |T|U(\ker T) = 0,$$

i.e., $\widetilde{T}^{\mathrm{D}}(\ker T) \subset \ker T$. Hence $\widetilde{T}^{\mathrm{D}}$ also has a nontrivial invariant subspace. $\blacksquare$

REMARK 2.25. For $T \in \mathcal{B}(\mathcal{H})$ with dense range, we have $\widetilde{T}^{\mathrm{D}} \neq 0$. If $\widetilde{T}^{\mathrm{D}} = 0$, then $\widetilde{T}^{\mathrm{D}}(\mathcal{H}) = |T|U(\mathcal{H}) = 0$, i.e., $U(\mathcal{H}) \subseteq \ker(|T|)$. Thus, we have

$$T(\mathcal{H}) = U|T|(\mathcal{H}) \subseteq U(\mathcal{H}) \subseteq \ker(|T|),$$

i.e.,

$$T(\mathcal{H}) \subseteq \ker(|T|) = \ker T.$$

Since $T$ has a dense range, $\ker T = \mathcal{H}$, i.e., $T = 0$. This is a contradiction to the fact that $T$ has dense range. Therefore, we have $\widetilde{T}^{\mathrm{D}} \neq 0$ as desired.

In [19], it was proved that for a commuting $n$-tuple of operators with dense ranges $\mathbf{T} = (T_1, \ldots, T_n)$, $\mathbf{T}$ has a common nontrivial invariant subspace if and only if $\widehat{\mathbf{T}}$ does. Our next result shows that it is true for $\widehat{\mathbf{T}}^{\mathrm{D}}$. Now, we have the following theorem.

THEOREM 2.26. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting $n$-tuple of operators with dense ranges. Then, $\widehat{\mathbf{T}}^{\mathrm{D}}$ has a common nontrivial invariant subspace if and only if $\mathbf{T}$ does.*

*Proof.* We let $\mathrm{Lat}(\mathbf{T})$ be the set of common invariant subspaces for $\mathbf{T} = (T_1, \ldots, T_n)$ and $\mathrm{Lat}(\widehat{\mathbf{T}}^{\mathrm{D}})$ for $\widehat{\mathbf{T}}^{\mathrm{D}}$.

*Case* 1. $\mathbf{T}$ is a commuting $n$-tuple of quasi-affinities.

($\Rightarrow$) Let $\mathcal{M}$ be a common nontrivial invariant subspace for $\widehat{\mathbf{T}}^{\mathrm{D}}$. We want to show that

$$\overline{(V_1 P V_2 \cdots P V_n \mathcal{M})}$$

is a common nontrivial invariant subspace for $\mathbf{T}$, where $\overline{(V_1 P V_2 \cdots P V_n \mathcal{M})}$ means the smallest closed set containing $V_1 P V_2 \cdots P V_n \mathcal{M}$. First, we assume that

$$V_1 P V_2 \cdots P V_n \mathcal{M} = \{0\}.$$

By (2.1), we have that

$$V_2 P V_3 P \cdots P V_n P (V_1 \mathcal{M}) = T_2 \cdots T_n (V_1 \mathcal{M}) = \{0\}.$$

Since each $T_i$ is one-to-one, we have that

$$V_1 \mathcal{M} = \{0\}. \tag{2.22}$$

By (2.1) and the one-to-one property of $T_i$ again, we have that

$$T_1 T_3 \cdots T_n (V_2 \mathcal{M}) = \{0\} \Rightarrow V_2 \mathcal{M} = \{0\}.$$

Repeating this argument, we have $V_i \mathcal{M} = \{0\}$ for each $i$. Thus, we get that

$$\mathcal{M} \subseteq \ker(V_1) \cap \cdots \cap \ker(V_n) = \ker \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} = \ker \mathbf{T} = \{0\},$$

which is a contradiction, because $\mathcal{M}$ is nontrivial. Thus, we have

$$V_1 P V_2 \cdots P V_n \mathcal{M} \neq \{0\}.$$

Since $\mathcal{M} \in \mathrm{Lat}(PV_i)$ for $i = 1, 2, \ldots, n$, we know have

$$P(V_1 PV_2 \cdots PV_n \mathcal{M}) \subseteq \mathcal{M}.$$

Since $P$ is injective, $\overline{(V_1 PV_2 \cdots PV_n \mathcal{M})} \neq \mathcal{H}$. Therefore, $\overline{(V_1 PV_2 \cdots PV_n \mathcal{M})}$ is nontrivial.

Now, by (2.1), we have that for each $i$,

$$\begin{aligned} T_i(V_1 PV_2 \cdots PV_n \mathcal{M}) &= V_i P(V_1 PV_2 \cdots PV_n \mathcal{M}) \\ &= V_1 PV_2 \cdots PV_n (PV_i)\mathcal{M} = (V_1 PV_2 \cdots PV_n)(\widehat{T}_i^{\mathrm{D}} \mathcal{M}) \\ &\subseteq V_1 PV_2 \cdots PV_n \mathcal{M} \subseteq \overline{(V_1 PV_2 \cdots PV_n \mathcal{M})}, \end{aligned}$$

i.e.,

$$T_i(\overline{(V_1 PV_2 \cdots PV_n \mathcal{M})}) \subseteq \overline{(V_1 PV_2 \cdots PV_n \mathcal{M})}.$$

Hence, $\overline{(V_1 PV_2 \cdots PV_n \mathcal{M})}$ is a nontrivial invariant subspace for $\mathbf{T}$.

($\Leftarrow$) Let $\mathcal{N}$ be a common nontrivial invariant subspace for $\mathbf{T} = (T_1, \ldots, T_n)$. Then, since $P$ is injective, $\overline{(P\mathcal{N})} \neq \{0\}$. Next, suppose that $\overline{(P\mathcal{N})} = \mathcal{H}$. Since $T_1$ is a quasi-affinity, $V_1\mathcal{H}$ must be dense in $\mathcal{H}$. So $V_1(\overline{(P\mathcal{N})}) = V_1\mathcal{H}$ is dense in $\mathcal{H}$. On the other hand, since $T_1\mathcal{N} \subseteq \mathcal{N}$, we have that $V_1(P\mathcal{N}) \subseteq \mathcal{N}$. So $\mathcal{H} \subseteq \mathcal{N}$, which is a contradiction. Hence, $\overline{(P\mathcal{N})} \neq \mathcal{H}$.

Now, for each $j$, we consider

$$\widehat{T}_j^{\mathrm{D}}(P\mathcal{N}) = PV_j P(\mathcal{N}) = PT_j(\mathcal{N}) \subseteq P\mathcal{N} \subseteq \overline{(P\mathcal{N})},$$

i.e.,

$$\widehat{T}_j^{\mathrm{D}}(\overline{(P\mathcal{N})}) \subseteq \overline{(P\mathcal{N})}.$$

Therefore, $\overline{(P\mathcal{N})}$ is a nontrivial common invariant subspace for $\widehat{\mathbf{T}}^{\mathrm{D}}$.

*Case* 2. Suppose $\ker(T_i) \neq \{0\}$ for some $i \in \{1, 2, \ldots, n\}$. Since $T_i$ and $T_j$ commute for $i, j = 1, 2, \ldots, n$,

$$T_i T_j x = T_j T_i x = 0 \quad \text{for all } x \in \ker(T_i),$$

i.e.,

$$T_j(\ker(T_i)) \subseteq \ker(T_i) \quad \text{for } j = 1, 2, \ldots, n.$$

Therefore, $\ker(T_i)$ is a common invariant subspace for $\mathbf{T}$.

On the other hand, we consider two subcases, that is, $\ker(P) \neq \{0\}$ or $\ker(P) = \{0\}$. Let $\ker(P) \neq \{0\}$. Since $\ker(V_1) \cap \cdots \cap \ker(V_n) = \ker \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} = \ker P$,

$$\widehat{T}_j^{\mathrm{D}}(\ker P) = PV_j(\ker P) = 0 \subseteq \ker P$$

for $j = 1, 2, \ldots, n$. Hence, $\widehat{\mathbf{T}}^{\mathrm{D}}$ has a common nontrivial invariant subspace. Let $\ker(P) = \{0\}$. Then, $P(\ker T_i) \neq \{0\}$. If $\overline{(P(\ker(T_i)))} = \mathcal{H}$, then

$$V_i[P(\ker(T_i))] = T_i(\ker(T_i)) = 0,$$

so that, $V_i = 0$, that is, $T_i = 0$. Thus, we have derived a contradiction to our assumption that $T_i \neq 0$. Therefore, $\overline{(P(\ker(T_i)))} \neq \mathcal{H}$. Therefore, we have

$$\overline{(P(\ker(T_i)))} \neq \{0\}, \mathcal{H}.$$

Now, we have for $j = 1, 2, \ldots, n$,

$$\begin{aligned}\widehat{T}_j^{\mathrm{D}}(P(\ker(T_i))) &= (PV_j)(P(\ker(T_i))) = PT_j(\ker(T_i)) \\ &\subseteq P(\ker(T_i)) \subseteq \overline{(P(\ker(T_i)))}.\end{aligned}$$

Hence, $\overline{(P(\ker(T_i)))}$ is a common nontrivial invariant subspace for $\widehat{\mathbf{T}}^{\mathrm{D}}$. Therefore, we have the desired result. ∎

REMARK 2.27. (i) By Proposition 2.3, we can see that $\widetilde{\mathbf{T}}^{\mathrm{D}}$ has a common nontrivial invariant subspace if and only if $\mathbf{T}$ does, whenever $\mathbf{T}$ is a doubly-commuting $n$-tuple of quasi-affinities. To see this, let $\mathbf{T}$ be a doubly commuting $n$-tuple of quasi-affinities. Lemma 2.2 shows that for $i = 1, 2, \ldots, n$

$$(U_1 \cdots U_n)\widetilde{T}_i^{\mathrm{D}} = T_i(U_1 \cdots U_n) \quad \text{and} \tag{2.23}$$

$$\widetilde{T}_i^{\mathrm{D}}(|T_1||T_2|\cdots|T_n|) = (|T_1||T_2|\cdots|T_n|)U_i|T_i| = (|T_1||T_2|\cdots|T_n|)T_i. \tag{2.24}$$

If $\mathcal{K} \in \mathrm{Lat}(\widetilde{\mathbf{T}}^{\mathrm{D}})$ is nontrivial, then by (2.23), we have that for $i = 1, 2, \ldots, n$,

$$T_i(U_1 \cdots U_n\mathcal{K}) = (U_1 \cdots U_n)\widetilde{T}_i^{\mathrm{D}}\mathcal{K} \subseteq U_1 \cdots U_n\mathcal{K} \Rightarrow \overline{(U_1 \cdots U_n\mathcal{K})} \in \mathrm{Lat}(\mathbf{T}).$$

Since $U_1, \ldots, U_n$ are unitary, $\overline{(U_1 \cdots U_n\mathcal{K})}$ is nontrivial. On the other hand, if $\mathcal{L} \in \mathrm{Lat}(\mathbf{T})$ is nontrivial, then by (2.24) we obtain

$$\overline{(|T_1||T_2|\cdots|T_n|\mathcal{L})} \in \mathrm{Lat}(\widetilde{\mathbf{T}}^{\mathrm{D}}).$$

Since each $|T_i|$ is quasi-affinity, $|T_1||T_2|\cdots|T_n|\mathcal{L} \neq \{0\}$. Lemma 2.2 shows that

$$T_1 \cdots T_n\mathcal{L} \subseteq \mathcal{L} \Rightarrow U_1 \cdots U_n(|T_1|\cdots|T_n|\mathcal{L}) \subseteq \mathcal{L}.$$

Since $U_1 \cdots U_n$ is unitary, $|T_1||T_2|\cdots|T_n|\mathcal{L}$ can not be dense in $\mathcal{H}$. Thus,

$$\overline{(|T_1||T_2|\cdots|T_n|\mathcal{L})}$$

is a common nontrivial invariant subspace for $\widetilde{\mathbf{T}}^{\mathrm{D}}$.

(ii) We also see that $\widetilde{\mathbf{T}}^{\mathrm{D}}$ has a common nontrivial invariant subspace if and only if $\mathbf{T}$ does, whenever $\mathbf{T}$ is an *isometrically* commuting $n$-tuple of operators of quasi-affinities. To see this, let $\mathbf{T}$ be an *isometrically* commuting $n$-tuple of operators of quasi-affinities. If $\mathcal{W}$ is a common nontrivial invariant subspace for $\widetilde{\mathbf{T}}^{\mathrm{D}}$, then by Proposition 2.3 and similar arguments introduced in Theorem 2.26 and the above (i), we can show that $\overline{(U_1|T_1|U_2|T_2|\cdots U_n\mathcal{W})}$ is a common nontrivial invariant subspace for $\mathbf{T}$. On the other hand, if $\mathcal{V}$ is a common trivial invariant subspace for $\mathbf{T}$, then we can also show that $\overline{(|T_1|\mathcal{V})}$ is a common nontrivial invariant subspace for $\widetilde{\mathbf{T}}^{\mathrm{D}}$.

(iii) If we assume that $\widetilde{\mathbf{T}}^{\mathrm{D}}$ is commuting, then by Remark 2.25, the above results (i) and (ii) can be extended to a commuting $n$-tuple of operators with dense ranges.

We next study $\mathrm{Lat}(\mathbf{T})$, $\mathrm{Lat}(\widehat{\mathbf{T}}^{\mathrm{D}})$, and $\mathrm{Lat}(\widetilde{\mathbf{T}}^{\mathrm{D}})$. The following examples show that there are many commuting pairs $\mathbf{T} = (T_1, T_2)$ such that $\mathrm{Lat}(\mathbf{T})$ and $\mathrm{Lat}(\widehat{\mathbf{T}}^{\mathrm{D}})$ are isomorphic. However, we can prove that there are examples in which $\mathrm{Lat}(\mathbf{T})$ and $\mathrm{Lat}(\widetilde{\mathbf{T}}^{\mathrm{D}})$ are not isomorphic.

EXAMPLE 2.28. (i) Consider a Hilbert space $\mathcal{H}$ and a set $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$. Let the commutant of $\mathcal{B}$ denoted by

$$\mathcal{B}^{'} := \{A \in \mathcal{B}(\mathcal{H}) : [A, B] = 0 \text{ for all } B \in \mathcal{B}\}.$$

Let $\mathbf{T} = (T_1, T_2)$ be a commuting pair of operators, where $T_2$ is an isometry, (e.g., the unilateral shift) and $T_1 \in \{T_2\}^{'}$. Then,

$$P^2 = T_1^*T_1 + T_2^*T_2 = \sqrt{T_1^*T_1 + I}$$

is invertible. Hence, the polar decomposition for $\mathbf{T}$ is

$$\mathbf{T} = (V_1P, V_2P) = \left(T_1\sqrt{T_1^*T_1 + I}^{\,-1}\sqrt{T_1^*T_1 + I}, T_2\sqrt{T_1^*T_1 + I}^{\,-1}\sqrt{T_1^*T_1 + I}\right) \quad \text{and}$$

$$\widehat{\mathbf{T}}^{\mathrm{D}} = (PV_1, PV_2) = \left(\sqrt{T_1^*T_1 + I}T_1\sqrt{T_1^*T_1 + I}^{\,-1}, \sqrt{T_1^*T_1 + I}T_2\sqrt{T_1^*T_1 + I}^{\,-1}\right).$$

Therefore, we get

$$\begin{aligned}\mathrm{Lat}\,(\widehat{\mathbf{T}}^{\mathrm{D}}) &= \mathrm{Lat}\left(\sqrt{T_1^*T_1 + I}T_1\sqrt{T_1^*T_1 + I}^{\,-1}\right) \cap \mathrm{Lat}\left(\sqrt{T_1^*T_1 + I}T_2\sqrt{T_1^*T_1 + I}^{\,-1}\right)\\ &= \mathrm{Lat}\,(T_1) \cap \mathrm{Lat}(T_2) = \mathrm{Lat}\,(\mathbf{T}).\end{aligned}$$

(ii) We consider the polar decomposition of $A = U|A| \in \mathcal{B}(\mathcal{H})$. Let

$$T_1 = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{on } \mathcal{H} \oplus \mathcal{H}.$$

Then, $T_1^*T_1 = \left(\begin{smallmatrix} 0 & 0 \\ A^*A & 0 \end{smallmatrix}\right)$, and so $|T_1| = \left(\begin{smallmatrix} 0 & 0 \\ |A| & 0 \end{smallmatrix}\right)$. Hence,

$$T_1 = \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}\begin{pmatrix} |A| & 0 \\ 0 & 0 \end{pmatrix}$$

is the polar decomposition of $T_1$ and

$$T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is the polar decomposition of $T_2$, respectively. Note that $\widetilde{T}_i^{\mathrm{D}} = 0$ $(i = 1, 2)$ and $T_1T_2 = T_2T_1 = 0$, so that $(T_1, T_2)$ is a commuting pair and $\widetilde{\mathbf{T}}^{\mathrm{D}} = (\widetilde{T}_1^{\mathrm{D}}, \widetilde{T}_2^{\mathrm{D}})$ has all subspaces of $\mathcal{H} \oplus \mathcal{H}$ as common invariant subspaces. However, $\mathbf{T}$ does not have $\mathcal{H} \oplus \{0\}$ as a common invariant subspace. Thus, $\mathrm{Lat}(\mathbf{T})$ and $\mathrm{Lat}(\widetilde{\mathbf{T}}^{\mathrm{D}})$ are not isomorphic.

We conclude this section with problems of independent interest. On the basis of Remark 2.27 and Example 2.28(i), it is natural to pose the following conjecture.

CONJECTURE 2.29. For a commuting $n$-tuple $\mathbf{T}$, we have:
(i) $\widetilde{\mathbf{T}}^{\mathrm{D}}$ has a common nontrivial invariant subspace if and only if $\mathbf{T}$ does;
(ii) $\mathrm{Lat}(\mathbf{T})$ and $\mathrm{Lat}(\widehat{\mathbf{T}}^{\mathrm{D}})$ are isomorphic.

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