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A NOTE ON RELATIVE AMENABILITY OF FINITE VON NEUMANN ALGEBRAS

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ABSTRACT. Let M be a finite von Neumann algebra (respectively, a type II_1 factor) and let $N \subset M$ be a II_1 factor (respectively, $N \subset M$ have an atomic part). We prove that if the inclusion $N \subset M$ is amenable, then implies the identity map on M has an approximate factorization through $M_m(\mathbb{C}) \otimes N$ via trace preserving normal unital completely positive maps, which is a generalization of a result of Haagerup. We also prove two permanence properties for amenable inclusions. One is weak Haagerup property, the other is weak exactness.

KEYWORDS: II_1 *factors, finite von Neumann algebras, relative amenability, trace preserving normal unital completely positive maps, Haagerup property, weak Haagerup property, weak exactness.*

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INTRODUCTION

To study an operator algebras analogue to the rigidity phenomena in representation of groups and ergodic theory, Connes [12], [13], [14] introduced the key concept of correspondences between two von Neumann algebras, which can be thought of as the representation theory for von Neumann algebras. He also observed that there are many ways to look at these correspondences. For example, we can construct a correspondence H_ϕ from a normal completely positive map ϕ (on a finite von Neumann algebra) using Stinespring dilation and vice versa. Later on, Popa [27] systematically developed the theory of correspondences to get new insights in the structure of von Neumann algebras, especially in the study of type II_1 factors.

In this paper, we are interested in a relative notion of amenability Popa introduced using the correspondence framework. Recall that for a von Neumann subalgebra N of a finite von Neumann algebra M, we say that the inclusion $N \subset M$ is *amenable* (or M is *amenable relative* to N, or N is *co-amenable* in M) if

H_{id} is weakly contained in H_{E_N}, where E_N is the trace preserving normal conditional expectation from M onto N. Here are some examples of amenable inclusions. If M is a finite von Neumann algebra, then M is amenable if and only if the inclusion $\mathbb{C}1 \subset M$ is amenable. If $N \subset M$ is an inclusion of II_1 factors, and the Jones index $[M : N] < \infty$, then the inclusion $N \subset M$ is amenable. If M is a cocycle crossed product of a finite von Neumann algebra N by a cocycle action of a discrete group G, then the inclusion $N \subset M$ is amenable if and only if G is an amenable group. If N is a finite von Neumann algebra and $G \curvearrowright N$ is a weakly compact action, then the inclusion $\mathrm{L}G \subset N \rtimes G$ is amenable by Proposition 3.2 of [26].

There are some permanence results for amenable inclusions. In [8], Bédos proved that if G is a discrete amenable group with a free action α on a von Neumann algebra M and M has property Γ, then $M \rtimes_\alpha G$ has property Γ. He also proved that if G is a discrete amenable group with a free action α on a type II_1 factor M and M is McDuff, then $M \rtimes_\alpha G$ is McDuff. Bannon and Fang [7] proved that if the inclusion of finite von Neumann algebras $N \subset M$ is amenable and N has the Haagerup property, then M also has the Haagerup property.

Just as many other conditions are equivalent to amenability, Popa showed the relative amenability can be characterized by the corresponding "relative type" conditions, see Theorem 3.23 of [27]. Since semidiscreteness is equivalent to amenability for von Neumann algebras, Popa asked whether a good analogue notion exists for relative amenability. This was answered affirmatively by Mingo in [23] for finite von Neumann algebras using normal completely positive maps, which is close to the definition of semidiscreteness in spirit. More precisely, he showed that for a finite von Neumann algebra M and two normal completely positive maps $\phi, \varphi : M \to M$, H_ϕ is weakly contained in H_φ if and only if ϕ can be approximately factored by φ. Later on, Anantharaman-Delaroche extended Mingo's result to all von Neumann algebras using correspondences in [3].

Applying Mingo's above result and the definition of approximate factorization, it is not difficult to deduce the following proposition.

PROPOSITION 2.3. *Let M be a finite von Neumann algebra with a faithful normal trace τ, and let $N \subset M$ be a von Neumann subalgebra. If the inclusion $N \subset M$ is amenable, then there exists a net of normal u.c.p. maps $\varphi_i : M \to M_{n_i}(\mathbb{C}) \otimes N$, a net of normal u.c.p. maps $\phi_i : M_{n_i}(\mathbb{C}) \otimes N \to M$ and a net of positive elements $h_i \in M_{n_i}(\mathbb{C}) \otimes N$ such that for all $x \in M$, $y \in M_{n_i}(\mathbb{C}) \otimes N$,*

(i) *$\phi_i \circ \varphi_i(x) \to x$ in the $\|\cdot\|_2$-norm topology;*

(ii) *$\tau \circ \phi_i(y) = (\mathrm{tr}_{n_i} \otimes \tau)(h_i y)$.*

We may try to apply Proposition 1.1 to study permanence properties for amenable inclusions, i.e., we try to prove if some approximation property holds for a von Neumann subalgebra N, then it also holds for the finite von Neumann algebra M assuming the inclusion $N \subset M$ is amenable. However, it turns out that in several situations, we need to assume h_i to be the identity; in other words, we expect the normal u.c.p. maps ϕ_i, φ_i can be chosen to be trace preserving. In fact, this issue also appears in Haagerup's proof that semidiscreteness $\Rightarrow$ hyperfiniteness for a II_1 factor, see [16]. Under certain assumptions on the two algebras, we show ϕ_i, φ_i could be chosen to be trace preserving.

The following are our main theorems.

THEOREM 3.2. *Let M be a finite von Neumann algebra with a faithful normal tracial state τ, and let $N \subset M$ be a type II_1 factor. Let the inclusion $N \subset M$ be amenable. Let $\{x_1, \ldots, x_n\}$ be a finite set in M and let $\varepsilon > 0$. Then there exists an $m \in \mathbb{N}$, and two normal c.p. maps $S : M \to M_m(\mathbb{C}) \otimes N$, $T : M_m(\mathbb{C}) \otimes N \to M$, such that:*

(i) *S and T are unital;*

(ii) $(\mathrm{tr}_m \otimes \tau) \circ S = \tau, \tau \circ T = \mathrm{tr}_m \otimes \tau;$

(iii) $\|T \circ S(x_k) - x_k\|_2 < \varepsilon, k = 1, \ldots, n.$

THEOREM 3.1. *Let M be a type II_1 factor with a faithful normal tracial state τ, and let $N \subset M$ be a von Neumann subalgebra having an atomic part. Let the inclusion $N \subset M$ be amenable. Let $\{x_1, \ldots, x_n\}$ be a finite set in M and let $\varepsilon > 0$. Then there exists an $m \in \mathbb{N}$, and two normal c.p. maps $S : M \to M_m(\mathbb{C}) \otimes N$, $T : M_m(\mathbb{C}) \otimes N \to M$, such that:*

(i) *S and T are unital;*

(ii) $(\mathrm{tr}_m \otimes \tau) \circ S = \tau, \tau \circ T = \mathrm{tr}_m \otimes \tau;$

(iii) $\|T \circ S(x_k) - x_k\|_2 < \varepsilon, k = 1, \ldots, n.$

Since M is amenable if and only if the inclusion $\mathbb{C}1 \subset M$ is amenable (c.f. 3.23 of [27] or Proposition 5 of [24]), Theorem 1.3 generalizes a result of Haagerup ([16], Proposition 3.5) which corresponds to the case $N = \mathbb{C}1$.

Using these two theorems, we could prove some permanence results for amenable inclusions.

COROLLARY 4.1. *Let M be a finite von Neumann algebra and let $N \subset M$ be a type II_1 factor. If the inclusion $N \subset M$ is amenable and N has the Haagerup property, then M also has the Haagerup property.*

COROLLARY 4.2. *Let M be a finite von Neumann algebra and let $N \subset M$ be a type II_1 factor. If the inclusion $N \subset M$ is amenable and N is weakly exact, then M is also weakly exact.*

COROLLARY 4.3. *Let M be a finite von Neumann algebra and let $N \subset M$ be a type II_1 factor. If the inclusion $N \subset M$ is amenable and N has the weak Haagerup property, then M also has the weak Haagerup property.*

Bannon and Fang [7] proved a permanence result for the Haagerup property for amenable inclusions of finite von Neumann algebras in the framework of correspondences. In this paper, we prove Corollary 4.1 from the point of view of normal u.c.p. maps.

This paper is organised as follows. In Section 1, we present some preliminaries. In Section 2, we prove that the amenability of the inclusion $N \subset M$ of finite von Neumann algebras implies that the identity map on M has an approximate factorization through $M_m(\mathbb{C}) \otimes N$ via normal unital completely positive maps. In Section 3, we use some matrix techniques and the results in Section 2 to show that the above normal unital completely positive maps can be chosen to be trace preserving in two cases: when M is a finite von Neumann algebra and $N \subset M$ is a II_1 factor, and when M is a II_1 factor and $N \subset M$ has an atomic part. In the last section, we present three permanence properties for some amenable inclusions.

1. PREMIMINARIES

In this section, we recall briefly some basic concepts that will be used later. For more details and results on correspondences, relative amenability, and completely positive maps, we refer the reader to [1], [2], [3], [4], [5], [22], [23], [27].

CORRESPONDENCES. Let M and N be von Neumann algebras. Recall that *a correspondence from M to N* is a $*$-representation of $N \otimes M^{\mathrm{op}}$ on a Hilbert space H, which is normal when restricted to both $N = N \otimes 1$ and $M^{\mathrm{op}} = 1 \otimes M^{\mathrm{op}}$.

Let M be a finite von Neumann algebra with a faithful normal trace τ. Given a normal completely positive map $\phi : M \to M$, we can use the Stinespring dilation to construct a correspondence which is denoted by H_ϕ. Define on the linear space $H_0 = M \otimes M$ a sesquilinear form $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_\phi = \tau(\phi(x_2^* x_1) y_1 y_2^*)$, $\forall x_1, y_1, x_2, y_2 \in M$. It is easy to check that the complete positivity of ϕ is equivalent to the positivity of $\langle \cdot, \cdot \rangle_\phi$. Let H_ϕ be the completion of $H_0 / \sim$, where $\sim$ is the equivalence modulo the null space of $\langle \cdot, \cdot \rangle_\phi$. Then H_ϕ is a correspondence of M and the bimodule structure is given by $x(x_1 \otimes y_1)y = xx_1 \otimes y_1 y$. We call H_ϕ the correspondence of M associated to ϕ, see [27].

RELATIVE AMENABILITY. Regarding correspondences as $*$-representations, we can define a topology on these correspondences which is just the usual topology on the set of equivalent classes of representations of $N \otimes M^{\mathrm{op}}$. Under this topology, we say that a correspondence H_1 *is weakly contained in* H_2 if H_1 is in the closure of H_2.

Let M be a finite von Neumann algebra with a trace τ, and let N be a von Neumann subalgebra of M. Then the inclusion $N \subset M$ *is amenable* if H_{id} is weakly contained in H_{E_N}, where id is the identity map from M to M and E_N is the faithful normal conditional expectation from M onto N preserving trace τ. Popa has given several equivalent conditions for relative amenability in 3.23 of [27] and Proposition 5 of [24].

Here are some examples of amenable inclusions. If M is a finite von Neumann algebra, then M is amenable if and only if the inclusion $\mathbb{C}1 \subset M$ is amenable.

If $N \subset M$ is an inclusion of II_1 factors, and the Jones index $[M : N] < \infty$, then the inclusion $N \subset M$ is amenable. If M is a cocycle crossed product of a finite von Neumann algebra N by a cocycle action of a discrete group G, then the inclusion $N \subset M$ is amenable if and only if G is an amenable group. If N is a finite von Neumann algebra and $G \curvearrowright N$ is a weakly compact action, then the inclusion $LG \subset N \rtimes G$ is amenable by Proposition 3.2 of [26].

APPROXIMATE FACTORIZATION. Let $\psi : M \to M$ be completely positive and $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$. Define

$$\Theta : M \to M,$$
$$x \mapsto \sum_{i,j=1}^{n} b_i^* \psi(a_i^* x a_j) b_j.$$

Let

$$A = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}^{\mathrm{t}}.$$

Then Θ is completely positive by the commutativity of the diagram

$$\begin{tikzcd} M \arrow[rr, "\Theta"] \arrow[dr, "\varphi"'] & & M, \\ & M_n(\mathbb{C}) \otimes M \arrow[ur, "\phi"'] & \end{tikzcd}$$

where $\varphi(x) = (\mathrm{id}_n \otimes \psi)(A^* x A)$, $\phi(y) = B^* y B$, $x \in M$ and $y \in M_n(\mathbb{C}) \otimes M$.

We shall say that *a c.p. map Θ can be factored by ψ* if it is of the above form, see [23]. We shall denote by F_ψ the set of finite sums of such maps.

Let $\phi, \varphi : M \to M$ be normal c.p. maps. That *φ may be approximately factored by ϕ* if there is a bounded net $(\phi_{\mathrm{r}}(x)) \subset F_\phi$ such that for each $x \in M$, $\phi_{\mathrm{r}}(x)$ converges to $\varphi(x)$ σ-weakly for all $x \in M$, see [23].

HAAGERUP PROPERTY. Let M be a finite von Neumann algebra with a faithful normal trace τ. For each $x \in M$, denote $\|x\|_2^2 = \tau(x^* x)$.

A finite von Neumann algebra M with a faithful normal trace τ has the *Haagerup property* if there exists a net $(\phi_i)_{i \in I}$ of normal completely positive maps from M to M which satisfy the following conditions:

(i) $\tau \circ \phi_i \leqslant \tau$;
(ii) each ϕ_i induces a compact bounded operator on $L^2(M)$;
(iii) for every $x \in M$, $\lim_i \|\phi_i(x) - x\|_2 = 0$.

Note that a normal c.p. map $\phi_i : M \to M$ with $\tau \circ \phi_i \leqslant \tau$ can induce a bounded linear operator on $L^2(M)$. To see this, $\|\phi_i(x)\|_2^2 = \tau(\phi_i(x)^* \phi_i(x)) \leqslant \tau(\phi_i(x^* x)) \leqslant \tau(x^* x) = \|x\|_2^2$. Thus ϕ_i can be extended to a bounded linear operator on $L^2(M)$.

WEAK HAAGERUP PROPERTY [21]. Let M be a von Neumann algebra with a faithful normal trace τ. M has the *weak Haagerup property* if there exist a constant $C > 0$ and a net $(\phi_i)_{i\in I}$ of normal completely bounded maps on M such that:

(i) $\|\phi_i\|_{\text{c.b.}} \leqslant C$ for every i;
(ii) $\langle \phi_i(x), y\rangle_\tau = \langle x, \phi_i(y)\rangle_\tau$ for every $x, y \in M$;
(iii) each ϕ_i induces a compact bounded operator on $L^2(M)$;
(iv) for every $x \in M$, $\lim\limits_i \|\phi_i(x) - x\|_2 = 0$.

WEAKLY EXACT VON NEUMANN ALGEBRAS [10]. Let B be an arbitrary unital C^*-algebra and $J \lhd B$ be a non-unital closed two-sided ideal. The canonical quotient map will be denoted by $Q : B \to B/J$.

A von Neumann algebra M is said to be *weakly exact* if for any ideal $J \lhd B$ and any $*$-representation $\pi : M \otimes B \to B(H)$ with $M \otimes J \subset \ker\pi$ and $\pi|_{M\otimes\mathbb{C}1}$ being normal, the induced representation $\widetilde{\pi} : M \odot (B/J) \to B(H)$ is continuous with respect to the minimal tensor norm.

THEOREM 1.1 ([25]). *Let M be a von Neumann algebra. The following conditions are equivalent:*

(i) *M is weakly exact;*
(ii) *for any finite dimensional operator system E in M, there exist two nets of u.c.p. maps $\phi_i : E \to M_n(\mathbb{C})$ and $\psi_i : \phi_i(E) \to M$ such that the net $(\psi_i \circ \phi_i)_{i\in I}$ converges to* id_E *in the point-σ-weak operator topology.*

REMARK 1.2. Assume that M is a finite von Neumann algebra with a trace τ. Note that the above $(\psi_i \circ \phi_i)_{i\in I}$ are u.c.p. maps. Then the choice of topology in which the net $(\psi_i \circ \phi_i)_{i\in I}$ converges to the identity map on E could be one of many topologies without affecting the results. The topologies are the point-weak operator topology, the point-σ-weak operator topology, the point-strong operator topology and the pointwise $\|\cdot\|_2$-norm topology.

2. APPROXIMATE FACTORIZATION OF THE IDENTITY MAP VIA UNITAL COMPLETELY POSITIVE MAPS

As the main result of this section, we prove Proposition 2.3. It is based on a result of Mingo [23] on the relation between approximate factorization and weak containment of correspondences.

THEOREM 2.1 ([23]). *Let M be a finite von Neumann algebra with a trace τ and let $\phi, \varphi : M \to M$ be normal c.p. maps. Then φ can be approximately factored by ϕ if and only if H_φ is weakly contained in H_ϕ.*

For a finite von Neumann algebra M with a faithful normal trace τ, denote by $L^1(M)$ the completion of M with respect to the norm $\|x\|_1 = \tau(|x|)$, $x \in M$.

Note that for the above normal c.p. map $\phi : M_n(\mathbb{C}) \otimes N \to M$, we have $\tau \circ \phi(x) = (\mathrm{tr}_n \otimes \tau)(hx)$, where $\mathrm{tr}_n \otimes \tau$ is the normal trace on $M_n(\mathbb{C}) \otimes N$, and h is a positive element in $L^1(M_n(\mathbb{C}) \otimes N)$.

Note that the convergent topology in approximate factorization is the σ-weak operator topology. The aim of this section is to show that the normal completely positive maps ϕ and φ in Proposition 2.3 can be chosen to be unital, the convergent topology can be the pointwise $\|\cdot\|_2$-norm topology, and the positive element h can be chosen to be invertible in $M_n(\mathbb{C}) \otimes N$.

We first need the following lemma.

LEMMA 2.2. *Let M be a finite von Neumann algebra with a trace τ and let $N \subset M$ be a von Neumann subalgebra. Then the inclusion $N \subset M$ is amenable if and only if there exists a net of normal c.p. maps $\varphi_i : M \to M_{n_i}(\mathbb{C}) \otimes N$ and a net of normal c.p. maps $\phi_i : M_{n_i}(\mathbb{C}) \otimes N \to M$ such that:*

(i) $\varphi_i(x) = \bigoplus_{j=1}^{l_i} (\mathrm{id}_{i_j} \otimes E)(A_{ij}^* x A_{ij})$ *for* $x \in M$, $l_i, i_j \in \mathbb{N}$, $A_{ij} \in M_{1\times i_j}(M)$, $\sum_{j=1}^{l_i} i_j = n_i$ *and E is the trace preserving normal conditional expectation from M onto N;*

(ii) $\phi_i(y) = B_i^* y B_i$ *for* $y \in M_{n_i}(\mathbb{C}) \otimes N$, $B_i \in M_{n_i \times 1}(M)$;

(iii) $\phi_i \circ \varphi_i(1) \leqslant 1$;

(iv) $\phi_i \circ \varphi_i(x) \to x$ *in the* $\|\cdot\|_2$*-norm topology for all* $x \in M$.

Proof. By Theorem 2.1, we know that the inclusion $N \subset M$ is amenable if and only if the identity map id can be approximately factored by the normal conditional expectation E.

For each element Θ in F_E, $\Theta(x) = \sum_{k=1}^{n} \theta_k(x)$, where

$$\theta_k(x) = \sum_{i,j=1}^{m_k} b_{ki}^* E(a_{ki}^* x a_{kj}) b_{kj}, \quad a_{ki}, b_{kj} \in M.$$

For simplicity, we may assume $n = 2$. Let

$$A_1 = \begin{pmatrix} a_{11} & \cdots & a_{1m_1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{21} & \cdots & a_{2m_2} \end{pmatrix},$$
$$B = \begin{pmatrix} b_{11} & \cdots & b_{1m_1} & b_{21} & \cdots & b_{2m_2} \end{pmatrix}^{\mathrm{t}}.$$

Let

$$\varphi(x) = \bigoplus_{i=1}^{2} (\mathrm{id}_{m_i} \otimes E)(A_i^* x A_i), \quad x \in M,$$
$$\psi(y) = B^* y B, \quad y \in M_{m_1+m_2}(\mathbb{C}) \otimes N.$$

Note that φ and ψ are normal completely positive maps from M to $M_{m_1+m_2}(\mathbb{C}) \otimes N$ and $M_{m_1+m_2}(\mathbb{C}) \otimes N$ to M respectively, with $\Theta(x) = \psi \circ \varphi(x)$.

It is clear that F_E is a convex set and $b^*\Theta(\cdot)b \in F_E$ for $b \in M$, $\Theta \in F_E$. Then by Lemma 2.2 of [3] and Theorem 2.1, we can choose a net $(\Theta_i) \subset F_E$ such that $\Theta_i(1) \leqslant 1$ and $\Theta_i(x) \to x$ σ-weakly for all $x \in M$.

Let $F_E' = \{\Theta \in F_E : \Theta(1) \leqslant 1\}$. Obviously, F_E' is convex. Note that for a convex set of $CP(M)$, where $CP(M)$ denotes the set of c.p. maps on M, the closure in the point-σ-weak operator topology and the closure in the point-σ-strong operator topology are the same. And since F_E' is bounded, we deduce that $\|\Theta_i(x) - x\|_2 \to 0$ for all $x \in M$ for a net $(\Theta_i) \subset F_E'$. Actually, the choice of topology in which the net (Θ_i) converges to the identity map on M could be one of many topologies without affecting the results. The topologies are the point-weak operator topology, the point-σ-weak operator topology, the point-strong operator topology and the point-wise $\|\cdot\|_2$-norm topology. ∎

PROPOSITION 2.3. *Let M be a finite von Neumann algebra with a trace τ and let $N \subset M$ be a von Neumann subalgebra. If the inclusion $N \subset M$ is amenable, then there exists a net of normal u.c.p. maps $\varphi_i : M \to M_{n_i}(\mathbb{C}) \otimes N$, a net of normal u.c.p. maps $\phi_i : M_{n_i}(\mathbb{C}) \otimes N \to M$ and a net of positive invertible elements $h_i \in M_{n_i}(\mathbb{C}) \otimes N$ such that for all $x \in M$, $y \in M_{n_i}(\mathbb{C}) \otimes N$,*

(i) *$\phi_i \circ \varphi_i(x) \to x$ in the $\|\cdot\|_2$-norm topology;*

(ii) *$\tau \circ \phi_i(y) = (\mathrm{tr}_{n_i} \otimes \tau)(h_i y)$.*

Proof. By Lemma 2.2, there exists a net of normal c.p. maps $\widetilde{\psi}_i : M \to M_{n_i}(\mathbb{C}) \otimes N$ and a net of normal c.p. maps $\widetilde{\phi}_i : M_{n_i}(\mathbb{C}) \otimes N \to M$ such that $\widetilde{\phi}_i \circ \widetilde{\psi}_i(x) \to x$ in the $\|\cdot\|_2$-norm topology for all $x \in M$ and $\widetilde{\phi}_i \circ \widetilde{\psi}_i(1) \leqslant 1$.

We can choose $(\eta_i), (\epsilon_i) \subset \mathbb{R}_+$, such that $\eta_i \to 1$, $\epsilon_i \widetilde{\phi}_i(1) \to 0$ in the operator norm topology, and $0 < \epsilon_i \widetilde{\phi}_i(1) + \eta_i < 1$. Then we have $\widetilde{\phi}_i \circ (\eta_i \widetilde{\psi}_i(x) + \epsilon_i) \to x$ in the $\|\cdot\|_2$-norm topology for all $x \in M$ and $\widetilde{\phi}_i \circ (\eta_i \widetilde{\psi}_i(1) + \epsilon_i) < 1$. Define $\widetilde{\varphi}_i(x) := \eta_i \widetilde{\psi}_i(x) + \epsilon_i$ and $\varphi_i(x) := \widetilde{\varphi}_i(1)^{-1/2} \widetilde{\varphi}_i(x) \widetilde{\varphi}_i(1)^{-1/2}$. Then φ_i is a normal u.c.p. map from M to $M_{n_i}(\mathbb{C}) \otimes N$.

Let $b_i = 1 - \widetilde{\phi}_i \circ \widetilde{\varphi}_i(1)$. Since $\widetilde{\phi}_i \circ \widetilde{\varphi}_i(1) < 1$, we have $b_i > 0$ and $b_i \to 0$ in the $\|\cdot\|_2$-norm topology.

Define the linear maps $\phi_i : M_{n_i}(\mathbb{C}) \otimes N \to M$ by

$$\phi_i(y) = (\mathrm{tr}_{n_i} \otimes \tau)(y) b_i + \widetilde{\phi}_i(\widetilde{\varphi}_i(1)^{1/2} y \widetilde{\varphi}_i(1)^{1/2}).$$

Then the ϕ_i's are normal u.c.p. maps. Since $b_i \to 0$, it follows that $\phi_i \circ \varphi_i(x) \to x$ in the $\|\cdot\|_2$-norm topology.

By Lemma 2.2, $\widetilde{\phi}_i(y) = B_i^* y B_i$ for $y \in M_{n_i}(\mathbb{C}) \otimes N$, $B_i \in M_{n_i \times 1}(M)$.

For simplicity, write $n = n_i$ and $\widetilde{\phi}_i$ from $M_n(\mathbb{C}) \otimes N$ to M in the following form:

$$\widetilde{\phi}_i(y) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}^* \begin{pmatrix} y_{11} & \dots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \dots & y_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i,j=1}^{n} b_i^* y_{ij} b_j,$$

where b_i is in M and $y = (y_{ij})_{n \times n}$ is in $M_n(\mathbb{C}) \otimes N$.

Let $h_{ij} = nb_j b_i^* \in M$ and put $\widetilde{h} = (h_{ij})_{n\times n} \in M_n(\mathbb{C}) \otimes M$. Then we have $\widetilde{h} \geqslant 0$ and

$$(\mathrm{tr}_n \otimes \tau)(\widetilde{h}y) = \sum_{i,j=1}^{n} \tau\Big(\frac{h_{ij}}{n} y_{ji}\Big) = \sum_{i,j=1}^{n} \tau(b_j b_i^* y_{ij}) = \tau \circ \widetilde{\phi}_i(y).$$

Since conditional expectation preserves the trace and y is in $M_n(\mathbb{C}) \otimes N$, we have

$$(\mathrm{tr}_n \otimes \tau)(\widetilde{h}y) = (\mathrm{tr}_n \otimes \tau)(E_{M_n(\mathbb{C})\otimes N}(\widetilde{h}y)) = (\mathrm{tr}_n \otimes \tau)(E_{M_n(\mathbb{C})\otimes N}(\widetilde{h})y).$$

Note that

$$\begin{aligned}\tau \circ \phi_i(y) =& \tau(b_i)(\mathrm{tr}_n \otimes \tau)(y) + \tau \circ \widetilde{\phi}_i(\widetilde{\varphi}_i(1)^{1/2} y \widetilde{\varphi}_i(1)^{1/2}) \\ =& (\mathrm{tr}_n \otimes \tau)(\tau(b_i)y + \widetilde{\varphi}_i(1)^{1/2} E_{M_n(\mathbb{C})\otimes N}(\widetilde{h})\widetilde{\varphi}_i(1)^{1/2} y).\end{aligned}$$

Let $h = \tau(b_i) + \widetilde{\varphi}_i(1)^{1/2} E_{M_n(\mathbb{C})\otimes N}(\widetilde{h})\widetilde{\varphi}_i(1)^{1/2}$. Since $\widetilde{\varphi}_i(1) \in M_n(\mathbb{C}) \otimes N$, $\widetilde{h} \geqslant 0$ and $b_i > 0$, we have that $h \in M_n(\mathbb{C}) \otimes N$ is positive and invertible. Hence, we finish the proof. ∎

3. MAIN RESULTS

In this section, we extend Proposition 3.5 of [16] to amenable inclusions in two cases, either the subalgebra N has an atomic part and the ambient algebra M is a II_1 factor or N is a II_1 factor.

The first case follows quite easily from Proposition 3.5 of [16], while the second case is quite involved.

Recall that a von Neumann algebra N has an atomic part means that there exists a nonzero projection $p \in N$ such that $pNp = \mathbb{C}p$.

THEOREM 3.1. *Let M be a type II_1 factor with a faithful normal tracial state τ, and let $N \subset M$ be a von Neumann subalgebra having an atomic part. Let the inclusion $N \subset M$ be amenable. Let $\{x_1, \ldots, x_n\}$ be a finite set in M and let $\varepsilon > 0$. Then there exists an $m \in \mathbb{N}$, and two normal c.p. maps $S : M \to M_m(\mathbb{C}) \otimes N$, $T : M_m(\mathbb{C}) \otimes N \to M$, such that:*

(i) *S and T are unital;*

(ii) *$(\mathrm{tr}_m \otimes \tau) \circ S = \tau, \tau \circ T = \mathrm{tr}_m \otimes \tau$;*

(iii) *$\|T \circ S(x_k) - x_k\|_2 < \varepsilon, k = 1, \ldots, n$.*

Proof. Assume p is a projection in N such that $pNp = \mathbb{C}p$. By Theorem 3.23 of [27], we have that $\mathbb{C}p \subset pMp$ is amenable, which shows that pMp is a hyperfinite type II_1 factor. We can find a projection e in M such that $e \leqslant p$ and $\tau(e) = \frac{1}{k}$ for some positive integer k. It follows that M is a hyperfinite type II_1 factor, since $M = M_k(\mathbb{C}) \otimes eMe$ and eMe is a hyperfinite type II_1 factor.

Let $\{x_1, \ldots, x_n\}$ be a finite set in M and let $\varepsilon > 0$. By Proposition 3.5 of [16], there exists an $m \in \mathbb{N}$, and two normal u.c.p. maps $S_1 : M \to M_m(\mathbb{C}), T_1 :$

$M_m(\mathbb{C}) \to M$, such that $\mathrm{tr}_m \circ S_1 = \tau, \tau \circ T_1 = \mathrm{tr}_m$ and $\|T_1 \circ S_1(x_k) - x_k\|_2 < \varepsilon, k = 1, \ldots, n$.

Define two normal unital c.p. maps S_2 from $M_m(\mathbb{C})$ to $M_m(\mathbb{C}) \otimes N$ and T_2 from $M_m(\mathbb{C}) \otimes N$ to $M_m(\mathbb{C})$ respectively, by

$$S_2(x) = x \otimes 1, \quad T_2(y \otimes z) = \tau(z)y, \quad x, y \in M_m(\mathbb{C}), z \in M.$$

Put $S = S_2 \circ S_1$, $T = T_1 \circ T_2$. Then $S : M \to M_m(\mathbb{C}) \otimes N, T : M_m(\mathbb{C}) \otimes N \to M$ are two normal unital c.p. maps.

Note that for $x \in M$, $y \in M_m(\mathbb{C})$ and $z \in N$,

$$(\mathrm{tr}_m \otimes \tau)(S(x)) = (\mathrm{tr}_m \otimes \tau)(S_1(x) \otimes 1) = \mathrm{tr}_m \circ S_1(x) = \tau(x) \quad \text{and}$$
$$\tau \circ T(y \otimes z) = \tau \circ T_1(y\tau(z)) = \tau(z)\tau(T_1(y)) = (\mathrm{tr}_m \otimes \tau)(y \otimes z).$$

Moreover, $\|T \circ S(x) - x\|_2 = \|T_1 \circ S_1(x) - x\|_2$. Hence we finish the proof. ∎

THEOREM 3.2. *Let M be a finite von Neumann algebra with a faithful normal tracial state τ, and let $N \subset M$ be a type* II_1 *factor. Let the inclusion $N \subset M$ be amenable. Let $\{x_1, \ldots, x_n\}$ be a finite set in M and let $\varepsilon > 0$. Then there exists an $m \in \mathbb{N}$, and two normal c.p. maps $S : M \to M_m(\mathbb{C}) \otimes N$, $T : M_m(\mathbb{C}) \otimes N \to M$, such that:*

(i) *S and T are unital;*

(ii) $(\mathrm{tr}_m \otimes \tau) \circ S = \tau, \tau \circ T = \mathrm{tr}_m \otimes \tau$;

(iii) $\|T \circ S(x_k) - x_k\|_2 < \varepsilon, k = 1, \ldots, n.$

For the sake of proving Theorem 3.2, we introduce the following definitions.

For any normal state ϕ on a von Neumann algebra M, we put

$$\|x\|_\phi^\sharp = \phi\Big(\frac{x^*x + xx^*}{2}\Big)^{1/2}, \quad \text{for } x \in M.$$

A "good" simple operator in a type II_1 factor means an operator with the form $\sum_{i=1}^{n} \lambda_i e_i$, where $\lambda_i \in \mathbb{C}$ and $e_1, \ldots, e_n$ are equivalent mutually orthogonal projections with $\sum_{i=1}^{n} e_i = 1$. *A rational positive "good" simple operator* is a positive "good" simple operator with rational numbers as coefficients. A "good" simple operator h in $M_m(\mathbb{C}) \otimes N$ is of *"scalar form"* if $h = \sum_{1 \leqslant i \leqslant m} f_{ii} \otimes \lambda_{i,i} 1_N$, where $\{f_{ij}\}_{1 \leqslant i,j \leqslant m}$ are the matrix units in $M_m(\mathbb{C})$, $\lambda_{i,i} \in \mathbb{C}$ and 1_N is the identity operator in N.

Our strategy to prove Theorem 3.2 is to mimic Haagerup's proof of Proposition 3.5 of [16]. To use Haagerup's techniques, we first need Lemma 3.3 and Lemma 3.4.

Using Proposition 2.3 in our paper, we deduce that for any $\varepsilon > 0$, there exist two normal u.c.p. maps $S : M \to M_n(\mathbb{C}) \otimes N, T : M_n(\mathbb{C}) \otimes N \to M$ such that for all $x \in M$, $\|T \circ S(x) - x\|_2 < \varepsilon$ and $\tau \circ T(x) = (\mathrm{tr}_n \otimes \tau)(hx)$, where h is a positive invertible element in $M_n(\mathbb{C}) \otimes N$. Then, using a result of Kadison in [18], we can assume h is of diagonal form in $M_n(\mathbb{C}) \otimes N$. In Haagerup's situation, $N = \mathbb{C}$, so h is always of scalar form, but in general, this h may not be of scalar form. Note that in Haagerup's assumptions, he dealt with $h \in M_m(\mathbb{C})$, which is of scalar form. If N is a diffuse finite factor, then we can assume that h is a "good" simple operator and we can also make a perturbation of h to assume its coefficients to be rational, this is our Lemma 3.3. In Lemma 3.4, we amplify $M_n(\mathbb{C}) \otimes N$ to $M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes N$, and in this larger algebra, h can be written in scalar form.

LEMMA 3.3. *Let M be a finite von Neumann algebra with a faithful normal trace τ_M, and let N be a type* II_1 *factor with trace τ_N. Let $T : N \to M$ be a normal u.c.p. map such that*

$$\tau_M \circ T(y) = \tau_N(yh), \quad \forall y \in N,$$

and let h be an invertible positive operator in N. For any $y_1, \dots, y_n \in N$ and any $\varepsilon > 0$, there exists a normal u.c.p. map T' from N to M such that

$$\|T(y_i) - T'(y_i)\|_2 < \varepsilon \quad \textit{and} \quad \tau_M \circ T'(y) = \tau_N(h'y)$$

for $1 \leqslant i \leqslant n$ and all $y \in N$, where h' is an invertible rational positive "good" simple operator in N.

Proof. Since h is an invertible positive operator in the type II_1 factor N, we can identify h with a positive function $h(t)$, $0 \leqslant t \leqslant 1$ and assume that $h(t) \geqslant \delta > 0$ for all t. Since N is a type II_1 factor, there exists a sequence of "good" simple operators $h_k = h_k(t)$ with the property that

(i) $\delta \leqslant h_k(t) \leqslant h(t)$ for all t, $0 \leqslant t \leqslant 1$;
(ii) $\lim_{k\to\infty} h_k(t) = h(t)$ for almost all t, $0 \leqslant t \leqslant 1$.

Assume $\|h - h_k\|_1 < \varepsilon$ for some $\varepsilon > 0$. Let $b_k = b_k(t) = \frac{h_k(t)}{h(t)}$. Then $0 < b_k(t) \leqslant 1$ for all $0 \leqslant t \leqslant 1$. Note that

$$\|1 - b_k\|_1 = \tau(1 - b_k) = \int_0^1 \frac{h(t) - h_k(t)}{h(t)} \mathrm{d}t \leqslant \frac{1}{\delta}\|h - h_k\|_1 < \frac{\varepsilon}{\delta}, \quad \text{and}$$

$$\|1 - b_k\|_2^2 = \tau((1 - b_k)^2) = \int_0^1 \frac{(h(t) - h_k(t))^2}{(h(t))^2} \mathrm{d}t \leqslant \frac{2\|h\|}{\delta^2}\|h - h_k\|_1 < \frac{2\|h\|}{\delta^2}\varepsilon.$$

Define $T_k : N \to M$ by

$$T_k(y) = T(b_k^{1/2} y b_k^{1/2}) + \tau_N(y) T(1 - b_k), \quad \text{for } y \in N.$$

Then T_k is a normal u.c.p. map. Note that b_k commutes with h, so for $y \in N$, we deduce

$$\begin{aligned} \tau_M \circ T_k(y) &= \tau_M \circ T(b_k^{1/2} y b_k^{1/2}) + \tau_N(y)\tau_M(T(1 - b_k)) \\ &= \tau_N(h b_k^{1/2} y b_k^{1/2}) + \tau_N(y)\tau_N(h(1 - b_k)) = \tau_N(h_k' y), \end{aligned}$$

where $h_k' = hb_k + \tau_N(h(1 - b_k))1 = h_k + \tau_N(h(1 - b_k))1$ is an invertible positive "good" simple operator.

By the Schwarz inequality for c.p. maps, we have for $y \in N$,

$$\|T(y)\|_2 = \tau_M(T(y^*)T(y))^{1/2} \leqslant \tau_M(T(y^*y))^{1/2} = \tau_N(hy^*y)^{1/2} \leqslant \|h\|^{1/2}\|y\|_2.$$

By Proposition 1.2.1 of [11], we have $\|1-b_k^{1/2}\|_2 \leqslant \|1-b_k\|_1^{1/2}$. Moreover, for $1 \leqslant i \leqslant n$,

$$\begin{aligned}\|T_k(y_i)-T(y_i)\|_2 &\leqslant \|T(y_i - b_k^{1/2}y_ib_k^{1/2})\|_2 + |\tau_N(y_i)|\|T(1-b_k)\|_2\\ &\leqslant \|T(y_i(1-b_k^{1/2}))\|_2 + \|T((1-b_k^{1/2})y_ib_k^{1/2})\|_2\\ &\qquad + |\tau_N(y_i)|\|T(1-b_k)\|_2\\ &\leqslant \|h\|^{1/2}(\|y_i(1-b_k^{1/2})\|_2 + \|(1-b_k^{1/2})y_ib_k^{1/2}\|_2 + |\tau_N(y_i)|\|1-b_k\|_2)\\ &\leqslant 2\|h\|^{1/2}\|y_i\|\|1-b_k\|_1^{1/2} + |\tau_N(y_i)|\|h\|^{1/2}\|1-b_k\|_2 \to 0.\end{aligned}$$

Next we want to make a perturbation of the invertible positive "good" simple operator h'_k to get rational coefficients.

Note that $h'_k \in N$ is an invertible positive "good" simple operator and $\tau_M \circ T_k(1) = \tau_N(h'_k) = 1$. Let $\lambda_1, \dots, \lambda_m$ be the diagonal elements of h'_k. Then we have $\lambda_i > 0$ and $\sum_{i=1}^{m} \lambda_i = m$.

Choose rational numbers $q_1, \dots, q_m$ such that $(1-\varepsilon)\lambda_i < q_i < \lambda_i$. Put $u_i = \frac{q_i}{\lambda_i}$ for $i = 1, \dots, m$. Moreover, let s be the diagonal matrix with the diagonal elements $u_1, \dots, u_m$. Then $1-\varepsilon < s < 1$. Define a map T' from N to M by

$$T'(x) = T_k(s^{1/2}xs^{1/2}) + \tau_N(x)T_k(1-s).$$

Then T' is a normal u.c.p. map and

$$\begin{aligned}\|T_k(x)-T'(x)\| &\leqslant \|s^{1/2}xs^{1/2} - x\| + \|1-s\|\|x\|\\ &= \frac{1}{2}\|(1+s^{1/2})x(1-s^{1/2}) + (1-s^{1/2})x(1+s^{1/2})\| + \|1-s\|\|x\|\\ &\leqslant (\|1+s^{1/2}\|\|1-s^{1/2}\| + \|1-s\|)\|x\| < 3\varepsilon\|x\|.\end{aligned}$$

We have

$$\|T' - T_k\| \to 0 \quad \text{and} \quad (\tau \circ T')(x) = \tau_N(h'x),$$

where $h' = s^{1/2}h'_ks^{1/2} + \tau_N((h_2(1-s))$. Let $l_1, \dots, l_m$ be the diagonal elements of h'. Note that $\tau_N(h'_ks) = \sum_{i=1}^{m} \frac{q_i}{m}$. Then we have $l_i = q_i + \left(1 - \sum_{i=1}^{m} \frac{q_i}{m}\right) > 0$ and is rational.

Then for $1 \leqslant i \leqslant n$, we get

$$\|T(y_i) - T'(y_i)\|_2 \leqslant \|T(y_i) - T_k(y_i)\|_2 + \|T_k(y_i) - T'(y_i)\|_2 \to 0.$$

Hence we finish the proof. ∎

LEMMA 3.4. *Let M be a finite von Neumann algebra with a faithful normal tracial state τ, and let $N \subset M$ be a type* II_1 *factor . Let the inclusion $N \subset M$ be amenable. Let*

$\{x_1, \dots, x_m\}$ be a finite set in M and let $\varepsilon > 0$. Then there exists an $n \in \mathbb{N}$, and two normal u.c.p. maps $S : M \to M_n(\mathbb{C}) \otimes N$, $T : M_n(\mathbb{C}) \otimes N \to M$, such that:

(i) *$\tau \circ T(y) = (\mathrm{tr}_n \otimes \tau)(hy)$, where $y, h \in M_n(\mathbb{C}) \otimes N$ and h is an invertible rational positive "good" simple operator, furthermore, it is of "scalar form";*

(ii) *$\|T \circ S(x_i) - x_i\|_2 < \varepsilon, i = 1, \dots, m$.*

Proof. By Proposition 2.3, for any $\varepsilon > 0$ we can find two normal u.c.p. maps $S_1 : M \to M_n(\mathbb{C}) \otimes N$, $T_1 : M_n(\mathbb{C}) \otimes N \to M$, such that $\tau \circ T_1(y) = (\mathrm{tr}_n \otimes \tau)(h_1 y)$, $\|T_1 \circ S_1(x_i) - x_i\|_2 < \varepsilon$, where $h_1, y \in M_n(\mathbb{C}) \otimes N$, h_1 is an invertible positive operator and $i = 1, \dots, m$. By Lemma 3.3, we have a normal u.c.p. map $T_2 : M_n(\mathbb{C}) \otimes N \to M$, with $\tau \circ T_2(y) = (\mathrm{tr}_n \otimes \tau)(h_2 y)$, where $h_2, y \in M_n(\mathbb{C}) \otimes N$ and h_2 is an invertible rational positive "good" simple operator.

By the definition of "good" simple operators, assume $h_2 = \sum_{i=1}^{k} \lambda_i e_i$, where $\{\lambda_i\}$ are positive rational numbers and $\{e_i\}$ are equivalent mutually orthogonal projections with $\sum_{i=1}^{k} e_i = 1$. Note that there exists a transform U of $M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes N$ which turns $I_k \otimes h_2$ into a "scalar form". Write $U(z) = vzv^*$, where $v, z \in M_k(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes N$, v is some unitary element, and $h := U(I_k \otimes h_2)$. Then h is an invertible rational positive "good" simple operator; furthermore, it is of "scalar form".

Define $T = T_2 \circ (\mathrm{tr}_k \otimes \mathrm{id}_{M_n(\mathbb{C})\otimes N}) \circ U^{-1}$ and $S = U \circ (\mathrm{id}_k \otimes \mathrm{id}_{M_n(\mathbb{C})\otimes N}) \circ S_1$, where $\mathrm{id}_{M_n(\mathbb{C})\otimes N}$ is the identity map on $M_n(\mathbb{C}) \otimes N$, id_k is the identity map on $M_k(\mathbb{C})$. It is clear that $\|T \circ S(x_i) - x_i\|_2 < \varepsilon, i = 1, \dots, m$.

Let $v = \sum_{1 \leqslant i,j \leqslant k} e_{ij} \otimes x_{ij}$, where $\{e_{ij}\}_{1 \leqslant i,j \leqslant k} \subset M_k(\mathbb{C})$ are the matrix units and $x_{ij} \in M_n(\mathbb{C}) \otimes N$. Then for $a \in M_k(\mathbb{C}), x \in M_n(\mathbb{C}) \otimes N$, we have

$$\begin{aligned}
\tau \circ T(a \otimes x) &= (\mathrm{tr}_n \otimes \tau)(h_2(\mathrm{tr}_k \otimes \mathrm{id}_{M_n(\mathbb{C})\otimes N})U^{-1}(a \otimes x)) \\
&= (\mathrm{tr}_n \otimes \tau)\Big(h_2(\mathrm{tr}_k \otimes \mathrm{id}_{M_n(\mathbb{C})\otimes N})\Big(\sum_{i,j,s,t} e_{ji} a e_{st} \otimes x_{ij}^* x x_{st}\Big)\Big) \\
&= (\mathrm{tr}_n \otimes \tau)\Big(h_2 \sum_{i,j,s} \mathrm{tr}_k(e_{si} a) x_{ij}^* x x_{sj}\Big) \\
&= \sum_{i,j,s} \mathrm{tr}_k(e_{si} a)(\mathrm{tr}_n \otimes \tau)(h_2 x_{ij}^* x x_{sj}), \\
(\mathrm{tr}_k \otimes \mathrm{tr}_n \otimes \tau)(h(a \otimes x)) &= (\mathrm{tr}_k \otimes \mathrm{tr}_n \otimes \tau)(v(I_k \otimes h_2)v^*(a \otimes x)) \\
&= (\mathrm{tr}_k \otimes \mathrm{tr}_n \otimes \tau)\Big(\sum_{i,j,s} e_{is} \otimes x_{ij} h_2 x_{sj}^* (a \otimes x)\Big) \\
&= \sum_{i,j,s} \mathrm{tr}_k(e_{si} a)(\mathrm{tr}_n \otimes \tau)(h_2 x_{ij}^* x x_{sj}).
\end{aligned}$$

Thus we have $\tau(T(a \otimes x)) = (\mathrm{tr}_k \otimes \mathrm{tr}_n \otimes \tau)(h(a \otimes x))$, where $a \in M_k(\mathbb{C})$, $x \in M_n(\mathbb{C}) \otimes N$. Let $m = nk$. Hence we finish the proof. ∎

With the help of the above two lemmas, we will mimic Lemma 3.1, Lemma 3.2 of [16] to prove the following two lemmas which also generalise Lemma 3.1, Lemma 3.2 of [16]. We should mention that the proofs are not trivial. We have to overcome some new difficulties since under our assumptions we deal with $M_m(\mathbb{C}) \otimes N$ where N is a von Neumann algebra, while Haagerup dealt with $M_m(\mathbb{C})$.

The difficulty of Lemma 3.5 is Claim 1, i.e., S maps M into $M_m(\mathbb{C}) \otimes N$, and it is normal.

LEMMA 3.5. *Let M be a finite von Neumann algebra with a faithful normal trace τ and $N \subset M$ be a von Neumann subalgebra. Let $m \in \mathbb{N}$ and T be a normal u.c.p. map from $M_m(\mathbb{C}) \otimes N$ to M such that $(\tau \circ T)(x) = (\mathrm{tr}_m \otimes \tau)(hx)$, where h is an invertible positive element in $M_m(\mathbb{C}) \otimes N$. Put $\phi(x) = \tau \circ T(x)$, for $x \in M_m(\mathbb{C}) \otimes N$. Then*

(i) *There is a unique normal u.c.p. map S from M to $M_m(\mathbb{C}) \otimes N$ such that*

$$(\mathrm{tr}_m \otimes \tau)(h^{1/2} S(y) h^{1/2} x^*) = \tau(y T(x)^*)$$

for all $y \in M$ and all $x \in M_m(\mathbb{C}) \otimes N$. Moreover, $\phi \circ S(y) = \tau(y)$for $y \in M$.

(ii) *For all $x \in M_m(\mathbb{C}) \otimes N$, $\|T(x)\|_2^2 \leqslant (\mathrm{tr}_m \otimes \tau)(h^{1/2} x h^{1/2} x^*)$.*

Proof. (i) If S_1, S_2 satisfy the condition in (i), then for $y \in M$,

$$(\mathrm{tr}_m \otimes \tau)(h^{1/2} S_1(y) h^{1/2} x^*) = (\mathrm{tr}_m \otimes \tau)(h^{1/2} S_2(y) h^{1/2} x^*)$$

for all $x \in M_m(\mathbb{C}) \otimes N$. This implies that $h^{1/2} S_1(y) h^{1/2} = h^{1/2} S_2(y) h^{1/2}$ and consequently $S_1(y) = S_2(y)$ since h is invertible.

Let s be the inner product on $M_m(\mathbb{C}) \otimes N$ defined for $x_1, x_2 \in M_m(\mathbb{C}) \otimes N$, by $s(x_1, x_2) = (\mathrm{tr}_m \otimes \tau)(h^{1/2} x_1 h^{1/2} x_2^*)$.

Note that s is positive definite because

$$s(x_1, x_2) = (\mathrm{tr}_m \otimes \tau)((h^{1/4} x_1 h^{1/4})(h^{1/4} x_2 h^{1/4})^*).$$

For $x \in M_m(\mathbb{C}) \otimes N$, we have

$$\|T(x)\|_2^2 = \tau(T^*(x) T(x)) \leqslant \tau(T(x^* x)) = (\mathrm{tr}_m \otimes \tau)(h x^* x).$$

Moreover,

$$\begin{aligned}
(\mathrm{tr}_m \otimes \tau)(h x^* x) &= (\mathrm{tr}_m \otimes \tau)(h^{1/2} x^* h^{1/4} h^{-1/2} h^{1/4} x h^{1/2}) \\
&\leqslant \|h^{-1/2}\| (\mathrm{tr}_m \otimes \tau)(h^{1/2} x^* h^{1/4} h^{1/4} x h^{1/2}) \\
&= \|h^{-1/2}\| (\mathrm{tr}_m \otimes \tau)(h^{1/2} h^{1/4} x^* h^{1/4} h^{1/4} x h^{1/4}) \\
&\leqslant \|h^{-1/2}\| \|h^{1/2}\| (\mathrm{tr}_m \otimes \tau)(h^{1/4} x^* h^{1/4} h^{1/4} x h^{1/4}) \\
&= \|h^{-1/2}\| \|h^{1/2}\| \|x\|_s^2.
\end{aligned}$$

Denote by $(M_m(\mathbb{C}) \otimes N, s)$ the completion of $M_m(\mathbb{C}) \otimes N$ with respect to the norm induced by the inner product s. Thus there exists a bounded linear map T_0 from the Hilbert space $(M_m(\mathbb{C}) \otimes N, s)$ to the Hilbert space $L^2(M, \tau)$ with the restriction to be T on $M_m(\mathbb{C}) \otimes N$.

Let $T_0^* : L^2(M,\tau) \to (M_m(\mathbb{C}) \otimes N, s)$ be the adjoint operator and let S be the restriction of T_0^* to M.

Claim 1. S is a normal map which maps M into $M_m(\mathbb{C}) \otimes N$.

Proof of Claim 1. For $x \in (M_m(\mathbb{C}) \otimes N)_+, y \in M_+$,

$$\begin{aligned}\tau(yT(x)) &= \tau(T(x)^{1/2}yT(x)^{1/2}) \leqslant \|y\|\tau \circ T(x)\\ &= \|y\|(\mathrm{tr}_m \otimes \tau)(hx) = \|y\|(\mathrm{tr}_m \otimes \tau)(x^{1/2}hx^{1/2}) \leqslant \|y\|\|h\|(\mathrm{tr}_m \otimes \tau)(x).\end{aligned}$$

Note that for any fixed y in M_+, $\tau(yT(x))$ and $(\mathrm{tr}_m \otimes \tau)(x)$ are normal positive linear functionals on $M_m(\mathbb{C}) \otimes N$. By Theorem 7.3.6 of [19], there exists a positive element z in $M_m(\mathbb{C}) \otimes N$ such that $\tau(yT(x)) = (\mathrm{tr}_m \otimes \tau)(xz)$. Besides, since h is invertible, we have

$$(\mathrm{tr}_m \otimes \tau)(xz) = (\mathrm{tr}_m \otimes \tau)(h^{1/2}h^{-1/2}zh^{-1/2}h^{1/2}x) = s(h^{-1/2}zh^{-1/2}, x^*).$$

For $x \in M_m(\mathbb{C}) \otimes N, y \in M$,

$$s(S(y), x) = s(T_0^*(y), x) = (y, T_0(x))_\tau = \tau(yT(x^*)).$$

Then we can obtain that for $x \in (M_m(\mathbb{C}) \otimes N)_+, y \in M_+$,

$$s(S(y), x) = \tau(yT(x)) = s(h^{-1/2}zh^{-1/2}, x),$$

which implies $S(y) = h^{-1/2}zh^{-1/2}$ and hence S is normal. Since h and z are both in $M_m(\mathbb{C}) \otimes N$, S maps all the elements of M into $M_m(\mathbb{C}) \otimes N$. This ends the proof of Claim 1. ∎

It is clear that

$$(\mathrm{tr}_m \otimes \tau)(h^{1/2}S(1)h^{1/2}x^*) = s(S(1), x) = \tau(T(x)^*) = (\mathrm{tr}_m \otimes \tau)(hx^*),$$

hence $S(1) = 1$ since h is invertible. For $y \in N$, we have

$$\phi \circ S(y) = \tau \circ T \circ S(y) = (\mathrm{tr}_m \otimes \tau)(hS(y)) = s(S(y), 1) = \tau(y).$$

To prove that S is completely positive, we will need the fact that an operator x in a finite von Neumann algebra B is positive if and only if $\tau_B(xy) \geqslant 0$ for any $y \in B_+$. Here, τ_B is a faithful normal tracial state on B.

Let $n \in \mathbb{N}$, $(e_{ij})_{i,j=1,\ldots,n}$ be the matrix units in $M_n(\mathbb{C})$. Let I_n be the identity in $M_n(\mathbb{C})$. Put $S^{(n)} = I_n \otimes S$, $T^{(n)} = I_n \otimes T$. We shall prove that $S^{(n)}$ is a positive map for all $n \in \mathbb{N}$. Let $a = \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij} \in M_n(\mathbb{C}) \otimes M$, and $b = \sum_{i,j=1}^{n} e_{ij} \otimes b_{ij} \in M_n(\mathbb{C}) \otimes (M_m(\mathbb{C}) \otimes N)$.

Then

$$\begin{aligned}&(\mathrm{tr}_n \otimes (\mathrm{tr}_m \otimes \tau))((I_n \otimes h^{1/2})S^{(n)}(a)(I_n \otimes h^{1/2})b^*)\\ &\quad = (\mathrm{tr}_n \otimes (\mathrm{tr}_m \otimes \tau))\Big(\Big(\sum_{i,j=1}^{n} e_{ij} \otimes h^{1/2}S(a_{ij})h^{1/2}\Big)\Big(\sum_{s,t=1}^{n} e_{ts} \otimes b_{st}^*\Big)\Big)\end{aligned}$$

$$= \frac{1}{n}(\mathrm{tr}_m \otimes \tau)\Big(\sum_{i,j=1}^{n} h^{1/2}S(a_{ij})h^{1/2}b_{ij}^*\Big)$$
$$= \frac{1}{n}\sum_{i,j=1}^{n} s(S(a_{ij}), b_{ij}) = \frac{1}{n}\sum_{i,j=1}^{n} \tau(a_{ij}T(b_{ij}^*)) = (\mathrm{tr}_n \otimes \tau)(aT^{(n)}(b)^*).$$

For all $a \in (M_n(\mathbb{C}) \otimes M)_+$ and $b \in (M_n(\mathbb{C}) \otimes (M_m(\mathbb{C}) \otimes N))_+$, we have $(I_n \otimes h^{1/2})S^{(n)}(a)(I_n \otimes h^{1/2}) \in (M_n(\mathbb{C}) \otimes (M_m(\mathbb{C}) \otimes N))_+$ since $T^{(n)}$ is positive. Hence $S^{(n)}$ is a positive map.

(ii) The composed map $T \circ S$ is a normal u.c.p. map from M to M and $\tau \circ (T \circ S) = \phi \circ S = \tau$. Then $\|T \circ S(x)\|_2 \leqslant \|x\|_2$ using the Schwarz inequality for c.p. maps. Hence $\|T_0 \circ T_0^*\| \leqslant 1$, where T_0 is the map T considered as a linear map from the Hilbert space $(M_m(\mathbb{C}) \otimes N, s)$ to $L^2(N, \tau)$. Thus $\|T_0\|^2 = \|T_0 \circ T_0^*\| \leqslant 1$, i.e. $\|T(x)\|_2^2 \leqslant s(x, x) = (\mathrm{tr}_m \otimes \tau)(h^{1/2}xh^{1/2}x^*)$, $x \in M_m(\mathbb{C}) \otimes N$. ∎

To prove Lemma 3.6, we first use the same method as Haagerup did to prove Step 1. The difficulty in our proof is Step 2. In Haagerup's proof, he first constructed a u.c.p. map $T : M_m(\mathbb{C}) \to M_q(\mathbb{C})$ which is Step 1 in our proof, then he used Lemma 3.1 of [16] to get a u.c.p. map $S : M_q(\mathbb{C}) \to M_m(\mathbb{C})$. Since this S is defined abstractly, to estimate $S \circ T(e_{ij})$, he used the fact that $x \in M_m(\mathbb{C})$ is determined once we know $\mathrm{tr}_m(xe_{ij})$ for all the matrix units $\{e_{ij}\}_{1 \leqslant i,j \leqslant m}$ in $M_m(\mathbb{C})$. However in our situation, this method does not work. Instead, to prove Step 2, we directly construct a normal u.c.p. map $S : M_q(\mathbb{C}) \otimes N \to M_m(\mathbb{C}) \otimes N$ such that for $x_{ij} \in N$, $S \circ T(e_{ij} \otimes x_{ij})$ can be estimated.

LEMMA 3.6. *Let M be a finite von Neumann algebra with a faithful normal trace τ and let $N \subset M$ be a von Neumann subalgebra. Let ϕ be a normal state on $M_m(\mathbb{C}) \otimes N$ of the form*

$$\phi(x) = (\mathrm{tr}_m \otimes \tau)(hx),$$

where h is an invertible rational positive "good" simple operator, and it is of "scalar form" in $M_m(\mathbb{C}) \otimes N$. Then there exists a $q \in \mathbb{N}$, and two normal u.c.p. maps $T : M_m(\mathbb{C}) \otimes N \to M_q(\mathbb{C}) \otimes N$, $S : M_q(\mathbb{C}) \otimes N \to M_m(\mathbb{C}) \otimes N$ such that:

(i) $\phi \circ S = \mathrm{tr}_q \otimes \tau$, $(\mathrm{tr}_q \otimes \tau) \circ T = \phi$;

(ii) $\|S \circ T(x) - x\|_\phi^\sharp \leqslant \|h^{1/2}x - xh^{1/2}\|_2$, $x \in M_m(\mathbb{C}) \otimes N$.

Proof. Step 1. There exists a normal unital completely positive map $T : M_m(\mathbb{C}) \otimes N \to M_q(\mathbb{C}) \otimes N$ such that $(\mathrm{tr}_q \otimes \tau) \circ T = \phi$.

Proof of Step 1. Assume h is of the diagonal form with diagonal elements $\lambda_1, \ldots, \lambda_m$, where $\lambda_i' s$ are strictly positive rational numbers. Then we can choose positive integers $p_1, \ldots, p_m$ and q such that $\frac{\lambda_i}{m} = \frac{p_i}{q}$, $i = 1, \ldots, m$. Since $(\mathrm{tr}_m \otimes \tau)(h) = 1$, we have $\sum_{i=1}^{m} p_i = q$.

A $q \times q$-matrix y can be represented by a block matrix $y = (y_{ij})_{i,j=1,\dots,m}$, where each y_{ij} is a $p_i \times p_j$-matrix. Let F_{ij} denote the $p_i \times p_j$-matrix given by

$$(F_{ij})_{k,l} = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases}$$

and let f_{ij} denote the $q \times q$-matrix with block matrix

$$(f_{ij})_{i'j'} = \begin{cases} F_{ij} & \text{if } (i', j') = (i, j), \\ 0 & \text{otherwise.} \end{cases}$$

Note that the number 1 occurs $\min\{p_i, p_j\}$ times in F_{ij} and f_{ij}. Let $(e_{ij})_{i,j=1,\dots,m}$ be the matrix units in $M_m(\mathbb{C})$ and define a linear map T from $M_m(\mathbb{C}) \otimes N$ to $M_q(\mathbb{C}) \otimes N$ by $T\Big(\sum\limits_{i,j=1}^{m} e_{ij} \otimes x_{ij}\Big) = \sum\limits_{i,j=1}^{m} f_{ij} \otimes x_{ij}, x_{ij} \in N$. Then T is unital. Moreover, for $i \neq j$, we have

$$(\mathrm{tr}_q \otimes \tau)(T(e_{ij} \otimes x_{ij})) = (\mathrm{tr}_q \otimes \tau)(f_{ij} \otimes x_{ij}) = (\mathrm{tr}_m \otimes \tau)(h(e_{ij} \otimes x_{ij})) = 0,$$
$$(\mathrm{tr}_q \otimes \tau)(T(e_{ii} \otimes x_{ii})) = \mathrm{tr}_q(f_{ii})\tau(x_{ii}) = \frac{\lambda_i}{m}\tau(x_{ii}) = (\mathrm{tr}_m \otimes \tau)(h(e_{ii} \otimes x_{ii})).$$

Hence, $(\mathrm{tr}_q \otimes \tau) \circ T(x) = (\mathrm{tr}_m \otimes \tau)(hx) = \phi(x), x \in M_m(\mathbb{C}) \otimes N$. To see that T is completely positive, put $p = \max\{p_1, \dots, p_m\}$ and let $\widetilde{f}_{ij}$ be the element in $M_{mp}(\mathbb{C})$ given by the $m \times m$-block matrix

$$(\widetilde{f}_{ij})_{i'j'} = \begin{cases} I_p & \text{if } (i', j') = (i, j), \\ 0 & \text{otherwise.} \end{cases}$$

Here I_p is the $p \times p$-unit matrix. The map $\widetilde{T}$ from $M_m(\mathbb{C}) \otimes N$ to $M_{mp}(\mathbb{C}) \otimes N$ by $\widetilde{T}\Big(\sum\limits_{i,j=1}^{m} e_{ij} \otimes x_{ij}\Big) = \sum\limits_{i,j=1}^{m} \widetilde{f}_{ij} \otimes x_{ij}, x_{ij} \in N$, is a $*$-representation and therefore completely positive. It is not difficult to see that there exists a projection e in $M_{mp}(\mathbb{C}) \otimes N$ such that $e(M_{mp}(\mathbb{C}) \otimes N)e = M_q(\mathbb{C}) \otimes N$ and $T(x) = e\widetilde{T}(x)e, x \in M_m(\mathbb{C}) \otimes N$. Hence T is normal and completely positive. This ends the proof of Step 1. ∎

Step 2. There is a normal u.c.p. map $S : M_q(\mathbb{C}) \otimes N \to M_m(\mathbb{C}) \otimes N$ such that $\phi \circ S = \mathrm{tr}_q \otimes \tau$ and $S \circ T(e_{ij} \otimes x_{ij}) = \frac{\min\{p_i, p_j\}}{\sqrt{p_i p_j}} e_{ij} \otimes x_{ij}$.

Proof of Step 2. For any $s, t \in \mathbb{N}$, define a linear map D from $M_{s\times t}(\mathbb{C}) \otimes N$ to N by

$$D\Big(\sum_{1\leqslant i\leqslant s, 1\leqslant j\leqslant t} l_{ij} \otimes h_{ij}\Big) = \sum_{i=1}^{\min\{s,t\}} h_{ii},$$

where $(l_{ij})_{1\leqslant i\leqslant s, 1\leqslant j\leqslant t}$ is the matrix units in $M_{s\times t}(\mathbb{C})$ and h_{ij} is in N for any $1 \leqslant i \leqslant s, 1 \leqslant j \leqslant t$. Let $(k_{st})_{s,t=1,\dots,q}$ be the matrix units in $M_q(\mathbb{C})$. For $x = \sum\limits_{i,j=1}^{m} e_{ij} \otimes$

$x_{ij} \in M_m(\mathbb{C}) \otimes N, y = \sum_{i,j=1}^{q} k_{ij} \otimes y_{ij} \in M_q(\mathbb{C}) \otimes N$, define a linear map S' from $M_q(\mathbb{C}) \otimes N$ to $M_m(\mathbb{C}) \otimes N$ by

$$S'(y) = \sum_{i,j=1}^{m} e_{ij} \otimes \frac{1}{\sqrt{p_i p_j}} D(f_{ii} y f_{jj}).$$

For $1 \leqslant i, j \leqslant m$, put $a_{ij} = \frac{1}{\sqrt{p_i p_j}} D(f_{ii} y f_{jj})$ and $p_0 = 0$, then

$$a_{ij} = \frac{1}{\sqrt{p_i p_j}} \sum_{k=1}^{\min\{p_i, p_j\}} y_{p_1+p_2+\cdots+p_{i-1}+k, p_1+p_2+\cdots+p_{j-1}+k}.$$

Note that

$$\begin{aligned}(\mathrm{tr}_m \otimes \tau)(h^{1/2} S'(y) h^{1/2} x^*) &= (\mathrm{tr}_m \otimes \tau)\Big(\Big(\sum_{i,j=1}^{m} e_{ij} \otimes \sqrt{\lambda_i \lambda_j} a_{ij}\Big)\Big(\sum_{k,l=1}^{m} e_{lk} \otimes x_{kl}^*\Big)\Big) \\ &= \sum_{i,j=1}^{m} \frac{\tau\Big(\sqrt{\lambda_i \lambda_j} a_{ij} x_{ij}^*\Big)}{m} = \sum_{i,j=1}^{m} \frac{\tau\Big(\sqrt{p_i p_j} a_{ij} x_{ij}^*\Big)}{q} \\ &= \sum_{i,j=1}^{m} \sum_{k=1}^{\min\{p_i, p_j\}} \frac{\tau(y_{p_1+p_2+\ldots+p_{i-1}+k, p_1+p_2+\ldots+p_{j-1}+k} x_{ij}^*)}{q}.\end{aligned}$$

Note that $f_{ij} = \sum_{k=1}^{\min\{p_i, p_j\}} k_{p_1+p_2+\cdots+p_{i-1}+k, p_1+p_2+\cdots+p_{j-1}+k}$, then we have

$$\begin{aligned}&(\mathrm{tr}_q \otimes \tau)(yT(x)^*) \\ &= (\mathrm{tr}_q \otimes \tau)\Big(\Big(\sum_{s,t=1}^{q} k_{st} \otimes y_{st}\Big)\Big(\sum_{i,j=1}^{m} f_{ji} \otimes x_{ij}^*\Big)\Big) \\ &= \sum_{i,j=1}^{m} \sum_{s,t=1}^{q} (\mathrm{tr}_q \otimes \tau)(k_{st} f_{ji} \otimes y_{st} x_{ij}^*) \\ &= \sum_{i,j=1}^{m} \sum_{k=1}^{\min\{p_i, p_j\}} \sum_{s,t=1}^{q} \mathrm{tr}_q(k_{st} k_{p_1+p_2+\cdots+p_{j-1}+k, p_1+p_2+\cdots+p_{i-1}+k}) \circ \tau(y_{st} x_{ij}^*) \\ &= \sum_{i,j=1}^{m} \sum_{k=1}^{\min\{p_i, p_j\}} \frac{\tau(y_{p_1+p_2+\cdots+p_{i-1}+k, p_1+p_2+\cdots+p_{j-1}+k} x_{ij}^*)}{q}.\end{aligned}$$

By Lemma 3.5, there exists a unique normal u.c.p. map S from $M_q(\mathbb{C}) \otimes N$ to $M_m(\mathbb{C}) \otimes N$ such that for $x \in M_m(\mathbb{C}) \otimes N, y \in M_q(\mathbb{C}) \otimes N$,

$$(\mathrm{tr}_m \otimes \tau)(h^{1/2} S(y) h^{1/2} x^*) = (\mathrm{tr}_q \otimes \tau)(yT(x)^*),$$

so it follows that $S = S'$ and $\phi \circ S = \mathrm{tr}_q \otimes \tau$.

Since $T(e_{ij} \otimes x_{ij}) = f_{ij} \otimes x_{ij}$, by the definition of $S' = S$ we have

$$S \circ T(e_{ij} \otimes x_{ij}) = \frac{\min\{p_i, p_j\}}{\sqrt{p_i p_j}} e_{ij} \otimes x_{ij}.$$

This ends the proof of Step 2. ∎

Now we check that $\|S \circ T(x) - x\|_\phi^\sharp \leqslant \|h^{1/2}x - xh^{1/2}\|_2, x \in M_m(\mathbb{C}) \otimes N$. For any $x = \sum_{i,j=1}^{m} x_{ij} \otimes e_{ij} \in M_m(\mathbb{C}) \otimes N$,

$$\begin{aligned}(\|x\|_\phi^\sharp)^2 &= \phi\Big(\frac{xx^* + x^*x}{2}\Big) = (\mathrm{tr}_m \otimes \tau)\Big(\frac{h(xx^* + x^*x)}{2}\Big) \\ &= \frac{1}{2m} \sum_{i,j=1}^{m} (\lambda_i + \lambda_j)\|x_{ij}\|_2^2 = \frac{1}{2q} \sum_{i,j=1}^{m} (p_i + p_j)\|x_{ij}\|_2^2.\end{aligned}$$

Hence $(\|S \circ T(x) - x\|_\phi^\sharp)^2 = \frac{1}{2q} \sum_{i,j=1}^{m} (p_i + p_j)\Big(1 - \frac{\min\{p_i, p_j\}}{\sqrt{p_i p_j}}\Big)^2 \|x_{ij}\|_2^2$.

If $p_i \leqslant p_j$,

$$\Big(1 - \frac{\min\{p_i, p_j\}}{\sqrt{p_i p_j}}\Big)^2 = \Big(1 - \Big(\frac{p_i}{p_j}\Big)^{1/2}\Big)^2 = \frac{1}{p_j}(p_i^{1/2} - p_j^{1/2})^2 \leqslant \frac{2}{p_i + p_j}(p_i^{1/2} - p_j^{1/2})^2.$$

By symmetry, the formula also holds for $p_j \leqslant p_i$. Hence

$$(\|S \circ T(x) - x\|_\phi^\sharp)^2 \leqslant \frac{1}{q} \sum_{i,j=1}^{m} (p_i^{1/2} - p_j^{1/2})^2 \|x_{ij}\|_2^2.$$

On the other hand, the (i,j)-th element of the matrix $h^{1/2}x - xh^{1/2}$ is $(\lambda_i^{1/2} - \lambda_j^{1/2})x_{ij}$. Thus

$$\|h^{1/2}x - xh^{1/2}\|_2^2 = \frac{1}{m} \sum_{i,j=1}^{m} (\lambda_i^{1/2} - \lambda_j^{1/2})^2 \|x_{ij}\|_2^2 = \frac{1}{q} \sum_{i,j=1}^{m} (p_i^{1/2} - p_j^{1/2})^2 \|x_{ij}\|_2^2.$$

Then we finish the proof. ∎

With the help of the above four lemmas, we now proceed to prove Theorem 3.2. Actually, the proof of Theorem 3.2 is adapted from Lemma 3.4 and Proposition 3.5 of [16]. For the reader's convenience, we include the proof below.

Proof of Theorem 3.2. It is sufficient to consider unitary operators $u_1, \ldots, u_n \in M$.

Claim 1. There exists a $q \in \mathbb{N}$, a normal u.c.p. map T from $M_q(\mathbb{C}) \otimes N$ to M, and n operators $y_1, \ldots, y_n \in M_q(\mathbb{C}) \otimes N$, such that $\|y_k\| \leqslant 1$, $\tau \circ T = \mathrm{tr}_q \otimes \tau$ and $\|T(y_k) - u_k\|_2 < \varepsilon, k = 1, \ldots, n$.

Proof of Claim 1. Let $\varepsilon > 0$. By Lemma 3.4, there exists an $m \in \mathbb{N}$, and normal u.c.p. maps $S_1 : M \to M_m(\mathbb{C}) \otimes N$ and $T_1 : M_m(\mathbb{C}) \otimes N \to M$ such that $\|T_1 \circ S_1(u_k) - u_k\|_2 < \varepsilon, k = 1, \ldots, n$, and $\tau \circ T_1(x) = (\mathrm{tr}_m \otimes \tau)(hx)$, where h is an invertible rational positive "good" simple operator, which is of scalar form. Put $x_k = S_1(u_k), k = 1, \ldots, n$. Note that $\|x_k\| \leqslant 1$ and

$$\|T_1(x_k) - u_k\|_2 < \varepsilon, \quad k = 1, \ldots, n.$$

Put $\phi(x) = (\mathrm{tr}_m \otimes \tau)(hx), x \in M_m(\mathbb{C}) \otimes N$. By Lemma 3.6, there exists a $q \in \mathbb{N}$, normal u.c.p. maps $T_2 : M_m(\mathbb{C}) \otimes N \to M_q(\mathbb{C}) \otimes N$ and $S_2 : M_q(\mathbb{C}) \otimes N \to M_m(\mathbb{C}) \otimes N$ such that $\phi \circ S_2 = \mathrm{tr}_q \otimes \tau$, $(\mathrm{tr}_q \otimes \tau) \circ T_2 = \phi$, and $\|S_2 \circ T_2(x) - x\|_\phi^\sharp \leqslant \|h^{1/2}x - xh^{1/2}\|_2, x \in M_m(\mathbb{C}) \otimes N$.

For $k = 1, \ldots, n$,

$$\begin{aligned}\|h^{1/2}x_k - x_kh^{1/2}\|_2^2 &= (\mathrm{tr}_m \otimes \tau)(hx_kx_k^* + hx_k^*x_k - 2h^{1/2}x_kh^{1/2}x_k^*)\\ &= \phi(x_kx_k^*) + \phi(x_k^*x_k) - 2(\mathrm{tr}_m \otimes \tau)(h^{1/2}x_kh^{1/2}x_k^*)\\ &\leqslant 2 - 2(\mathrm{tr}_m \otimes \tau)(h^{1/2}x_kh^{1/2}x_k^*).\end{aligned}$$

By Lemma 3.5(ii),

$$(\mathrm{tr}_m \otimes \tau)(h^{1/2}x_kh^{1/2}x_k^*) \geqslant \|T_1(x_k)\|_2^2 \geqslant (\|u_k\|_2 - \|u_k - T_1(x_k)\|_2)^2 > (1-\varepsilon)^2 > 1 - 2\varepsilon.$$

Then we have $\|S_2 \circ T_2(x_k) - x_k\|_\phi^\sharp < 2\varepsilon^{1/2}$.

Put $y_k = T_2(x_k), k = 1, \ldots, n$ and $T = T_1 \circ S_2$. Then T is a normal u.c.p. map such that $\tau \circ T = (\tau \circ T_1) \circ S_2 = \phi \circ S_2 = \mathrm{tr}_q \otimes \tau$.

By the Schwarz inequality for c.p. maps, we have for $x \in M_m(\mathbb{C}) \otimes N$,

$$\|T_1(x)\|_2^2 \leqslant \frac{1}{2}\tau(T_1(x^*x) + T_1(xx^*)) = (\|x\|_\phi^\sharp)^2.$$

Note that

$$\|T(y_k) - T_1(x_k)\|_2 = \|T_1(S_2(y_k) - x_k)\|_2 \leqslant \|S_2(y_k) - x_k\|_\phi^\sharp < 2\varepsilon^{1/2}.$$

Then we have $\|T(y_k) - u_k\|_2 < 3\varepsilon^{1/2}, k = 1, \ldots, n$. This ends the proof of Claim 1. ∎

By Lemma 3.5(i), there is a unique normal u.c.p. map S from M to $M_q(\mathbb{C}) \otimes N$ such that $(\mathrm{tr}_q \otimes \tau)(S(y)x^*) = \tau(yT(x)^*)$, for $y \in M$, $x \in M_q(\mathbb{C}) \otimes N$, and $(\mathrm{tr}_q \otimes \tau) \circ S = \tau$.

Note that

$$\|T(x)\|_2^2 \leqslant \tau(T(x^*x)) = (\mathrm{tr}_q \otimes \tau)(x^*x) = \|x\|_2.$$

Similarly we get $\|S(y)\|_2 \leqslant \|y\|_2, y \in M$.

For $k = 1, \ldots, n$,

$$\begin{aligned}|(\mathrm{tr}_q \otimes \tau)(S(u_k)y_k^*)| &= |\tau(u_kT(y_k)^*)| = |\tau(1) - \tau(u_k(u_k - T(y_k))^*)|\\ &\geqslant 1 - \|u_k\|_2\|u_k - T(y_k)\|_2 > 1 - 3\varepsilon^{1/2},\end{aligned}$$

$$\begin{aligned}\operatorname{Im} \tau(u_k T(y_k)^*) &= \frac{1}{2}|\tau(u_k T(y_k)^*) - \tau(u_k^* T(y_k))| \\ &= \frac{1}{2}|\tau(u_k(T(y_k) - u_k)^*) - \tau(u_k^*(T(y_k) - u_k))| \\ &\leqslant \|T(y_k) - u_k\|_2 < 3\varepsilon^{1/2}.\end{aligned}$$

Then we conclude that $\operatorname{Re} \tau(u_k T(y_k)^*) > \sqrt{(1-3\varepsilon^{1/2})^2 - (3\varepsilon^{1/2})^2} > 1 - 6\varepsilon^{1/2}$.

Thus, we obtain that

$$\begin{aligned}\|S(u_k) - y_k\|_2^2 &= \|S(u_k)\|_2^2 + \|y_k\|_2^2 - 2\operatorname{Re}(\mathrm{tr}_q \otimes \tau)(S(u_k)y_k^*) \\ &< 2 - 2(1 - 6\varepsilon^{1/2}) = 12\varepsilon^{1/2}.\end{aligned}$$

Hence,

$$\|T \circ S(u_k) - u_k\|_2 = \|T(S(u_k) - y_k)\|_2 + \|T(y_k) - u_k\|_2 < 4\varepsilon^{1/4} + 3\varepsilon^{1/2}. \quad \blacksquare$$

4. PERMANENCE PROPERTIES FOR AMENABLE INCLUSIONS

In this section, we apply our main theorems to study permanence properties for amenable inclusions.

HAAGERUP PROPERTY. In [17], it was shown that if the basic construction $\langle M, e_N \rangle$ is a finite von Neumann algebra and N has the Haagerup property, then M also has the Haagerup property. Anantharaman-Delaroche [5] showed that if $\mathrm{L}H \subset \mathrm{L}G$ is an amenable inclusion of group von Neumann algebras and $\mathrm{L}H$ has the Haagerup property, then $\mathrm{L}G$ also has the Haagerup property. In [28], Popa asked if the inclusion of finite von Neumann algebras $N \subset M$ is amenable, and N has the Haagerup property, does M also have the Haagerup property? Bannon and Fang settled the question in the affirmative in [7]. Their proof is based on an equivalent characterization of the Haagerup property using correspondences.

Since the definition of the Haagerup property involves normal c.p. maps, it is natural to expect a proof using normal c.p. maps rather than correspondences. As an application of our main results, we can give such a proof of certain cases of Bannon-Fang's result.

COROLLARY 4.1. *Let M be a finite von Neumann algebra (respectively, a type* II_1 *factor) with a faithful normal tracial state τ, and let $N \subset M$ be a type* II_1 *factor (respectively, N have an atomic part). If the inclusion $N \subset M$ is amenable and N has the Haagerup property, then M also has the Haagerup property.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a finite set in M and let $\varepsilon > 0$. By Theorem 3.2 (respectively, Theorem 3.1), there exists an $m \in \mathbb{N}$, and normal u.c.p. maps $S : M \to M_m(\mathbb{C}) \otimes N, T : M_m(\mathbb{C}) \otimes N \to M$, such that $(\mathrm{tr}_m \otimes \tau) \circ S = \tau, \tau \circ T = \mathrm{tr}_m \otimes \tau$ and $\|T \circ S(x_i) - x_i\|_2 < \varepsilon$, $i = 1, \ldots, n$. Since N has the Haagerup property, we can find a normal c.p. map $L : M_m(\mathbb{C}) \otimes N \to M_m(\mathbb{C}) \otimes N$, such that $(\mathrm{tr}_m \otimes \tau) \circ L \leqslant \mathrm{tr}_m \otimes \tau$, $\|L(S(x_i)) - S(x_i)\|_2 < \varepsilon$, $i = 1, \ldots, n$, and L induces a compact bounded operator on $L^2(M)$. It is easy to check that $T \circ L \circ S$ satisfies the subtracial condition $\tau \circ T \circ L \circ S \leqslant \tau$, and it induces a compact bounded operator on $L^2(M)$. Moreover, we have

$$\begin{aligned}\|T \circ L \circ S(x_i) - x_i\|_2 &= \|T \circ L \circ S(x_i) - T \circ S(x_i) + T \circ S(x_i) - x_i\|_2 \\ &\leqslant \|T\| \|L \circ S(x_i) - S(x_i)\|_2 + \|T \circ S(x_i) - x_i\|_2 < 2\varepsilon.\end{aligned}$$

Let $\Lambda = \{(E, \varepsilon) : E$ is a finite subset in M and $\varepsilon > 0\}$. For (E, ε), $(F, \epsilon) \in \Lambda$, define $(E, \varepsilon) \prec (F, \epsilon)$ if $E \subseteq F$ and $\varepsilon \geqslant \epsilon$. Then Λ is a directed set. Thus $(T \circ L \circ S_{(\{x_1,\ldots,x_n\},\varepsilon)})_{(\{x_1,\ldots,x_n\},\varepsilon)\in\Lambda}$ is the net which proves the corollary. ∎

WEAK EXACTNESS. The theory of exact C^*-algebras was introduced and studied intensively by Kirchberg. It has been playing a significant role in the development of C^*-algebras, e.g. in the classification of C^*-algebras (see [20], [29]) and in the theory of noncommutative topological entropy (see [9], [31], [32]). Hence it is natural to explore an analogue of this notion for von Neumann algebras. The concept of weakly exact von Neumann algebras was also introduced by Kirchberg [20]. He proved that a von Neumann algebra M is weakly exact if it contains a dense weakly exact C^*-algebra. Ozawa in [25] gave a local characterization of weak exactness and proved that a discrete group is exact if and only if its group von Neumann algebra is weakly exact. Weak exactness also passes to a von Neumann subalgebra which is the range of a normal conditional expectation. Hence, every von Neumann subalgebra of a weakly exact finite von Neumann algebra is again weakly exact. It is left open whether the ultrapower R^ω of the hyperfinite type II_1 factor R is weakly exact or not. For more details and results on weak exactness, we refer the reader to [10], [25].

As the second application of our main results Theorem 3.2 and Theorem 3.1, we prove a permanence result for weak exactness.

COROLLARY 4.2. *Let M be a finite von Neumann algebra (respectively, a type II_1 factor) with a faithful normal tracial state τ, and let $N \subset M$ be a type II_1 factor (respectively, N have an atomic part). If the inclusion $N \subset M$ is amenable and N is weakly exact, then M is also weakly exact.*

Proof. Let E be a finite dimensional operator system in M. Since the inclusion $N \subset M$ is amenable, by Theorem 3.2 (respectively, Theorem 3.1), there exist two nets of trace preserving normal u.c.p. maps $S_i : M \to M_{n_i}(\mathbb{C}) \otimes N$ and $T_i : M_{n_i}(\mathbb{C}) \otimes N \to M$, such that for all $x \in M$, $T_i \circ S_i(x) \to x$ in the $\|\cdot\|_2$-norm topology. By Corollary 14.1.5 of [10], $M_{n_i}(\mathbb{C}) \otimes N$ is weakly exact. Note that $S_i(E) \subset \widetilde{E}$ for some finite-dimensional operator system $\widetilde{E}$ in $M_{n_i}(\mathbb{C}) \otimes N$. By p. 2 of [25] and Remark 1.2, there exist two nets of u.c.p. maps $S_j' : \widetilde{E} \to M_{n_i}(\mathbb{C})$ and $T_j' : S_j'(\widetilde{E}) \to M_{n_i}(\mathbb{C}) \otimes N$ such that the net $(T_j' \circ S_j')$ converges to $\mathrm{id}_{\widetilde{E}}$ in the point-wise $\|\cdot\|_2$-norm topology. For $x \in E$, we have

$$\|T_i \circ T_j' \circ S_j' \circ S_i(x) - x\|_2 \leqslant \|T_i(T_j' \circ S_j' \circ S_i(x) - S_i(x))\|_2 + \|T_i \circ S_i(x) - x\|_2$$

$$\leqslant \|T_j' \circ S_j' \circ S_i(x) - S_i(x)\|_2 + \|T_i \circ S_i(x) - x\|_2 \to 0.$$

The second inequality follows from the fact that T_i is a trace preserving u.c.p. map. Thus $(S_i \circ S_j')$ and $(T_j' \circ T_i)$ are two nets of u.c.p. maps witnessing the weak exactness of M. ∎

WEAK HAAGERUP PROPERTY. Knudby [21] introduced the weak Haagerup property for both locally compact groups and finite von Neumann algebras. He proved that a discrete group has the weak Haagerup property if and only if its group von Neumann algebra does and several hereditary results for the weak Haagerup property. We should mention that the weak Haagerup property of a von Neumann algebra does not depend on the choice of faithful normal traces by Proposition 8.4 of [21], hence we omit the mention of the trace below.

Note that the weak Haagerup property requires normal completely bounded maps. Our main results give a description of relative amenability using normal unital completely positive maps, which are naturally completely bounded. Thus, as the third application of our main results, we add one more permanence property.

COROLLARY 4.3. *Let M be a finite von Neumann algebra (respectively, a type II_1 factor) with a faithful normal tracial state τ, and let $N \subset M$ be a type II_1 factor (respectively, N have an atomic part). If the inclusion $N \subset M$ is amenable and N has the weak Haagerup property, then M also has the weak Haagerup property.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a finite set in the unit ball of M and let $\varepsilon > 0$. By Theorem 3.2 (respectively, Theorem 3.1), there exists an $m \in \mathbb{N}$, and two normal u.c.p. maps $S : M \to M_m(\mathbb{C}) \otimes N$, $T : M_m(\mathbb{C}) \otimes N \to M$, such that $(\mathrm{tr}_m \otimes \tau) \circ S = \tau$, $\tau \circ T = \mathrm{tr}_m \otimes \tau$ and $\|T \circ S(x_k) - x_k\|_2 < \varepsilon, k = 1, \ldots, n$. By Lemma 2.5 of [6], there exist two normal u.c.p. maps $S' : M_m(\mathbb{C}) \otimes N \to M$ and $T' : M \to M_m(\mathbb{C}) \otimes N$ such that $\langle S(x), a\rangle_{\mathrm{tr}_m \otimes \tau} = \langle x, S'(a)\rangle_\tau$ and $\langle T(a), y\rangle_\tau = \langle a, T'(y)\rangle_{\mathrm{tr}_m \otimes \tau}$ for all $x, y \in M$ and $a \in M_m(\mathbb{C}) \otimes N$. Since N has the weak Haagerup property, there exists a constant $C > 0$ and a normal completely bounded map L on $M_m(\mathbb{C}) \otimes N$ with $\|L\|_{\mathrm{c.b.}} \leqslant C$ such that $\langle L(a), b\rangle_{\mathrm{tr}_m \otimes \tau} = \langle a, L(b)\rangle_{\mathrm{tr}_m \otimes \tau}$ for $a, b \in M_m(\mathbb{C}) \otimes N$, L induces a compact bounded map on $L^2(M_m(\mathbb{C}) \otimes N)$, and for $i, j = 1, \ldots, n$, $|\langle L \circ S(x_i) - S(x_i), T'(x_j)\rangle_{\mathrm{tr}_m \otimes \tau}| < \varepsilon$, $|\langle L \circ T'(x_i) - T'(x_i), S(x_j)\rangle_{\mathrm{tr}_m \otimes \tau}| < \varepsilon$ following from Remark 7.5 of [21].

Define $\widetilde{T} = \frac{1}{2}(T \circ L \circ S + S' \circ L \circ T')$. It is clear that $\widetilde{T}$ is a normal completely bounded map with $\|\widetilde{T}\|_{\mathrm{c.b.}} \leqslant C$, since T, T', S, S' are normal u.c.p. maps and L is a normal completely bounded map with $\|L\|_{\mathrm{c.b.}} \leqslant C$.

We check that $\langle \widetilde{T}(x), y\rangle_\tau = \langle x, \widetilde{T}(y)\rangle_\tau$ for $x, y \in M$. Note that

$$\langle T \circ L \circ S(x), y\rangle_\tau = \langle L \circ S(x), T'(y)\rangle_{\mathrm{tr}_m \otimes \tau} = \langle S(x), L \circ T'(y)\rangle_{\mathrm{tr}_m \otimes \tau} = \langle x, S' \circ L \circ T'(y)\rangle_\tau.$$

Clearly, this implies $\langle \widetilde{T}(x), y\rangle_\tau = \langle x, \widetilde{T}(y)\rangle_\tau$ for $x, y \in M$. It is easy to see that $\widetilde{T}$ induces a compact operator on $L^2(M)$, since L induces a compact operator.

We check that $|\langle \widetilde{T}x_i - x_i, x_j\rangle_\tau| < 2\varepsilon$ for $i, j = 1, \ldots, n$. Since $\|T \circ S(x_i) - x_i\|_2 < \varepsilon$ and x_i is in the unit ball of M, it follows that $|\langle T \circ S(x_i) - x_i, x_j\rangle_\tau| < \varepsilon$, for $i, j = 1, \ldots, n$.

Thus we have

$$\begin{aligned}
&|\langle T \circ L \circ S(x_i) - x_i, x_j\rangle_\tau| \\
&\qquad = |\langle L \circ S(x_i), T'(x_j)\rangle_{\mathrm{tr}_m \otimes \tau} - \langle x_i, x_j\rangle_\tau| \\
&\qquad \leqslant |\langle L \circ S(x_i) - S(x_i), T'(x_j)\rangle_{\mathrm{tr}_m \otimes \tau} + \langle S(x_i), T'(x_j)\rangle_{\mathrm{tr}_m \otimes \tau} - \langle x_i, x_j\rangle_\tau| < 2\varepsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&|\langle S' \circ L \circ T'(x_i) - x_i, x_j\rangle_\tau| \\
&\qquad = |\langle L \circ T'(x_i), S(x_j)\rangle_{\mathrm{tr}_m \otimes \tau} - \langle x_i, x_j\rangle_\tau| \\
&\qquad \leqslant |\langle L \circ T'(x_i) - T'(x_i), S(x_j)\rangle_{\mathrm{tr}_m \otimes \tau} + \langle T'(x_i), S(x_j)\rangle_{\mathrm{tr}_m \otimes \tau} - \langle x_i, x_j\rangle_\tau| < 2\varepsilon.
\end{aligned}$$

Let $\Lambda = \{(E, \varepsilon) : E$ is a finite subset in the unit ball of M and $\varepsilon > 0\}$. For (E, ε), $(F, \epsilon) \in \Lambda$, define $(E, \varepsilon) \prec (F, \epsilon)$ if $E \subseteq F$ and $\varepsilon \geqslant \epsilon$. Then Λ is a directed set. Thus $(\widetilde{T}_{(\{x_1,\ldots,x_n\},\varepsilon)})_{(\{x_1,\ldots,x_n\},\varepsilon)\in\Lambda}$ is the net which proves the corollary. ∎

CONCLUSION REMARK. Recall that a type II_1 factor M with a trace τ is said to have property Γ if, given any $\varepsilon > 0$ and $x_1, \ldots, x_n \in M$, there exists a trace zero unitary $u \in M$ such that $\|ux_i - x_i u\|_2 < \varepsilon$, $1 \leqslant i \leqslant n$. In Problem 3.3.2 of [27], Popa asked, if $N \subset M$ are type II_1 factors with trace τ, the inclusion $N \subset M$ is amenable, and N has property Γ, does this imply that M has property Γ? In [2], Bédos proved that if G is a discrete amenable group with a free action α on a von Neumann algebra N and N has property Γ, then $M := N \rtimes_\alpha G$ has property Γ. We tried to use our Theorem 3.2 to attack this problem, but did not succeed. The reason is as follows. Following the above ideas, assume $x_1, \ldots, x_n$ are finite elements in the unit ball of M. By Theorem 3.2, for any $\varepsilon > 0$, there exists an $m \in \mathbb{N}$, and two normal u.c.p. maps $S : M \to M_m(\mathbb{C}) \otimes N$, $T : M_m(\mathbb{C}) \otimes N \to M$, such that $(\mathrm{tr}_m \otimes \tau) \circ S = \tau$, $\tau \circ T = \mathrm{tr}_m \otimes \tau$ and $\|T \circ S(x_k) - x_k\|_2 < \varepsilon, k = 1, \ldots, n$. Since N has property Γ, we can find a unitary operator $\widetilde{u} \in M_m(\mathbb{C}) \otimes N$ with $(\mathrm{tr}_m \otimes \tau)(\widetilde{u}) = 0$ such that $\|S(x_i)\widetilde{u} - \widetilde{u}S(x_i)\|_2 < \varepsilon$. It follows that $\|T(S(x_i)\widetilde{u} - \widetilde{u}S(x_i))\|_2 < \varepsilon$ and $\tau \circ T(\widetilde{u}) = (\mathrm{tr}_m \otimes \tau)(\widetilde{u}) = 0$, since T is a trace preserving normal u.c.p map. Then, we run into two problems. One is that this normal u.c.p. map T is not a homomorphism on the algebra $M_m(\mathbb{C}) \otimes N$. If so, then we would have $\|x_i T(\widetilde{u}) - T(\widetilde{u})x_i\|_2 < 2\varepsilon, 1 \leqslant i \leqslant n$ and $\tau \circ T(\widetilde{u}) = 0$, but we do not know this $T(\widetilde{u})$ is a unitary operator or not, or it can be approximated by trace zero unitaries in M.

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