THE CASE OF EQUALITY IN YOUNG'S INEQUALITY FOR THE s-NUMBERS IN SEMI-FINITE VON NEUMANN ALGEBRAS

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ABSTRACT. For a semi-finite von Neumann algebra $\mathcal{A}$, we study the case of equality in Young's inequality of s-numbers for a pair of τ-measurable operators a, b, and we prove that equality is only possible if $|a|^p = |b|^q$. We also extend the result to unbounded operators affiliated with $\mathcal{A}$, and relate this problem with other symmetric norm Young inequalities.

KEYWORDS: *Measurable operator, τ-compact operator, semi-finite von Neumann algebra, Young's inequality, s-number.*

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1. INTRODUCTION

The well-known inequality, valid for $p > 1$ and $1/p + 1/q = 1$, named after W.H. Young, is usually stated as

$$\alpha\beta \leqslant \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q$$

for any $\alpha, \beta \in \mathbb{R}^+$, *with equality if and only if $\alpha^p = \beta^q$.*

In this paper, we establish an analogue for *the case of equality* in the setting of operators affiliated to semi-finite von Neumann algebras. For more references and further discussion on the subject of Young's inequality for matrices and operators, we refer the reader to [2] for the proof for compact, trace class operators, and to [14] where the proof is given for the case of compact operators in $\mathcal{B}(H)$, the discrete (or atomic measure) case, of this fact. We also remark that it was the fundamental paper by T. Ando [1] which initiated the study of Young's inequality for the singular values of $n \times n$ matrices.

The emphasis in this paper is in the measure theoretic approach to operators affiliated with a semi-finite von Neumann algebra, since the approach by

induction used in [14] is not at hand. The inequality for s-numbers of operators a, b affiliated with a semi-finite von Neuman algebra $\mathcal{A}$, is stated as

$$(1.1) \qquad \mu_s(ab^*) \leqslant \mu_s\left(\frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q\right), \quad s > 0$$

and extended here to unbounded operators; we are interested in the case of equality.

  We remark that this result includes all semi-finite von Neumann algebras $\mathcal{A}$, since by a standard tensor product technique ([9], p. 286) we can always embed $\mathcal{A}$ into the diffuse algebra $\mathcal{A} \otimes \mathscr{L}^\infty([0,1], \mathrm{d}t)$ without altering the s-numbers.

  This paper is organized as follows: Section 2 presents the general facts about s-numbers recalling the well-known and establishing some simple lemmas used later. Section 3 deals with some simplifications and reductions of the problem to deal with it in full generality. Section 4, after certain technical propositions, contains the main result of this paper, Theorem 4.7, which states that equality holds for all s-numbers in (1.1) if and only if $|a|^p = |b|^q$, or equivalently, if equality of norms

$$\|ab^*\|_E = \left\|\frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q\right\|_E$$

holds for some *strictly increasing symmetric norm* $\|\cdot\|_E$ (definition given in Section 4.1, just before the main theorem).

2. SINGULAR NUMBERS IN VON NEUMANN ALGEBRAS

  In this paper $\mathcal{A}$ stands for a finite or semi-finite von Neumann algebra with faithful normal trace τ, which when convenient we will assume represented in a complex Hilbert space $\mathcal{H}$. The set of (self-adjoint) projections in $\mathcal{A}$ will be denoted by $\mathcal{P}(\mathcal{A})$.

  We consider the topology of *convergence in measure* in $\mathcal{A}$: a neighbourhood of 0 is given by

$$V(\varepsilon, \delta) = \{x \in \mathcal{A} : \exists\, p \in \mathcal{P}(\mathcal{A}) \text{ such that } \tau(1-p) < \delta \text{ and } \|xp\| < \varepsilon\}.$$

We will denote by $\widetilde{\mathcal{A}}$ the completion in measure of $\mathcal{A}$, therefore $\widetilde{\mathcal{A}}$ is *the ring of τ-measurable operators affiliated with $\mathcal{A}$*.

  If x is a closed, densely defined operator and $B \subset \mathbb{R}_{\geqslant 0}$ is a Borelian set, we denote $p^x(B) = \chi_B(|x|)$. Let $p^x(s, +\infty) = \chi_{(s,+\infty)}(|x|)$ denote the range projections of x. Then from the very definition of $\widetilde{\mathcal{A}}$, if $a \in \widetilde{\mathcal{A}}$ the number $\tau(p^a(s, +\infty))$ is eventually finite, and moreover $\tau(p^a(s, +\infty)) \to 0$ when $s \to \infty$.

  Hence if the polar decomposition of a closed densely defined operator a is $a = u|a|$, we have $a \in \widetilde{\mathcal{A}}$ if and only if the partial isometry $u \in \mathcal{A}$, and all the spectral projections $p^a(B) = \chi_B(|a|)$ are in $\widetilde{\mathcal{A}}$ (in particular $|a|^p \in \widetilde{\mathcal{A}}$ for all

$1 \leqslant p < \infty$, see for instance Nelson's paper [17], Section 3, and the paragraph before it).

We remark that in the atomic case, convergence in measure reduces to the norm topology, therefore $\widetilde{\mathcal{A}} = \mathcal{A}$ in that case.

2.1. s-NUMBERS. For $x \in \widetilde{\mathcal{A}}$ and $s > 0$, we denote the s-th singular number of x by $\mu_s(x)$:

$$\mu_s(x) = \inf\{\| |x| p\| : p \in \mathcal{P}(\mathcal{A}) \text{ with } \tau(1 - p) \leqslant s\}.$$

The standard reference on the subject of s-numbers is the paper by Fack and Kosaki [9].

We remark that $\lim_{s \to 0^+} \mu_s(x) = \|x\|$ including the posibility of $+\infty$ when x is unbounded.

We comment here on some useful characterizations (Proposition 3.1 in [11] and Proposition 2.2, Lemma 2.5, Proposition 3.1 in [9]).

(•) The variational (min-max) characterization:

$$(2.1) \quad \mu_s(x) = \inf_{p \in \mathcal{P}(\mathcal{A}), \tau(1-p) \leqslant s} \left[\sup_{\xi \in \text{Ran}(p), \|\xi\|=1} \langle x\xi, \xi \rangle \right] = \sup_{p \in \mathcal{P}(\mathcal{A}), \tau(p) \geqslant s} \left[\inf_{\xi \in \text{Ran}(p), \|\xi\|=1} \|x\xi\| \right].$$

(•) The distribution characterization: if $p^x(s, +\infty) = \chi_{(s,+\infty)}(|x|)$ denote the range projections of $|x|$, then

$$\mu_t(x) = \min\{s \geqslant 0 : \tau(p^x(s, +\infty)) \leqslant t\}.$$

(•) For $x \in \widetilde{\mathcal{A}}$, the following are equivalent:
   (i) $\tau(p^x(t, +\infty)) < +\infty$ for all $t > 0$;
   (ii) $\lim_{t \to \infty} \mu_t(x) \to 0$;
   (iii) there exists a sequence of bounded operators $x_n \in \mathscr{L}^1(\mathcal{A})$ such that $x_n \to x$ in the measure topology.

REMARK 2.1. If any of (i)–(iii) in the last characterization is fulfilled, we say that x is τ-compact; these operators form a complete bilateral ideal in $\widetilde{\mathcal{A}}$ that we will denote by $\mathscr{K}(\widetilde{\mathcal{A}})$; note that a τ-compact operator is not necessarily bounded. We will denote with $\mathscr{K}(\widetilde{\mathcal{A}})^+$ the positive ($x \geqslant 0$) τ-compact operators.

In the atomic case (when $\mathcal{A} = \mathcal{B}(\mathcal{H})$), we recover the ordinary compact operators $\mathscr{K}(\mathcal{H})$. If $\{\lambda_k(x)\}_{k \in \mathbb{N}_0}$ denotes the usual singular values of x (i.e. the eigenvalues of $|x|$), and we arrange them in a right-continuous decreasing function which is constant on $[k, k+1)$, then we obtain the distribution function $\mu_s(x)$ as follows:

$$\mu_s(x) = \sum_{k \in \mathbb{N}_0} \lambda_k \chi_{[k,k+1)}(s).$$

In this lemma we collect some other known facts on s-numbers that we will use later.

LEMMA 2.2. *Let $x, y \in \mathcal{A}$, $a, b \in \widetilde{\mathcal{A}}$. Then for each $s > 0$,*

(i) $\mu_s(xay) \leqslant \|x\| \, \|y\| \mu_s(a)$, *and if $a \leqslant b$ then $\mu_s(a) \leqslant \mu_s(b)$;*

(ii) $\mu_s(|ab^*|) = \mu_s(|\,|a|\,|b|\,|)$;

(iii) $\mu_{s+t}(a + b) \leqslant \mu_s(a) + \mu_t(b)$, $s, t \geqslant 0$;

(iv) *if $p \in \mathcal{P}(\mathcal{A})$ then $\mu_s(ap) = 0$ for each $s \geqslant \tau(p)$;*

(v) $\tau(|a|) = \int\limits_0^\infty \mu_s(a)\mathrm{d}s$;

(vi) *if $a, b \geqslant 0$, $a \in \mathcal{K}(\widetilde{\mathcal{A}})$, $ab = 0$ and $\mu_s(a + b) = \mu_s(a)$ for all $s > 0$, then $b = 0$.*

Proof. The first assertion is a consequence of the min-max characterization of the s-numbers. To prove the second, note that if $b = v|b|$ is the polar decomposition of b, a straightforward computation using the functional calculus shows that

$$|ab^*| = v \, |a| \, |b| \, |v^* \quad \text{and} \quad |\,|a| \, |b|\,| = v^* |ab^*| v.$$

Then by the first item we obtain $\mu_s(|ab^*|) = \mu_s(|\,|a|\,|b|\,|)$. The proof of the third, fourth and fifth assertion is due to Fack and Kosaki and can be found in their original paper ([9], Lemmas 2.5, 2.6 and Proposition 2.7). The final assertion seems evident, but requires some proof though. For $t > 0$, let $p^a[t, +\infty) = \chi_{[t,+\infty)}(a)$ be the spectral projections of a, and likewise for $b, a + b$. Then $\mu_s(a + b) = \mu_s(a)$ for all $s > 0$ implies (since a is τ-compact and $ab = 0$) that

$$\tau(p^a(t, +\infty)) = \tau(p^{a+b}(t, +\infty)) = \tau(p^a(t, +\infty)) + \tau(p^b(t, +\infty))$$

for all $t > 0$ (cf. Corollary 2.9 of [9]). Therefore $\tau(p^b(t, +\infty)) = 0$ for all $t > 0$, implying $b = 0$. ∎

3. DIFFUSE ALGEBRAS

Recall that an algebra is *diffuse* if it has no minimal projections. Following Fack and Kosaki ([9], p. 286) we can always embed $\mathcal{A}$ into the diffuse algebra $\mathcal{A} \otimes \mathcal{L}^\infty([0, 1], \mathrm{d}t)$ without altering the s-numbers. Then, the following remark ([9], Lemma 2.1) will be useful later.

REMARK 3.1. If $\mathcal{A}$ is diffuse and $x \geqslant 0$ is τ-measurable, then for each $t \geqslant 0$

$$\sup\{\tau(xp) : p \in \mathcal{P}(\mathcal{A}), \, \tau(p) \leqslant t\} = \int\limits_0^t \mu_s(x)\mathrm{d}s.$$

3.1. COMPLETE FLAGS. If $0 \leqslant x \in \mathcal{K}(\widetilde{\mathcal{A}})$ and $\mathcal{A}$ is a diffuse von Neumann algebra, there exists an increasing assignment $\mathbb{R}_{\geqslant 0} \ni t \mapsto e_t \in \mathcal{P}(\mathcal{A})$ ($e_s \leqslant e_t$ for $s \leqslant t$) such that $\tau(e_t) = t$ for all $t \geqslant 0$ and

$$x = \int\limits_0^\infty \mu_s(x)\mathrm{d}e(s).$$

Note the analogy with the atomic case, where $x = \sum\limits_{k\in\mathbb{N}} \lambda_k(x)p_k$ *with* p_k *the projection to the* λ_k *eigenspace of* x, *and we assume the eigenvalues are arranged in decrasing order.*

Since $e_0 = 0$, we denote $e(s,t) = e_t - e_s$ for $s \leqslant t \in \mathbb{R}_{\geqslant 0}$ and since $\mathcal{A}$ is diffuse,

$$e_t - e_s = e(s,t) = e[s,t) = e(s,t] = e[s,t].$$

The spectral resolution $\{e_t\}_{t\geqslant 0}$ is called a *complete flag* for x; for more details on this useful constructions in diffuse semi-finite algebras, we refer the reader to the papers [3], [4] by Argerami and Massey. In particular, for each $t > 0$,

$$\int_0^t \mu_s(x)\mathrm{d}s = \tau(xe_t).$$

3.2. EQUALITY OF SINGULAR NUMBERS, τ-COMPACT OPERATORS. Let $(\mathcal{A}, \tau)$ be a semi-finite von Neumann algebra with semi-finite trace ($\tau(1) = +\infty$ here). In Theorem 1 of [10] Farenick and Manjegani proved the remarkable Young's inequality for the s-numbers: if $p > 1, 1/p + 1/q = 1$, and $a, b \in \mathcal{A}$, then

$$(3.1) \qquad \mu_s(ab^*) \leqslant \mu_s\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)$$

for all $s \geqslant 0$. We now extend the inequality and the conjecture to unbounded operators.

THEOREM 3.2. *Let* $a, b \in \widetilde{\mathcal{A}}$, *then for each* $s > 0$

$$(3.2) \qquad \mu_s(ab^*) \leqslant \mu_s\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right).$$

Proof. Let $a = u|a|, b = v|b|$ be the polar decompositions of a, b. Approximating $|a|, |b|$ in measure from below with bounded operators $x_n, y_n \geqslant 0$, we have for each $s > 0$

$$\mu_s(x_n y_n) \leqslant \mu_s\left(\frac{1}{p}x_n^p + \frac{1}{q}y_n^q\right) \leqslant \mu_s\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)$$

by (3.1) applied to the pair x_n, y_n and Lemma 2.2(i). Since $x_n \leqslant |a|, y_n \leqslant |b|$, it is easy to check that $|x_n y_n| \leqslant | |a| |b| |$; since $|ab^*| = v| |a| |b| |v^*$, then $\mu_s(x_n y_n) \leqslant \mu_s(ab^*)$. Since $x_n y_n$ converges in measure to $|a| |b|$, then by Lemma 3.4 of [9],

$$\lim_n \mu_s(x_n y_n) = \mu_s(ab^*)$$

for each $s > 0$, proving the claim. ∎

The purpose of this paper is to attack the following conjecture.

Let $p, q > 1$ with $1/p + 1/q = 1$. Does

$$(3.3) \qquad \mu_s(ab^*) = \mu_s\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)$$

for all $s > 0$ imply $|a|^p = |b|^q$?

REMARK 3.3. If the algebra $\mathcal{A}$ is atomic, we have already answered in the affirmative the conjecture in Theorem 2.12 of [14]. There, we used the existence of eigenvectors for each non-trivial eigenvalue. In this paper we will be dealing with the continuous case (that contains the previous one, see Section 3), using continuous techniques.

3.3. SOME RESTRICTIONS AND SIMPLIFICATIONS. To make sense out of the conjecture (3.3), we should ask for a complete description of an operator in terms of its s-numbers. We therefore think that it is natural to confine the conjecture to the ideal $\mathcal{K}(\widetilde{\mathcal{A}})$ of τ-compact operators (Remark 2.1).

In fact, it is known that for $x \in \mathcal{K}(\widetilde{\mathcal{A}})^+$,

$$\sigma(x) = \text{clos}\{\mu_s(x) : s > 0\}$$

(see Theorem 4.10 of [19]). On the other hand, if e, f are disjoint and infinite projections ($\tau(e) = \tau(f) = \infty$), taking $x = e + (1/2)f$ shows that $\sigma(x) = \{1/2, 1\}$ while $\mu_s(x) = 1$ for all $s > 0$, therefore it is hopeless to recover x from the data in $\mu_s(x)$.

Exchanging a with b, we can always assume that $1 < p \leqslant 2$. Since $\lambda_s = \mu_s(|ab^*|) = \mu_s(|a|\,|b|)$, we can safely assume that $a, b \geqslant 0$. Moreover, we can assume (see Section 3) that $\mathcal{A}$ is diffuse and there exist complete flags $e_t, q_t \in \mathcal{P}(\mathcal{A})$ ($t \geqslant 0$, $\tau(e_t) = \tau(q_t) = t$) such that

$$(3.4) \qquad |ab| = \int_0^\infty \lambda_s \mathrm{d}e(s) \quad \text{and} \quad \frac{1}{p}a^p + \frac{1}{q}b^q = \int_0^\infty \lambda_s \mathrm{d}q(s),$$

since $a, b \in \mathcal{K}(\widetilde{\mathcal{A}})^+$ and τ-compact operators form a (closed in measure) ideal of $\widetilde{\mathcal{A}}$.

Our arguments will be based on continuous majorization. We are therefore interested in those operators that are locally integrable. More precisely, let $1 \leqslant p < \infty$, let $x \in \mathcal{A}$ and assume that there exists $\delta > 0$ such that

$$\int_0^\delta \mu_s(x)^p \mathrm{d}s < \infty$$

(hence the integral is finite for all finite $\delta > 0$). We will denote the set containing all these operators by $\mathscr{L}_{\text{loc}}^p(\mathcal{A}) \subset \widetilde{\mathcal{A}}$. Note that in particular, all bounded operators $a \in \mathcal{A}$ are of this class. Moreover,

$$\int_0^\delta \mu_s(x)^p \mathrm{d}s \geqslant \mu_\delta(x)^{p-1} \int_0^\delta \mu_s(x) \mathrm{d}s$$

shows that $\mathscr{L}_{\text{loc}}^p(\mathcal{A}) \subset \mathscr{L}_{\text{loc}}^1(\mathcal{A})$ for each $p \geqslant 1$.

LEMMA 3.4. *Let* $a \in \widetilde{\mathcal{A}}$ *and* $p \geqslant 1$. *Then* $a \in \mathscr{L}^p_{\mathrm{loc}}(\mathcal{A})$ *if and only if* $a \in \mathscr{L}^p(\mathcal{A}) + \mathcal{A}$, *and in that case the decomposition can be taken as follows for some* $r > 0$:

$$(3.5) \qquad a = ap^a(r, +\infty) + ap^a[0, r].$$

Proof. By polar decomposition, it suffices to consider $a \geqslant 0$. Note that $ap^a[0, r] \leqslant r \in \mathcal{A}$ and since $a \in \mathcal{A}$, eventually $\tau(p^a(r, +\infty)) < \infty$ for some $r > 0$. Likewise, for $p > 1$,

$$a^p = a^p p^a(r, +\infty) + a^p p^a[0, r].$$

These expressions imply the following (see Proposition 1.2 of [12]):

$$a^p \in \mathscr{L}^1_{\mathrm{loc}}(\mathcal{A}) \Leftrightarrow a^p \in \mathscr{L}^1(\mathcal{A}) + \mathcal{A} \Leftrightarrow \exists r > 0 \text{ such that } a^p p^a(r, +\infty) \in \mathscr{L}^1(\mathcal{A}).$$

Note that then $ap^a(r, +\infty) \in \mathscr{L}^p(\mathcal{A})$ for the same r, therefore $a \in \mathscr{L}^p(\mathcal{A}) + \mathcal{A}$ by (3.5). On the other hand, if $a = l + m \in \mathscr{L}^p(\mathcal{A}) + \mathcal{A}$, then taking $f(x) = x^p$ which is continuous, convex and increasing in $[0, +\infty)$,

$$\int_0^t \mu_s(a)^p \mathrm{d}s = \int_0^t \mu_s(l + m)^p \mathrm{d}s \leqslant \int_0^t (\mu_s(l) + \mu_s(m))^p \mathrm{d}s$$

by Lemma 4.4(iii) of [9]. Therefore for any $t > 0$

$$\left(\int_0^t \mu_s(a)^p \mathrm{d}s \right)^{1/p} \leqslant \left(\int_0^t (\mu_s(l) + \mu_s(m))^p \mathrm{d}s \right)^{1/p}$$

$$\leqslant \left(\int_0^t \mu_s(l)^p \mathrm{d}s \right)^{1/p} + \left(\int_0^t \mu_s(m)^p \mathrm{d}s \right)^{1/p}$$

$$\leqslant \left(\int_0^t \mu_s(l)^p \mathrm{d}s \right)^{1/p} + \|m\| t^{1/p} < \infty$$

by the classical Minkowski inequality, therefore $a \in \mathscr{L}^p_{\mathrm{loc}}(\mathcal{A})$. Take $r = \mu_t(a)$, and note that for all $s > 0$,

$$\mu_s(ap^a(r, +\infty)) = \begin{cases} \mu_s(a) & 0 < s < t, \\ 0 & s \geqslant t, \end{cases}$$

therefore (3.5) gives the stated decomposition. ∎

4. MAIN RESULTS

We start by examining the ranges of a, b. Throughout, p, q are positive with $1/p + 1/q = 1$.

PROPOSITION 4.1. *Let* $0 \leqslant a, b \in \mathcal{K}(\widetilde{\mathcal{A}})$ *with* $ab \in \mathcal{L}^2_{\mathrm{loc}}(\mathcal{A})$. *If* $p \neq 2$ *and*

$$\mu_s(ab) = \mu_s\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \quad \text{for all } s > 0$$

then $\overline{\mathrm{Ran}\,(a)} = \overline{\mathrm{Ran}\,(b)}$.

Proof. Exchanging a, b it will suffice to consider $1 < p < 2$. Let p_b be the projection onto the closure of the range of b. Let $b_\varepsilon = b + \varepsilon(1 - p_b)$, then $b_\varepsilon^q = b^q + \varepsilon^q(1 - p_b)$ and $b_\varepsilon^2 = b^2 + \varepsilon^2(1 - p_b)$. Fix $t > 0$, let $\{e_s\}$ be a complete flag for $|ba|$, then denoting $\lambda_s = \mu_s(ab)$ we have

$$\int_0^t \lambda_s^2 de(s) + \varepsilon^2 e_t a(1 - p_b)ae_t = e_t|ba|^2 e_t + \varepsilon^2 e_t a(1 - p_b)ae_t = e_t|b_\varepsilon a|^2 e_t.$$

Taking the trace, it follows that

$$\int_0^t \lambda_s^2 ds + \varepsilon^2 \tau(e_t a(1 - p_b)ae_t) = \tau(e_t|b_\varepsilon a|^2) \leqslant \int_0^t \mu_s(|b_\varepsilon a|)^2$$

by Remark 3.1. On the other hand, by (3.2) applied to a, b_ε,

$$\mu_s(|b_\varepsilon a|) \leqslant \mu_s\left(\frac{1}{p}a^p + \frac{1}{q}b_\varepsilon^q\right) = \mu_s\left(\frac{1}{p}a^p + \frac{1}{q}b^q + \frac{1}{q}\varepsilon^q(1 - p_b)\right)$$

$$\leqslant \mu_s\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) + \frac{1}{q}\varepsilon^q = \mu_s(ab) + \frac{1}{q}\varepsilon^q = \lambda_s + \frac{1}{q}\varepsilon^q.$$

Note that in particular, all the integrals computed up to now are finite by the hypothesis on ab, and

$$\int_0^t \lambda_s^2 ds + \varepsilon^2 \tau(e_t a(1 - p_b)ae_t) \leqslant \int_0^t \lambda_s^2 ds + \frac{1}{q^2}t\varepsilon^{2q} + \frac{2}{q}\varepsilon^q\int_0^t \lambda_s ds.$$

Canceling $\int_0^t \lambda_s^2 ds$ and dividing by ε^2, noting that $q > 2$ and letting $\varepsilon \to 0$ gives us that $\tau(e_t a(1 - p_b)ae_t) = 0$. Since the trace is faithful, we conclude that $(1 - p_b)ae_t = 0$ or equivalently, $ae_t = p_b ae_t$ for all $t > 0$. Then

$$p_b a|ba| = p_b a\int_0^\infty \lambda_t de(t) = \int_0^\infty \lambda_t p_b a de(t) = \int_0^\infty \lambda_t a de(t) = a|ba|,$$

that is $a(\mathrm{Ran}\,|ba|) \subset \overline{\mathrm{Ran}\,(b)}$.

Now if $\xi \in \mathcal{H}$, then $a|ba|\xi \in \overline{\mathrm{Ran}\,(b)}$, therefore $a^2|ba|\xi = a(a|ba|\xi) \in a\overline{\mathrm{Ran}\,(b)} \subset \overline{\mathrm{Ran}\,(ab)} = \overline{\mathrm{Ran}\,|ba|}$, and

$$a^3|ba|\xi = a(a^2|ba|\xi) \in a\overline{\mathrm{Ran}\,|ba|} \subset \overline{\mathrm{Ran}\,(b)}.$$

Iterating this argument, we arrive to the conclusion that

$$a^{2n+1}(\operatorname{Ran}|ba|) \subset \overline{\operatorname{Ran}(b)}$$

for all $n \in \mathbb{N}_0$. Using an approximation of $f = \chi_{\sigma(a)}$ by odd functions, we conclude that $p_a(\operatorname{Ran}|ba|) = f(a)(\operatorname{Ran}|ba|) \subset \overline{\operatorname{Ran}(b)}$ where p_a is the projection onto the closure of the range of a. Therefore $|ba|^2\xi = ab^2a\xi = p_aab^2a\xi = p_a|ba|^2\xi \subset \overline{\operatorname{Ran}(b)}$, which gives $\overline{\operatorname{Ran}|ba|} = \overline{\operatorname{Ran}(|ba|^2)} \subset \overline{\operatorname{Ran}(b)}$. But then

$$a\overline{\operatorname{Ran}(b)} = \overline{\operatorname{Ran}(ab)} = \overline{\operatorname{Ran}|ba|} \subset \overline{\operatorname{Ran}(b)}$$

which proves that the range of b is invariant for a; since $a \geqslant 0$ the same is true for the kernel of b. Therefore we can write $a = a_b + a_\perp$, with $a_b = p_bap_b \geqslant 0$ and $a_\perp = (1-p_b)a(1-p_b) \geqslant 0$. Note that $ba^2b = ba_b^2b$ and $a^p = a_b^p + a_\perp^p$, thus for all $s > 0$,

$$\mu_s\left(\frac{1}{p}a_b^p + \frac{1}{q}b^q\right) \leqslant \mu_s\left(\frac{1}{p}a_b^p + \frac{1}{q}b^q + \frac{1}{p}a_\perp^p\right) = \mu_s\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) = \mu_s(ab)$$

$$= \mu_s(a_bb) \leqslant \mu_s\left(\frac{1}{p}a_b^p + \frac{1}{q}b^q\right)$$

by the hypothesis and (3.2) applied to a_b, b. This proves that for all $s > 0$

$$\mu_s\left(\frac{1}{p}a_b^p + \frac{1}{q}b^q\right) = \mu_s\left(\frac{1}{p}a_b^p + \frac{1}{q}b^q + \frac{1}{p}a_\perp^p\right)$$

which (by Lemma 2.2.5) is only possible if $a_\perp = 0$, proving the assertion of the proposition. ∎

The following will be used twice throughout the proof of the main theorem, therefore we preferred to state it as a separate lemma.

LEMMA 4.2. *Let* $0 \leqslant x \in \mathscr{L}^1_{\mathrm{loc}}(\mathcal{A})$ *and* $p \in \mathcal{A}$ *be a projection with finite trace. Then*

$$\tau(px) = \int_0^{\tau(p)} \mu_s(x)\mathrm{d}s$$

implies $xp = px$.

Proof. Since p is a projection and $x \geqslant 0$,

$$(pxp)^2 = pxpxp \leqslant px^2p = |xp|^2.$$

Since the square root is operator monotone (see Theorem 2 of [13] for the details for unbounded operators), we have $pxp \leqslant |xp|$. Take the trace and invoke items (iv) and (v) of Lemma 2.2, then

$$\tau(|xp|) = \int_0^\infty \mu_s(xp)\mathrm{d}s = \int_0^{\tau(p)} \mu_s(xp)\mathrm{d}s \leqslant \int_0^{\tau(p)} \mu_s(x)\mathrm{d}s,$$

thus by the hypothesis pxp and $|xp|$ have equal (and finite) trace. Since the trace is faithful this is only possible if $pxp = |xp|$, or equivalently if $pxpxp = px^2p$. This implies that

$$\langle px, xp \rangle_2 = \tau(pxpx) = \tau(px^2p) = \|px\|_2^2 = \|px\|_2\|xp\|_2,$$

which by the case of equality in Cauchy–Schwarz inequality implies $xp = px$.   ∎

We will also need the following classical result on operator ranges, in fact the original proof of Douglas ([7], Theorem 2) is given for closed, densely defined operators and therefore it applies to elements of $\widetilde{\mathcal{A}}$.

REMARK 4.3 (Douglas' lemma). Let $x, y \in \widetilde{\mathcal{A}}$. If $xx^* \leqslant \lambda yy^*$ for some $\lambda \geqslant 0$, there exists a contraction c such that $x = yc$, therefore $\mathrm{Ran}\,(x) \subset \mathrm{Ran}\,(y)$.

With these tools at hand, we are now able to prove the main theorem.

THEOREM 4.4. Let $0 \leqslant a, b \in \mathcal{K}(\widetilde{\mathcal{A}})$ with $ab \in \mathscr{L}_{\mathrm{loc}}^2(\mathcal{A})$. If

$$\mu_s(ab) = \mu_s\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \quad \textit{for all } s > 0,$$

then $a^p = b^q$. When $p = q = 2$ it suffices to assume $ab \in \mathscr{L}_{\mathrm{loc}}^1(\mathcal{A})$.

Proof. Exchanging a, b it will suffice to consider $1 < p \leqslant 2$. Denoting $\lambda_s = \mu_s(ab)$ we write $ba^2b = |ab|^2 = \int_0^\infty \lambda_s^2 de(s)$ with a complete flag $\{e_s\}_{s \geqslant 0} \subset \mathcal{P}(\mathcal{A})$. For $I = [s, t] \subset [0, +\infty)$ denote $e_I = e_s - e_t = e(s, t)$, then $e_t = e_{[0,t]}$. Since λ_s is non-increasing,

$$(4.1) \quad ba^2b = ba(ba)^* = |ab|^2 = \int_0^\infty \lambda_s^2 de(s) \geqslant \int_I \lambda_s^2 de(s) \geqslant \lambda_t^2 \int_s^t de(s) = \lambda_t^2 e_I,$$

and the previous lemma ensures that $\mathrm{Ran}\,(e_I) \subset \mathrm{Ran}\,(|ab|) = \mathrm{Ran}\,(ba) \subset \mathrm{Ran}\,(b)$ for each interval $I = [s, t]$. Moreover, if $e = \bigvee_{s \geqslant 0} e_s$ is the join of the increasing projections, clearly $e = p_{|ab|}$, the projection onto the closure of the range of $|ab|$.

We now treat three cases separately.

Case $4/3 \leqslant p < 2$. By Proposition 4.1 we can consider $\mathcal{H} = \overline{\mathrm{Ran}\,(a)} = \overline{\mathrm{Ran}\,(b)}$, and the semi-finite von Neumann subalgebra $\mathcal{M} \subset \mathcal{A}$ generated by the (finitely supported) spectral projections of a, b, ab. We give $\mathcal{M}$ the inherited trace τ and identity $1 = 1_{\mathcal{M}} = P_{\mathcal{H}} = p_b$. All the operators involved a, b, ab, a^p, b^q are in $\widetilde{\mathcal{M}}$, and $\mathcal{M}$ can be faithfully represented in this $\mathcal{B}(\mathcal{H})$. Then we can safely assume that b is injective, $e, e_I \in \mathcal{M}$ for each interval I, and $\mathcal{M} \subset \mathcal{H}$ is a common core for all $x \in \widetilde{\mathcal{M}}$.

We remark that in what follows, *we will only use that b is injective*, or equivalently, that the range of b is dense.

Again for each interval I, let $\mathcal{H}_I$ be the closure of $b^{-1}\mathrm{Ran}\,(e_I) \subset \mathcal{H}$. Let $f_I = P_{\mathcal{H}_I}$ and $f = \bigvee_{s \geqslant 0} f_s$ the closed join of all projections, where $f_s = f_{[0,s]}$. We divide the proof in several smaller claims.

Claim 1. $f = p_a$, *the projection onto the closure of the range of a.* Let $\eta \in \mathrm{Ran}\,(f_s)$; then $\eta = \lim_n \eta_n$ with $b\eta_n = e_s \xi_n$; since $\mathrm{Ran}\,(e_s) \subset \mathrm{Ran}\,(|ab|) = \mathrm{Ran}\,(ba)$, it must be $b\eta_n = ba\psi_n$ for some $\psi_n \in \mathcal{H}$, and by the injectivity of b, we obtain $\eta_n \in \mathrm{Ran}\,(a)$, therefore $\eta \in \overline{\mathrm{Ran}\,(a)}$. This proves that $f \leqslant p_a$. On the other hand, if $\eta \in \mathrm{Ran}\,(a)$, then $b\eta \in \mathrm{Ran}\,(ba) = \mathrm{Ran}\,|ab| \subset \mathrm{Ran}\,(e)$, therefore $b\eta = \lim \xi_n$ with $\xi_n \in \mathrm{Ran}\,(e_{s_n})$ for some $s_n > 0$. Therefore $\xi_n = b\eta_n$ with $\eta_n \in \mathrm{Ran}\,(f_{s_n}) \subset \mathrm{Ran}\,(f)$ for each n. Now

$$|\langle \eta_n - \eta_m, b\xi \rangle| = |\langle b\eta_n - b\eta_m, \xi \rangle| = |\langle \xi_n - \xi_m, \xi \rangle| \leqslant \|\xi_n - \xi_m\| \|\xi\|$$

and since the range of b is dense, $\{\eta_n\}_n$ is a weak Cauchy sequence in $\mathrm{Ran}\,(f)$ which, being closed and linear, it is weakly closed. Therefore η_n converges weakly to some $\eta_0 \in \mathrm{Ran}\,(f)$. But for each $\xi \in \mathcal{H}$,

$$\langle b\eta, \xi \rangle = \lim_n \langle b\eta_n, \xi \rangle = \lim_n \langle \eta_n, b\xi \rangle = \langle \eta_0, b\xi \rangle = \langle b\eta_0, \xi \rangle,$$

which implies that $b\eta = b\eta_0$, and by the injectivity of b, we obtain $\eta = \eta_0 \in \mathrm{Ran}\,(f)$, thus $p_a \leqslant f$. This proves that $f = p_a$.

Claim 2. *There exists a closed operator $c_I = "b^{-1}e_I b^{-1}"$ defined on $\mathrm{Ran}\,(b)$ such that $0 \leqslant c_I \leqslant (1/\lambda_t^2)a^2$.* Since b is injective and $\mathrm{Ran}\,(e_I) \subset \mathrm{Ran}\,(b)$, for each $\xi \in \mathcal{H}$ there exists a unique $\eta \in \mathcal{H}_I$ such that $b\eta = e_I \xi$. Define c_I in $\mathrm{Ran}\,(b)$ as follows: $c_I b\xi = \eta$. We now compute

$$\langle c_I b\xi_1, b\xi_2 \rangle = \langle \eta_1, b\xi_2 \rangle = \langle b\eta_1, \xi_2 \rangle = \langle e_I \xi_1, \xi_2 \rangle$$

which shows that c_I is a symmetric operator on $\mathrm{Ran}\,(b)$. Moreover, since $e_I \leqslant (1/\lambda_t^2)ba^2 b$ (recall $I = [s,t]$ and equation (4.1)), it follows that

$$\langle c_I b\xi, b\xi \rangle = \langle \eta, b\xi \rangle = \langle e_I \xi, \xi \rangle \leqslant \frac{1}{\lambda_t^2} \langle ba^2 b\xi, \xi \rangle = \frac{1}{\lambda_t^2} \langle a^2 b\xi, b\xi \rangle,$$

therefore c_I has a self-adjoint extension (cf. Theorem 5.1.13 of [18], that we still denote c_I), and $0 \leqslant c_I \leqslant (1/\lambda_t^2)a^2$.

Claim 3. $c_I \in \widetilde{\mathcal{M}}$. Let $u \in \mathcal{M}'$, let $\xi \in \mathcal{H}$. Then there exist unique $\eta, \psi \in \mathcal{H}$ such that $e_I \xi = b\psi$ and $e_I(u\xi) = b\eta$. Now $e_I u\xi = ue_I \xi$ since $u \in \mathcal{M}'$, therefore

$$b\eta = e_I u\xi = ue_I \xi = ub\psi = bu\psi,$$

and since b is injective, $u\psi = \eta$. We now compute

$$c_I u(b\xi) = c_I b(u\xi) = \eta = u\psi = u(c_I b\xi) = uc_I(b\xi),$$

which shows that $c_I u = uc_I$, proving that $c_I \in \widetilde{\mathcal{M}}$.

Claim 4. $bc_I b = e_I$, $e_I bf_I = bf_I$ and $f_I b^2 f_I c_I = f_I$ *for each* I. From the very definition, $bc_I b = e_I$. On the other hand note that if $\eta \in f_I$ then $\eta = \lim_n \eta_n$ with $b\eta_n \in \mathrm{Ran}\,(e_I)$ and for any $\xi \in \mathcal{H}$

$$|\langle b\eta_n - b\eta, \xi\rangle| = |\langle \eta_n - \eta, b\xi| \leqslant \|\eta_n - \eta\|\|b\xi\|$$

therefore $b\eta_n \in \mathrm{Ran}\,(e_I)$ converges weakly to $b\eta$, therefore $b\eta \in \mathrm{Ran}\,(e_I)$ and we obtain $e_I bf_I = bf_I$. Taking adjoints, $f_I b = f_I be_I$, hence for each $\xi \in \mathcal{H}$,

$$f_I b^2 f_I c_I(b\xi) = f_I b^2 f_I \eta = f_I b^2 \eta = (f_I b)(b\eta) = (f_I b)(e_I \xi) = f_I(b\xi),$$

which proves that $f_I b^2 f_I c_I = f_I$ for any I.

Claim 5. $f_I c_J = c_J f_I$ *for any* I, J. Since when $e_I \xi = b\eta$, then $\eta \in \mathrm{Ran}\,(f_I)$, clearly $f_I c_I = c_I = c_I f_I$. Moreover it is not hard to see that $f_I c_J = c_{I \cap J}$ for any pair of intervals I, J by the injectivity of b, therefore $f_I c_J = c_J f_I$.

Claim 6. $f_I \sim e_I$. Inspection of the ranges shows that (again by the injectivity of b)
$$\mathrm{Ran}\,(e_I) = \overline{\mathrm{Ran}\,(bc_I)} \quad \text{and} \quad \mathrm{Ran}\,(f_I) = \overline{\mathrm{Ran}\,(c_I b)},$$

and since $c_I b = (bc_I)^*$, it follows that e_I is von Neumann equivalent to f_I, which implies that $f, f_I \in \mathcal{M}$ and moreover $\tau(f_I) = \tau(e_I)$ for each I.

Summing up our findings: for any interval $I = [s, t] \subset [0, +\infty)$, we have $e, e_I, f, f_I \in \mathcal{P}(\mathcal{M})$ with $f_I \simeq e_I$, $\tau(f_I) = \tau(e_I) = t - s$, $e_I bf_I = bf_I$, $f_I b = f_I be_I$. Moreover $c_I \in \widetilde{\mathcal{M}}$, $bc_I b = e_I$,

$$(4.2) \quad f_I c_J = c_{I \cap J} = c_J f_I, \quad f_I b^2 f_I c_I = f_I = c_I f_I b^2 f_I, \quad a^2 \geqslant \lambda_t^2 c_I \quad \text{and} \quad f_I a^2 f_I \geqslant \lambda_t^2 c_I.$$

Claim 7. $a^p f_s = f_s a^p$ *for all* $s > 0$. Let $\pi = \{I_i\}_{i=1\cdots n}$ with $I_i = [s_i, s_{i+1}]$ be a partition of $[0, +\infty)$, and denote $e_i = e_{I_i}$ and likewise with f_i, c_i. We have

$$ba^2 b \geqslant \sum_i \int_{I_i} \lambda_s^2 de(s) \geqslant \sum_i \lambda_{s_{i+1}}^2 e_i = \sum_i \lambda_{s_{i+1}}^2 bc_i b$$

which implies $a^2 \geqslant \sum_i \lambda_{s_{i+1}}^2 c_i$ since b is injective with dense range. Now refining the partition

$$\langle a^2 b\xi, b\xi\rangle = \langle ba^2 b\xi, \xi\rangle = \Big\langle \int_0^{+\infty} \lambda_s^2 de(s)\xi, \xi\Big\rangle = \lim_{|\pi|\to 0} \Big\langle \sum_i \lambda_{s_{i+1}}^2 e_i\xi, \xi\Big\rangle$$

$$= \lim_{|\pi|\to 0} \Big\langle \sum_i \lambda_{s_{i+1}}^2 c_i\, b\xi, b\xi\Big\rangle$$

for any $\xi \in \mathcal{H}$. Since the range of b is dense and the operators involved are positive, we conclude that $\lim_{|\pi|\to 0} \sum_i \lambda_{s_{i+1}}^2 c_i = a^2$ in the strong operator topology. Since $f_s c_i = c_i f_s (= c_{[0,s]\cap I_i})$, we conclude that $f_s a^2 = a^2 f_s$ for all $s \geqslant 0$, which implies that $a^p f_i = f_i a^p$ for all i.

We now take the $p/2$-th root in (4.2), which is a monotone operator function since $1 < p < 2$. Thus

(4.3) $$\lambda_{s_{i+1}}^p c_i^{p/2} \leqslant a^p \quad \text{and} \quad \lambda_{s_{i+1}}^p c_i^{p/2} \leqslant f_i a^p f_i = a^p f_i.$$

Since $4/3 \leqslant p < 2$, this implies that $2 < q \leqslant 4$. Then $t \mapsto t^{q/2}$ is operator convex ([18], Theorem 2.4) and $(f_i b^2 f_i)^{q/2} \leqslant f_i b^q f_i$. By Young's inequality in the commutative algebra generated by c_i, f_i ([8], Lemma 2.2) we have:

$$\lambda_{s_{i+1}} f_i = \lambda_{s_{i+1}} f_i^{1/2} = \lambda_{s_{i+1}} c_i^{1/2} (f_i b^2 f_i)^{1/2}$$
$$\leqslant \frac{1}{p} \lambda_{s_{i+1}}^p c_i^{p/2} + \frac{1}{q} (f_i b^2 f_i)^{q/2}$$
(4.4) $$\leqslant \frac{1}{p} f_i a^p f_i + \frac{1}{q} f_i b^q f_i = f_i \left(\frac{1}{p} a^p + \frac{1}{q} b^q \right) f_i = f_i D f_i,$$

where $D = (1/p) a^p + (1/q) b^q$ for short.

Claim 8. $f_t D = D f_t$ for all $t > 0$. Assume that π is a partition of $[0, t]$. Summing over i, we obtain $\sum_i \lambda_{s_{i+1}} f_i \leqslant \sum_i f_i D f_i$, and taking traces

$$\sum_i \lambda_{s_{i+1}} (s_{i+1} - s_i) \leqslant \sum_i \tau(f_i D) = \tau(D f_t) \leqslant \int_0^t \mu_s(D) \mathrm{d}s = \int_0^t \lambda_s \mathrm{d}s < \infty$$

by (3.4), Remark 3.1 and the assumption on a, b (recall $\mathscr{L}_{\mathrm{loc}}^2(\mathcal{A}) \subset \mathscr{L}_{\mathrm{loc}}^1(\mathcal{A})$). Refining the partition π, it follows that $\int_0^t \lambda_s \mathrm{d}s = \tau(D f_t)$. Since $\lambda_s = \mu_s(D)$ and $\tau(f_t) = t$, Lemma 4.2 implies that $f_t D = D f_t$.

Claim 9. $D = \int_0^\infty \lambda_s \mathrm{d}f(s)$. Since t was arbitrary, $f_i D = D f_i$ also holds. Returning to the previous inequality (4.4) we now sum over i to obtain

$$\sum_i \lambda_{s_{i+1}} f_i \leqslant \sum_i f_i D f_i = \sum_i f_i D = f_t D = f_t D.$$

Let $\overline{D} = \int_0^\infty \lambda_s \mathrm{d}f(s)$, then $(1 - f) \overline{D} = 0$ and $\overline{D} f_t = \int_0^t \lambda_s \mathrm{d}f(s)$. Refining the partition π of $[0, t]$ we obtain $\overline{D} f_t \leqslant f_t D$, and since $\tau(\overline{D} f_t) = \tau(f_t D) = \int_0^t \lambda_s \mathrm{d}s$, it must be $\overline{D} f_t = D f_t = f_t D$ for each $t > 0$. Recall $f = \bigvee_t f_t$ is the union of the projections f_t, then clearly $\overline{D} = D f = f D$; on the other hand

$$\mu_s(\overline{D}) = \lambda_s = \mu_s(D) = \mu_s(Df + (1 - f)D) = \mu_s(\overline{D} + (1 - f)D)$$

and by Lemma 2.2.3 it is only possible if $(1 - f)D = 0$, or equivalently $D = \overline{D}$.

Claim 10. *a commutes with b, b commutes with all f_t and $b^q f = b^q$. Since a^p commutes with all f_s, then a^p commutes with*

$$D = \int_0^\infty \lambda_s \mathrm{d}f(s) = \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Then a^p commutes with b^q or equivalently, a commutes with b. Note that since $f = p_a$, then

$$\frac{1}{p}a^p + \frac{1}{q}b^q = D = Df = \frac{1}{p}a^p + \frac{1}{q}b^q f,$$

therefore $b^q f = b^q$. Since a^p commutes with all f_t and $(1/q)b^q = D - (1/p)a^p$, then b^q commutes with all f_t.

Claim 11. *$f_t = e_t$ for all t.* Recall that for all t, $bf_t = e_t b f_t$, therefore $f_t b e_t = f_t b$. Since f_t commutes with b, $bf_t e_t = bf_t$; since b is injective, $f_t e_t = f_t$. Therefore $f_t = f_t e_t = e_t f_t e_t \leqslant e_t$, and since $\tau(f_t) = \tau(e_t) = t$, it must be $f_t = e_t$ for all t.

Finally,

$$|ab| = ab = \int_0^\infty \lambda_s \mathrm{d}f(s) = \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Let $a_t = af_t, b_t = bf_t$, then $(1/p)\mu_s(a_t)^p \leqslant \mu_s(D_t) = \lambda_s \in \mathscr{L}^1_{\mathrm{loc}}(\mathcal{A})$ for $0 < s < t$ and likewise with b. This means that $a_t^p, b_t^q \in \mathscr{L}^1_{\mathrm{loc}}(\mathcal{A})$ and

$$|a_t b_t| = a_t b_t = abf_t = \int_0^t \lambda_s \mathrm{d}f(s) = \frac{1}{p}a_t^p + \frac{1}{q}b_t^q,$$

and all the operators involved have *finite* trace. Farenick and Manjegani proved that in that case (see Theorem 3.1 of [10] or Theorem 2.1 of [16]), it must be $a^p f_t = a_t^p = b_t^q = b^q f_t$. We give here an alternative argument: taking traces

$$\frac{1}{p}\|a_t\|_p^p + \frac{1}{q}\|b_t\|_q^q = \tau\left(\frac{1}{p}a_t^p + \frac{1}{q}b_t^q\right) = \tau(|a_t b_t|) = \|a_t b_t\|_1$$

$$\leqslant \|a_t\|_p \|b_t\|_q \leqslant \frac{1}{p} + \|a_t\|_p^p + \frac{1}{q}\|b_t\|_q^q$$

by the operator Hölder inequality (applied to $\|a_t b_t\|_1$) and Young's numeric inequality (applied to $\|a_t\|_p, \|b_t\|_p$). This implies $\|a_t b_t\|_1 = \|a_t\|_p \|b_t\|_q$, and this is only possible if $a_t^p = b_t^q$ [5], [15]. Since this holds for all $t > 0$, $a^p = a^p f = b^q f = b^q$ as we claimed.

Case $1 < p < 4/3$. This implies that $q > 4$, but since the ranges of a and b still match by Proposition 4.1, we can assume that b is injective with dense range, and the computation goes through the same lines, modifying the step regarding the commutative operator Young inequality (4.4) according to Theorem 2 of [1] or Proposition 2.3 of [8].

Case $p = q = 2$. First note that

$$\frac{1}{2}\mu_s(a)^2 \leqslant \mu_s\left(\frac{1}{2}a^2 + \frac{1}{2}b^2\right) = \mu_s(ab) \in \mathscr{L}^2_{\text{loc}}(\mathcal{A}) \subset \mathscr{L}^1_{\text{loc}}(\mathcal{A}),$$

therefore $\mu_s(a) \in \mathscr{L}^2_{\text{loc}}(\mathcal{A})$ and likewise with b. Proposition 4.1 is of no use here, therefore it suffices to assume $\mu_s(ab) \in \mathscr{L}^1_{\text{loc}}(\mathcal{A})$.

Let $\widetilde{a} = (p_b a^2 p_b)^{1/2}$. Then $\text{Ran}\,(\widetilde{a}) \subset \text{Ran}\,(b)$ and $b\widetilde{a}^2 b = ba^2b$, therefore $|\widetilde{a}b| = |ab|$. Hence

$$\mu_s\left(\frac{1}{2}\widetilde{a}^2 + \frac{1}{2}b^2\right) = \mu_s\left(p_b\left(\frac{1}{2}a^2 + \frac{1}{2}b^2\right)p_b\right) \leqslant \mu_s\left(\frac{1}{2}a^2 + \frac{1}{2}b^2\right) = \mu_s(ab)$$

$$= \mu_s(\widetilde{a}b) \leqslant \mu_s\left(\frac{1}{2}\widetilde{a}^2 + \frac{1}{2}b^2\right)$$

by (3.2) applied to the pair $\widetilde{a}, b$. Therefore, for all $s \geqslant 0$,

$$\mu_s(\widetilde{a}b) = \mu_s\left(\frac{1}{2}\widetilde{a}^2 + \frac{1}{2}b^2\right).$$

Since $\text{Ran}\,(\widetilde{a}) \subset \overline{\text{Ran}\,(b)}$, we can assume that b is injective, and argumenting as in the previous cases, arrive to $\widetilde{a}^2 = b^2$, that is $p_b a^2 p_b = b^2$. In particular $\mu_s(b)^2 \leqslant \mu_s(a)^2$ for all $s > 0$. Reversing the argument, we also get $p_a b^2 p_a = a^2$, therefore $\mu_s(a) = \mu_s(b)$ for all $s > 0$.

Let $\{b_s\}_{s \geqslant 0}$ be a complete flag for $b = \int_0^\infty \mu_s(b)\mathrm{d}b(s)$ with $\tau(b_t) = t$. Then for all $t \geqslant 0$, b_t commutes with b, we have $b_t b = \int_0^t \mu_s(b)\mathrm{d}b(s)$ and since $b_t \leqslant p_b$, $b_t p_b = b_t$. Therefore from $p_b a^2 p_b = b^2$ we obtain $b_t a^2 b_t = b_t b^2$, which implies that

$$\tau(b_t a^2) = \tau(b_t b^2) = \int_0^t \mu_s(b)^2 \mathrm{d}s = \int_0^t \mu_s(a)^2 \mathrm{d}s < \infty.$$

By Lemma 4.2, this is only possible if a commutes with b_t. Therefore, if a commutes with b, then from $p_a b^2 p_a = a^2$ we have $b p_a = p_a b = a$. But

$$\mu_s(a) = \mu_s(b) = \mu_s(bp_a + (1 - p_a)b) = \mu_s(a + (1 - p_a)b)$$

implies (Lemma 2.2.6) $b = p_a b = a$. ∎

REMARK 4.5. As the proof goes, it suffices to consider $ab \in \mathscr{L}^1_{\text{loc}}(\mathcal{A})$ if either

$$\text{Ran}\,(a) \subset \overline{\text{Ran}\,(b)} \quad \text{or} \quad \text{Ran}\,(b) \subset \overline{\text{Ran}\,(a)}.$$

COROLLARY 4.6. *Let $0 \leqslant a, b \in \mathcal{A} \cap \mathscr{K}(\widetilde{\mathcal{A}})$ and assume*

$$\mu_s(ab) = \mu_s\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \quad \text{for all } s > 0.$$

Then $a^p = b^q$.

4.1. Symmetric norms. We close the paper putting this result in context with the theory of symmetric norms on $\widetilde{\mathcal{A}}$, see for instance [6] and the references therein.

We say that a symmetric norm $\|\cdot\|_E$ is *strictly increasing* if $x, y \in E \subset \widetilde{\mathcal{A}}$, $\mu_s(x) \leqslant \mu_s(y)$ for all $s > 0$ and $\|x\|_E = \|y\|_E$ implies $\mu_s(x) = \mu_s(y)$ for all $s > 0$. All $\mathscr{L}^p$-norms are strictly increasing for $1 \leqslant p < \infty$, while the uniform norm or the Ky–Fan norms $\|x\|_{(t)} = \int_0^t \mu_s(x) \mathrm{d}s$ are not.

THEOREM 4.7. *Let $a, b \in \mathscr{K}(\widetilde{\mathcal{A}}) \cap \mathscr{L}^2_{\mathrm{loc}}(\mathcal{A})$. If $p > 1$ and $1/p + 1/q = 1$, then the following are equivalent:*

(i) $|a|^p = |b|^q$;

(ii) $z|ab^*|z^* = (1/p)|a|^p + (1/q)|b|^q$ *for some contraction $z \in \mathcal{A}$;*

(iii) $\||z|ab^*|w\|_E = \|(1/p)|a|^p + (1/q)|b|^q\|_E$ *for a pair of contractions $z, w \in \mathcal{A}$ and $\|\cdot\|_E$ a strictly increasing symmetric norm;*

(iv) $\mu_s(ab^*) = \mu_s((1/p)|a|^p + (1/q)|b|^q)$ *for all $s > 0$.*

The proof is much like as in Theorem 2.13 of [14], therefore it is omitted.

As in Theorem 4.4, Remark 4.5 or Corollary 4.6, the hypothesis $ab \in \mathscr{L}^2_{\mathrm{loc}}(\mathcal{A})$ is unnecessary when ab is bounded, and can be relaxed to $ab \in \mathscr{L}^1_{\mathrm{loc}}(\mathcal{A})$ if $p = q = 2$ or if there is an inclusion of ranges.

REFERENCES

[1] T. Ando, Matrix Young inequalities, in *Operator Theory in Function Spaces and Banach Lattices*, Oper. Theory Adv. Appl., vol. 75, Birkhäuser, Basel 1995, pp. 33–38.

[2] M. Argerami, D.R. Farenick, Young's inequality in trace-class operators, *Math. Ann.* **325**(2003), 727–744.

[3] M. Argerami, P. Massey, A Schur–Horn theorem in II$_1$-factors, *Indiana Univ. Math. J.* **56**(2007), 2051–2059.

[4] M. Argerami, P. Massey, Schur–Horn theorems in II$_\infty$-factors, *Pacific J. Math.* **261**(2013), 283–310.

[5] J. Dixmier, Formes lineaires sur un anneau d'operateurs [French], *Bull. Soc. Math. France* **81**(1953), 9–39.

[6] P.G. Dodds, T.K. Dodds, B. de Pagter, Fully symmetric operator spaces, *Integral Equations Operator Theory* **15**(1992), 942–972.

[7] R.G. DOUGLAS, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.* **17**(1966), 413–415.

[8] J. ERLIJMAN, D.R. FARENICK, R. ZENG, Young's inequality in compact operators, in *Linear Operators and Matrices*, Oper. Theory Adv. Appl., vol. 130, Birkhäuser, Basel 2002, pp. 171–184.

[9] T. FACK, H. KOSAKI, Generalized s-numbers of τ-measurable operators, *Pacific J. Math.* **123**(1986), 269–300.

[10] D.R. FARENICK, S.M. MANJEGANI, Young's inequality in operator algebras, *J. Ramanujan Math. Soc.* **20**(2005), 107–124.

[11] T. HARADA, Klyachko's theorem in semi-finite von Neumann algebras, *Publ. Res. Inst. Math. Sci.* **43**(2007), 1125–1137.

[12] F. HIAI, Majorization and stochastic maps in von Neumann algebras, *J. Math. Anal. Appl.* **127**(1987), 18–48.

[13] T. KATO, Notes on some inequalities for linear operators, *Math. Ann.* **125**(1952), 208–212.

[14] G. LAROTONDA, Young's (in)equality for compact operators, *Studia Math.* **233**(2016), 169–181.

[15] G. LAROTONDA, The case of equality in Hölder's inequality for matrices and operators, *Math. Proc. Roy. Irish Acad. Sect. A* **118**(2018), 1–4.

[16] S.M. MANJEGANI, Hölder and Young inequalities for the trace of operators, *Positivity* **11**(2007), 239–250.

[17] E. NELSON, Notes on non-commutative integration, *J. Funct. Anal.* **15**(1974), 103–116.

[18] G.K. PEDERSEN, *Analysis Now*, Grad. Texts in Math., vol. 118, Springer-Verlag, New York 1989.

[19] A. STRÖH, G.P. WEST, τ-compact operators affiliated to a semifinite von Neumann algebra, *Proc. Roy. Irish Acad. Sect. A* **93**(1993), 73–86.

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