J. OPERATOR THEORY
81:1(2019), 225–254
doi: 10.7900/jot.2018jan19.2184

L^p-OPERATOR ALGEBRAS ASSOCIATED WITH ORIENTED GRAPHS

GUILLERMO CORTIÑAS and MARÍA EUGENIA RODRÍGUEZ

Communicated by Marius Dădârlat

ABSTRACT. For each $1 \leqslant p < \infty$ and each countable oriented graph Q we introduce an L^p-operator algebra $\mathcal{O}^p(Q)$, which contains the Leavitt path $\mathbb{C}$-algebra L_Q as a dense subalgebra, and is universal for those L^p-representations of L_Q which are spatial in the sense of N.C. Phillips. We prove that $\mathcal{O}^p(Q)$ is simple as an L^p-operator algebra if and only if L_Q is simple, in which case it is isometrically isomorphic to $\overline{\rho(L_Q)}$ for any nonzero spatial L^p-representation $\rho : L_Q \to \mathcal{L}(L^p(X))$. If moreover L_Q is purely infinite simple and $p \neq p'$, then there is no nonzero continuous homomorphism $\mathcal{O}^p(Q) \to \mathcal{O}^{p'}(Q)$.

KEYWORDS: *Oriented graph, Leavitt path algebra, L^p-operator algebra, spatial representation, simple, purely infinite, desingularization.*

MSC (2010): 47L10, 46L55, 16G20.

1. INTRODUCTION

The study of algebras of operators on L^p-spaces, for $p \in [1, \infty)$, can be traced back to the work of Carl Herz in the 60's and 70's on harmonic analysis on L^p-spaces. There has been a much more recent interest for this area, with an influx of ideas and motivations coming from operator algebras and specifically C^*-algebras ([9], [10], [11], [15], [16], [17]). These new ideas have led to the solution of some long standing open problems, and have given the area new impetus. In this context, it has proved to be very fruitful to study L^p-versions of well-established and useful notions in C^*-algebras: Cuntz algebras, crossed products, AF-algebras, groupoid algebras, etc. In the current paper, we introduce and study L^p-algebras associated to graphs, which are L^p-versions of C^*-graph algebras.

Let Q be a countable oriented graph, let Q^0 and Q^1 be the sets of vertices and edges, and let L_Q be the Leavitt path $\mathbb{C}$-algebra. For $1 \leqslant p < \infty$ we call a representation $\rho : L_Q \to \mathcal{L}(L^p(X))$ spatial if X is a σ-finite measure space and ρ

maps the elements of $Q^0 \sqcup Q^1 \sqcup (Q^1)^*$ to partial isometries which are spatial in the sense of Definition 6.4 in [15]. Each spatial representation ρ induces a seminorm on L_Q via $\|a\|_\rho = \|\rho(a)\|$; the supremum $\|\cdot\|$ of these seminorms is a norm (Proposition 4.11) and we write $\mathcal{O}^p(Q)$ for the completion of $(L_Q, \|\cdot\|)$. We show that $\mathcal{O}^2(Q)$ is the usual graph C^*-algebra (Proposition 7.9) and that for $p \neq 1$, $\mathcal{O}^p(Q)$ is the tight semigroup algebra of [9] (Proposition 7.12). We prove the following theorem.

THEOREM 1.1 (Simplicity theorem). *Let Q be a countable graph and let $p \in [1, \infty)$. The following are equivalent:*

(i) *L_Q is simple;*

(ii) *every nonzero spatial L^p-representation of L_Q is injective;*

(iii) *every nondegenerate contractive nonzero L^p-representation of $\mathcal{O}^p(Q)$ is injective.*

Furthermore, if either Q^0 is finite or $p > 1$, then the above conditions are also equivalent to:

(iv) *for every L^p-operator algebra B, every contractive, nonzero homomorphism $\mathcal{O}^p(Q) \to B$ is injective.*

Condition (iv) says that $\mathcal{O}^p(Q)$ is simple as an L^p-operator algebra. Since every L^p-operator algebra is isometrically embedded in $\mathcal{L}(L^p(X))$ for some σ-finite measure space X, simplicity as an L^p-operator algebra is equivalent to the condition that every contractive nonzero representation $\rho : \mathcal{O}^p(Q) \to \mathcal{L}(L^p(X))$, degenerate or not, be injective. For $p = 2$ any such contractive representation factors through a nondegenerate one, so (iii) and (iv) are equivalent in this case. We show (using a classical result of Andô [5] and a recent result of Gardella and Thiel [11]) that a similar result holds for $p \neq 2$ if either Q^0 is finite or $p > 1$; this allows us to prove that (iii) $\Leftrightarrow$ (iv).

To prove Theorem 1.1 we first show the following uniqueness theorem.

THEOREM 1.2 (Uniqueness theorem). *Let Q be a countable graph such that L_Q is simple. Let $p \in [1, \infty)$, X a σ-finite measure space and $\rho : L_Q \to \mathcal{L}(L^p(X))$ a nonzero spatial representation. Then the canonical map $\mathcal{O}^p(Q) \to \overline{\rho(L_Q)}$ is an isometric isomorphism.*

Specializing Theorem 1.2 to the case when Q has only one vertex recovers N.C. Phillips' uniqueness result for L^p-analogues of Cuntz algebras ([15], Theorem 8.7). We also show the following theorem (Theorem 11.2)).

THEOREM 1.3. *Let Q be a countable graph and let $p, p' \in [1, \infty)$, $p \neq p'$. If L_Q is purely infinite simple then there is no nonzero continuous homomorphism $\mathcal{O}^p(Q) \to \mathcal{O}^{p'}(Q)$.*

A similar result for L^p-Cuntz algebras was obtained by N.C. Phillips in Theorem 9.2 of [15].

The rest of this paper is organized as follows. In Section 2 we recall some definitions and basic facts on Leavitt path algebras and prove some elementary technical lemmas. In Section 3 we show (Lemma 3.1) that L_Q is the universal algebra for tight algebraic representations of the inverse semigroup $\mathcal{S}(Q)$ generated by Q. Spatial representations of the Leavitt path algebra L_Q of a countable graph Q are introduced in Section 4. We give examples of such representations and show in Proposition 4.11 that for every countable Q and $1 \leqslant p < \infty$, there is an injective, nondegenerate spatial representation $L_Q \to \mathcal{L}(\ell^p(\mathbb{N}))$. Spatial representations of matrix algebras $M_n L_Q$ for $1 \leqslant n \leqslant \infty$ are considered in Section 5 and it is shown that they are the same as spatial representations of the Leavitt path algebra over the graph $M_n Q$ (Remark 5.1). Morever, we prove that any such representation is equivalent to the matricial amplification $M_n\rho$ of a spatial representation ρ of L_Q (Lemma 5.3). Section 6 is concerned with a characterization of spatiality of representations in terms of norm estimates. We prove a spatiality criterion which we shall presently explain. The subalgebra $(L_Q)_{0,1} = \operatorname{span}\{v \in Q^0, ee^*, e \in Q^1\} \subset L_Q$ is a direct sum of, possibly infinite dimensional, matrix algebras and is thus naturally equipped with an L^p-operator norm. The spatiality criterion, Theorem 6.2, says that if $p \in [1,\infty) \setminus \{2\}$, then a nondegenerate representation $\rho : L_Q \to \mathcal{L}(L^p(X))$ is spatial if and only if its restriction to $(L_Q)_{0,1}$ is contractive and $\|\rho(x)\| \leqslant 1$ for every $x \in Q^1 \coprod (Q^1)^*$. Along the way we also prove a spatiality criterion for nondegenerate L^p-representations of matricial algebras (Proposition 6.1). Both spatiality criteria fail to be true if the nondegeneracy hypothesis is dropped (see Remark 6.3). By contrast, for a representation $\rho : L_Q \to \mathcal{L}(L^2(X))$, the condition that $\|\rho(x)\| \leqslant 1$ for every $x \in Q^0 \coprod Q^1 \coprod (Q^1)^*$ is equivalent to requiring that ρ be a $*$-homomorphism (Proposition 6.4). In Section 7 we define L^p-operator algebras and introduce the L^p-operator algebra $\mathcal{O}^p(Q)$. By definition, any spatial representation of $L_Q \to \mathcal{L}(L^p(X))$ factors uniquely through a contractive representation $\mathcal{O}^p(Q) \to \mathcal{L}(L^p(X))$ $(1 \leqslant p < \infty)$. Moreover we prove, using the spatiality criterion of Section 6, that for $p \neq 2$, any nondegenerate contractive representation $\mathcal{O}^p(Q) \to \mathcal{L}(L^p(X))$ induces a nondegenerate spatial representation $L_Q \to \mathcal{L}(L^p(X))$ (Theorem 7.6). We show that if moreover $p \neq 1$, then the nondegeneracy hypothesis may be dropped. We also prove that $\mathcal{O}^2(Q)$ is just the usual graph C^*-algebra $C^*(Q)$ (Proposition 7.9). It follows from this that a contractive L^2-representation of $C^*(Q)$ is equivalent to a $*$-representation of L_Q (Remark 7.10). We also show, using the material of Section 3, that if $p \in (1,\infty)$ then $\mathcal{O}^p(Q)$ is the same as the L^p-algebra $F^p_{\text{tight}}(\mathcal{S}(Q))$ introduced by E. Gardella and M. Lupini in [9] (Corollary 7.11 and Proposition 7.12). The latter is universal for those tight L^p-representations of $\mathcal{S}(Q)$ which are either spatial (if $p \neq 2$) or $*$-representations (if $p = 2$). In Section 8 we show that adding heads and tails to a graph Q to obtain a new graph Q' without sources, sinks or infinite emitters results in an isometric inclusion $\mathcal{O}^p(Q) \to \mathcal{O}^p(Q')$ (Corollary 8.2). Section 9 is devoted to the proof of Theorem 1.2 (Theorem 9.1). The technical result of the previous section is used here to reduce the proof to the case of graphs without sources, sinks or infinite emitters. After this reduction, the strategy of proof is similar to that of the analogous result for the L^p-Cuntz algebra ([15], Theorem 8.7), although it requires several nontrivial technical adjustments. The simplicity Theorem 1.1 is proved in Section 10. The last section of this article is Section 11, where we prove Theorem 11.2, of which Theorem 1.3 is a particular case.

NOTATION 1.4. In this paper $\mathbb{N} = \mathbb{Z}_{\geqslant 1}$ and $\mathbb{N}_0 = \mathbb{Z}_{\geqslant 0}$. All algebras, vector spaces, and tensor products are over $\mathbb{C}$. All identities pertaining measure spaces are to be interpreted up to sets of measure zero. For example we say that a family $\{X_n\}_{n\geqslant 1}$ of measurable sets in a measurable space $X = (X, \mathcal{B}, \mu)$ is *disjoint* if $X_n \cap X_m$ has measure zero for all $n \neq m$, and write $\coprod_n X_n$ for their union. In case the latter agrees with X up to measure zero, we write $X = \coprod_n X_n$. This reflects the fact that under the above hypothesis $(X, \mathcal{B}, \mu)$ is equivalent to the set theoretic coproduct $\coprod_n X_n$ equipped with the σ-algebra generated by $\coprod_n \mathcal{B}_n$ and the measure induced by the sequence of measures $\{\mu_{|_{X_n}}\}$. We write $L^0(X)$ for the vector space of classes of measurable functions $X \to \mathbb{C}$.

2. GRAPHS AND LEAVITT PATH ALGEBRAS

In this section we briefly recall some of the basics of Leavitt path algebras; a general reference for the subject is [1].

An *oriented graph* or *quiver* $Q = (Q^0, Q^1, r, s)$ consists of sets Q^0 and Q^1 of vertices and edges, and range and source functions $r, s : Q^1 \to Q^0$. We say that Q is *finite* or *countable* if Q^0 and Q^1 are both finite or countable. A vertex $v \in Q^0$ is an *infinite emitter* if $s^{-1}(v)$ is infinite, and is a *sink* if $s^{-1}(v) = \emptyset$. A vertex is *singular* if it is either a sink or an infinite emitter. We write $\operatorname{sing}(Q) = \operatorname{sink}(Q) \cup \inf(Q) \subset Q^0$ for the set of singular vertices and $\operatorname{reg}(Q) = Q^0 \setminus \operatorname{sing}(Q)$. We call Q *singular* if $\operatorname{sing}(Q) \neq \emptyset$ and *nonsingular* (or *regular*) otherwise. We call Q *row-finite* if it has no infinite emitters. A vertex v is a *source* if $r^{-1}(v) = \emptyset$; we write $\operatorname{sour}(Q) \subset Q^0$ for the set of sources.

Since all our graphs will be oriented, we shall use the term graph to mean oriented graph.

A *path* α is a (finite or infinite) sequence of edges $\alpha = e_1 \cdots e_i \cdots$ such that $r(e_i) = s(e_{i+1})$ $(i \geqslant 1)$. For such α, we write $s(\alpha) = s(e_1)$; if α is finite of *length* l, we put $|\alpha| = l$ and $r(\alpha) = r(e_l)$. Vertices are considered as paths of length 0. A finite path α is *closed* if $s(\alpha) = r(\alpha)$. A closed path $\alpha = \alpha_1 \cdots \alpha_n$ is a *cycle* if in addition $s(e_i) \neq s(e_j)$ if $i \neq j$. Let $\mathcal{P} = \mathcal{P}(Q)$ be the set of finite paths, and let $\mathcal{P}_n$ be the set of paths of length n. Thus,

$$\mathcal{P} = \coprod_{n \in \mathbb{N}_0} \mathcal{P}_n. \tag{2.1}$$

We consider the following preorder in $\mathcal{P}$:

$$\alpha \leqslant \beta \Leftrightarrow \exists\, \gamma \text{ such that } r(\beta) = s(\gamma) \text{ and } \alpha = \beta\gamma. \tag{2.2}$$

Observe that (2.2) also makes sense when α is an infinite path.

DEFINITION 2.1. Let Q be a graph. The *Leavitt path algebra* L_Q is the quotient of the free $\mathbb{C}$-algebra on $Q^0 \cup Q^1 \cup (Q^1)^*$, modulo the following relations:

(i) $vv' = \delta_{v,v'}v$ for all $v, v' \in Q^0$;
(ii) $s(e)e = er(e) = e$ for all $e \in Q^1$;
(iii) $r(e)e^* = e^*s(e) = e^*$ for all $e \in Q^1$;
(iv) (CK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in Q^1$;
(v) (CK2) $v = \sum\limits_{\{e \in Q^1 : s(e) = v\}} ee^*$, if $v \in \operatorname{reg}(Q)$.

The Leavitt path algebra is a $*$-algebra with involution determined by $v \mapsto v$, $e \mapsto e^*$. It has a $\mathbb{Z}$-grading where vertices have degree zero, edges have degree 1, and $|e^*| = -1$ for $e \in Q^1$ ([1], Corollary 2.1.5). We write

$$(L_Q)_n = \operatorname{span}\{\alpha\beta^* : |\alpha| - |\beta| = n\} \tag{2.3}$$

for the n-th homogeneous component with respect to this grading.

The elementary lemmas below shall be used later in the article.

LEMMA 2.2. *Let Q be a nonsingular graph and $a_1, \ldots, a_m \in L_Q$. Then there exist $n \in \mathbb{N}$, a finite set $F \subset \mathcal{P}$, and finitely supported functions $\lambda^i : F \times \mathcal{P}_n \to \mathbb{C}$, $(\alpha, \beta) \mapsto \lambda^i_{\alpha,\beta}$ ($i = 1, \ldots, m$, $\alpha \in F$, $\beta \in \mathcal{P}_n$), such that*

$$a_i = \sum_{\alpha \in F} \sum_{\beta \in \mathcal{P}_n} \lambda^i_{\alpha,\beta}\alpha\beta^*, \quad \textit{for all } i = 1, \ldots, m.$$

Proof. For each $i = 1, \ldots, m$, we may write $a_i = \sum\limits_{j=1}^{n_i} \lambda^i_j \alpha^i_j {\beta^i_j}^*$ with paths β^i_j of length $n := \max\limits_{i,j}\{|\beta^i_j|\}$, using relation (CK2) of Definition 2.1. Put $F_i := \{\alpha^i_j : j = 1, \ldots, n_i\}$, $G_i := \{\beta^i_j : j = 1, \ldots, n_i\}$ and $F := \bigcup\limits_{i=1}^{m} F_i$. Rewriting the sums for each i, we have $a_i = \sum\limits_{\alpha \in F} \sum\limits_{\beta \in \mathcal{P}_n} \lambda^i_{\alpha,\beta}\alpha\beta^*$ with $\lambda^i_{\alpha,\beta} = 0$ if $\alpha \notin F_i$ or $\beta \notin G_i$. ∎

LEMMA 2.3. *Let Q be a graph, B a $\mathbb{C}$-algebra, and $\rho : L_Q \to B$ a homomorphism. Let $u := \{u_v\}_{v \in Q^0} \subset B$ such that u_v is invertible in $\rho(v)B\rho(v)$ for all $v \in Q^0$. Then there is a unique homomorphism $\rho_u : L_Q \to B$ such that for all $e \in Q^1$, $v \in Q^0$*

$$\rho_u(e) = u_{s(e)}\rho(e), \quad \rho_u(e^*) = \rho(e^*)u^{-1}_{s(e)} \quad \textit{and} \quad \rho_u(v) = \rho(v).$$

Proof. One checks that for $x \in Q^0 \cup Q^1 \cup (Q^1)^*$ the elements $\rho_u(e)$, satisfy the relations of Definition 2.1. ∎

3. LEAVITT PATH ALGEBRAS AND SEMIGROUPS

Let Q be a graph and $\mathcal{P} = \mathcal{P}(Q)$ the set of finite paths. Write

$$\mathcal{S} = \mathcal{S}(Q) = \{0\} \cup \{\alpha\beta^* : \alpha, \beta \in \mathcal{P}\} \subset L_Q. \tag{3.1}$$

Then $\mathcal{S}$ is the inverse semigroup associated with Q. The *Cohn* algebra of Q is the semigroup algebra $C_Q = \mathbb{C}[\mathcal{S}]$ of $\mathcal{S}$; its elements are the finite linear combinations of the elements of $\mathcal{S}$ with multiplication induced by that of $\mathcal{S}$. Observe that L_Q is the quotient of C_Q modulo the relation $CK2$. Consider the sub-semigroup $\mathcal{E}$ of idempotent elements of $\mathcal{S}$:

$$\mathcal{S} \supset \mathcal{E} = \{0\} \cup \{\alpha\alpha^* : \alpha \in \mathcal{P}\}.$$

The set $\mathcal{E}$ is partially ordered by $p \leqslant q \Leftrightarrow pq = p$ and is a semilattice for this partial order. Observe that for the order of paths defined in (2.2), the bijection $\mathcal{P} \to \mathcal{E} \setminus \{0\}$, $\alpha \mapsto \alpha\alpha^*$ is a poset isomorphism. Note also that $p, q \in \mathcal{E}$ are incomparable if and only if $pq = 0$. Let $p \in \mathcal{E}$ and $Z \subset \{q \in \mathcal{E} : q \leqslant p\}$. We call Z a *cover* of p if for every $q \leqslant p$ there exists $z \in Z$ such that $zq \neq 0$. Let $(\mathrm{End}(\mathbb{V}), \circ)$ be the set of linear endomorphisms considered as a semigroup under composition. A *representation* of $\mathcal{S}$ on a vector space $\mathbb{V}$ is a semigroup homomorphism $\rho : \mathcal{S} \to (\mathrm{End}(\mathbb{V}), \circ)$. The image of $\mathcal{E}$ under a representation ρ generates a boolean algebra $\mathcal{B}_\rho$ with operations $p \wedge q = pq$, $p \vee q = p + q - pq$. By Proposition 11.8 of [8], the boolean representation $\rho : \mathcal{E} \to \mathcal{B}_\rho$ is *tight* in the sense of Definition 11.6 in [8] if and only if for every $p \in \mathcal{E}$ and every finite cover Z of p, we have

$$\bigvee_{z \in Z} \rho(z) = \rho(p). \tag{3.2}$$

Following Definition 13.1 of [8], we call the representation ρ of $\mathcal{S}$ tight if its restriction to $\mathcal{E}$ is tight.

Although the following lemma is well-known to experts, we have not been able to find it explicitly stated in the literature, so we include it here with proof. The particular case of Lemma 3.1 when Q has a single vertex is Lemma 7.5 of [11]. See also Corollary 5.3 of [22].

LEMMA 3.1. *Let $\rho : \mathcal{S}(Q) \to (\mathrm{End}(\mathbb{V}), \circ)$ be a representation. Then ρ is tight if and only if it extends to an algebra homomorphism $L_Q \to \mathrm{End}(\mathbb{V})$.*

Proof. If $v \in \mathrm{reg}(Q)$, then $Z = \{ee^* : e \in Q^1, s(e) = v\}$ is a finite cover of v and the supremum in (3.2) equals $\sum_{e \in Z} \rho(ee^*)$. It follows that if ρ is tight then it extends to an algebra homomorphism $L_Q \to \mathrm{End}(\mathbb{V})$. Assume conversely that ρ extends to L_Q. We have to prove that (3.2) holds. Since the supremum in (3.2) depends only on the maximal elements of Z, and any two of these are incomparable, we may assume that no two distinct elements of Z are comparable. Hence

$$\bigvee_{z\in Z}\rho(z)=\sum_{z\in Z}\rho(z).$$

If $\alpha\in\mathcal{P}$ and $r(\alpha)=v$, then $W=\alpha^*Z\alpha$ is a cover of v and $\sum\limits_{z\in Z}=\alpha\sum\limits_{w\in W}w\alpha^*$. Hence we may further assume that $\alpha=v$. We must then prove that for each finite cover Z of v in which no two distinct elements are comparable, the following identity holds in L_Q

$$\sum_{z\in Z}z=v.$$

We do this by induction on $n=m(Z)=\max\{|\alpha|:\alpha\alpha^*\in Z\}$. For $n=0$ this is trivial. Assume $n\geqslant 1$ and let $A=\{\alpha\in\mathcal{P}_n:\alpha\alpha^*\in Z\}$. Each $\alpha\in A$ can be written uniquely as $\widetilde{\alpha}e_\alpha$ where $|\widetilde{\alpha}|=n-1$ and $e_\alpha\in Q^1$. For $w\in B:=\{s(e_\alpha):\alpha\in A\}$, put $C_w=\{e_\alpha:s(e_\alpha)=w\}$; because Z is a cover, $C_w=s^{-1}(w)$. Hence

$$\sum_{\alpha\in A}\alpha\alpha^*=\sum_{\beta\in\widetilde{A}}\sum_{\widetilde{\alpha}=\beta}\alpha\alpha^*=\sum_{\beta\in\widetilde{A}}\sum_{s(e)=r(\beta)}\beta ee^*\beta^*=\sum_{\beta\in\widetilde{A}}\beta\beta^*.$$

Let $Z'=(Z\setminus A)\cup\widetilde{A}$; then $m(Z')=n-1$, any two distinct elements of Z' are incomparable, and by the calculation above, $\sum\limits_{z'\in Z'}z'=\sum\limits_{z\in Z}z$. This concludes the proof. ∎

4. SPATIAL REPRESENTATIONS OF L_Q

Let E be a Banach space. We write $\mathcal{L}(E)$ for the Banach algebra of bounded linear maps $E\to E$. A *representation* of L_Q on E is an algebra homomorphism $\rho:L_Q\to\mathcal{L}(E)$. We say that ρ is *nondegenerate* if $\rho(L_Q)E\subset E$ is dense. In this paper we shall be mostly concerned with *L^p-representations*, that is, with representations on Banach spaces of the form $L^p(X)$, $p\in[1,\infty)$, where $X=(X,\mathcal{B},\mu)$ is a σ-finite measure space. If $A\in\mathcal{B}$, we write $P(A)$ for the set of subsets of A and consider A as a measure space with σ-algebra $\mathcal{B}_A:=\mathcal{B}\cap P(A)$ and measure $\mu_{|_{\mathcal{B}_A}}$; thus

$$A=(A,\mathcal{B}_A,\mu_{|_{\mathcal{B}_A}}).$$

We write $\mathcal{N}(\mu)=\{A\in\mathcal{B}:\mu(A)=0\}$ and $\mathcal{B}_\mu=\mathcal{B}/\mathcal{N}(\mu)$.

In what follows, we need to borrow several definitions from [15], pertaining to (partial) isometries between L^p-spaces.

Let $X=(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ be σ-finite measure spaces. A *measurable set transformation* from X to Y is homomorphism of σ-algebras $S:\mathcal{B}_\mu\to\mathcal{C}_\nu$. If S is bijective, then $S_*(\mu)=\mu S^{-1}$ is a σ finite measure on $\mathcal{C}$, absolutely continuous with respect to ν. By Proposition 5.6 of [15], there is also a map $S_*:L^0(X)\to L^0(Y)$ such that $S_*(\chi_E)=\chi_{S(E)}$ $(E\in\mathcal{B}_\mu)$.

Let $1 \leqslant p < \infty$; to a bijective measurable set transformation S from X to Y and a measurable function $h : Y \to \mathbb{C}$ such that $|h(x)| = 1$ for almost every $x \in B$ one associates an isometric isomorphism $u : L^p(X) \to L^p(Y)$ as follows:

$$u(\xi)(y) = h(y)\Big(\Big[\frac{\mathrm{d}S_*(\mu)}{\mathrm{d}\nu(y)}\Big]\Big)^{1/p} S_*(\xi)(y) \quad \text{for all } \xi \in L^p(X), y \in X. \tag{4.1}$$

An isometric isomorphism $u : L^p(X) \to L^p(Y)$ is called *spatial* if there exist S and h such that u is of the form (4.1). If $p \neq 2$, then every isometric isomorphism in $\mathcal{L}(L^p(X), L^p(Y))$ is spatial, by the Banach–Lamperti theorem ([13], Theorem 3.1; see Theorem 6.9 and Lemma 6.15 of [15] for a detailed proof). A partial isometry $s : L^p(X) \to L^p(X)$ is *spatial* if there are $A, B \in \mathcal{B}_\mu$, called respectively the *domain* and the *range support* of s, and a spatial isometric isomorphism $u : L^p(A) \to L^p(B)$, such that for the projection $\pi_A : L^p(X) \to L^p(A)$ and the inclusion $\iota_B : L^p(B) \to L^p(X)$ we have a factorization

$$s = \iota_B u \pi_A. \tag{4.2}$$

If S and h are as in (4.1) we call s the spatial partial isometry associated with the *spatial system* (S, A, B, h); S and h are the *spatial realization* and the *phase factor* of the spatial system. Observe that whereas different choices of h and S may induce the same partial isometry s, π_A and π_B depend only on the latter. Indeed

$$\pi_A = \inf\{\pi_{A'} : A' \in \mathcal{B}, s\pi_{A'} = s\} \quad \text{and} \quad \pi_b = \inf\{\pi_{B'} : B' \in \mathcal{B}, \pi_{B'}s = s\}.$$

If s is as in (4.2), then $t = \iota_A u^{-1} \pi_B$ is the unique spatial partial isometry such that $ts = \pi_A$ and $st = \pi_B$; we call t the *reverse* of s. If $p = 2$ and s is a spatial partial isometry then the reverse of s is just its adjoint $t = s^*$.

EXAMPLE 4.1. Let $X = (X, \mathcal{B}, \mu)$ be a σ-finite measure space. Let $E \in \mathcal{B}$ and let χ_E be the characteristic function. Then the canonical projection $\pi_E : L^p(X) \to L^p(E) \subset L^p(X), \pi_E(\xi) = \chi_E \xi$ is a spatial partial isometry with spatial system $(\mathrm{Id}_{\mathcal{B}_E}, E, E, 1)$. Every idempotent spatial partial isometry is of this form, by Lemma 6.18 of [15].

EXAMPLE 4.2. Let X be as in the previous example and let $s : L^p(X) \to L^p(X)$ be a spatial partial isometry with spatial system (S, A, B, h). If $z \in \mathbb{S}^1$ then $\xi \mapsto zs(\xi)$ is again a spatial partial isometry with spatial system (S, A, B, zh).

REMARK 4.3. Spatial partial isometries in general and spatial idempotents in particular have norm 1. However the converse does not hold. For example,

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \in M_2 = \mathcal{B}(\ell^p(\{1,2\}))$$

is a norm one idempotent that is not spatial in our sense (which is that of [15]) for any $p \geqslant 1$ ([15], Example 7.3). However it is self-adjoint and therefore 2-spatial in the sense of Definition 4.6 in [9].

A representation $\rho : L_Q \to \mathcal{L}(L^p(X))$ is *spatial* if for each $v \in Q^0$, $\rho(v)$ is a spatial idempotent and for each $e \in Q^1$, $\rho(e)$ is a spatial partial isometry with reverse $\rho(e^*)$. If ρ is spatial then $\rho(x)$ is spatial for every $x \in \mathcal{S}(Q)$, whence by Lemma 3.1 a spatial representation of L_Q is the same as a tight *spatial representation of* $\mathcal{S}(Q)$, that is, a tight representation of $\mathcal{S}(Q)$ which takes values in the inverse semigroup $\mathcal{S}(L^p(X))$ of spatial partial isometries.

REMARK 4.4. As we explained above, the reverse of a spatial isometry $s \in L^2(X)$ is just its adjoint. Hence any spatial representation $L_Q \to \mathcal{L}(L^2(X))$ is a $*$-representation. The converse does not hold. For example $\mathbb{C}$ is the Leavitt path algebra of the graph consisting of a single vertex and no edges, and the representation $\rho : \mathbb{C} \to M_2 = \mathcal{L}(\ell^2(\mathbb{N}))$ that sends 1 to the self-adjoint idempotent of Remark 4.3 is a $*$-representation that is not spatial in our sense.

REMARK 4.5. If ρ is spatial and $\alpha, \beta \in \mathcal{P}(Q)$ are paths with $r(\alpha) = r(\beta)$, then $\rho(\alpha\beta^*)$ is a spatial partial isometry. In particular, $\rho(\alpha\alpha^*)$ is an idempotent spatial partial isometry, and thus by Example 4.1, there is $X_\alpha \in \mathcal{B}$ such that $\rho(\alpha\alpha^*)$ is the canonical projection $\pi_{X_\alpha} : L^p(X) \to L^p(X_\alpha) \subset L^p(X)$. If S_α is the measurable set transformation of $\rho(\alpha)$ then $X_\alpha = S_\alpha(X_{r(\alpha)})$, so the spatial system of $\rho(\alpha)$ is of the form

$$(S_\alpha, X_{r(\alpha)}, X_\alpha, g_\alpha)$$

for some $g_\alpha : X_\alpha \to \mathbb{C}$ such that $|g(x)| = 1$ for almost all $x \in X_\alpha$. If $\alpha \geqslant \beta$, say $\beta = \alpha\gamma$, then $X_\beta \subset X_\alpha$ because $X_\beta = S_\alpha(X_\gamma) \subset S_\alpha(X_{r(\alpha)}) = X_\alpha$. On the other hand if α and β are not comparable then X_α and X_β are disjoint. In particular, for each $v \in Q^0$ the family $\{X_e : s(e) = v\} \subset \mathcal{B} \cap P(X_v)$ consists of pairwise disjoint sets, and if v is regular its union is the whole X_v:

$$X_v = \coprod_{e \in s^{-1}(v)} X_e \quad \text{for all } v \in \operatorname{reg}(Q). \tag{4.3}$$

It follows from (4.3) that if Q is nonsingular then for each $l \geqslant 0$ we have

$$X_v = \coprod_{\alpha \in v\mathcal{P}_l(Q)} X_\alpha. \tag{4.4}$$

Conversely, if we are given disjoint families $\{X_v : v \in Q^0\} \subset \mathcal{B}$ and $\{X_e : e \in Q^1, s(e) = v\} \subset \mathcal{B} \cap P(X_v)$ for each $v \in Q^0$ satisfying (4.3) and a family $\{s_e : e \in Q^1\}$ of spatial partial isometries in $\mathcal{L}(L^p(X))$ with range and source projections π_{X_e} and π_{X_v}, then there exists a unique algebra homomorphism $\rho : L_Q \to \mathcal{L}(L^p(X))$ satisfying $\rho(v) = \pi_{X_v}$, $\rho(e) = s_e$, and sending e^* to the reverse of s_e.

LEMMA 4.6. *Let X be a σ-finite measure space. If $\rho : L_Q \to \mathcal{L}(L^p(X))$ is a spatial representation, then ρ is nondegenerate if and only if*

$$X = \coprod_{v \in Q^0} X_v. \tag{4.5}$$

Proof. Immediate from the fact that

$$\rho(L_Q)L^p(X) = \sum_{v\in Q^0} \rho(v)L^p(X) = \bigoplus_{v\in Q^0} L^p(X_v). \qquad \blacksquare$$

It follows from (4.4) and Lemma 4.6 that if Q is nonsingular and ρ is nondegenerate, then for each $l \geqslant 0$ we have

$$X = \coprod_{\alpha\in\mathcal{P}_l(Q)} X_\alpha. \tag{4.6}$$

LEMMA 4.7. *Let Q be a graph, $1 \leqslant p < \infty$, $X = (X, \mathcal{B}, \mu)$ a σ-finite measure space, and $\rho : L_Q \to \mathcal{L}(L^p(X))$ a spatial representation. Then there are $X' \in \mathcal{B}$ and a nondegenerate spatial representation $\rho' : L_Q \to \mathcal{L}(L^p(X'))$ such that ρ factors as ρ' followed by the inclusion $\mathcal{L}(L^p(X')) \subset \mathcal{L}(L^p(X))$.*

For the proof put $X' = \coprod_{v\in Q^0} X_v$.

The following example of a nondegenerate spatial representation is used in the proof of Theorem 10.1.

EXAMPLE 4.8. Let Q be a graph, and let

$$\mathfrak{X} = \mathfrak{X}_Q = \{\alpha : \text{ infinite path in } Q\} \cup \{\alpha \in \mathcal{P} : r(\alpha) \in \operatorname{sing}(Q)\}. \tag{4.7}$$

For $\alpha \in \mathcal{P}$, let

$$\mathfrak{X} \supset Z_\alpha = \{x \in \mathfrak{X} : \alpha \geqslant x\} = \alpha\mathfrak{X}.$$

The sets Z_α are the basis of a topology on $\mathfrak{X}$ which makes it a locally compact Hausdorff space; modulo our different conventions for ranges and sources, this is the space considered in page 3 of [6]. The inverse semigroup $\mathcal{S} = \mathcal{S}(Q)$ acts on $\mathfrak{X}$ by partial homeomorphisms; an element $u = \alpha\beta^* \in \mathcal{S}$ acts on $\mathfrak{X}$ with domain Z_β and range Z_α via

$$\alpha\beta^*(\beta x) = \alpha x. \tag{4.8}$$

Let $\mathcal{B}$ be the σ-algebra of all Borel subsets of $\mathfrak{X}$. The semigroup $\mathcal{S}$ of (3.1) acts on $\mathfrak{X}$ via (4.8). If $\alpha, \beta \in \mathcal{P}$ with $r(\alpha) = r(\beta)$, then

$$S_{\alpha\beta^*} : \mathcal{B}_{|Z_\beta} \to \mathcal{B}_{|Z_\alpha}, \quad A \mapsto \alpha\beta^*(A) \tag{4.9}$$

is a bijective homomorphism of σ-algebras. Let μ be a measure on $\mathcal{B}$; μ is *quasi-invariant* under $\alpha\beta^*$ if $\mu_{|Z_\beta}$ and $\mu_{|Z_\alpha} \circ \beta\alpha^*$ are equivalent measures (that is, if they are absolutely continuous with respect to each other); μ is quasi-invariant under $\mathcal{S}$ if it is quasi-invariant under any element of $\mathcal{S}$. One can show that $\mathfrak{X}$ always has a σ-finite measure that is quasi-invariant under $\mathcal{S}$. For example, in case $\mathfrak{X}$ is countable we can take μ to be the counting measure. Assume that μ is a σ-finite measure on the Borel subsets of $\mathfrak{X}$, quasi-invariant under $\mathcal{S}$, and let $s_{\alpha\beta^*}$ be the spatial isometry of (4.1) with spatial realization $S = S_{\alpha\beta^*}$ and constant phase factor $h = 1$. Then

$$\mathcal{S} \to \mathcal{L}(L^p(\mathfrak{X}, \mu)), \quad \alpha\beta^* \mapsto s_{\alpha\beta^*}$$

is a tight nondegenerate spatial representation of $\mathcal{S}$ and thus induces a nondegenerate spatial representation $\rho_\mu : L_Q \to \mathcal{L}(L^p(\mathfrak{X},\mu))$. There are graphs Q such that ρ_μ is not injective for any $p \in [1,\infty)$. For example, if Q consists of one vertex and one loop, then $L_Q \cong \mathbb{C}[t,t^{-1}]$ and ρ_μ is 1-dimensional.

REMARK 4.9. Each element $x \in \mathfrak{X}$ induces a tight Boolean representation $\phi_x : \mathcal{E} \to \{0,1\}$ (that is, a *tight character* in the sense of Definition 12.8 in [8]) so that $\phi_x(\alpha\alpha^*) = 1$ if and only if $x \in Z_\alpha$. One can show that the map $x \mapsto \phi_x$ is a homeomorphism between $\mathfrak{X}$ and the space $\widehat{\mathcal{E}}_{\text{tight}}$ of tight characters with the topology of point-wise convergence, and that the action (4.8) corresponds to the canonical action of Proposition 12.8 in [8].

CONSTRUCTION 4.10. Let X be a countable set, and let $\mathcal{I}(X)$ be the inverse semigroup of all partially defined injections

$$X \supset \operatorname{dom} f \xrightarrow{f} X.$$

Let Q be a countable graph, $\mathcal{S} = \mathcal{S}(Q)$ its associated inverse semigroup and $S : \mathcal{S} \to \mathcal{I}(X)$ a semigroup homomorphism. For each $\alpha \in \mathcal{P} = \mathcal{P}(Q)$, set $X_\alpha = \operatorname{dom}(S_\alpha)$. We shall assume that S is tight, i.e. that the identities (4.3) and (4.5) are satisfied. Let $\mathcal{G} = \mathcal{G}(\mathcal{S},X)$ be the groupoid of germs, as defined in Section 4 of [8]. The elements of $\mathcal{G}$ are equivalence classes $[\alpha\beta^*,x]$ where $r(\alpha) = r(\beta)$, $x \in X_\beta$; the equivalence relation is determined by the prescription that $[\alpha\beta^*,x] = [\alpha\gamma\gamma^*\beta^*,x]$ for any $\gamma \in \mathcal{P}$ with $s(\gamma) = r(\alpha)$. For $\alpha\beta^* \in \mathcal{S} \setminus \{0\}$, put

$$\Theta_{\alpha,\beta} = \{[\alpha\beta^*,x] : x \in X_\beta\} \subset \mathcal{G}.$$

Let $\mathcal{A}(\mathcal{G}) \subset \operatorname{map}(\mathcal{G},\mathbb{C})$ be the linear subspace generated by the characteristic functions $\chi_{\Theta_{\alpha,\beta}}$, $(\alpha\beta^* \in \mathcal{S} \setminus \{0\})$. One checks that $\mathcal{A}(\mathcal{G})$ is an algebra under the convolution product (it is in fact the Steinberg algebra of $\mathcal{G}$ [21]) and that

$$\psi : L_Q \to \mathcal{A}(\mathcal{G}), \quad \psi(\alpha\beta^*) = \chi_{\Theta_{\alpha,\beta}} \tag{4.11}$$

is an algebra homomorphism. Let

$$L : \mathcal{A}(\mathcal{G}) \to \mathcal{L}(\ell^p(\mathcal{G})), \quad L(f)(\xi)(h) = \sum_{g \in \mathcal{G}} f(g)\xi(g^{-1}h), \tag{4.12}$$

for $f \in \mathcal{A}(\mathcal{G})$, $\xi \in \ell^p(\mathcal{G})$ and $h \in \mathcal{G}$. This is well-defined because the domain and range functions are injective on each $\Theta_{\alpha,\beta}$. One checks that L is a monomorphism. Consider the composite

$$\rho = L\psi : L_Q \to \mathcal{L}(\ell^p(\mathcal{G})). \tag{4.13}$$

Let $\alpha\beta^* \in \mathcal{S}(Q)$ and consider the following subsets of $\mathcal{G}$:

$$A = \{[\gamma\delta^*,\delta x] : \beta \geqslant \gamma x\}, \quad B = \{[\alpha\beta^*\gamma\delta^*,\delta x] : \beta \geqslant \gamma x\}.$$

The map

$$A \to B,$$
$$[\gamma\delta^*, \delta x] \mapsto [\alpha\beta^*, \gamma x][\gamma\delta^*, \delta x] = [\alpha\beta^*\gamma\delta^*, \delta x]$$

is bijective and thus induces a cardinality preserving bijection $S_{\alpha,\beta} : P(A) \to P(B)$. One checks that $\rho(\alpha\beta^*)$ is the spatial isometry with spatial system $(S_{\alpha,\beta}, A, B, 1)$. Hence ρ is a spatial, nondegenerate representation.

LEMMA 4.10. *Assume that in Construction* 4.10*, one has* $X_v \neq \emptyset$ *for all* $v \in Q^0$*. Then* (4.11) *is an isomorphism and* (4.13) *is an injective, nondegenerate spatial representation.*

Proof. Put $\mathcal{A}(\mathcal{G})_n = \operatorname{span}\{\psi(\alpha\beta^*) : |\alpha\beta^*| = n\}$; we have

$$\mathcal{A}(\mathcal{G}) = \sum_n \mathcal{A}(\mathcal{G})_n. \tag{4.14}$$

Let $c : \mathcal{G} \to \mathbb{Z}$, $c([\alpha\beta^*, x]) = |\alpha\beta^*|$; note that the elements of $\mathcal{A}(\mathcal{G})_n$ are supported in $c^{-1}(\{n\})$. It follows from this that the sum in (4.14) is direct. Moreover, because c is a groupoid homomorphism, we have $\mathcal{A}(\mathcal{G})_n\mathcal{A}(\mathcal{G})_m \subset \mathcal{A}(\mathcal{G})_{n+m}$. Thus ψ is a homogeneous homomorphism of graded algebras. For $v \in Q^0$, $\psi(v)$ is the characteristic function of $\{[v, x] : x \in X_v\}$ which is nonempty by hypothesis, so $\psi(v) \neq 0$. By Theorem 2.2.15 of [1] this implies that ψ is an isomorphism. ∎

PROPOSITION 4.11. *Let* Q *be a countable graph and let* $p \in [1, \infty)$*. Then* L_Q *has an injective, nondegenerate spatial representation* $L_Q \to \mathcal{L}(\ell^p(\mathbb{N}))$*.*

Proof. Let X be any countably infinite set. Because X is infinite and $\#Q^0 \leqslant \#X$, there exists a bijection $\phi : X \to Q^0 \times X$. For $v \in Q^0$, set $X_v = \phi^{-1}(\{v\} \times X)$; observe that (4.5) is satisfied by construction. Put $Q^1(v, -) = s^{-1}(\{v\}) \subset Q^1$ and let

$$R_v = \begin{cases} Q^1(v, -) & v \in \operatorname{reg}(Q), \\ \{v\} \coprod Q^1(v, -) & v \in \operatorname{sing}(Q). \end{cases}$$

Because $\#X_v = \#X$ is infinite and $\#R_v \leqslant \#X_v$, there is a bijection $\zeta_v : X_v \to R_v \times X$. Set $X_e = \zeta_{s(e)}^{-1}(\{e\} \times X_{s(e)})$ $(e \in Q^1)$. By construction, (4.3) is satisfied. For $e \in Q^1$, let $r^{-1} \times 1 : \{r(e)\} \times X \to \{e\} \times X$ be the obvious bijection. Define a semigroup homomorphism $S : \mathcal{S}(Q) \to \mathcal{I}(X)$ by setting

$$S_v = 1_{X_v}, \quad S_e = \zeta_{s(e)}^{-1}(r^{-1} \times 1)\phi : X_{r(e)} \to X_e, \quad S_{e^*} = S_e^{-1} \quad \text{for } v \in Q^0, e \in Q^1.$$

Let $\mathcal{G}$ be the groupoid of germs associated to this action of S on X, and consider the nondegenerate spatial representation $\rho : L_Q \to \mathcal{L}(\ell^p(\mathcal{G}))$ of (4.13). Then ρ is injective by Lemma 4.10; furthermore, $\#\mathcal{G} = \aleph_0$ and any bijection $\mathcal{G} \cong \mathbb{N}$ induces a spatial isometric isomorphism $\ell^p(\mathcal{G}) \cong \ell^p(\mathbb{N})$. ∎

5. MATRIX ALGEBRAS AND SPATIAL REPRESENTATIONS

Let $1 \leqslant n \leqslant \infty$ and let A be an algebra. Write M_n for the algebra of $n \times n$-matrices with finitely many nonzero entries, and $M_n A = M_n \otimes A$. If $i, j \in \mathbb{N}$, we write $E_{i,j}$ for the canonical matrix unit. Let Q be a countable graph, X a σ-finite measure space, and $p \in [1, \infty)$. Call a representation $\rho : M_n(L_Q) \to \mathcal{L}(L^p(X))$ *spatial* if for every $x \in Q^0 \cup Q^1$ and i, j, $\rho(E_{i,j} \otimes x)$ is a spatial partial isometry with reverse $\rho(E_{j,i} \otimes x^*)$.

REMARK 5.1. Let $n \leqslant \infty$ and let $M_n Q$ be the graph obtained by adding a head

$$\cdots \longrightarrow v_i \xrightarrow{e_i^v} v_{i-1} \xrightarrow{e_{i-1}^v} \cdots \xrightarrow{e_2^v} v_1 \xrightarrow{e_1^v} v$$

for each $v \in Q^0$ and $i < n$. By Propositions 9.3 and 9.8 of [3], there is a $*$-isomor–phism

$$\begin{gathered} L_{M_n Q} \xrightarrow{\cong} M_n L_Q, \\ v \mapsto E_{1,1} \otimes v, \quad v_i \mapsto E_{i+1,i+1} \otimes v \\ e \mapsto E_{1,1} \otimes e, \quad e_i^v \mapsto E_{i+1,i} \otimes e. \end{gathered} \tag{5.1}$$

It is clear that a representation $M_n L_Q \to \mathcal{L}(L^p(X))$ is spatial in the matricial sense above if and only if its composition with the map (5.1) is a spatial representation of $L_{M_n Q}$.

EXAMPLE 5.2. Let $\sigma : L_Q \to \mathcal{L}(L^p(X))$ be a spatial representation. Let $I = \{1, \ldots, n\}$ if n is finite, and $I = \mathbb{N}$ if $n = \infty$. We have a canonical isometric isomorphism $L^p(I \times X) \cong \ell^p(I, L^p(X))$. Let

$$\sigma_I : M_n L_Q \to \mathcal{L}(\ell^p(I, L^p(X))), \quad \sigma_I(E_{i,j} \otimes a)(\xi)(k) = \delta_{k,i} \sigma(a)(\xi(j)).$$

Then σ_I is spatial. Indeed if $a \in \mathcal{S}(Q)$ and $\sigma(a)$ is a spatial isometry with domain support A and rank support B, then $\sigma_I(E_{i,j} \otimes a)$ is a spatial isometry with domain support $\{j\} \times A$ and range support $\{i\} \times B$. We remark that for $I = \{1, \ldots, n\}$, σ_I is the representation induced by the *amplification* of σ in the sense of Definition 4.10 in [9].

LEMMA 5.3. *Let Q be a countable graph, I a countable set, X a σ-finite measure space, $p \in [1, \infty)$, and $\rho : M_{|I|} L_Q \to \mathcal{L}(L^p(X))$ a nondegenerate spatial representation. Then there exist a σ-finite measure space Y, a spatial representation $\sigma : L_Q \to \mathcal{L}(L^p(Y))$ and a spatial isometric isomorphism $u : \ell^p(I, L^p(Y)) \to L^p(X)$ such that $\rho(a) = u \sigma_I(a) u^{-1}$ for all $a \in L_Q$.*

Proof. For each $i \in I$ and $v \in Q^0$, let $X_{i,v}$ be the domain support of the spatial idempotent $\rho(E_{i,i} \otimes v)$. Set $X_i = \coprod_{v \in Q^0} X_{i,v}$; we have $X = \coprod_{i \in I} X_i$. Hence we have L^p-direct sum decompositions $L^p(X) = \bigoplus_{i \in I} L^p(X_i)$ and $L^p(X_i) = \bigoplus_{v \in Q^0} L^p(X_{i,v})$.

Choose $i_0 \in I$, and let $Y = X_{i_0}$. Then $u = \bigoplus_{i,v} \rho(E_{i,i_0} \otimes v)$ is a spatial isometric isomorphism $\ell^p(I, L^p(Y)) = \bigoplus_{i \in I} L^p(Y) \to L^p(X)$. Let $\sigma : L_Q \to \mathcal{L}(L^p(Y))$, $\sigma(a) = \rho(E_{i_0,i_0} \otimes a)$. One checks that u conjugates σ_I to ρ, concluding the proof. ∎

6. A SPATIALITY CRITERION

We have a natural identification $M_n = \mathcal{L}(\ell^p(\{1, \dots, n\})$ for $n < \infty$ and a natural embedding $M_\infty \to \mathcal{L}(\ell^p(\mathbb{N}))$; by pulling back the operator norm, we get a norm $\|\cdot\|_p$ on M_n, for $n \in [1, \infty]$, which makes the latter into a normed algebra M_n^p. If I is a set and

$$\underline{n} = (n_i)_{i \in I} \tag{6.1}$$

is a family with $n_i \in [1, \infty]$, we write

$$M_{\underline{n}}^p = \bigoplus_{i \in I} M_{n_i}^p \tag{6.2}$$

for the algebraic direct sum equipped with the supremum norm

$$\|(a_i)\| = \sup_{i \in I} \|a_i\|_p.$$

We write $E_{a,b}^i$ $(i \in I)$, $1 \leqslant a, b \leqslant n_i$ for the canonical matrix unit.

The following proposition generalizes Theorem 7.2 of [15]; its proof is adapted from loc. cit.

PROPOSITION 6.1. *Let $p \in [1, \infty) \setminus \{2\}$, I a countable set, $\underline{n}$ as in* (6.1), *and $M_{\underline{n}}^p$ as in* (6.2). *Let $X = (X, \mathcal{B}, \mu)$ be a σ-finite measure space with $\mu \neq 0$. The following are equivalent for a nondegenerate representation $\rho : M_{\underline{n}}^p \to \mathcal{L}(L^p(X))$:*

(i) *$\rho(E_{a,b}^i)$ is a spatial partial isometry for all $i \in I$ and, $a, b \in [1, n_i]$;*

(ii) *ρ is contractive.*

Proof. Assume that (i) holds. Then each $\rho(E_{a,a}^i)$ is a spatial idempotent, whence by Example 4.1 there is $X_a^i \in \mathcal{B}$ such that $\rho(E_{a,a}^i) = \pi_{X_a^i}$ is the canonical projection. For each $i \in I$ put $\mathcal{N}_i = \mathbb{N}$ if $n_i = \infty$ and $\mathcal{N}_i = \{1, \dots, n_i\}$ if $n_i < \infty$. Because ρ is nondegenerate, we have $X = \coprod_{i \in I} \coprod_{a \in \mathcal{N}_i} X_a^i$. Put $X^i = \coprod_{a \in \mathcal{N}_i} X_a^i$. By restriction, we obtain a nondegenerate representation $\rho^i : M_{n_i} \to \mathcal{L}(L^p(X^i))$ satisfying (i); hence we may assume that $I = \{1\}$ has only one element. If $n < \infty$, nondegeneracy implies that $\rho(1) = 1$, so ρ is contractive by Theorem 7.2 of [15]. Assume $n = \infty$. Proceed as in loc. cit., using the partial isometries $\rho(E_{1,a}) : L^p(X_a) \to L^p(X_1)$ to construct an isometry $u : L^p(X) \to \ell^p(\mathbb{N}, L^p(X_1)) = \ell^p(\mathbb{N}) \otimes_p L^p(X_1)$ (the L^p-tensor product) that conjugates ρ to the contractive representation $T \mapsto T \otimes 1$. It follows that ρ is contractive, concluding the proof that (i) $\Rightarrow$ (ii).

Assume now that (ii) holds. Then $\{\rho(E^i_{a,a}) : i \in I, a \in \mathcal{N}_i\}$ is a family of orthogonal idempotents. Let $B^i_a = \rho(E^i_{a,a})L^p(X)$; then the algebraic direct sum $B = \bigoplus_{i,a} B^i_a$ is dense in $L^p(X)$. For each $z \in \mathbb{S}^1$, $i \in I$ and $a \in \mathcal{N}_i$ define an operator $u_{i,a}(z) : B \to B$ as multiplication by z on B^i_a and the identity on every other summand. Because ρ is contractive, $u_{i,a}(z)$ has norm 1, so it extends to a norm 1 operator $u_{i,a}(z) \in \mathcal{L}(L^p(X))$. Since this also holds for $u_{i,a}(z^{-1})$, $u_{i,a}(z)$ is a bijective isometry. Hence it is spatial, by the Banach–Lamperti theorem. Now proceed as in page 42 of [15] to deduce that $\rho(E^i_{a,a}) = (1 - u_{i,a}(-1))/2$ is a spatial idempotent. Hence there exists $X^i_a \in \mathcal{B}$ such that $B^i_a = L^p(X^i_a)$ and $X = \coprod X^i_a$. Since $\rho(E^i_{a,b})$ is an isometry $B^i_b \to B^i_a$, another application of the Banach–Lamperti theorem shows that it is spatial. ∎

Recall that the Leavitt path algebra is equipped with a $\mathbb{Z}$-grading $L_Q = \bigoplus_n (L_Q)_n$ where $(L_Q)_n$ is as in (2.3). Write $(L_Q)_{0,n} \subset (L_Q)_0$ for the subalgebra linearly spanned by the elements of the form $\alpha\beta^*$ with $r(\alpha) = r(\beta)$ and $|\alpha| = |\beta| \leqslant n$. We have an increasing union

$$(L_Q)_0 = \bigcup_{n=0}^{\infty} (L_Q)_{0,n}.$$

Each $(L_Q)_{0,n}$ is isomorphic to a direct sum of (possibly infinite dimensional) matrix algebras.

THEOREM 6.2 (cf. Theorem 7.7 of [9]). *Let $X = (X, \mathcal{B}, \mu)$ be a σ-finite measure space with $\mu \neq 0$, $p \in [1, \infty) \setminus \{2\}$, and Q a countable graph. The following are equivalent for a nondegenerate representation $\rho : L_Q \to \mathcal{L}(L^p(X))$:*

(i) *ρ is spatial;*

(ii) *$\|\rho(e)\|, \|\rho(e^*)\| \leqslant 1$ ($e \in Q^1$) and the restriction of ρ to $((L_Q)_{0,1}, \|\cdot\|_p)$ is contractive.*

Proof. The implication (i) ⇒ (ii) is clear using Proposition 6.1. Assume that (ii) holds; then $\rho(e)$ is a bijective isometry $\rho(r(e))L^p(X) \to \rho(ee^*)L^p(X)$ with inverse $\rho(e^*)$. By Proposition 6.1, $\rho(v)$ and $\rho(ee^*)$ are spatial idempotents for all $v \in Q^0$ and $e \in Q^1$. Hence it follows from the Banach–Lamperti theorem, Theorem 6.9 in [15] and from Lemma 6.15 of [15] that $\rho(e)$ and $\rho(e^*)$ are spatial. This concludes the proof. ∎

REMARK 6.3. The assumption that ρ be nondegenerate in necessary in both Proposition 6.1 and Theorem 6.2. For example the trivial graph on one vertex has Leavitt algebra $\mathbb{C}$, which equals M^p_1 for all $1 \leqslant p < \infty$, and the representation $\mathbb{C} \to M^p_2$ that maps 1 to the idempotent of Remark 4.3 is contractive but not spatial. The correct version of Theorem 6.2 for $p = 2$ is Proposition 6.4.

PROPOSITION 6.4. *Let $(X, \mathcal{B}, \mu)$ and Q be as in Theorem 6.2. The following are equivalent for a representation $\rho : L_Q \to \mathcal{L}(L^2(X))$:*

(i) *ρ is a $*$-homomorphism;*

(ii) *$\|\rho(e)\|, \|\rho(e^*)\|, \|\rho(v)\| \leqslant 1$ for every $e \in Q^1$ and $v \in Q^0$.*

Proof. Recall that an idempotent $\pi \in \mathcal{L}(L^2(X))$ is contractive if and only if it is self-adjoint. It follows that if two elements $s, t \in \mathcal{L}(L^2(X))$ satisfy $sts = s$ and $tst = t$ then $\|s\| \leqslant 1 \geqslant \|t\|$ if and only if $t = s^*$. The proposition is immediate from this observation applied to $\pi = \rho(v)$, $s = \rho(e)$ and $t = \rho(e^*)$ for all $v \in Q^0$ and $e \in Q^1$. ∎

7. THE L^p-OPERATOR ALGEBRA $\mathcal{O}^p(Q)$

DEFINITION 7.1. Let $p \in [1, \infty)$. An *L^p-operator algebra* is a Banach algebra B together with a norm on each M_nB that makes into a Banach algebra in such a way that there exists a nondegenerate representation $\rho : B \to \mathcal{L}(L^p(X))$ for some σ-finite measure space X, such that $M_n\rho : M_nB \to M_n\mathcal{L}(L^p(X)) = \mathcal{L}\Big(L^p\Big(\coprod_{i=1}^{n} X\Big)\Big)$ is isometric for each $1 \leqslant n < \infty$. We call B *standard* if X can be chosen to be a standard Borel space. A homomorphism $f : A \to B$ between L^p-operator algebras is *p-completely contractive* (respectively *isometric*) if M_nf is contractive (respectively isometric) for every n.

REMARK 7.2. By Proposition 1.25 of [16], any separable L^p-operator algebra admits an isometric representation on a separable, whence standard, L^p-space. Thus a separable L^p-operator algebra is automatically standard.

REMARK 7.3. If either $p \in [1, \infty)$ and B has a contractive unit or $p \neq 1$ and B has a contractive approximate unit, then the condition that the isometric representation in Definition 7.1 be nondegenerate can be dropped, by Theorem 4 of [5] and Theorem 3.19 of [11].

Let Q be a countable graph and let $p \in [1, \infty)$. A *spatial* p-seminorm on its Leavitt path algebra is a seminorm $h : L_Q \to \mathbb{R}_{\geqslant 0}$ such that there exist a σ-finite measure space X and spatial representation $\rho : L_Q \to \mathcal{L}(L^p(X))$ such that $h(a) = \|\rho(a)\|$ for all $a \in L_Q$. Observe that by Lemma 4.7, every spatial seminorm is induced by a nondegenerate spatial representation. Put

$$\|a\| = \sup\{h(a) : h \text{ is a spatial } p\text{-seminorm}\}. \tag{7.1}$$

By Proposition 4.11, $\| \cdot \|$ is a norm.

DEFINITION 7.4. Let Q be a countable graph and let $p \in [1, \infty)$. Write $\mathcal{O}^p(Q) = \overline{L}_Q^{\|\cdot\|}$ for the completion of L_Q with respect to the norm (7.1). By definition, $\mathcal{O}^p(Q)$ is a Banach algebra; we shall see in Proposition 7.5 below that furthermore, $\mathcal{O}^p(Q)$ is an L^p-operator algebra. We call $\mathcal{O}^p(Q)$ the *L^p-operator algebra of Q*.

Observe that the canonical map $L_Q \to \mathcal{O}^p(Q)$ is injective, by Proposition 4.11. Since Q is countable, there is a countable family $\{\rho_n\}$ of σ-finite nondegenerate spatial representations such that $\|\cdot\|$ is the norm associated to the L^p-direct sum

$$\rho = \bigoplus_n \rho_n : L_Q \to \mathcal{L}\Big(L^p\Big(\coprod_n X_n\Big)\Big) \tag{7.2}$$

which is a nondegenerate spatial representation. Hence $\mathcal{O}^p(Q)$ is isometrically isomorphic to the closure of $\rho(L_Q)$.

PROPOSITION 7.5. *Let Q be a countable graph. Then $\mathcal{O}^p(Q)$ has a canonical structure of L^p-operator algebra such that there is an isometric isomorphism $M_n\mathcal{O}^p(Q) \cong \mathcal{O}^p(M_nQ)$ for all $n \in [1,\infty)$.*

Proof. By Remark 5.1, every L^p-representation of M_nL_Q which is spatial in the sense of Section 5 factors uniquely through the canonical map $M_nL_Q \cong L_{M_nQ} \to \mathcal{O}^p(M_nQ)$. By Lemma 5.3 and the discussion above, for each n there is a spatial representation $\sigma_n : L_Q \to \mathcal{L}(L^p(Y_n))$ such that $\|\cdot\|_n := \|M_n\sigma_n(\cdot)\|$ is the supremum of all p-spatial norms on $L_{M_n(Q)}$. Let $Y = \coprod_n Y_n$ and let $\sigma = \bigoplus_n \sigma_n : L_Q \to \mathcal{L}(L^p(Y))$ be the L^p-direct sum. Then $\|M_n\sigma(\cdot)\| = \|\cdot\|_n$ for all $n \geqslant 1$, and we have isometric isomorphisms

$$\mathcal{O}^p(M_nQ) \cong \overline{\sigma(M_n(L_Q))} = \overline{M_n(\sigma(L_Q))} = M_n(\overline{\sigma(L_Q)}) \cong M_n\mathcal{O}^p(Q). \quad \blacksquare$$

THEOREM 7.6. *Let X be a σ-finite measure space with nonzero measure, $p \in [1,\infty)\setminus\{2\}$, Q a countable graph, $\widehat{\rho} : \mathcal{O}^p(Q) \to \mathcal{L}(L^p(X))$ a representation and $\rho : L_Q \to \mathcal{L}(L^p(X))$ the restriction of $\widehat{\rho}$. If ρ is nondegenerate, then the following conditions are equivalent:*

(i) *$\widehat{\rho}$ is contractive;*

(ii) *ρ is spatial.*

Further assume either that $p \neq 1$ or that Q^0 is finite. Then for any, possibly degenerate representation, condition (i) *is equivalent to*

(ii′) *there exist a σ-finite measure space Y, an isometry $\iota : L^p(Y) \to L^p(X)$, a norm 1 operator $\pi : L^p(X) \to L^p(Y)$ such that $\pi\iota = 1$, and a spatial representation $\rho' : L_Q \to \mathcal{L}(L^p(Y))$, such that for $f : \mathcal{L}(L^p(Y)) \to \mathcal{L}(L^p(X))$, $f(T) = \iota T\pi$, the following diagram commutes*

$$\begin{tikzcd} L_Q \arrow[rr, "\rho"] \arrow[dr, "\rho'"'] & & \mathcal{L}(L^p(X)) \\ & \mathcal{L}(L^p(Y)) \arrow[ur, "f"'] & \end{tikzcd}$$

Proof. First assume that ρ is nondegenerate. If ρ is spatial then it induces a contractive homomorphism $\widehat{\rho}' : \mathcal{O}^p(Q) \to \mathcal{L}(L^p(X))$ which agrees with ρ on L_Q; since $\widehat{\rho}$ does the same, we must have $\widehat{\rho} = \widehat{\rho}'$. This proves that (ii) $\Rightarrow$ (i). Conversely if (i) holds, then ρ is spatial by Theorem 6.2. Next observe that if ρ is any (possibly degenerate) representation that satisfies (ii′) then ρ' factors through a contractive representation $\widehat{\rho'} : \mathcal{O}^p(Q) \to \mathcal{L}(L^p(Y))$. Thus $f\widehat{\rho'} = \widehat{\rho}$ is contractive. Assume conversely that (i) holds. Let $E \subset L^p(X)$ be the closure of $\rho(L_Q)(L^p(X))$. If Q^0 is finite then $\mathcal{O}^p(Q)$ is unital with unit $1 = \sum_{v \in Q^0} v$ which has norm 1; thus E is the image of the contractive idempotent $\rho(1)$. For general Q, the family $\left\{ \sum_{v \in F} v \right\}$ indexed by the finite subsets of Q^0 is a contractive approximate unit of $\mathcal{O}^p(Q)$; hence if $p \neq 1$, then again E is the image of a contractive idempotent, by Corollary 3.13 of [11]. Hence under either hypothesis, by Theorem 4 of [5] there are a contractive projection $\pi' : L^p(X) \to E$ and an isometric isomorphism $h : E \to L^p(Y)$ for some standard Borel space Y. Put $\pi = h\pi'$, let ι be h^{-1} followed by the inclusion $E \subset L^p(X)$, and set $\rho'(a) = h\rho(a)h^{-1}$. It is clear that the diagram commutes; moreover, ρ' is spatial by Theorem 6.2. ∎

REMARK 7.7. The argument of the proof that (ii) $\Rightarrow$ (i) in Theorem 7.6 still works for $p = 2$. The proof of the converse uses Theorem 6.2, which in turn relies on the Banach–Lamperti theorem. Since the latter does not hold for $p = 2$, the proof above does not apply.

Recall that the circle group $\mathbb{S}^1$ acts on L_Q via the *gauge action*, which associates to each $z \in \mathbb{S}^1$ an automorphism

$$\gamma_z : L_Q \to L_Q. \tag{7.3}$$

This action is characterized by the fact that $\gamma_z(a) = z^n a$ whenever $a \in (L_Q)_n$.

LEMMA 7.8. *Let Q be a countable graph, $p \in [1, \infty)$ and $z \in \mathbb{S}^1$. Then the map* (7.3) *extends to an isometric isomorphism $\widehat{\gamma}_z : \mathcal{O}^p(Q) \to \mathcal{O}^p(Q)$. Moreover, the map*

$$\widehat{\gamma}(a) : \mathbb{S}^1 \to \mathcal{O}^p(Q), \quad w \mapsto \widehat{\gamma}_w(a)$$

is continuous for each fixed $a \in \mathcal{O}^p(Q)$.

Proof. Let $\iota : L_Q \to \mathcal{O}^p(Q)$ be the inclusion and let $\rho : \mathcal{O}^p(Q) \to \mathcal{L}(L^p(X))$ be an isometric embedding. If $z \in \mathbb{S}^1$, then $\rho\iota\gamma_z$ is a spatial representation by Example 4.2, hence it gives rise to a contractive homomorphism $\widehat{\gamma}_z : \mathcal{O}^p(Q) \to \mathcal{O}^p(Q)$. Because $\mathbb{S}^1 \to \operatorname{Aut}(L_Q)$, $z \mapsto \gamma_z$ is a group homomorphism, $\widehat{\gamma}_z$ is an isometric isomorphism with inverse $\widehat{\gamma}_{\bar{z}}$. This proves the first assertion of the lemma. The second assertion follows as in the proof of Proposition 2.1 in [19]. ∎

As pointed out above, the reverse of a 2-spatial partial isometry is just its adjoint. It follows from this that any 2-spatial representation of L_Q is a $*$-representa–tion. Hence $\mathcal{O}^2(Q)$ is a C^*-algebra and we have a canonical $*$-homomorphism

$$\pi_Q : C^*(Q) \to \mathcal{O}^2(Q). \tag{7.4}$$

We shall show that π_Q is an isomorphism. For the proof we need the desingularization of a singular graph whose definition we shall presently recall. Let Q be a countable, singular graph. Recall from Section 5 of [2] that the *desingularization* of Q is a nonsingular graph $Q_\mathfrak{d}$ obtained from Q as follows. For each sink v, add an infinite tail

$$v = v_0 \xrightarrow{f_1} v_1 \xrightarrow{f_2} v_2 \xrightarrow{f_3} \cdots . \tag{7.5}$$

For each infinite emitter v, number the elements of $s^{-1}(v) = \{e_1, e_2, \dots\}$ and add a tail (7.5) and an arrow $g_i : v_i \to r(e_i)$ $(1 \leqslant i)$. There is a canonical $*$-monomorphism ([2], Proposition 5.5)

$$\phi_\mathfrak{d} : L_Q \to L_{Q_\mathfrak{d}}, \tag{7.6}$$
$$\phi_\mathfrak{d}(v) = v, \quad \phi_\mathfrak{d}(e) = \begin{cases} e & s(e) \in \operatorname{reg}(Q), \\ f_1 \cdots f_i g_i & e = e_i. \end{cases}$$

PROPOSITION 7.9. *Let Q be a countable graph. Then the map π_Q in* (7.4) *is a C^*-algebra isomorphism.*

Proof. The image of π_Q is a closed subalgebra containing the image L_Q, which is dense, so the map is surjective. If moreover, Q is row-finite, then π_Q is injective by Lemma 7.8 and the gauge invariant uniqueness theorem ([19], Theorem 2.2). Hence for general Q, π_{Q_δ} is an isomorphism. Thus the top row of the following commutative diagram, whose columns are induced by (7.6), is an isomorphism:

$$\begin{tikzcd} C^*(Q_\mathfrak{d}) \arrow[r, "\pi_{Q_\mathfrak{d}}"] & \mathcal{O}^2(Q_\mathfrak{d}) \\ C^*(Q) \arrow[u] \arrow[r, "\pi_Q"'] & \mathcal{O}^2(Q). \arrow[u] \end{tikzcd} \tag{7.7}$$

Moreover, the first vertical map is injective by Theorem 2.11 of [7]. It follows that the bottom row of (7.7) is injective. This concludes the proof. ∎

REMARK 7.10. Let $\widehat{\rho} : \mathcal{O}^2(Q) \to \mathcal{L}(L^2(X))$ be a representation and ρ its restriction to L_Q. It follows from Propositions 6.4 and 7.9 and the universal property of $C^*(Q)$, that $\widehat{\rho}$ is contractive if and only if ρ is a $*$-homomorphism if and only of $\widehat{\rho}$ is a $*$-homomorphism.

Let $\mathcal{S}(Q)$ be the semigroup of (3.1) and let $p \in (1, \infty)$. Let $F^p_{\mathrm{tight}}(\mathcal{S}(Q))$ be the standard L^p-operator algebra of Definition 6.7 in [9]; $F^p_{\mathrm{tight}}(\mathcal{S}(Q))$ is universal for tight L^p-representations of $\mathcal{S}(Q)$ which are spatial in the sense of Definition 4.6 in [9] and take values in L^p-spaces of standard Borel spaces.

COROLLARY 7.11. *Let $\mathcal{S}(Q)$ be the semigroup generated by Q. Then there is a C^*-algebra isomorphism $F^2_{\mathrm{tight}}(\mathcal{S}(Q)) \cong \mathcal{O}^2(Q)$.*

Proof. A partial isometry $s : L^2(X) \to L^2(X)$ of a standard L^2-space is spatial in the sense of Definition 4.6 in [9] if and only if $s = s^*ss^*$. Hence $F^2_{\mathrm{tight}}(\mathcal{S}(Q))$ is universal for tight $*$-representations of $\mathcal{S}(Q)$ on Hilbert space, which by Lemma 3.1 are the same as the $*$-representations of L_Q. Since $C^*(Q)$ has the same universal property, we have $C^*(Q) \cong F^2_{\mathrm{tight}}(Q)$. Now apply Proposition 7.9. ∎

As pointed out above, the spatiality notion of [9] agrees with ours for $p \neq 2$. Hence by Lemma 3.1 and the universal property of $\mathcal{O}^p(Q)$, for $p \in (1, \infty) \setminus \{2\}$, we have a canonical contractive homomorphism

$$\mathcal{O}^p(Q) \to F^p_{\mathrm{tight}}(S(Q)). \tag{7.8}$$

Moreover, since the p-operator space structure on $F^p_{\mathrm{tight}}(S(Q))$ is defined in [9] so that $M_n(F^p_{\mathrm{tight}}(S(Q)) = F^p_{\mathrm{tight}}(S(M_nQ))$, the induced map $M_n(\mathcal{O}^p(Q)) \to M_n(F^p_{\mathrm{tight}}(S(Q)))$ is also contractive, by Proposition 7.5. In other words (7.8) is p-completely contractive.

PROPOSITION 7.12. *Let $p \in [1, \infty) \setminus \{2\}$. Then the map* (7.8) *is a p-completely isometric isomorphism.*

Proof. It suffices to show that $F^p_{\mathrm{tight}}(\mathcal{S}(Q))$ is universal for all σ-finite spatial representations. Let X be a σ-finite measure space and let $\rho : L_Q \to \mathcal{L}(L^p(X))$ be a spatial representation; we have to show that ρ factors through $L_Q \to F^p_{\mathrm{tight}}(\mathcal{S}(Q))$. An argument similar to that of the proof of Theorem 7.6 shows that ρ factors through a nondegenerate representation $\rho' : L_Q \to \mathcal{L}(L^p(Y))$ with Y standard Borel. Thus ρ factors through $L_Q \to F^p_{\mathrm{tight}}(\mathcal{S}(Q))$, as required. ∎

8. SPATIAL SEMINORMS, DESINGULARIZATION, AND SOURCE REMOVAL

If Q is a graph such that $\mathrm{sour}(Q) \neq \emptyset$, we may embed it in the source-free graph $Q_{\mathfrak{r}}$ obtained by adding an infinite head

$$w = w_0 \overset{f_1}{\leftarrow} w_1 \overset{f_2}{\leftarrow} w_2 \overset{f_3}{\leftarrow} \cdots \tag{8.1}$$

at each $w \in \operatorname{sour}(Q)$. The obvious inclusion $Q \subset Q_{\mathfrak{r}}$ induces an algebra mono–morphism

$$\phi_{\mathfrak{r}} : L_Q \to L_{Q_{\mathfrak{r}}}. \tag{8.2}$$

Recall from Section 7 that if Q is a singular graph, we write $Q_{\mathfrak{d}}$ for its desingularization.

PROPOSITION 8.1. *Let Q be a countable graph, $p \in [1, \infty)$, and $\# \in \{\mathfrak{r}, \mathfrak{d}\}$. Then for every nonzero spatial representation $\rho : L_Q \to \mathcal{L}(L^p(X))$ there exist a spatial representation $\rho_{\#} : L_{Q_{\#}} \to \mathcal{L}(L^p(Y))$ and a spatial isometry $s : L^p(X) \to L^p(Y)$ with reverse t, with both Y and s depending on ρ and $\#$, such that for the map $\sigma : \mathcal{L}(L^p(X)) \to \mathcal{L}(L^p(Y))$, $\sigma(A) = sAt$, the following diagram commutes:*

$$\begin{tikzcd} L_Q \arrow[r, "\rho"] \arrow[d, "\phi_{\#}"'] & \mathcal{L}(L^p(X)) \arrow[d, "\sigma"] \\ L_{Q_{\#}} \arrow[r, "\rho_{\#}"'] & \mathcal{L}(L^p(Y)). \end{tikzcd} \tag{8.3}$$

Proof. We begin by the case $\# = \mathfrak{r}$. If $\alpha \in \mathcal{P}(Q)$, we write X_α for the support of the spatial projection $\rho(\alpha\alpha^*)$. Regard $\mathbb{N}$ as a measure space with counting measure; set $Y := X \sqcup \coprod_{w \in \operatorname{sour}(Q)} (X_w \times \mathbb{N})$. Let s and t be the inverse isometries induced by the inclusion $X \subset Y$. The canonical identification $X_w \to X_w \times \{n\}$ induces an isometric spatial isomorphism $\tau_n : L^p(X_w) \to L^p(X_w \times \{n\})$. Extend ρ along $\phi_{\mathfrak{r}}$ to a map $\rho_{\mathfrak{r}} : L_{Q_{\mathfrak{r}}} \to \mathcal{L}(L^p(Y))$ by setting $\rho_{\mathfrak{r}}(w_n) := \operatorname{Id}_{L^p(X_w \times \{n\})}$, $\rho_{\mathfrak{r}}(f_n) := \tau_n \tau_{n-1}^{-1}$, $\rho_{\mathfrak{r}}(f_n^*) = \tau_{n-1}\tau_n^{-1}$. One checks that $\rho_{\mathfrak{r}}$ is well-defined and makes (8.3) commute.

Next we consider the case $\# = \mathfrak{d}$. The measure space Y will be a coproduct

$$Y = X \sqcup \coprod_{v \in \operatorname{sing}(Q), n \geqslant 1} Y_{v_n};$$

the isometries s, t will be those induced by the inclusion $X \subset Y$. For $v \in \operatorname{sink}(Q)$, we set $Y_{v_n} = X_v \times \{n\}$, $\tau_n : X_v \xrightarrow{\cong} X_v \times \{n\}$ the obvious bijection, and put $\rho_{\mathfrak{d}}(f_n) = \tau_{n-1}\tau_n^{-1}$. If $v \in \inf(Q)$ and $X'_v = X_v \setminus \coprod_{i=1}^{\infty} X_{e_i}$, we set

$$Y_{v_n} = X'_v \sqcup \coprod_{i \geqslant n} X_{e_i}$$

and let $\rho_{\mathfrak{d}}(f_n)$ be induced by the inclusion $Y_{v_n} \subset Y_{v_{n-1}}$ and $\rho_{\mathfrak{d}}(g_n)$ by the composite of $\rho(e_n) : L^p(X_{r(e_n)}) \to L^p(X_{e_n})$ followed by the inclusion $L^p(X_{e_n}) \subset L^p(Y_{v_n})$. One checks that this prescription defines a spatial representation $\rho_{\mathfrak{d}} : L_{Q_{\mathfrak{d}}} \to \mathcal{L}(L^p(Y))$ that makes (8.3) commute. ∎

COROLLARY 8.2. *The canonical homomorphisms* (7.6) *and* (8.2) *induce isometric homomorphisms $\mathcal{O}^p(Q) \to \mathcal{O}^p(Q_{\mathfrak{d}})$ and $\mathcal{O}^p(Q) \to \mathcal{O}^p(Q_{\mathfrak{r}})$.*

REMARK 8.3. One may wonder whether other standard graph transformations also induce isometric embeddings of the corresponding L^p-operator algebras. We show in Remark 9.7 that such is indeed the case of the standard graph transformations described in Chapter 6, Section 3 of [1], at least in the simple case.

9. A UNIQUENESS THEOREM

The purpose of this section is to prove the following theorem.

THEOREM 9.1. *Let Q be a countable graph, $p \in [1, \infty)$ and $\rho : L_Q \to \mathcal{L}(L^p(X))$ a nonzero spatial representation. If L_Q is simple, then the natural map is an isometric isomorphism*

$$\mathcal{O}^p(Q) \xrightarrow{\cong} \overline{\rho(L_Q)}.$$

The proof of Theorem 9.1 will be given at the end of the section, after a series of propositions, definitions, and lemmas, which adapt and extend those in Section 8 of [15].

DEFINITION 9.2. Let Q be a countable row-finite graph, $p \in [1, \infty)$, $X = (X, \mathcal{B}, \mu)$ a σ-finite measure space, and $\rho : L_Q \to \mathcal{L}(L^p(X))$ a representation.

(i) We say that ρ is *free* if there is a partition $X = \bigsqcup_{m \in \mathbb{Z}} E_m$, $E_m \in \mathcal{B}$, such that for all $m \in \mathbb{Z}$, $e \in Q^1$, we have

$$\rho(e)(L^p(E_m)) \subset L^p(E_{m+1}) \quad \text{and} \quad \rho(e^*)(L^p(E_m)) \subset L^p(E_{m-1}). \tag{9.1}$$

(ii) We say that ρ is *approximately free* if for every $N \in \mathbb{N}$, there are $n \geqslant N$ and a partition $X = \bigsqcup_{m=0}^{n-1} E_m$, $E_m \in \mathcal{B}$, such that for $m = 0, \dots, n-1$ and all $e \in Q^1$ (9.1) holds if we set $E_n = E_0$ and $E_{-1} = E_{n-1}$.

LEMMA 9.3. *Let $p \geqslant 1$, $X = (X, \mathcal{B}, \mu)$ and $Y = (Y, \mathcal{C}, \nu)$ be σ-finite measure spaces, Q a row-finite graph, $\rho : L_Q \to \mathcal{L}(L^p(X))$ a representation, and $u \in \mathcal{L}(L^p(Y))$ an invertible operator. Then, there is a unique representation $\rho^u : L_Q \to \mathcal{L}(L^p(X \times Y))$ such that, for all $e \in Q^1$, we have $\rho^u(e) = \rho(e) \otimes u$ and $\rho^u(e^*) = \rho(e^*) \otimes u^{-1}$.*

Moreover, ρ^u has the following properties:

(i) *if $\alpha \in L_Q$ is homogeneous of degree k with respect to the $\mathbb{Z}$-grading of* (2.3), *then $\rho^u(\alpha) = \rho(\alpha) \otimes u^k$;*

(ii) *if u is isometric, $p \neq 2$ and ρ is spatial, then ρ^u is spatial;*

(iii) *if there is a partition $Y = \coprod_{m \in \mathbb{Z}} F_m$, $F_m \in \mathcal{C}$, such that $u(L^p(F_m)) = L^p(F_{m+1})$ for all $m \in \mathbb{Z}$, then ρ^u is free in the sense of Definition* 9.2.

The proof is analogous to that of Lemma 8.2 in [15] using Lemma 2.3 instead of Lemmas 2.18, 2.19 and 2.20 in [15].

PROPOSITION 9.4. *Let p, X, Q, and ρ be as in Lemma 9.3. Let $u \in \mathcal{L}(\ell^p(\mathbb{Z}))$ be the shift operator, $(u(x))(m) := x(m-1)$ ($x \in \ell^p(\mathbb{Z})$). Let ρ^u be as in Lemma 9.3. Then, for all $a \in L_Q$, we have $\|\rho^u(a)\| \geqslant \|\rho(a)\|$.*

The proof is analogous to that of Proposition 8.3 in [15], using Lemma 9.3 instead of Lemma 8.2 in [15].

LEMMA 9.5. *Let Q be a nonsingular countable graph such that L_Q is simple. Let $X = (X, \mathcal{B}, \mu)$ be a σ-finite measure space. Let $\{X_v\}_{v\in Q^0} \subset \mathcal{B}$ be a family of sets of nonzero measure, $\{X_e\}_{e\in Q^1} \subset \mathcal{B}$ a disjoint family such that $X = \coprod_{v\in Q^0} X_v$ and $X_v = \coprod_{\{e:s(e)=v\}} X_e$ (for all $v \in Q^0$), and*

$$S_e : (X_{r(e)}, \mathcal{B}_{|X_{r(e)}}, \mu_{|X_{r(e)}}) \to (X_e, \mathcal{B}_{|X_e}, \mu_{|X_e}) \quad (e \in Q^1)$$

a bijective measurable set transformation. If $\alpha = \alpha_1 \cdots \alpha_m$ is a path, write $S_\alpha = S_{\alpha_1} \circ \cdots \circ S_{\alpha_m}$. Then, for each $n \geqslant 0$ and each $v \in Q^0$ there is a set $E_v \in \mathcal{B}_{|X_v}$ such that $\mu(E_v) \neq 0$, and such that the following family is disjoint:

$$\{S_\alpha(E_v) : r(\alpha) = v, |\alpha| \leqslant n\}.$$

Proof. We shall use the fact that, because L_Q is simple, Q is cofinal, i.e. for every $v \in Q^0$ and each cycle c there is a path starting at v and ending at some vertex in c (see Theorem 2.9.7 of [1]). Let $v \in Q^0$. If $v \in Q^0$ is not in any cycle, we set $E_v = X_v$; observe that $\mu(E_v) \neq 0$ by hypothesis. Because v is not in any cycle, any two distinct paths ending in v are incomparable, and so E_v satisfies the disjointness condition of the lemma. Next assume that v belongs to a cycle. Let $\alpha := \alpha_v$ be a cycle based at v and let β be a closed path with $s(\beta) = v$ that agrees with α up to an exit, goes out following the exit, returns to c (which is possible by cofinality) and follows it till it gets back to v. Consider the infinite path

$$\gamma := \alpha\beta\alpha\alpha\beta\beta\alpha\alpha\alpha\beta\beta\beta\cdots .$$

It is long, but straightforward to check that

$$\nexists\, \theta \in \mathcal{P}(Q) \quad \text{such that } \theta\theta \geqslant \gamma. \tag{9.2}$$

Let $n \in \mathbb{N}$ and $v \in E^0$. For $i \geqslant 1$, let γ_i be the i-th edge of γ. Put

$$\mathcal{B} \ni E_v := X_{\gamma_1\cdots\gamma_{2n}}.$$

Then $\mu(E_v) \neq 0$ because $\mu(X_w) \neq 0$ for all $w \in Q^0$. Let η and τ be different paths such that $r(\eta) = r(\tau) = v$, of lengths k and l respectively ($k \leqslant l \leqslant n$). We have to check that $S_\eta(E_v)$ and $S_\tau(E_v)$ are disjoint. If $k = 0$ this is clear from Remark 4.5, because $S_\tau(E_v) = X_{\tau\gamma_1\cdots\gamma_{2n}}$ and the paths $\tau\gamma_1\cdots\gamma_{2n}$ and $\gamma_1\cdots\gamma_{2n}$ are incomparable, by (9.2). So assume that $0 < k \leqslant l$; if η and τ are incomparable, we are done. Otherwise, we must have $\eta \geqslant \tau$; say $\tau = \eta\delta$. Hence $S_\eta(E_v) \cap S_\tau(E_v) = S_\eta(E_v \cap S_\delta(E_v))$ has measure zero because $E_v \cap S_\delta(E_v)$ does. ∎

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $\tau_1, \dots, \tau_n \in \mathcal{L}(L^p(X))$ spatial partial isometries with reverses $\sigma_1, \dots, \sigma_n$. Call $\tau_1, \dots, \tau_n$ *orthogonal* if $\tau_j\sigma_i = \sigma_i\tau_j = 0$ whenever $i \neq j$.

LEMMA 9.6. *Let X be a σ-finite measure space, $p \in [1, \infty)$, $\tau_1, \dots, \tau_n \in \mathcal{L}(L^p(X))$ orthogonal spatial partial isometries, $\lambda \in \mathbb{C}^n$, and $\tau_\lambda = \sum_{i=1}^{n} \lambda_i\tau_i$. Then $\|\tau_\lambda\| = \|\lambda\|_\infty$.*

The proof is straightforward.

REMARK 9.7. Let Q and Q' be countable graphs and let $\phi : L_Q \to L_{Q'}$ be a $*$-homomorphism with the property that for any $x \in Q^0 \coprod Q^1$, $\phi(x)$ decomposes as a sum $\sum x_i$ of elements of $\mathcal{S}(Q')$ such that $x_i x_j^* = x_j^* x_i = 0$ whenever $i \neq j$. Then for every spatial representation $\rho' : L_{Q'} \to \mathcal{L}(L^p(X))$, $\rho'\phi$ is spatial, by Lemma 9.6. Hence ϕ induces a contractive homomorphism $\widehat{\phi} : \mathcal{O}^p(Q) \to \mathcal{O}^p(Q')$, which, by Theorem 9.1 is isometric whenever L_Q is simple. This applies, in particular, when Q is finite, L_Q is simple, and Q' is obtained from Q by any of the three expansive standard graph transformations described in Definitions 6.3.1, 6.3.17, 6.3.20 and 6.3.23 of [1], and ϕ is the canonical homomorphism.

PROPOSITION 9.8. *Let Q be a nonsingular countable graph without sources. Let $p \in [1, \infty) \setminus \{2\}$, and let X and Y be measure spaces and $\rho : L_Q \to \mathcal{L}(L^p(X))$ and $\phi : L_Q \to \mathcal{L}(L^p(Y))$ spatial representations. Assume that L_Q is simple and that ρ is approximately free. Then*

$$\|\rho(a)\| \leqslant \|\phi(a)\| \quad (a \in L_Q).$$

Proof. This proposition generalizes Proposition 8.6 of [15]; we shall adapt the argument therein using Lemma 9.5 instead of Lemma 8.5 in [15]. Let

$$X' = \coprod_{v \in Q^0} X_v;$$

observe that the corestriction ρ' of ρ to $\mathcal{L}(L^p(X'))$ is approximately free. Hence by Lemma 4.7 we may assume that ρ and ϕ are both nondegenerate. For each $\alpha \in \mathcal{P} = \mathcal{P}(Q)$, let R_α and S_α be the bijective measurable set transformations $X_{r(\alpha)} \to X_\alpha$, $Y_{r(\alpha)} \to Y_\alpha$ associated to $\rho(\alpha)$ and $\phi(\alpha)$, as in Remark 4.5. We have to show that if $a \in L_Q$ is such that $\|\rho(a)\| = 1$, then $\|\phi(a)\| \geqslant 1$. By Lemma 2.2, there are $N_0 \geqslant 0$, a finite set $F_0 \subset \mathcal{P}$ and a finitely supported function $\lambda^0 : F_0 \times \mathcal{P}_{N_0} \to \mathbb{C}$ such that

$$a = \sum_{\alpha \in F_0} \sum_{\beta \in \mathcal{P}_{N_0}} \lambda^0_{\alpha,\beta} \alpha\beta^*.$$

Because $\mathrm{sour}(Q) = \emptyset$ by hypothesis, for each $v \in s(F_0)$ we may choose a path $\tau_v \in \mathcal{P}_{N_0}$ with $r(\tau_v) = v$. Put

$$x = \sum_{v \in s(F_0)} \tau_v, \quad b = xa.$$

Because every path in the set $\tau_{F_0} = \{\tau_v : v \in s(F_0)\}$ is of length N_0, any two of them are incomparable. Hence by Remark 4.5, the elements of $\rho(\tau_{F_0})$ are orthogonal spatial partial isometries. Therefore $\|\rho(x)\| = 1$, by Lemma 9.6; similarly, $\|\rho(x^*)\| = 1$. Hence $\|\rho(b)\| = \|\rho(a)\| = 1$ and by the same argument, $\|\phi(b)\| = \|\phi(a)\|$. Therefore it suffices to show that for every $\varepsilon > 0$,

$$\|\phi(b)\| > 1 - \varepsilon. \tag{9.3}$$

For $\beta \in \mathcal{P}_{N_0}$ and $\alpha \in F_0$, let

$$\lambda_{\tau_{s(\alpha)}\alpha,\beta} = \lambda^0_{\alpha,\beta}.$$

Put $F = \{\tau_{s(\alpha)}\alpha : \alpha \in F_0\}$; the map $F_0 \to F$, $\alpha \mapsto \tau_{s(\alpha)}\alpha$ is clearly surjective. Moreover, because $\tau_v \in \mathcal{P}_{N_0}$ for all $v \in s(F_0)$, it is also injective. Using this in the third step, we obtain

$$\begin{aligned} b &= \Big(\sum_{v \in s(F_0)} \tau_v\Big)\Big(\sum_{\alpha \in F_0}\sum_{\beta \in \mathcal{P}_{N_0}} \lambda^0_{\alpha,\beta}\alpha\beta^*\Big) \\ &= \sum_{v \in s(F_0)}\ \sum_{s(\alpha)=v,\, \alpha \in F_0}\ \sum_{\beta \in \mathcal{P}_{N_0}} \lambda_{\tau_v\alpha,\beta}\alpha\beta^* \\ &= \sum_{\alpha \in F}\sum_{\beta \in \mathcal{P}_{N_0}} \lambda_{\alpha,\beta}\alpha\beta^*. \end{aligned}$$

Let $N_1 = \max\{|\alpha| : \alpha \in F_0\}$; then $N_0 \leqslant |\alpha| \leqslant N_0 + N_1$ for all $\alpha \in F$. If $N_0 = N_1 = 0$, then b is a linear combination of vertices, $b = \sum_v \lambda_v v$, whence by Lemma 9.6 we have

$$\|\phi(b)\| = \|\lambda\|_\infty = \|\rho(b)\| = 1.$$

Hence (9.3) holds in this case. So we may assume $N_0 + N_1 > 0$, and take $\mathrm{j} > (N_0 + N_1)(2/\varepsilon)^p$. By our hypothesis on ρ, there are $N \geqslant \mathrm{j}(N_0 + N_1)$ and a partition

$$X = \coprod_{n=0}^{N-1} D_n \tag{9.4}$$

such that for the remainder $\overline{n}$ of n modulo N, we have $\rho(e)(L^p(D_{\overline{n}})) \subset L^p(D_{\overline{n+1}})$ and $\rho(e^*)(L^p(D_{\overline{n}})) \subset L^p(D_{\overline{n-1}})$. By the argument of pages 54–55 in [15], after cyclic permutation of the D_n if necessary, there exists

$$\xi = \sum_{m=0}^{N-1} \xi_m \in \bigoplus_{m=0}^{N-1} L^p(D_m) = L^p(X)$$

with $\xi_m = 0$ for $m \leqslant N_0 - 1$ and for $m \geqslant N - N_1$, and such that $\|\xi\| \leqslant 1$ and $\|\rho(b)\xi\| > 1 - \varepsilon$. For each $\gamma \in \mathcal{P}$, put

$$D_\gamma = R_\gamma(X_{r(\gamma)} \cap D_0) = D_{|\gamma|} \cap X_\gamma.$$

Because Q is nonsingular by hypothesis, and because we have assumed that ρ is nondegenerate, for each $l \geqslant 0$ we have a decomposition (4.6). It follows from this that

$$D_m = \coprod_{|\gamma|=m} D_\gamma \quad \text{for all } m \in [0, N-1]. \tag{9.5}$$

Let $W = \mathcal{P}_{\leqslant N-1} = \coprod_{0\leqslant l \leqslant N-1} \mathcal{P}_l$. It follows from (9.4) and (9.5) that $X = \coprod_{\gamma\in W} D_\gamma$. Hence we can write any $\eta \in L^p(X)$ as a sum $\eta = \sum_{\gamma\in W} \eta_\gamma$ with $\eta_\gamma \in L^p(D_\gamma)$. Next, by Lemma 9.5, for each $v \in Q^0$ there is a measurable set $E_v \subset Y$ of nonzero measure such that the family $\{S_\gamma(E_{r(\gamma)}) : \gamma \in W\}$ is disjoint. Choose a norm-one element $\zeta_v \in L^p(E_v)$ for each $v \in Q^0$. Let

$$\begin{aligned} &u : L^p(X) \to L^p(X \times Y), \\ &u\eta = \sum_{\gamma\in W} \rho(\gamma)\eta_\gamma \otimes \phi(\gamma)\zeta_{r(\gamma)}. \end{aligned}$$

One checks, as in the proof of Proposition 8.6 in [15], that u is an isometry. Let $\psi = 1 \otimes \phi : L_Q \to \mathcal{L}(L^p(X \times Y))$, be as in Lemma 9.3. Observe that

$$\|\psi(b)\| = \|\phi(b)\|. \tag{9.6}$$

A calculation similar to that of the proof of Proposition 8.6 in [15] shows that for ξ as above,

$$u\rho(b)\xi = \psi(b)u\xi. \tag{9.7}$$

It follows from (9.6) and (9.7) that (9.3) holds. This completes the proof. ∎

Proof of Theorem 9.1. Because L_Q is simple by hypothesis, the C^*-algebra $C^*(Q)$ is simple; thus every nonzero $*$-representation $L_Q \to \mathcal{L}(L^2(X))$ induces the same norm. But by Remark 4.4 every spatial representation is a $*$-representation, so the theorem is clear for $p = 2$. Assume $p \neq 2$. By Proposition 8.1 and Corollary 8.2, we may assume that Q is nonsingular and has no sources. By Lemma 9.3 and Propositions 9.4 and 9.8, every spatial seminorm is associated to a free spatial representation. Applying Proposition 9.8 again, we get that any two nonzero approximately free spatial representations induce the same seminorm. ∎

10. A SIMPLICITY THEOREM

THEOREM 10.1. *Let $p \in [1, \infty)$. The following are equivalent for a countable graph Q:*

(i) *L_Q is simple;*

(ii) *every spatial nonzero L^p-representation of L_Q is injective;*

(ii′) *every spatial nonzero representation $L_Q \to \mathcal{L}(\ell^p(\mathbb{N}))$ is injective;*

(ii'') *every nondegenerate spatial nonzero representation $L_Q \to \mathcal{L}(\ell^p(\mathbb{N}))$ is injective;*

(iii) *every nondegenerate, contractive, nonzero L^p-representation of $\mathcal{O}^p(Q)$ is injective;*

(iii′) *every nondegenerate, contractive, nonzero representation $\mathcal{O}^p(Q) \to \mathcal{L}(\ell^p(\mathbb{N}))$ is injective.*

If in addition we assume either that $p \neq 1$ or that Q^0 is finite, then the above conditions are also equivalent to the following:

(iv) *every nonzero contractive homomorphism from $\mathcal{O}^p(Q)$ to another L^p-operator algebra is injective.*

Proof. If either $p \neq 1$ or Q^0 is finite, then (iii) and (iv) are equivalent, by Theorem 7.6 and Remark 7.10. For $p \in [1, \infty) \setminus \{2\}$ the implication (i) $\Rightarrow$ (iii) follows from Theorems 7.6 and 9.1. It is well-known that $C^*(Q)$ is simple if and only if L_Q is; using this and Remark 7.10 we obtain (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) for $p = 2$. It follows from Lemma 4.7 and Theorem 7.6 that (iii) $\Rightarrow$ (ii) for $p \neq 2$. Similarly, (iii′) $\Rightarrow$ (ii"). It is clear that (ii) $\Rightarrow$ (ii′) $\Rightarrow$ (ii") and that (iii) $\Rightarrow$ (iii′).

It remains to show that (ii") $\Rightarrow$ (i). By Theorem 2.9.1 of [1], L_Q is simple if and only if Q^0 is the only nonempty hereditary and saturated subset of vertices, and every cycle in Q has an exit. We shall show that if any of these two conditions does not hold, then (ii") does not hold either.

So suppose there is a proper hereditary and saturated subset $H \subset Q^0$. Let Q/H be the quotient graph described in Definition 2.4.11 of [1]. Then the natural map $\pi : L_Q \to L_{Q/H}$ is a nonzero surjection with nonzero kernel the ideal generated by H. Hence if ρ is an injective nondegenerate spatial representation $L_{Q/H} \to \mathcal{L}(\ell^p(\mathbb{N}))$ (which exists by Proposition 4.11) then $\rho\pi$ is a nondegenerate nonzero spatial representation $L_Q \to \mathcal{L}(\ell^p(\mathbb{N}))$ which is not injective.

So assume that Q^0 is the only nonempty saturated and hereditary set of vertices, or equivalently, by Lemma 2.9.6 of [1], that Q is cofinal in the sense of Definitions 2.9.4 in [1] and that it has a cycle c without exits. Cofinality implies that c is the only cycle of Q modulo cycle rotation (by Lemma 2.7.1 and Theorem 2.7.3 of [1]), and that $\mathrm{sink}(Q) = \emptyset$ (by Lemma 2.9.5 of [1]). Moreover, Q cannot have any infinite emitters. For this suppose $v \in \inf(Q)$; then v cannot be in any cycle, since any cycle containing v would have exits. In particular if $e \in Q^1$ and $s(e) = v$ then $r(e) \neq v$ and by Lemma 2.0.7 of [1] the hereditary and saturated closure of $\{r(e)\}$ does not contain v, a contradiction. Hence $Q = \mathrm{reg}(Q)$, and therefore the space $\mathfrak{X}$ of (4.7) consists of the infinite paths of Q. If $s(c) = w$, then any such path is of the form αc^∞ for some finite path $\alpha \in \mathcal{P}$ with $r(\alpha) = w$. In particular $\mathfrak{X}$ is countable and $\mathfrak{X}_w = \mathfrak{X}_{c^n} = \{c^\infty\}$ for all $n \geqslant 1$. Hence for the counting measure μ on $\mathfrak{X}$, there is a spatial isometric isomorphism $L^p(\mathfrak{X}, \mu) \cong \ell^p(\mathbb{N})$, and the nondegenerate representation ρ_μ of Example 4.8 maps $c - c^2$ to zero, so it is not injective. This concludes the proof. ∎

REMARK 10.2. By [10], an L^p-operator algebra may admit Banach algebra quotients which are not again L^p-operator algebras. Thus Phillips' theorem that the L^p-Cuntz algebra $\mathcal{O}_d^p$ is simple as a Banach algebra for $d \in [2,\infty)$ ([17], Theorem 5.14) does not follow from Theorem 10.1 above. We expect $\mathcal{O}^p(Q)$ to be simple as a Banach algebra whenever L_Q is simple. We intend to investigate this in a forthcoming joint paper.

11. $\mathcal{O}^p(Q)$ VS. $\mathcal{O}^{p'}(Q)$

For each integer $n \in [1,\infty]$, let $\mathcal{R}_n$ be the countable graph with exactly one vertex and n loops. We write $L_n = L(\mathcal{R}_n)$, $\mathcal{O}_n^p = \mathcal{O}^p(\mathcal{R}_n)$. In particular,

$$L_\infty = \mathbb{C}\{x_i, x_i^* : 1 \leqslant i\}/\langle x_i^* x_j - \delta_{i,j}\rangle.$$

LEMMA 11.1. *Let Q be a countable graph and let $p \in [1,\infty)$. Assume that L_Q is purely infinite simple. Then there is a homomorphism $L_\infty \to L_Q$ which induces an isometry $\mathcal{O}_\infty^p \to \mathcal{O}^p(Q)$.*

Proof. Let α be a cycle in Q and let $v = s(\alpha)$. Choose a closed path β with $s(\beta) = v$ so that α and β are not comparable under the preorder of paths, as in the proof of Lemma 9.5. Then $\beta^*\alpha = \alpha^*\beta = 0$ and, of course, $\alpha^*\alpha = \beta^*\beta = v$. Hence there is a $*$-homomorphism $\phi : L_\infty \to L_Q$ such that $\phi(x_i) = \beta^i\alpha$. Observe that if $\rho : L_Q \to \mathcal{L}(L^p(X))$ is any spatial representation, then $\rho\phi$ is again spatial. Hence ϕ induces a contractive homomorphism $\widehat{\phi} : \mathcal{O}_\infty^p \to \mathcal{O}^p(Q)$. By Theorem 9.1, if $\rho : L_Q \to \mathcal{L}(L^p(X))$ is a nonzero spatial representation, then $\widehat{\phi}$ agrees, up to isometric isomorphism, with the isometric inclusion $\overline{\rho\phi(L_\infty)} \subset \overline{\rho(L_Q)}$. ∎

THEOREM 11.2. *Let Q, Q' be countable graphs and let $p, p' \in [1,\infty)$, $p \neq p'$. Assume that L_Q is purely infinite simple. If in addition, any of the following conditions holds, then there is no nonzero continuous homomorphism $\mathcal{O}^p(Q) \to \mathcal{O}^{p'}(Q')$:*

(i) *$L_{Q'}$ is simple;*
(ii) *$p' \leqslant 2$ and $p \notin (p', 2]$;*
(iii) *$p' > 2 \neq p$.*

Proof. Assume that there is a nonzero continuous homomorphism $f : \mathcal{O}^p(Q) \to \mathcal{O}^{p'}(Q')$. Because the inclusion $L_Q \subset \mathcal{O}^p(Q)$ is dense, $f(L_Q) \neq 0$, which in view of the simplicity of L_Q implies that f is injective on L_Q. Let $\phi : L_\infty \to L_Q$ be as in Lemma 11.1. Then $f\phi$ is injective, whence $f\widehat{\phi} : \mathcal{O}_\infty^p \to \mathcal{O}^{p'}(Q')$ is a nonzero continuous homomorphism. Hence by Theorem 9 of Chapter 15, Section 3 of [20] there exists $X \in \{\mathbb{N}, [0,1]\}$ and a spatial representation $\rho' : L_{Q'} \to \mathcal{L}(L^{p'}(X))$ such that $\widehat{\rho'} f\widehat{\phi} : \mathcal{O}_\infty^p \to \mathcal{L}(L^{p'}(X))$ is nonzero. By Lemma 9.1 of [15] this implies that $L^{p'}(X)$ contains a subspace isomorphic to $\ell^p(\mathbb{N})$. If $X = \mathbb{N}$, this cannot be, as noted in the proof of Theorem 9.2 in [15] and by page 54 in [14]; if $X = [0,1]$ and either (ii) or (iii) holds, this cannot happen either, by Theorem 6.4.19 of [4].

Thus parts (ii) and (iii) of the theorem are proved. Part (i) also follows, using Proposition 4.11 and Theorem 9.1. ∎

Acknowledgements. This article has evolved from the Ph.D. Dissertation of the second named author [18]. We are indebted to Chris Phillips for discussions on his paper [15]. Thanks also to our colleague Daniel Carando for several useful discussions and references on L^p-spaces. The first named author also wishes to thank Eusebio Gardella for an enlightening email exchange including several useful comments on a previous version of this paper. Thanks also to the anonymous referee, whose careful reading and useful suggestions have helped improve the paper.

REFERENCES

[1] G. ABRAMS, P. ARA, M. SILES MOLINA, *Leavitt Path Algebras*, Lecture Notes in Math., vol. 2191, Springer, London 2017.

[2] G. ABRAMS, G. ARANDA PINO, The Leavitt path algebras of arbitrary graphs, *Houston J. Math.* **34**(2008), 423–442.

[3] G. ABRAMS, M. TOMFORDE, Isomorphism and Morita equivalence of graph algebras, *Trans. Amer. Math. Soc.* **363**(2011), 3733–3767.

[4] F. ALBIAC, N.J. KALTON, *Topics in Banach Space Theory*, Grad. Texts in Math., vol. 233, Springer, New York 2006.

[5] T. ANDÔ, Contractive projections in L_p spaces, *Pacific J. Math.* **17**(1966), 391–405.

[6] L.O. CLARK, A. SIMS, Equivalent groupoids have Morita equivalent Steinberg algebras, *J. Pure Appl. Algebra* **219**(2015), 2062–2075.

[7] D. DRINEN, M. TOMFORDE, Computing K-theory and Ext for graph C^*-algebras, *Illinois J. Math.* **46**(2002), 81–91.

[8] R. EXEL, Inverse semigroups and combinatorial C^*-algebras, *Bull. Braz. Math. Soc. (N.S.)* **39**(2008), 191–313.

[9] E. GARDELLA, M. LUPINI, Representations of étale groupoids on L^p-spaces, *Adv. Math.* **318**(2017), 233–278.

[10] E. GARDELLA, H. THIEL, Quotients of Banach algebras acting on L^p-spaces, *Adv. Math.* **296**(2016), 85–92.

[11] E. GARDELLA, H. THIEL, Extending representations of Banach algebras to their biduals, arXiv:1703.00882v1 [math.FA].

[12] C. HERZ, Harmonic synthesis for subgroups, *Ann. Inst. Fourier (Grenoble)* **23**(1973), 91–123.

[13] J. LAMPERTI, On the isometries of certain function-spaces, *Pacific J. Math.* **8**(1958), 459–466.

[14] J. LINDENSTRAUSS, L. TZAFRIRI, *Classical Banach Spaces*. I. *Sequence Spaces*, Ergeb. Math. Grenzgeb., vol. 92, Springer-Verlag, Berlin-New York 1977.

[15] N.C. PHILLIPS, Analogs of Cuntz algebras on L^p-spaces, arXiv:1201.4196 [math.FA].

[16] N.C. PHILLIPS, Crossed products of L^p-operator algebras and the K-theory of Cuntz algebras on L^p-spaces, preprint, arXiv:1309.6406 [math.FA].

[17] N.C. PHILLIPS, Simplicity of UHF and Cuntz algebras on L^p-spaces, arXiv:1309.0115 [math.FA].

[18] M.E. RODRÍGUEZ, Álgebras de operadores en espacios L^p asociadas a grafos orientados, Ph.D. Dissertation, University Buenos Aires, Buenos Aires 2016.

[19] I. RAEBURN, *Graph Algebras*, CBMS Reg. Conf. Ser. Math., vol. 103, Amer. Math. Soc., Providence, RI 2005.

[20] H.L. ROYDEN, *Real Analysis*, Macmillan Publ. Co., New York 1988.

[21] B. STEINBERG, A groupoid approach to discrete inverse semigroup algebras, *Adv. Math.* **223**(2010), 689–727.

[22] B. STEINBERG, Simplicity, primitivity and semiprimitivity of étale groupoid algebras with applications to inverse semigroup algebras, *J. Pure Appl. Algebra* **220**(2016), 1035–1054.

GUILLERMO CORTIÑAS, DEPARTAMENTO DE MATEMÁTICA-INSTITUTO SANTALÓ, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES, CIUDAD UNIVERSITARIA (1428) BUENOS AIRES, ARGENTINA
E-mail address: gcorti@dm.uba.ar

MARÍA EUGENIA RODRÍGUEZ, DEPARTAMENTO DE CIENCIAS EXACTAS , CICLO BÁSICO COMÚN, UNIVERSIDAD DE BUENOS AIRES, CIUDAD UNIVERSITARIA, (1428) BUENOS AIRES, ARGENTINA
E-mail address: merodrig@dm.uba.ar

Received January 19, 2018; revised March 28, 2018.