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THE DIXMIER–DOUADY CLASSES OF CERTAIN GROUPOID C^*-ALGEBRAS WITH CONTINUOUS TRACE

MARIUS IONESCU, ALEX KUMJIAN, AIDAN SIMS, and DANA P. WILLIAMS

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ABSTRACT. Given a locally compact abelian group G, we give an explicit formula for the Dixmier–Douady invariant of the C^*-algebra of the groupoid extension associated to a Čech 2-cocycle in the sheaf of germs of continuous G-valued functions. We then exploit the blow-up construction for groupoids to extend this to some more general central extensions of étale equivalence relations.

KEYWORDS: *Dixmier–Douady class, groupoid C^*-algebras, continuous-trace C^*-algebras*

MSC (2010): Primary 46L05, 46L55.

1. INTRODUCTION

This article provides explicit formulas for the Dixmier–Douady invariants of a large class of continuous-trace C^*-algebras arising from groupoid extensions. Continuous-trace C^*-algebras are amongst the best understood and most intensively studied classes of Type I C^*-algebras. A C^*-algebra A is a continuous-trace C^*-algebra if it has Hausdorff spectrum and is locally Morita equivalent to a commutative C^*-algebra. Alternatively, A is of continuous trace if it has Hausdorff spectrum and every irreducible representation of A admits a neighbourhood U and an element a of A such that the image of a under any element of U is a rank-one projection. The celebrated Dixmier–Douady theorem [3] associates to each continuous-trace C^*-algebra A with spectrum X an element $\delta(A)$ of $H^3(X, \mathbb{Z})$ such that A is Morita equivalent to an abelian C^*-algebra if and only if $\delta(A) = 0$. Indeed, the collection of Morita-equivalence classes of continuous-trace C^*-algebras with spectrum X forms a group under a balanced tensor-product operation, and $A \mapsto \delta(A)$ induces a group isomorphism of this group with $H^3(X, \mathbb{Z})$; see Theorem 6.3 of [25].

As a result, there has been a great deal of work characterizing when C^*-algebras associated to dynamical systems have continuous trace. For example,

[4], [6], [16], [22], [24], and [30] investigate when a transformation group C^*-algebra is continuous-trace; the epic [5] deals with general crossed product C^*-algebras; and [15], [17], [18], [19], and [20] study when groupoid C^*-algebras have continuous trace.

However, there are few results that provide tools for calculating the Dixmier–Douady invariant of a given continuous-trace C^*-algebra; and the results that do address this question are not entirely satisfactory. For example, in the examples appearing in [6] and [17], the Dixmier–Douady class is always trivial. And the formula developed in [24] is somewhat unwieldy. There are, on the other hand, some intriguing formulas in [22], and an explicit computation in Example 4.6 in [8] based on one of these.

There is a simple reason for the dearth of results that compute $\delta(A)$: the Dixmier–Douady invariant is difficult to compute except on an ad hoc basis. In this note, we take some steps towards addressing this lack of computable examples, by developing usable formulas for the Dixmier–Douady class of significant classes of groupoid C^*-algebras. Starting with a locally compact abelian group G and a space X, we first consider groupoids constructed directly from a Čech cocycle on X taking values in the sheaf of germs of continuous G-valued functions on X. We then extend this to central groupoid extensions of equivalence relations constructed from local homeomorphisms between locally compact Hausdorff spaces. Not surprisingly, our most elegant results are obtained under more restrictive hypotheses; but even in the more general situation our computation is fairly concrete. In any case, the subject of operator algebras, and the study of C^*-algebras associated to dynamical systems in particular, is short on concrete examples, so we think that extra hypotheses are worth it.

The main results of the paper are as follows. Consider a second-countable locally compact Hausdorff space X and a second-countable locally compact abelian group G. Take a Čech 2-cocycle c on X, relative to a locally finite open cover $\mathscr{U} = \{U_i : i \in I\}$ of X, taking values in the sheaf $\mathscr{G}$ of germs of continuous G-valued functions on X. The associated Raeburn–Taylor groupoid $\Gamma_{\mathscr{U}}$ consists of triples (i, x, j) such that $x \in U_{ij} \subseteq X$. The cocycle c determines a natural central extension Σ_c of $\Gamma_{\mathscr{U}}$ by G: as a set Σ_c is just a copy of $\Gamma_{\mathscr{U}} \times G$, but the composition in the second coordinate is twisted by the cocycle c as in equation (3.1). Our first main result, Theorem 3.4, says that $C^*(\Sigma_c)$ is a continuous-trace algebra with spectrum $\widehat{G} \times X$, and computes its Dixmier–Douady invariant as follows: write $\mathscr{S}$ for the sheaf of germs of continuous $\mathbb{T}$-valued functions on $\widehat{G} \times X$, and let $\mathscr{V}$ be the cover $\{\widehat{G} \times U_i\}_{i \in I}$ of $\widehat{G} \times X$. Then the cocycle c determines a cocycle $\nu^c \in Z^2(\widehat{G} \times X, \mathscr{S})$ relative to $\mathscr{V}$ such that $\nu^c_{ijk}(\tau, x) = \overline{\tau(c_{ijk}(x))}$ for all $i, j, k \in I$, $x \in U_{ijk}$ and $\tau \in \widehat{G}$. The assignment $c \mapsto \nu^c$ descends to a homomorphism

$$m_* : H^2(X, \mathscr{G}) \to H^2(\widehat{G} \times X, \mathscr{S}),$$

and Theorem 3.4 shows that, under the canonical isomorphism of $H^3(\widehat{G} \times X, \mathbb{Z})$ with $H^2(\widehat{G} \times X, \mathscr{S})$, the Dixmier–Douady class of $C^*(\Sigma_c)$ is carried to the cohomology class $[\nu^c]$. The set-up and proof of this theorem occupy Sections 3–5.

We then build upon Theorem 3.4 in Section 6 to describe a method for computing the Dixmier–Douady invariants of more general central extensions. We start with a local homeomorphism $\psi : Y \to X$ of second-countable locally compact Hausdorff spaces, and form the equivalence relation $R(\psi)$ on Y consisting of pairs with identical image under ψ. We consider a central extension Σ of $R(\psi)$ by $G \times Y$. We assume that X is locally G-trivial in the sense that every open cover of X admits a refinement such that, on double overlaps, every principal G-bundle is trivial. This hypothesis ensures that a suitable blow-up Σ' of the extension Σ admits a continuous section for the surjection onto the corresponding blow-up R' of $R(\psi)$. It follows that Σ' is determined by a continuous G-valued groupoid 2-cocycle on R'. A little more work puts us back in the situation of Theorem 3.4, and we can use this to compute the Dixmier–Douady invariant of $C^*(\Sigma')$. The blow-up operation determines an equivalence of extensions, and hence a Morita equivalence of their C^*-algebras, yielding a computation of $\delta(C^*(\Sigma))$.

For the reader's convenience we include an appendix with background on central extensions of groupoids by locally compact abelian groups and, in particular, those that arise from continuous 2-cocycles (see Appendix A).

2. CENTRAL ISOTROPY

In the sequel, Σ will always be a second-countable locally compact Hausdorff groupoid with a Haar system $\{\lambda^u\}_{u \in \Sigma^{(0)}}$. The *isotropy groupoid* of Σ is the closed subgroupoid

$$\mathcal{I}(\Sigma) = \{\, \gamma \in \Sigma : s(\gamma) = r(\gamma) \,\}.$$

Note that $\mathcal{I}(\Sigma)$ is a group bundle over $\Sigma^{(0)}$ and that $\mathcal{I}(\Sigma)$ admits a Haar system if and only if the isotropy map

$$u \mapsto \Sigma(u) = \{\, \gamma \in \mathcal{I}(\Sigma) : s(\gamma) = u = r(\gamma) \,\}$$

is continuous from $\Sigma^{(0)}$ into the locally compact Hausdorff space $\mathcal{C}(\Sigma)$ of closed subgroups of Σ ([27], Lemma 1.3). In the sequel we need to assume not only that each $\Sigma(u)$ is abelian, but that the isotropy is central in the following sense.

DEFINITION 2.1. Let Σ be a groupoid, $\mathcal{I}(\Sigma)$ its isotropy subgroupoid, and $q : \Sigma^{(0)} \to \Sigma\backslash\Sigma^{(0)}$ the quotient map. We say that Σ has *central isotropy* if $\Sigma\backslash\Sigma^{(0)}$ is Hausdorff and there is an abelian group bundle $\mathcal{A}$ over $\Sigma\backslash\Sigma^{(0)}$ and a groupoid isomorphism $\iota : q^*\mathcal{A} \to \mathcal{I}(\Sigma)$ such that $\iota|_{\Sigma^{(0)}} = \mathrm{id}$ and such that

$$\iota(r(\gamma), a)\gamma = \gamma\iota(s(\gamma), a) \quad \text{for all } a \in A([r(\gamma)]).$$

The use of the word "central" is partially justified by the following example.

EXAMPLE 2.2. Let (G, X) be a transformation group with $G\backslash X$ Hausdorff and such that each stability group $G_x = \{ h \in G : h \cdot x = x \}$ is central in G. Let $\Sigma = G \times X$ be the corresponding transformation groupoid. Note that $G_{h \cdot x} = G_x$ for all $h \in G$ and $x \in X$. Then

$$\mathcal{A} = \{ (G \cdot x, h) \in G\backslash X \times G : h \in G_x \}$$

is a group bundle over $G\backslash X$. If $q : X \to G\backslash X$ is the orbit map, then Σ has central isotropy with respect to the isomorphism $\iota(x, G \cdot x, h) = (h, x)$.

An important class of examples comes from $\mathbb{T}$-*groupoids* or *twists* as introduced by the second author ([11], [12]). Recall that a $\mathbb{T}$-groupoid Σ over a groupoid $\mathcal{R}$ is a unit-space-preserving groupoid extension

$$\begin{tikzcd} \Sigma^{(0)} \times \mathbb{T} \arrow[r, hook, "\iota"] \arrow[dr, shift left] \arrow[dr, shift right] & \Sigma \arrow[r, two heads, "\pi"] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{R} \arrow[dl, shift left] \arrow[dl, shift right] \\ & \Sigma^{(0)} & \end{tikzcd} \tag{2.1}$$

such that

$$\iota(r(\gamma), z)\gamma = \gamma\iota(s(\gamma), z) \quad \text{for all } \gamma \in \Sigma \text{ and } z \in \mathbb{T}.$$

Given a $\mathbb{T}$-groupoid Σ over $\mathcal{R}$, there is a free and proper $\mathbb{T}$-action on Σ such that $z \cdot \gamma = \iota(s(\gamma), z)\gamma$. We can identify $\mathcal{R}$ with the orbit space $\mathbb{T}\backslash\Sigma$ and π with the orbit map; see Section 3 of [19]. See Appendix A for more background on extensions of groupoids by locally compact abelian groups.

EXAMPLE 2.3. Recall that a groupoid $\mathcal{R}$ is *principal* if the isotropy groupoid $\mathcal{I}(\mathcal{R})$ is just $\mathcal{R}^{(0)}$. Let Σ be a $\mathbb{T}$-groupoid over a principal groupoid $\mathcal{R}$. Then $\iota(\Sigma^{(0)} \times \mathbb{T}) = \mathcal{I}(\Sigma)$, and if $\Sigma\backslash\Sigma^{(0)}$ is Hausdorff, then Σ has central isotropy.

REMARK 2.4. In previous studies of $\mathbb{T}$-groupoids, the emphasis was on the quotient C^*-algebra $C^*(\mathcal{R}; \Sigma)$. Here we focus on $C^*(\Sigma)$.

A central player in much of the work on the Effros–Hahn theory for groupoids, as in [10] and [27], is the *equivalence relation* $\mathcal{R}$ *associated to* Σ. By definition, this $\mathcal{R}$ is the image of the map $\pi : \Sigma \to \Sigma^{(0)} \times \Sigma^{(0)}$ given by $\pi(\gamma) = (r(\gamma), s(\gamma))$. Since the relative product topology on $\mathcal{R}$ is unlikely to be useful in general, it is common to equip $\mathcal{R}$ with the quotient topology, which is finer (often strictly finer) than the relative product topology. Even then, $\mathcal{R}$ need not be a tractable topological space. However, the isotropy groupoid $\mathcal{I}(\Sigma)$ acts on the right and left of Σ, and with respect to the quotient topologies

$$\mathcal{R} \cong \mathcal{I}(\Sigma)\backslash\Sigma = \Sigma/\mathcal{I}(\Sigma).$$

As observed above, $\mathcal{I}(\Sigma)$ has a Haar system precisely when $u \mapsto \Sigma(u)$ is continuous. In this case, the orbit map $k : \Sigma \to \mathcal{I}(\Sigma)\backslash\Sigma$ is open (see Lemma 2.1 in [19]), and then $\mathcal{R}$ is locally compact and Hausdorff. The subtlety here is that the orbit map for an action by a groupoid Γ is open *provided* the range map of Γ is open; this is automatic if Γ has a Haar system.

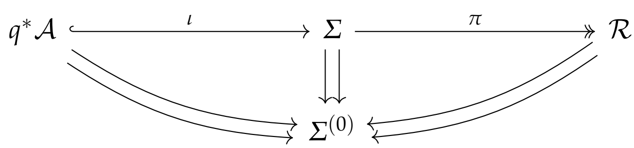
If $\mathcal{R}$ is locally compact, then we can ask for $\mathcal{R}$ to act properly on its unit space, which we identify with $\Sigma^{(0)}$. In that case, Lemma 2.1 in [17] shows that we can identify $\mathcal{R}$ with $\pi(\Sigma)$ under the relative topology inherited from $\Sigma^{(0)} \times \Sigma^{(0)}$.

Hence we work under three key assumptions:

(A1) Σ has central isotropy;
(A2) the isotropy map $u \mapsto \Sigma(u)$ is continuous;
(A3) the action of $\mathcal{R}$ on $\Sigma^{(0)}$ is proper.

If Σ has central isotropy and $C^*(\Sigma)$ has continuous trace, then (A2) and (A3) are automatically satisfied ([15], Theorem 1.1).

In any event, we have a unit-preserving short exact sequence of groupoids



similar to (2.1). We study special cases of these sorts of groupoids in the next four sections.

REMARK 2.5 (Irreducible representations induced from characters). In our main results, we will make considerable use of irreducible representations of $C^*(\Sigma)$ induced from characters on an abelian stability group as described in Section 2 of [9]. If τ is a character on the stability group $\Sigma(u)$, then we will write $\operatorname{Ind}^{\Sigma}(u,\tau)$, or simply $\operatorname{Ind}(u,\tau)$ when there is no ambiguity about Σ, for the induced represenation $\operatorname{Ind}_{\Sigma(u)}^{\Sigma}(\tau)$. Then $\operatorname{Ind}(u,\tau)$ is irreducible by Theorem 5 in [9]. If $\{\lambda^u\}_{u\in\Sigma^{(0)}}$, is a Haar system on Σ and μ is a Haar measure on $\Sigma(u)$, then $\operatorname{Ind}(u,\tau)$ acts by convolution on the completion of $C_c(\Sigma_u)$ with respect to to the pre-inner product

$$(f_1 \mid f_2) = \int_{\Sigma(u)} \int_{\Sigma} \overline{f_2(\sigma^{-1})} f_1(\sigma^{-1}g)\, d\lambda^u(\sigma)\tau(g)\, d\mu(g). \tag{2.2}$$

If, as will always be the case here, the orbits $[u] = \Sigma \cdot u$ are closed, then every irreducible representation of $C^*(\Sigma)$ factors through a restriction $\Sigma([u]) = \Sigma_{[u]}^{[u]}$. Since the latter is equivalent as a groupoid to the isotropy group $\Sigma(u)$, it is easy to see directly that $\operatorname{Ind}(u,\tau)$ is irreducible. If all the isotropy groups are abelian, then it is also clear that every irreducible representation of $C^*(\Sigma)$ is of this form for some $u \in \Sigma^{(0)}$ and $\tau \in \Sigma(u)^\wedge$. Furthermore, if $[u] = [v]$ and $\sigma \in \Sigma_u^v$ then $\operatorname{Ind}(u,\tau)$ is easily seen to be equivalent to $\operatorname{Ind}(v, \sigma\cdot\tau)$ where $\sigma\cdot\tau(g) = \tau(\sigma^{-1}g\sigma)$. If we have central isotropy, so that we can identify $\Sigma(u)$ and $\Sigma(\sigma\cdot u)$, then the spectrum of $C^*(\Sigma)$ is parameterized by $\{\, ([u],\tau) : u \in \Sigma^{(0)} \text{ and } \tau \in \Sigma(u)^\wedge \,\}$.

3. A CLASS OF EXAMPLES

Let X be a second-countable locally compact Hausdorff space and G a second-countable locally compact abelian group. Let $\mathscr{G}$ be the sheaf of germs of G-valued functions on X (see Section 4.1 of [25]). Let $\mathfrak{a}$ be an element in the sheaf cohomology group $H^2(X, \mathscr{G})$. Then $\mathfrak{a}$ is represented by a two cocycle $c \in Z^2(\mathscr{U}, \mathscr{G})$ for some *locally finite* cover $\mathscr{U} = \{ U_i \}_{i \in I}$ of X by precompact open sets. We say that c is *normalized* if $c_{iii}(x) = 0$ for all i and all $x \in U_i$. It is easy to see that every 2-cocycle is cohomologous to a normalized one and we will assume all our cocycles are normalized. We record some elementary facts about normalized cocycles for reference.

LEMMA 3.1. *Let $c \in Z^2(\mathscr{U}, \mathscr{G})$ be normalized. Then for all $i, j, k \in I$,*
(i) $c_{iij}(x) = c_{ijj}(x) = 0$;
(ii) $c_{iji}(x) = c_{jij}(x)$;
(iii) $c_{ijk}(x) = -c_{jik}(x) + c_{iji}(x)$;
(iv) $c_{ijk}(x) = -c_{ikj}(x) + c_{jkj}(x)$;
(v) $c_{iji}(x) + c_{jki}(x) = -c_{ikj}(x) + c_{iki}(x) + c_{jkj}(x)$.

Proof. These are all straightforward consequences of the cocycle identity. For example, (i) follows from computations like

$$0 = \delta(c)_{iiji}(x) = c_{iji}(x) - c_{iji}(x) + c_{iii}(x) - c_{iij}(x).$$

For (ii), consider $\delta(c)_{ijij}(x)$ and use (i). For (iii), consider $\delta(c)_{ijik}(x)$ and (iv) follows from $\delta(c)_{ijkj}(x)$. We get (v) using (iii) and (iv):

$$\begin{aligned} c_{ikj}(x) &= -c_{ijk}(x) + c_{jkj}(x) = c_{jik}(x) - c_{iji}(x) + c_{jkj}(x) \\ &= -c_{jki}(x) + c_{iki}(x) - c_{iji}(x) + c_{jkj}(x). \end{aligned}$$ ∎

Given $\mathscr{U}$, we can form the blow-up groupoid (the terms "pull-back" and "ampliation" are also used; blow-ups are defined and discussed in Section 3.3 of [31]) $\Gamma_{\mathscr{U}}$ with respect to the natural map of $\coprod U_i$ onto X:

$$\Gamma_{\mathscr{U}} = \{ (i, x, j) : x \in U_{ij} := U_i \cap U_j \}$$

with $(i, x, j)(j, x, k) = (i, x, k)$ and $(i, x, j)^{-1} = (j, x, i)$. In particular, $\Gamma_{\mathscr{U}}$ is a principal groupoid with unit space $\Gamma_{\mathscr{U}}^{(0)} = \coprod U_i$, and is equivalent to the space X (see [7] and [23]).

Let Σ_c be the groupoid extension equal as a topological space to $G \times \Gamma_{\mathscr{U}}$ endowed with the operations

$$(g, (i, x, j))(h, (j, x, k)) = (g + h + c_{ijk}(x), (i, x, k)) \tag{3.1}$$

and

$$(g, (i, x, j))^{-1} = (-g - c_{iji}(x), (j, x, i)).$$

Since

$$(g,(i,x,j))(g,(i,x,j))^{-1} = (g,(i,x,j))(-g-c_{iji}(x),(j,x,i)) = (0,(i,x,i)),$$

and similarly

$$(g,(i,x,j))^{-1}(g,(i,x,j)) = (0,(j,x,j)),$$

we can identify the unit space of Σ_c with $\coprod U_i$. Let μ be a Haar measure on G. We equip Σ_c with the Haar system $\lambda = \{\lambda^{(i,x)}\}$ given by

$$\lambda^{(i,x)}(f) = \sum_j \int_G f(g,(i,x,j))\,\mathrm{d}\mu(g).$$

Then using Lemma 3.1,

$$\begin{aligned} f * f'(g,(i,x,j)) &= \sum_k \int_G f(h,(i,x,k))f'(g-h-c_{iki}(x)+c_{kij}(x),(k,x,j))\,\mathrm{d}\mu(h) \\ &= \sum_k \int_G f(h,(i,x,k))f'(g-h-c_{ikj}(x),(k,x,j))\,\mathrm{d}\mu(h) \end{aligned} \tag{3.2}$$

while

$$f^*(g,(i,x,j)) = \overline{f(-g-c_{iji}(x),(j,x,i))}.$$

If we define $\iota : G \times \coprod U_i \to \Sigma_c$ by

$$\iota(g,(i,x)) = (g,(i,x,i)),$$

and $\pi : \Sigma_c \to \Gamma_{\mathscr{U}}$ by

$$\pi(g,(i,x,j)) = (i,x,j),$$

then we obtain a groupoid extension

$$\begin{array}{ccccc} G \times \coprod U_i & \overset{\iota}{\hookrightarrow} & \Sigma_c & \overset{\pi}{\twoheadrightarrow} & \Gamma_{\mathscr{U}}. \\ & \searrow\!\searrow & \downarrow\downarrow & \swarrow\!\swarrow & \\ & & \coprod U_i & & \end{array} \tag{3.3}$$

We think of Σ_c as a generalized twist in which $\mathbb{T}$ has been replaced by G. As in Remark 2.5, we can identify the spectrum of $C^*(\Sigma_c)$ as a set with $\widehat{G} \times X$ via $(\tau, x) \mapsto [\operatorname{Ind}((i,x),\tau)]$ for any i such that $x \in U_i$.

LEMMA 3.2. *Let Σ_c be as above. Let $I(x) = \{\, j \in I : x \in U_j \,\}$. Then* $\operatorname{Ind}((i,x),\tau)$ *is equivalent to the representation L on $\ell^2(I(x))$ where $L(f)$ is given by multiplication by the matrix $A = (a_{jk})$ with*

$$a_{jk} = \tau(c_{ijk}(x)) \int_G f(g,(j,x,k))\tau(g)\,\mathrm{d}\mu(g).$$

Proof. We have $(\Sigma_c)_{(i,x)} = \{ (g,(j,x,i) : j \in I(x) \}$. Then $\operatorname{Ind}((i,x),\tau)$ acts by convolution on the completion of $C_c((\Sigma_c)_{(i,x)})$ with respect to the inner product given in (2.2):

$$\begin{aligned}(f_1 \mid f_2) &= \sum_j \int_G \int_G \overline{f_2(-h-c_{iji}(x),(j,x,i))} f_1(-h-c_{iji}(x)+g,(j,x,i))\tau(g)\,\mathrm{d}\mu(h)\,\mathrm{d}\mu(g)\\ &= \sum_j U(f_1)(j)\overline{U(f_2)(j)},\end{aligned}$$

where

$$U(f)(j) = \int_G f(g - c_{iji}(x),(j,x,i))\tau(g)\,\mathrm{d}\mu(g).$$

Hence U is a unitary from the space of $\operatorname{Ind}((i,x),\tau)$ onto $\ell^2(I(x))$. But

$$\begin{aligned}&U(f_1 * f_2)(j)\\ &\quad= \int_G f_1 * f_2(g - c_{iji}(x),(j,x,i))\tau(g)\,\mathrm{d}\mu(g)\\ &\quad= \sum_k \int_G \int_G f_1(h,(j,x,k)) f_2(-h+g-c_{iji}(x)-c_{jki}(x),(k,x,i))\tau(g)\,\mathrm{d}\mu(h)\,\mathrm{d}\mu(g)\end{aligned}$$

which, since $c_{iji}(x) + c_{jki}(x) = c_{ijk}(x) + c_{iki}(x)$, is

$$= \sum_k \int_G \int_G f_1(h,(j,x,k)) f_2(-h+g-c_{ijk}(x)-c_{iki}(x),(k,x,i))\tau(g)\,\mathrm{d}\mu(h)\,\mathrm{d}\mu(g)$$

$$= \sum_k \tau(c_{ijk}(x)) \int_G f_1(h,(j,x,k))\tau(h)\,\mathrm{d}\mu(h) U(f_2)(k) = \sum_k a_{jk} U(f_2)(k).$$

Thus U intertwines $\operatorname{Ind}((i,x),\tau)$ with multiplication by $A = (a_{jk})$ as claimed. ∎

REMARK 3.3. There is a continuous groupoid 2-cocycle $\varphi_c \in Z^2(\Gamma_{\mathscr{U}}, G)$ given by the formula $\varphi_c((i,x,j),(j,x,k)) = c_{ijk}(x)$ for $x \in U_{ijk}$. For this cocycle, Σ_c is equal to the extension $\Sigma(\Gamma_{\mathscr{U}}, \varphi_c)$ described in Notation A.4 under the natural identification.

Our first goal is to determine the Dixmier–Douady class of $C^*(\Sigma_c)$. To do so, we need the following construction. Given a Čech 2-cocycle $c \in Z^2(\mathscr{U}, \mathscr{G})$, where $\mathscr{U} = \{U_i\}_{i\in I}$ is a locally finite open cover of X, the cover $\mathscr{V} = \{\widehat{G} \times U_i\}_{i\in I}$ of $\widehat{G} \times X$ is locally finite and supports a normalized 2-cocycle ν^c such that

$$\nu^c_{ijk}(\tau, x) = \overline{\tau(c_{ijk}(x))}. \tag{3.4}$$

There is a well-defined homomorphism

$$m_* : H^2(X, \mathscr{G}) \to H^2(\widehat{G} \times X, \mathscr{S})$$

such that $m_*([c]) = [\nu^c]$ for all $c \in Z^2(X, \mathscr{G})$.

Our first main theorem is the following computation of the Dixmier–Douady class of $C^*(\Sigma_c)$.

THEOREM 3.4. *Suppose that X is a second-countable locally compact Hausdorff space and that G is a second-countable locally compact abelian group. Let $\mathscr{G}$ be the sheaf of germs of continuous G-valued functions on X. Suppose that $c \in Z^2(\mathscr{U}, \mathscr{G})$ is a normalized cocycle on a locally finite cover $\mathscr{U}$ by precompact open sets representing a class $\mathfrak{a} \in H^2(X, \mathscr{G})$, and let Σ_c be the associated groupoid extension* (3.3). *Then $(\tau, x) \mapsto [\mathrm{Ind}((i, x), \tau)]$, $x \in U_i$, is a homeomorphism of $\widehat{G} \times X$ with the spectrum $C^*(\Sigma_c)^\wedge$. Furthermore $C^*(\Sigma_c)$ has continuous trace, and with respect to this identification of the spectrum with $\widehat{G} \times X$, its Dixmier–Douady class $\delta(C^*(\Sigma_c))$ is equal to the image of $m_*(\mathfrak{a})$ in $H^3(\widehat{G} \times X, \mathbb{Z})$.*

The proof of Theorem 3.4 requires some preparation, including the Raeburn–Taylor construction described in the next section. So we defer the proof to Section 5.

4. THE RAEBURN–TAYLOR ALGEBRA

For our computation of the Dixmier–Douady class of $C^*(\Sigma_c)$, we want a slight modification of the *Raeburn–Taylor algebra* based on their original construction in [23] and reproduced in Proposition 5.40 in [25]. Specifically, given a *normalized* 2-cocycle $\nu = \{\nu_{ijk}\} \in Z^2(\mathscr{U}, \mathscr{S})$ defined on a *locally finite* cover $\mathscr{U}$ by precompact open sets, we want to produce a concrete C^*-algebra $A(\nu)$ with finite-dimensional representations such that $\delta(A(\nu))$ is the image of $[\nu]$ in $H^3(X, \mathbb{Z})$. In Proposition 5.40 of [25] and in [23], it is assumed for convenience that ν is alternating. Although every cocycle is cohomologous to an alternating one by Proposition 4.41 in [25], it will simplify our arguments here to observe that basically the same constructs work in the normalized case. We supply some of the details for completeness and also since we will have to push the envelope a bit further in Section 5.

REMARK 4.1. It is not stated in [23] that their cocycle is alternating, but it is needed for their constructions. Furthermore, there is a subtle caveat in the Raeburn–Taylor construction. They use the shrinking lemma (see Lemma 4.32 in [25]) to replace $\mathscr{U}$ by a cover $\mathscr{V} = \{V_i\}_{i \in I}$ *with the same index set* such that $\overline{V_i} \subset U_i$. Replacing $\mathscr{U}$ by $\mathscr{V}$ allows them to assume that each ν_{ijk} extends to $\overline{U}_{ijk}$. The exposition here, taken from [25], avoids this technicality.

To construct $A(\nu)$, we begin by forming the algebra $A_1(\nu)$ which is the set of sparse $I \times I$ matrices

$$f = (f_{ij})_{i,j \in I}$$

where each $f_{ij} \in C_0(X)$ and vanishes off U_{ij}. For each $x \in X$, the set $I(x) := \{ i \in I : x \in U_i \}$ is finite. Thus if $n_x = |I(x)|$, then $(f_{ij}(x))_{i,j \in I}$ is an $n_x \times n_x$ matrix. We define a multiplication on $A_1(\nu)$ by twisting the usual matrix multiplication with ν:

$$(f_{ij})(g_{lk}) = (h_{ik})$$

where

$$h_{ik}(x) = \sum_j \overline{\nu_{ijk}(x)} f_{ij}(x) g_{jk}(x). \tag{4.1}$$

To see that the sum in (4.1) is meaningful, observe that it is always a finite sum:

$$h_{ik}(x) = \begin{cases} \sum_{\{ j \in I : x \in U_{ijk} \}} \overline{\nu_{ijk}(x)} f_{ij}(x) g_{jk}(x) & \text{if } x \in U_{ijk} \text{ for some } j, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

As in Lemma 5.39 in [25], using the local finiteness of the cover and the compactness of the $\overline{U}_i$, it is not hard to see that each h_{ik} is continuous and vanishes off U_{ik}. (This will also be a special case of Lemma 5.1.)

To get an involution, we have to adjust the definition of the involution on $A(\nu)$ given by Raeburn and Taylor to account for the fact that our cocycle might not be alternating: we define

$$f^* = (f_{ij})^* = (g_{ij}) \quad \text{where } g_{ij}(x) = \nu_{iji}(x) \overline{f_{ji}(x)}. \tag{4.2}$$

This map is involutive in view of Lemma 3.1(ii). To see that it is anti-multiplicative, we require Lemma 3.1(v):

$$(f * g)^*_{ij}(x) = \nu_{iji}(x) \overline{(f * g)_{ji}(x)} = \nu_{iji}(x) \sum_k \nu_{jki}(x) \overline{f_{jk}(x) g_{ki}(x)}$$

which, using Lemma 3.1(v), is

$$= \sum_k \overline{\nu_{ikj}(x)} \nu_{iki}(x) \overline{g_{ki}(x)} \nu_{jkj}(x) \overline{f_{jk}(x)} = \sum_k \overline{\nu_{ikj}(x)} g^*_{ik}(x) f^*_{kj}(x) = (g^* * f^*)_{ij}(x).$$

We now follow the discussion preceding Proposition 5.40 in [25] mutatis mutandis.

For each pair $(i, x) \in \coprod U_i$, define a representation $\pi_{(i,x)}$ of $A_1(\nu)$ on $\ell^2(I(x))$ by letting $\pi_{(i,x)}(f)$ act by multiplication by the matrix

$$(\overline{\nu_{ikl}(x)} f_{kl}(x))_{kl}.$$

Using some straightforward cocycle identities, we see that $\pi_{(i,x)}$ is multiplicative and $*$-preserving. The representations $\pi_{(i,x)}$ and $\pi_{(j,x)}$ are equivalent, so we get a seminorm

$$\|f\|_x = \|\pi_{(i,x)}(f)\| \quad \text{for any } i \text{ such that } x \in U_i.$$

Just as in Proposition 5.40 in [25] and Theorem 1 in [23], we can let

$$A(\nu) = \{ f \in A_1(\nu) : x \mapsto \|f\|_x \text{ vanishes at infinity on } X \}.$$

Then $\|f\| = \sup_x \|f\|_x$ is a complete norm on $A(\nu)$. Furthermore, with respect to this norm, $A(\nu)$ is a continuous-trace C^*-algebra with spectrum $X = \{ [\pi_{(i,x)}] : x \in X \}$ and Dixmier–Douady class $[\nu]$. (Here and elsewhere we will often write $[\nu]$ for its image in $H^3(X, \mathbb{Z})$.)

REMARK 4.2 (The Raeburn–Taylor groupoid). Nowadays we favor realizing $A(\nu)$ as a groupoid C^*-algebra twisted by a continuous 2-cocycle φ_ν. The groupoid is the blow-up $\Gamma_{\mathscr{U}}$ associated to the cover $\mathscr{U}$ of X corresponding to ν and the cocycle φ_ν in $Z^2(\Gamma_{\mathscr{U}}, \mathbb{T})$ is given by

$$\varphi_\nu((i,x,j),(j,x,k)) = \overline{\nu_{ijk}(x)}. \tag{4.3}$$

(The complex conjugate in (4.3) is missing from the formula in [23].) Note that φ_ν is normalized as in Appendix A since ν is. The operations in $C_c(\Gamma_{\mathscr{U}}, \varphi_\nu)$ are given by

$$\begin{aligned} f * g(i,x,k) &= \sum_j f(i,x,j)g(j,x,k)\varphi_\nu((i,x,j),(j,x,k)) \\ &= \sum_j f(i,x,j)g(j,x,k)\overline{\nu_{ijk}(x)}, \quad \text{and} \\ f^*(i,x,j) &= \overline{f(j,x,i)\varphi_\nu((i,x,j),(j,x,i))} = \nu_{iji}(x)\overline{f(j,x,i)}. \end{aligned}$$

Looking over these formulas, it is immediate that we get a $*$-homomorphism $\Phi : C_c(\Gamma_{\mathscr{U}}, \varphi_\nu) \to A(\nu)$ given by

$$\Phi(f)_{ij}(x) = f(i,x,j).$$

Just as observed in Remark 3 in [23], this map extends to an isomorphism.

PROPOSITION 4.3 ([23], Remark 3). *The map Φ extends to an isomorphism of $C^*(\Gamma_{\mathscr{U}}, \varphi_\nu)$ onto $A(\nu)$.*

The proof is essentially the same, but easier, than the proof of Theorem 3.4 that is given below.

5. THE DIXMIER–DOUADY CLASS OF $C^*(\Sigma_c)$

Given a locally compact space X, a locally compact abelian group G and a 2-cocycle $c \in Z^2(\mathscr{U}, \mathscr{G})$, let ν^c be as in (3.4). We can form the associated Raeburn–Taylor twisted groupoid $(\Gamma_{\mathscr{V}}, \varphi_{\nu^c})$. We want to verify that $C^*(\Gamma_{\mathscr{V}}, \varphi_{\nu^c})$ is a continuous-trace C^*-algebra with Dixmier–Douady class $[\nu^c] = [m_*([c])]$, and then to realize $C^*(\Gamma_{\mathscr{V}}, \varphi_{\nu^c})$ concretely as $A(\nu^c)$. Unfortunately we cannot refer directly to Proposition 5.40 in [25] for this because $\mathscr{V}$ is not a cover of $\Gamma_{\mathscr{U}} \times X$ by precompact sets and the proof of Proposition 5.40 in [25] assumes this, even if it is not mentioned in the statement. So in this section we show that $C^*(\Gamma_{\mathscr{V}}, \varphi_{\nu^c})$ is continuous-trace with the desired Dixmier–Douady class for the given cover $\mathscr{V}$, and then use this to prove Theorem 3.4.

We let $A_1(\nu^c)$ be defined just as in Section 4. We want to define $(f_{ij})(g_{ij})$ to be the matrix (h_{ij}) defined by the equation (4.1). To see that this determines a binary operation on $A_1(\nu^c)$ we need the following analogue of Proposition 5.39 in [25].

LEMMA 5.1. *Let $f = (f_{ij})$ and $g = (g_{ij})$ be elements of $A_1(\nu^c)$. Define*

$$h_{ik}(\tau, x) = \sum_j f_{ij}(\tau, x) g_{jk}(\tau, x) \overline{\nu^c_{ijk}(\tau, x)}.$$

Then $h_{ik} \in C_0(\widehat{G} \times X)$ and vanishes off V_{ik}.

Proof. Clearly, h_{ik} vanishes off V_{ik}. Since each point x in the compact set $\overline{U}_{ij}$ has a neighborhood W that meets only finitely many U_l, there is a finite set F such that

$$h_{ik}(\tau, x) = \sum_{j \in F} f_{ij}(\tau, x) g_{jk}(\tau, x) \overline{\nu^c_{ijk}(\tau, x)}$$

for all $(\tau, x) \in V_{ij} = \widehat{G} \times U_{ij}$.

Since each summand is in $C_0(\widehat{G} \times X)$, it will suffice to see that h_{ik} is continuous on $\widehat{G} \times X$. Suppose that $(\tau_n, x_n) \to (\tau_0, x_0)$. It is enough to show that $h_{ik}(\tau_n, x_n) \to h_{ik}(\tau_0, x_0)$. For this, it suffices to consider each summand

$$a_j(\tau, x) = \begin{cases} f_{ij}(\tau, x) g_{jk}(\tau, x) \overline{\nu^c_{ijk}(\tau, x)} & \text{if } x \in U_{ijk}, \\ 0 & \text{otherwise.} \end{cases}$$

We clearly have $h_{ik}(\tau_n, x_n) \to h_{ik}(\tau_0, x_0)$ if $(\tau_0, x_0) \in V_{ijk}$ or if $(\tau_0, x_0) \notin \overline{V}_{ijk}$. So we suppose that $(\tau_0, x_0) \in \overline{V}_{ijk} \setminus V_{ijk} = \widehat{G} \times (\overline{U}_{ijk} \setminus U_{ijk})$.

We suppose that $a_j(\tau_n, x_n) \not\to a_j(\tau_0, x_0) = 0$ and derive a contradiction. By passing to a subsequence, we can assume that $|a_j(\tau_n, x_n)| \geqslant \varepsilon > 0$ for all n. Since $x_0 \notin U_{ijk}$, we can assume by symmetry that $x \notin U_{ij}$ and hence that $x_0 \in \overline{U}_{ij} \setminus U_{ij}$. Since f_{ij} is continuous on X and vanishes off U_{ij}, we have $f_{ij}(x_n) \to 0$. Hence $a_j(\tau_n, x_n) \to 0$, a contradiction. ∎

Since there is no difficulty with the involution as defined in (4.2), we see that $A_1(\nu^c)$ is a $*$-algebra just as in Section 4. We get seminorms $\|f\|_{(\tau,x)} = \|\pi_{(i,(\tau,x))}(f)\|$ almost exactly as in Proposition 5.40 in [25], and let

$$A(\nu^c) := \{f \in A_1(\nu^c) : (\tau, x) \mapsto \|f\|_{(\tau,x)} \text{ vanishes at infinity}\}.$$

Just as in the proof of Proposition 5.40 in [25], $A(\nu^c)$ is complete with respect to the norm

$$\|f\| = \sup_{(\tau,x)} \|f\|_{(\tau,x)},$$

and has spectrum identified (topologically) with $\widehat{G} \times X$. Moreover, we have the following analogue of the Raeburn–Taylor result.

LEMMA 5.2. *The C^*-algebra $A(\nu^c)$ has continuous trace with spectrum $\widehat{G} \times X$ and Dixmier–Douady class $\delta(A(\nu^c)) = [\nu^c]$.*

Proof. We have already seen that $A(\nu^c)$ is a C^*-algebra with Hausdorff spectrum. We continue by making the necessary modifications to the proof of Proposition 5.40 in [25]. Let $\{O_n\}$ be a locally finite cover of $\widehat{G}$ with open, precompact sets. To show that $A(\nu^c)$ has continuous trace, we show that it has local rank-one projections as required by Definition 5.13 in [25]. Let $(\tau, x) \in \widehat{G} \times X$, say with $(\tau, x) \in O_n \times U_i$. Let $\phi \in C_c^+(O_n \times U_i)$ be such that $\phi \equiv 1$ near (τ, x). Let

$$p_{jk}(x) = \begin{cases} \phi(x) & \text{if } j = i = k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then $p = (p_{jk}) \in A(\nu^c)$ and

$$\pi_{(i,(\tau',x'))}(p)$$

is a rank-one projection for (τ', x') near (τ, x). This suffices.

We will calculate $\delta(A(\nu^c))$ using Lemma 5.28 in [25]. Using the shrinking lemma (see Lemma 4.32 in [25]), we can find compact sets $F_{n,i}$ in $W_{n,i} := O_n \times U_i$ such that the interiors of the $F_{n,i}$ cover $\widehat{G} \times X$. Since $\mathscr{W} = \{W_{n,i}\}$ is a refinement of $\mathscr{V}$, $[\nu^c]$ is represented by the cocycle in $Z^2(\mathscr{W}, \mathscr{S})$ given by

$$\widetilde{\nu^c}_{(n,i)(m,j)(l,k)}(\tau, x) = \nu^c_{ijk}(\tau, x).$$

Let $\phi_{n,i} \in C_c^+(W_{n,i})$ be such that $\phi_{n,i} \equiv 1$ on $F_{n,i}$. Then as above, we get $p(n,i) \in A(\nu^c)$ such that

$$\pi_{(i,(\tau,x))}(p(n,i))$$

is a rank-one projection for all $(\tau, x) \in F_{n,i}$. Similarly, let $\phi_{(n,i)(m,j)}$ be an element of $C_c^+(W_{(n,i)(m,j)})$ which is identically one on $F_{(n,i)(m,j)}$. Then we get $v((n,i),(m,j)) \in A(\nu^c)$ with

$$v((n,i),(m,j))_{rs}(\tau, x) = \begin{cases} \phi_{(n,i)(m,j)}(\tau, x) & \text{if } r = i \text{ and } s = j, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then we check that

$$\pi_{(i,(\tau,x))}(v((n,i),(m,j))v((n,i),(m,j))^*) = \pi_{(i,(\tau,x))}(p(n,i)),$$

while, for all $(\tau,x) \in F_{(n,i)(m,j)}$,

$$\pi_{(i,(\tau,x))}(v((n,i),(m,j))^*v((n,i),(m,j))) = \pi_{(i,(\tau,x))}(p(m,j)).$$

Note that the situation is symmetric as $\pi_{(j,(\tau,x))}$ is equivalent to $\pi_{(i,(\tau,x))}$ if $x \in U_{ij}$.

If $(\tau,x) \in F_{(n,i)(m,j)(l,k)}$, then

$$\phi_{(n,i)(m,j)}(\tau,x) = \phi_{(m,j),(l,k)}(\tau,x) = \phi_{(n,i),(l,k)}(\tau,x) = 1.$$

Thus we have the following that completes the proof:

$$\begin{aligned}
&[v((n,i),(m,j))v((m,j),(l,k))]_{rs}(\tau,x) \\
&= \sum_a \overline{\nu^c_{rsa}(\tau,x)}v((n,i),(m,j))_{ra}(\tau,x)v((m,j),(l,k))_{as}(\tau,x) \\
&= \begin{cases} 0 & \text{if } r \neq i \text{ or } s \neq l, \text{ and} \\ \overline{\nu^c_{ijk}(\tau,x)}\phi_{(n,i)(m,j)}(\tau,x)\phi_{(m,j)(l,k)}(x,\tau) & \text{if } r = i \text{ and } s = l; \end{cases} \\
&= \begin{cases} 0 & \text{if } r \neq i \text{ or } s \neq l, \text{ and} \\ \overline{\nu^c_{ijk}(\tau,x)}\phi_{(n,i)(l,k)}(\tau,x) & \text{if } r = i \text{ and } s = l; \end{cases} \\
&= \overline{\nu^c_{ijk}(\tau,x)}v((n,i),(l,k))_{rs}(\tau,x) = \overline{\widetilde{\nu^c}_{(n,i)(m,j)(l,k)}(\tau,x)}v((n,i),(l,k))(\tau,x). \qquad \blacksquare
\end{aligned}$$

With these modifications in place, we can prove Theorem 3.4.

Proof of Theorem 3.4. By Lemma 5.2, it suffices to produce an isomorphism $\Phi : C^*(\Sigma_c) \to A(\nu^c)$ that intertwines each $\mathrm{Ind}((i,x),\tau)$ with $\pi_{(i,(\tau,x))}$. We use the Fourier transform. If $f \in C_c(\Sigma_c)$, then we define

$$\Phi(f)(i,(\tau,x),j) = \int_G \tau(g)f(g,(i,x,j))\,\mathrm{d}\mu(g).$$

To see that $\Phi(f) \in A(\nu^c)$, first suppose that there exist $\phi \in C_c(G)$ and $h \in C_c(\Gamma_{\mathscr{U}})$ such that $f(g,(i,x,j)) = \phi(g)h(i,x,j)$. Then

$$\Phi(f)(i,(\tau,x),j) = \widehat{\phi}(\tau)h(i,x,j),$$

and $\Phi(f) \in A(\nu^c)$ because $\widehat{\phi} \in C_0(\widehat{G})$. Since finite sums of such functions are dense in the inductive limit topology, we deduce that $\Phi(f) \in A(\nu^c)$ for all f.

Note

$$\begin{aligned}
\Phi(f^*(i,(\tau,x),j)) &= \int_G \tau(g)\overline{f(-g-c_{iji}(x),(j,x,i))}\,\mathrm{d}\mu(g) \\
&= \overline{\int_G \tau(g)f(g-c_{iji}(x),(j,x,i))\,\mathrm{d}\mu(g)} \\
&= \overline{\tau(c_{iji}(x))}\,\overline{\int_G \tau(g)f(g,(j,x,i))\,\mathrm{d}\mu(g)} \\
&= \nu^c_{iji}(\tau,x)\overline{\Phi(f)(j,(\tau,x),i))} = \Phi(f)^*(i,(\tau,x),j).
\end{aligned}$$

Hence Φ is $*$-preserving.

To see that it is multiplicative, we use (3.2) at the second equality, and Fubini's theorem and invariance of Haar measure at the third to calculate:

$$\begin{aligned}
&\Phi(f*f')(i,(\tau,x),j) \\
&\quad= \int_G \tau(g)f*f'(g,(i,x,j))\,\mathrm{d}\mu(g) \\
&\quad= \sum_k \int_G\int_G \tau(g)f(h,(i,x,k))f'(g-h-c_{ikj}(x),(k,x,j))\,\mathrm{d}\mu(h)\,\mathrm{d}\mu(g) \\
&\quad= \sum_k \int_G\int_G \tau(g+h+c_{ikj}(x))f(h,(i,x,k))f'(g,(k,x,j))\,\mathrm{d}\mu(g)\,\mathrm{d}\mu(h) \\
&\quad= \sum_k \tau(c_{ikj}(x))\Phi(f)(i,(\tau,x),k)\Phi(f')(k,(\tau,x),j) \\
&\quad= \sum_k \Phi(f)(i,(\tau,x),k)\Phi(f')(k,(\tau,x),j)\varphi_{\nu^c}((i,(\tau,x),k),(k,(\tau,x),j)) \\
&\quad= \Phi(f)*\Phi(f')(i,(\tau,x),j).
\end{aligned}$$

It remains to see that Φ is isometric and surjective. It follows from Lemma 3.2, that $\mathrm{Ind}((i,x),\tau)(f)$ is equivalent to multiplication by the matrix

$$[\tau(c_{ijk})(x)\Phi(f)].$$

Since $\tau(c_{ijk}(x)) = \overline{\nu^c_{ijk}(\tau,x)}$, we see that

$$\mathrm{Ind}((i,x),\tau)(f) = \pi_{(i,(\tau,x))}(\Phi(f)).$$

This shows immediately that Φ is isometric. It also shows that the image $\Phi(C^*(\Sigma_c))$ is a rich subalgebra (in the sense of Definition 11.1.1 in [2]) of the continuous-trace C^*-algebra $A(\nu^c)$. It follows that Φ is surjective by Proposition 11.1.6 in [2]. ∎

6. GROUPOIDS ASSOCIATED TO LOCAL HOMEOMORPHISMS

In this section we extend our results from Section 3 to a more general setting. Let $\psi : Y \to X$ be a local homeomorphism and form the principal groupoid

$$R(\psi) = \{\, (x,y) \in Y \times Y : \psi(x) = \psi(y) \,\}.$$

Let G be a locally compact abelian group. Given a unit-space-preserving groupoid extension

$$\begin{array}{ccccc} G \times Y & \xhookrightarrow{\quad\iota\quad} & \Sigma & \xrightarrow{\quad\pi\quad} & R(\psi) \\ & \searrow\!\searrow & \downarrow\downarrow & \swarrow\!\swarrow & \\ & & Y & & \end{array} \tag{6.1}$$

such that $\iota(g, r(\sigma))\sigma = \sigma\iota(g, s(\sigma))$ for all $g \in G$ and $\sigma \in \Sigma$, the groupoid Σ is a groupoid with central isotropy. As described in detail in Appendix A, Σ is a G-twist over $R(\psi)$. So Σ is a principal G-bundle over $R(\psi)$ with G action

$$g \cdot \sigma = \iota(g, r(\sigma))\sigma = \sigma\iota(g, s(\sigma)) = \sigma \cdot g.$$

We endow Σ with the Haar system $\{\lambda^y\}$ given by

$$\int_\Sigma f(\sigma)\,\mathrm{d}\lambda^y(\sigma) := \sum_{r(\sigma)=y} \int_G f(g \cdot \sigma)\,\mathrm{d}\mu(g) = \sum_{r(\sigma)=y} \int_G f(\sigma \cdot g)\,\mathrm{d}\mu(g). \tag{6.2}$$

If π has a continuous section $\kappa : R(\psi) \to \Sigma$ (this is equivalent to π being trivial as a principal G-bundle), then Proposition A.6 shows that Σ is properly isomorphic to the extension $\Sigma(R(\psi), \varphi)$ constructed from a continuous (normalised) G-valued 2-cocycle $\varphi \in Z^2(R(\psi), G)$ as in Notation A.4.

To proceed, we need to assume that the map π in (6.1) has local sections: that is, for each $(x, y) \in R(\psi)$, there is a neighbourhood U of (x, y) on which there is a continuous map $s : U \to \Sigma$ satisfying $\pi \circ s = \mathrm{id}_U$. If G is a Lie group, then this is automatic due to the Palais slice theorem ([21], Section 4.1). But in addition, we need to guarantee that the collection of local sections is sufficiently robust to allow us to build an equivalent groupoid with a global section. To this end, we assume that X is *locally G-trivial*: every open cover of X has a refinement $\{W_i\}$ such that each $H^1(W_{ij}, \mathscr{G}) = \{0\}$. Equivalently, all locally trivial principle G-bundles over the double-overlaps W_{ij} are trivial. A special case where these assumptions automatically hold is when G is a Lie group and X admits good covers in the sense that every open over of X admits a refinement in which all nontrivial overlaps are contractible. This suffices: locally trivial principle G-bundles over a space Z have a classifying space BG so that bundle classes are parameterized by homotopy classes $[Z, BG]$; so if Z is contractible, then all bundles over Z are trivial. All differentiable manifolds admit good covers by Corollary I.5.2 in [1].

For our main result, we will need to verify that the Morita equivalences we will use preserve the identification of the spectra with $\widehat{G} \times X$ in each case. In particular, recall that a subset $U \subset Y = \Sigma^{(0)}$ is *full* if it meets every orbit:

equivalently, $\Sigma \cdot U = \Sigma^{(0)}$. In that case, Σ_U is a $(\Sigma, \Sigma(U))$-equivalence. Then induction from $C^*(\Sigma(U))$ to $C^*(\Sigma)$, Σ_U-Ind, induces the Rieffel homeomorphism of $C^*(\Sigma(U))^\wedge$ onto $C^*(\Sigma)^\wedge$. (For the basics on induced representations in this context, see Section 2 of [9].) Both these C^*-algebras have spectrum identified with $\widehat{G} \times X$, and the observation that the Rieffel homeomorphism is the identity with respect to these identifications follows immediately from the next lemma.

LEMMA 6.1. *If $y \in U$ and $\tau \in \widehat{G}$, then $\Sigma_U\text{-Ind}(\operatorname{Ind}^{\Sigma(U)}(y,\tau))$ is equivalent to $\operatorname{Ind}^{\Sigma}(y,\tau)$.*

Proof. As on p. 12 of [14] or equation (1.3) in [28], the $C_c(\Sigma(U))$-valued inner product on $C_c(\Sigma_U)$ is given by

$$\langle f_1 \,,\, f_2\rangle_{\Sigma(U)}(\gamma) = \int_{\Sigma} \overline{f_1(\sigma^{-1})} f_2(\sigma^{-1}\gamma)\, \mathrm{d}\lambda^{r(\gamma)}(\sigma).$$

Then $\Sigma_U\text{-Ind}\operatorname{Ind}^{\Sigma(U)}(y,\tau)$ acts by convolution on the completion of $C_c(\Sigma_U) \odot C_c(\Sigma(U)_y)$ with respect to the inner product

$$(f_1 \otimes k_1 \mid f_2 \otimes k_2) = (\langle f_2 \,,\, f_1\rangle_{\Sigma(U)} * k_1 \mid k_2)$$

which, by (2.2), is

$$= \int_G \int_{\Sigma(U)} \overline{k_2(\sigma^{-1})} \langle f_2 \,,\, f_1\rangle_{\Sigma(U)} * k_1(\sigma^{-1}\cdot g)\tau(g)\, \mathrm{d}\lambda_U^y(\sigma)\, \mathrm{d}\mu(g)$$

$$= \int_G \int_{\Sigma(U)} \int_{\Sigma(U)} \overline{k_2(\sigma^{-1})} \langle f_2 \,,\, f_1\rangle_{\Sigma(U)}(\eta) k_1(\eta^{-1}\sigma^{-1}\cdot g)\tau(g) \qquad \mathrm{d}\lambda_U^{s(\sigma)}(\eta)\tau(g)\, \mathrm{d}\lambda_U^y(\sigma)\, \mathrm{d}\mu(g)$$

which, after sending $\eta \mapsto \sigma^{-1}\eta$, is

$$= \int_G \int_{\Sigma(U)} \int_{\Sigma(U)} \overline{k_2(\sigma^{-1})} \langle f_2 \,,\, f_1\rangle_{\Sigma(U)}(\sigma^{-1}\eta) k_1(\eta^{-1}\cdot g)\tau(g) \qquad \mathrm{d}\lambda_U^{y}(\eta)\tau(g)\, \mathrm{d}\lambda_U^y(\sigma)\, \mathrm{d}\mu(g)$$

$$= \int_G \int_{\Sigma(U)} \int_{\Sigma(U)} \int_{\Sigma} \overline{f_2(\gamma^{-1}) k_2(\sigma^{-1})} f_1(\gamma^{-1}\sigma^{-1}\eta) k_1(\eta^{-1}\cdot g)\tau(g) \qquad \mathrm{d}\lambda^{s(\sigma)}(\gamma)\, \mathrm{d}\lambda_U^{y}(\eta)\, \mathrm{d}\lambda_U^y(\sigma)\, \mathrm{d}\mu(g)$$

which, after $\gamma \mapsto \sigma^{-1}\gamma$, is

$$= \int_G \int_{\Sigma(U)} \int_{\Sigma(U)} \int_{\Sigma} \overline{f_2(\gamma^{-1}\sigma) k_2(\sigma^{-1})} f_1(\gamma^{-1}\eta) k_1(\eta^{-1}\cdot g)\tau(g) \qquad \mathrm{d}\lambda^{y}(\gamma)\, \mathrm{d}\lambda_U^{y}(\eta)\, \mathrm{d}\lambda_U^y(\sigma)\, \mathrm{d}\mu(g)$$

$$= \int_G \int_\Sigma \overline{W(f_2 \otimes k_2)(\sigma^{-1})} W(f_1 \otimes k_1)(\sigma^{-1} \cdot g) \, d\lambda^y(\sigma) \tau(g) \, d\mu(g),$$

where

$$W(f \otimes k)(\sigma) := \int_{\Sigma(U)} f(\sigma\eta) k(\eta^{-1}) \, d\lambda^y_U(\eta).$$

It follows that W defines an isometry from the space of Σ_U-Ind$(\mathrm{Ind}^{\Sigma(U)}(y,\tau)$ into the space of $\mathrm{Ind}^\Sigma(y,\tau)$ that intertwines the two representations. Since the representations are irreducible, W must be a unitary and the representations must be equivalent. ∎

We also will need to examine the case of blowing up the unit space $\Sigma^{(0)}$ with respect to a locally finite cover $\mathscr{U} = \{U_i\}$. This gives us the equivalent groupoid

$$\Sigma' = \{ (i,\sigma,j) : \sigma \in \Sigma, r(\sigma) \in U_i \text{ and } s(\sigma) \in U_j \},$$

where the (Σ, Σ')-equivalence is given by

$$Z := \coprod \Sigma_{U_i} = \{ (i,\sigma) : \sigma \in \Sigma \text{ and } s(\sigma) \in U_i \}.$$

We endow Σ' with a Haar system $\underline{\lambda} = \{\underline{\lambda}^{(i,x)}\}$ just as in (6.2). As above, every irreducible representation of $C^*(\Sigma')$ is equivalent to one of the form $\mathrm{Ind}^{\Sigma'}((i,y),\tau)$ for $y \in \Sigma^{(0)}$ and $\tau \in \widehat{G}$. As before, we want the Rieffel homeomorphism induced by Z-Ind to preserve the identification of these spectra with $\widehat{G} \times X$. This is verified in the next lemma which is analogous to Lemma 6.1.

LEMMA 6.2. *With $\mathscr{U}$ and Σ' as above, we have* Z-Ind$(\mathrm{Ind}^{\Sigma'}((i,y),\tau)$ *equivalent to* $\mathrm{Ind}^\Sigma(y,\tau)$ *for all $y \in U_i$ and $\tau \in \widehat{G}$.*

Proof. The $C_c(\Sigma')$-valued inner product on $C_c(Z)$ is given by

$$\langle f_1 \,, f_2 \rangle_{\Sigma'}(i,\gamma,j) = \int_\Sigma \overline{f_1(i,\sigma^{-1})} f_2(j,\sigma^{-1}\gamma) \, d\lambda^{r(\gamma)}(\sigma).$$

Thus Z-Ind$(\mathrm{Ind}^{\Sigma'}((i,y),\tau))$ acts by convolution on the completion of $C_c(Z) \odot C_c(\Sigma'_{(i,y)})$ with respect to the inner product

$$(f_1 \otimes k_1 \mid f_2 \otimes k_2) = (\langle f_2 \,, f_1 \rangle_{\Sigma'} * k_1 \mid k_2)$$

which, in view of (2.2), is

$$= \int_G \int_{\Sigma'} \overline{k_2(j,\sigma^{-1},j)} \langle f_2 \,, f_1 \rangle_{\Sigma'} * k_1(j,\sigma^{-1} \cdot g, i) \tau(g) \, d\underline{\lambda}^{(i,y)}(i,\sigma,j) \, d\mu(g)$$

$$= \int_G \int_{\Sigma'} \int_{\Sigma'} \overline{k_2(j,\sigma^{-1},j)} \langle f_2 \,, f_1 \rangle_{\Sigma'}(j,\eta,l) k_1(l,\eta^{-1}\sigma^{-1}\cdot g, i)\tau(g) \, \mathrm{d}\underline{\lambda}^{(j,s(\sigma))}(j,\eta,l) \, \mathrm{d}\underline{\lambda}^{(i,y)}(i,\sigma,j) \, \mathrm{d}\mu(g)$$

which, after invoking left-invariance, is

$$\begin{aligned}
&\int_G \int_{\Sigma'} \int_{\Sigma'} \overline{k_2(j,\sigma^{-1},j)} \langle f_2 \,, f_1 \rangle_{\Sigma'}(j,\sigma^{-1}\eta,l) k_1(l,\eta^{-1}\cdot g, i)\tau(g) \, \mathrm{d}\underline{\lambda}^{(i,y)}(i,\eta,l) \, \mathrm{d}\underline{\lambda}^{(i,y)}(i,\sigma,j) \, \mathrm{d}\mu(g) \\
&= \int_G \int_{\Sigma'} \int_{\Sigma'} \int_{\Sigma} \overline{k_2(j,\sigma^{-1},j) f_2(j,\gamma^{-1})} f_1(l,\gamma^{-1}\sigma^{-1}\eta) k_1(l,\eta^{-1}\cdot g, i)\tau(g) \, \mathrm{d}\lambda^{s(\sigma)}(\gamma) \, \mathrm{d}\underline{\lambda}^{(i,y)}(i,\eta,l) \, \mathrm{d}\underline{\lambda}^{(i,y)}(i,\sigma,j) \, \mathrm{d}\mu(g) \\
&= \int_G \int_{\Sigma'} \int_{\Sigma'} \int_{\Sigma} \overline{k_2(j,\sigma^{-1},j) f_2(j,\gamma^{-1}\sigma)} f_1(l,\gamma^{-1}\eta) k_1(l,\eta^{-1}\cdot g, i)\tau(g) \, \mathrm{d}\lambda^{y}(\gamma) \, \mathrm{d}\underline{\lambda}^{(i,y)}(i,\eta,l) \, \mathrm{d}\underline{\lambda}^{(i,y)}(i,\sigma,j) \, \mathrm{d}\mu(g) \\
&= \int_G \int_{\Sigma} \overline{W(f_2 \otimes k_2)(\gamma^{-1})} W(f_1 \otimes k_1)(\gamma^{-1}\cdot g) \, \mathrm{d}\lambda^{y}(\gamma)\tau(g) \, \mathrm{d}\mu(g),
\end{aligned}$$

where

$$W(f \otimes k)(\gamma) = \int_{\Sigma'} f((i,\gamma)\cdot(i,\sigma,j)) k(j,\sigma^{-1},i) \, \mathrm{d}\underline{\lambda}^{(i,y)}(i,\sigma,j).$$

As in the proof of Lemma 6.1, W extends to an intertwining unitary implementing the desired equivalence. ∎

THEOREM 6.3. *Let Y and X be second-countable locally compact Hausdorff spaces with X locally G-trivial as defined above. Suppose that $\psi : Y \to X$ is a local homeomorphism, and let Σ be a groupoid extension as in* (6.1). *Then $C^*(\Sigma)$ has continuous trace with spectrum identified with $\widehat{G} \times X$ via $(\tau, x) \mapsto \mathrm{Ind}((i,y),\tau)$ for any $y \in \psi^{-1}(x)$. Furthermore, there is a locally finite open covering $\mathscr{W} = \{W_j\}_{j\in J}$ of X and a cocycle $c \in Z^2(\mathscr{W},\mathscr{G})$ (given in* (6.3) *below) such that the Dixmier–Douady invariant of $C^*(\Sigma)$ is given by the image of $m_*([c])$ in $H^3(\widehat{G} \times X, \mathbb{Z})$.*

Proof. Let $\{U_i\}$ be a family of open subsets of Y such that each $\psi|_{U_i}$ is injective, and the sets $\{\psi(U_i)\}$ cover X. (For example, any cover $\{U_i\}$ of Y by sets on which ψ is injective.) Since X is locally G-trivial, there is a locally finite refinement $\mathscr{W} = \{W_j\}_{j\in J}$ of the cover $\{\psi(U_i)\}$ of X such that $H^1(W_{ij},\mathscr{G}) = \{0\}$ for all i and j. Fix $r : J \to I$ such that each $W_j \subset \psi(U_{r(j)})$. For each j, let $V_j = \psi^{-1}(W_j) \cap U_{r(j)}$ so that ψ restricts to a homeomorphism of V_j onto W_j. Let $Y' = \bigcup V_j \subseteq Y$. Then Y' is open and meets every orbit in Y. Hence $\Sigma(Y')$ is equivalent to Σ and we can apply Lemma 6.1. (The Σ-orbits and $R(\psi)$-orbits coincide on Y.) We can blow up $\Sigma(Y')$ with respect to the cover $\{V_j\}$ to get an equivalent groupoid

$$\Sigma' = \{\, (i,\sigma,j) : \sigma \in \Sigma, r(\sigma) \in V_i \text{ and } s(\sigma) \in V_j \,\}$$

and then apply Lemma 6.2. If we let

$$R' = \{\, (i,(x,y),j) : \psi(x) = \psi(y), x \in V_i \text{ and } y \in V_j \,\},$$

then we obtain a generalised twist

$$\begin{array}{ccccc} G \times \coprod V_j & \overset{\iota'}{\hookrightarrow} & \Sigma' & \overset{\pi'}{\twoheadrightarrow} & R'. \\ & \searrow\!\!\searrow & \downarrow\!\downarrow & \swarrow\!\!\swarrow & \\ & & \coprod V_j & & \end{array}$$

The map $(x,y) \mapsto \psi(x)$ is a homeomorphism of $R(\psi) \cap (V_i \times V_j)$ onto W_{ij}. Hence $\pi^{-1}(R(\psi) \cap (V_i \times V_j))$ is a trivial bundle by our assumption on the $\{W_i\}$ and there is a local section κ_{ij} defined on $R(\psi) \cap (V_i \times V_j)$ (we may choose κ_{ii} to respect the identification of unit spaces). By definition of R' and Σ' each κ_{ij} determines a section $\kappa'_{ij} : \{i\} \times (R(\psi) \cap (V_i \times V_j)) \times \{j\} \to \Sigma'$ satisfying

$$\kappa'_{ij}(i,(x,y),j) = (i,\kappa_{ij}(x,y),j).$$

Since the domains of the κ'_{ij} are topologically disjoint in R', these κ'_{ij} assemble into a global section $\kappa' : R' \to \Sigma'$ for π'. Hence there is a continuous G-valued normalised cocycle $\varphi \in Z^2(R', G)$ such that $\Sigma' \cong \Sigma(R', \varphi)$ (see Proposition A.6 for details). There is a groupoid homomorphism $\tau : R' \to \Gamma_{\mathscr{W}}$ such that

$$\tau(i,(x,y),j) = (i,\psi(x),j)$$

and

$$\tau^{-1}(i,w,j) = (i,(x,y),j) \quad \text{if } x \in V_i \text{ and } y \in V_j \text{ satisfy } \psi(x) = w = \psi(y).$$

So, defining $\widetilde{\varphi} := \varphi \circ (\tau^{-1} \times \tau^{-1}) \in Z^2(\Gamma_{\mathscr{W}}, G)$, we obtain an isomorphism $\Sigma' \cong \Sigma(\Gamma_{\mathscr{W}}, \widetilde{\varphi})$. We define $c \in Z^2(\mathscr{W}, \mathscr{G})$ by

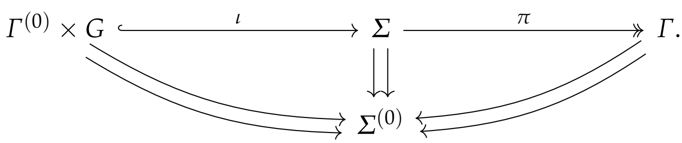
$$c_{ijk}(w) = \widetilde{\varphi}((i,w,j),(j,w,k)). \tag{6.3}$$

Then $\Sigma(\Gamma_{\mathscr{W}}, \widetilde{\varphi}) = \Sigma_c$.

The isomorphism of Σ' and Σ_c clearly intertwines the two Haar systems (see Proposition A.6), and therefore it intertwines the representations $\operatorname{Ind}^{\Sigma'}((i,\psi(x))$ and $\operatorname{Ind}^{\Sigma_c}((i,x),\tau)$. Combining this with Lemmas 6.2 and 6.1, the Dixmier–Douady class of $C^*(\Sigma)$ can be identified with that of $C^*(\Sigma_c)$. The result now follows from Theorem 3.4. ∎

Appendix A. EXTENSIONS AND COCYCLES

Let G be a locally compact Hausdorff abelian group and let Γ be a locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in\Gamma^{(0)}}$. Following [8], [13], and [29], we define an *extension* (or twist) by G over Γ to be a central groupoid extension $\Gamma^{(0)} \times G \xrightarrow{\iota} \Sigma \xrightarrow{\pi} \Gamma$ where $\Sigma^{(0)} = \Gamma^{(0)}$, ι is a groupoid homeomorphism onto a closed subgroupoid of Σ such that $\iota(u, 0_G) = u$ for all $u \in \Gamma^{(0)}$, π is an open, surjective groupoid homomorphism such that $\pi(u) = u$ for all $u \in \Gamma^{(0)}$, such that $\pi^{-1}(\Gamma^{(0)}) = \iota(\Gamma^{(0)} \times G)$, and such that $\iota(r(\sigma), g)\sigma = \sigma\iota(s(\sigma), g)$ for all $\sigma \in \Sigma$ and $g \in G$. We summarize all of this by drawing the diagram



Two twists by G are *properly isomorphic* if there is a groupoid isomorphism between them which preserves the inclusions of $\Gamma^{(0)} \times G$ and intertwines the surjections onto Γ. If $\Gamma^{(0)} \times G \xrightarrow{\iota} \Sigma \xrightarrow{\pi} \Gamma$ is an extension by G over Γ, then G acts freely and properly on Σ via $g\sigma := \iota(r(\sigma), g)\sigma$. Hence $\pi : \Sigma \to \Gamma$ is a principal G-bundle. Moreover, $G \setminus \Sigma \simeq \Gamma$ and π can be identified with the quotient map.

REMARK A.1. For completeness, we check that the action of G on Σ is proper. We need to show that the map $(g, \sigma) \mapsto (g\sigma, \sigma)$ is proper. Let K be a compact subset of Σ and let $\{(g_n, \sigma_n)\}_n$ be a sequence in the preimage of $K \times K$. Then $\{g_n\sigma_n\}_n \subset K$ and $\{\sigma_n\}_n \subset K$. Hence there is a subsequence $\{\sigma_{n_k}\}_{k\geqslant 1}$ such that $\sigma_{k_n} \to \sigma \in K$ and $g_{k_n}\sigma_{k_n} \to \sigma' \in K$. It follows that $\pi(\sigma) = \pi(\sigma')$. Hence there exists $g \in G$ such that $\sigma' = g\sigma$. Therefore

$$\iota(r(\sigma_{k_n}), g_{k_n}) = \iota(r(\sigma_{k_n}), g_{k_n})\sigma_{k_n}\sigma_{k_n}^{-1} \to g\sigma\sigma^{-1} = \iota(r(\sigma), g).$$

Since ι is a homeomorphism, it follows that $g_{k_n} \to g$, so the action of G on Σ is proper.

As in [29], the Baer sum $\Sigma_1 * \Sigma_2$ of two extensions $\Gamma^{(0)} \times G \longrightarrow \Sigma_i \longrightarrow \Gamma$ is the extension $\Gamma^{(0)} \times G \longrightarrow \Sigma_1 * \Sigma_2 \longrightarrow \Gamma$ with

$$\Sigma_1 * \Sigma_2 = \{(\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 \mid \pi_1(\sigma_1) = \pi_2(\sigma_2)\}/\sim,$$

where $(g\sigma_1, \sigma_2) \sim (\sigma_1, g\sigma_2)$. The map $\pi : \Sigma \to G$ is given by $\pi[(\sigma_1, \sigma_2)] = \pi_1(\sigma_1) = \pi_2(\sigma_2)$, and the inclusion $\iota : \Gamma^{(0)} \times G \to \Sigma$ is $\iota(u, g) = [(\iota_1(u, g), u)] = [(u, \iota_2(u, g))]$. The inverse of the extension $\Gamma^{(0)} \times G \xrightarrow{\iota} \Sigma \xrightarrow{\pi} \Gamma$ is the extension $\Gamma^{(0)} \times G \xrightarrow{\iota'} \widetilde{\Sigma} \xrightarrow{\pi} \Gamma$, where $\widetilde{\Sigma} = \Sigma$ as a groupoid, but $\iota'(g) = \iota(-g)$. The semidirect product $\Gamma \times G$ is called the *trivial twist*. The collection $T_\Gamma(G)$ of proper isomorphism classes of twists by G forms an abelian group under $*$ with neutral element $[\Gamma \times G]$.

REMARK A.2. Let Σ be an extension by G over a principal groupoid Γ; that is, the map $\gamma \mapsto (r(\gamma), s(\gamma))$ is injective; equivalently, $\mathcal{I}(\Gamma) = \Sigma^{(0)}$. Then $\iota(\Sigma^{(0)} \times G) = \mathcal{I}(\Sigma)$: $\iota(\Sigma^{(0)} \times G) \subseteq \mathcal{I}(\Sigma)$ is clear; and if $\sigma \in \mathcal{I}(\Sigma)$ then $\pi(\sigma) \in \mathcal{I}(\Gamma) = \Sigma^{(0)}$ and, hence $\sigma \in \iota(\Sigma^{(0)} \times G)$. As in the case of $\mathbb{T}$-groupoids, if in addition $\Sigma \setminus \Sigma^{(0)}$ is Hausdorff, then Σ has central isotropy.

A continuous 2-cocycle $\varphi : \Gamma^{(2)} \to G$ is a continuous function such that for all $(\gamma_0, \gamma_1, \gamma_2) \in \Gamma^{(3)}$,

$$\varphi(\gamma_0, \gamma_1) + \varphi(\gamma_0\gamma_1, \gamma_2) = \varphi(\gamma_1, \gamma_2) + \varphi(\gamma_0, \gamma_1\gamma_2).$$

A 2-cocycle φ is *normalised* provided that $\varphi(u, \gamma) = 0 = \varphi(\gamma, u)$ for all $\gamma \in \Gamma, u \in \Sigma^{(0)}$. The collection of all normalized cocycles forms a group denoted $Z^2(\Gamma, G)$. A 1-cochain is a continuous function $f : \Gamma \to G$. The associated 2-coboundary is the map $d^1 f \in Z^2(\Gamma, G)$ given by $(d^1 f)(\gamma_0, \gamma_1) = f(\gamma_0) + f(\gamma_1) - f(\gamma_0\gamma_1)$.

REMARK A.3. If Γ is étale, then any continuous 2-cocycle is cohomologous to a normalized continuous 2-cocycle. Indeed, note first that $\varphi(\gamma_0, u) = \varphi(u, \gamma_1)$ for all $(\gamma_0, u, \gamma_1) \in \Gamma^{(3)}$. Define a 1-cochain $f : \Gamma \to G$ by $f(\gamma) = 0_G$ for all $\gamma \notin \Sigma^{(0)}$ and $f(u) = \varphi(u, u)$ for $u \in \Sigma^{(0)}$. Since Γ is étale, $\Sigma^{(0)}$ is open so f is continuous. So $\psi(\gamma_0, \gamma_1) := \varphi(\gamma_0, \gamma_1) - (d^1 f)(\gamma_0, \gamma_1)$ defines a continuous normalized 2-cocycle cohomologous to φ.

NOTATION A.4. Recall from Lemma I.1.14 in [26] that given a normalized 2-cocycle φ on Γ, there is an extension $\Sigma(\Gamma, \varphi)$ of Γ by G given by

$$\Sigma(\Gamma, \varphi) := G \times \Gamma$$

with (g, γ_1) and (h, γ_2) composable if and only if γ_1 and γ_2 are composable, and $(g, \gamma_1)(h, \gamma_2) = (g + h + \varphi(\gamma_1, \gamma_2), \gamma_1\gamma_2)$. Then $(g, \gamma)^{-1}$ is given by $(-g - \varphi(\gamma^{-1}, \gamma), \gamma^{-1})$. The maps $\iota_\varphi : \Gamma^{(0)} \times G \to \Sigma(\Gamma, \varphi)$ and $\pi_\varphi : \Sigma(\Gamma, \varphi) \to \Gamma$ are defined by $\iota_\varphi(u, g) = (g, u)$ and $\pi_\varphi(g, \gamma) = \gamma$.

REMARK A.5. One can prove that the proper-isomorphism class of $\Sigma(\Gamma, \varphi)$ depends only on the cohomology class of φ. Indeed, if $\varphi_2 = \varphi_1 + d^1 f$, then the map $\psi : \Sigma(\Gamma, \varphi_1) \to \Sigma(\Gamma, \varphi_2)$ defined via $\psi(g, \gamma) = (g - f(\gamma), \gamma)$ is a proper isomorphism.

The following result generalizes the discussion on pages 130–131 of [18] (see also [26], Lemma I.1.14).

PROPOSITION A.6. *An extension Σ is properly isomorphic to $\Sigma(\Gamma, \varphi)$ for some continuous normalized 2-cocycle $\varphi \in Z^2(\Gamma, G)$ if and only if the map π admits a continuous cross section τ.*

Proof. Assume that $i : \Sigma(\Gamma, \varphi) \to \Sigma$ is a proper isomorphism, where φ is a continuous 2-cocycle. Then one can define a continuous cross section τ of π by $\tau(\gamma) = i(0_G, \gamma)$. Conversely, assume that $\tau : \Gamma \to \Sigma$ is a continuous cross section of π. By replacing τ with the map $\gamma \mapsto \tau(r(\gamma))^{-1}\tau(\gamma)$, we can assume without loss of generality that $\tau(u) = u$ for all $u \in \Sigma^{(0)}$. Then $\tau(\gamma_1)\tau(\gamma_2)\tau(\gamma_1\gamma_2)^{-1} \in \pi^{-1}(\Gamma^{(0)}) = \iota(\Gamma^{(0)} \times G)$ for all $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$. Since ι is a homeomorphism onto its image, there is a unique $(u, g) \in \Gamma^{(0)} \times G$ such that $\iota(u, g) = \tau(\gamma_1)\tau(\gamma_2)\tau(\gamma_1\gamma_2)^{-1}$. Note that $u = r(\gamma_1)$. Define $\varphi : \Gamma^{(2)} \to G$ by $\iota(r(\gamma_1), \varphi(\gamma_1, \gamma_2)) = \tau(\gamma_1)\tau(\gamma_2)\tau(\gamma_1\gamma_2)^{-1}$. Then φ is continuous. To see that φ is a 2-cocycle, first note that $\iota(r(\gamma_1), \varphi(\gamma_1, \gamma_2))\tau(\gamma_1, \gamma_2) = \tau(\gamma_1)\tau(\gamma_2)$ and that Σ is an extension. So if $(\gamma_0, \gamma_1, \gamma_2) \in \Gamma^{(3)}$, then

$$\begin{aligned}\iota(r(\gamma_0), \varphi(\gamma_1, \gamma_2) + \varphi(\gamma_0, \gamma_1\gamma_2)) &= \tau(\gamma_0)\tau(\gamma_1)\tau(\gamma_2)\tau(\gamma_0\gamma_1\gamma_2)^{-1} \\ &= \iota(r(\gamma_0), \varphi(\gamma_0\gamma_1, \gamma_2) + \varphi(\gamma_0, \gamma_1)).\end{aligned}$$

Hence φ is a 2-cocycle because ι is injective. Moreover φ is normalized since $\tau(u) = u$ for all $u \in \Sigma^{(0)}$. The map $\psi : \Sigma(\Gamma, \varphi) \to \Sigma$ defined by $\psi(g, \gamma) := g \cdot \tau(\gamma) = \iota(r(\gamma), g)\tau(\gamma)$ is a homeomorphism and a groupoid morphism. ∎

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MARIUS IONESCU, DEPARTMENT OF MATHEMATICS, UNITED STATES NAVAL ACADEMY, ANNAPOLIS, MD 21402 U.S.A.
E-mail address: ionescu@usna.edu

ALEX KUMJIAN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEVADA, RENO NV 89557 U.S.A.
E-mail address: alex@unr.edu

AIDAN SIMS, SCHOOL OF MATHEMATICS AND APPLIED STATISTICS, UNIVERSITY OF WOLLONGONG, NSW 2522, AUSTRALIA
E-mail address: asims@uow.edu.au

DANA P. WILLIAMS, DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NH 03755-3551 U.S.A.
E-mail address: dana.williams@Dartmouth.edu

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