J. OPERATOR THEORY
81:2(2019), 433–479
doi: 10.7900/jot.2018mar08.2219

REALIZATION OF RIGID C^*-TENSOR CATEGORIES VIA TOMITA BIMODULES

LUCA GIORGETTI and WEI YUAN

Communicated by Stefaan Vaes

ABSTRACT. Starting from a (small) rigid C^*-tensor category $\mathscr{C}$ with simple unit, we construct von Neumann algebras. These algebras are factors of type II or III$_\lambda$, $\lambda \in (0,1]$. The choice of type is tuned by the choice of Tomita structure (defined in the paper) on certain bimodules we use in the construction. If the spectrum is infinite we realize the whole tensor category as endomorphisms of these algebras. Furthermore, if the Tomita structure is trivial, the algebras that we get are an amplification of the free group factors with infinitely (possibly uncountably) many generators.

KEYWORDS: *C^*-tensor category, pre-Hilbert C^*-bimodule, full Fock space construction, free group factor.*

MSC (2010): 18D10, 46L08, 46L10, 46L54.

INTRODUCTION

Tensor categories (also called monoidal categories, see [16], [43]) are abstract mathematical structures which naturally arise in different ways when dealing with operator algebras on a Hilbert space $\mathcal{H}$. In this context, they are typically unitary and C^* [20], i.e., they come equipped with an involution $t \mapsto t^*$ and a norm $t \mapsto \|t\|$ on arrows, describing respectively the adjunction and the operator norm in $\mathcal{B}(\mathcal{H})$. Examples come from categories of endomorphisms of von Neumann algebras with tensor structure given on objects by the composition of endomorphisms, or from categories of bimodules with the relative tensor product (Connes fusion), or from representation categories of quantum groups, or again from the analysis of superselection sectors in quantum field theory (in the algebraic formulation of QFT due to Haag and Kastler) where the tensor product describes the composition of elementary particle states. See [45] and references therein. In the theory of subfactors [30], [33], tensor categories (or more generally a 2-category) can be associated to a given (finite index) subfactor by looking at its fusion graphs.

One of the most exciting additional structures that can be given on top of an abstract (C^*-)tensor category is an intrinsic notion of dimension [42], see also [10], [11], which associates a real number $d_X \geqslant 1$ to each object X of the category. The dimension is intrinsic in the sense that it is formulated by means of objects and arrows in the category only, more precisely by means of conjugate objects and (solutions of) the conjugate equations (also called zig-zag equations). In the case of the category of finite-dimensional Hilbert spaces V, the intrinsic dimension coincides with the usual notion of dimension of V as a vector space. The finiteness of the Jones index of a subfactor [38], [42], and the finiteness of the statistics of a superselection sector in QFT [40] are as well instances of the intrinsic dimension applied to concrete C^*-tensor categories. A C^*-tensor category whose objects have finite dimension (i.e., admit conjugate objects in the sense of the conjugate equations) is called rigid. Similar notions appearing in the literature on tensor categories, sometimes available in slightly more general contexts than ours, are pivotality, sphericality, see, e.g., [16], [45], and Frobenius duality [68].

The question (motivated by the previous discussion) of how to realize abstract rigid C^*-tensor categories as endomorphisms (or more generally bimodules) of operator algebras can be traced back to the seminal work of Jones on the index of type II_1 subfactors [30] and it has been studied by many authors over the years. In [27], Hayashi and Yamagami realize categories admitting an "amenable" dimension function as bimodules of the hyperfinite type II_1 factor. Later on, a different realization of arbitrary categories with countable spectrum (the set of isomorphism classes of simple objects) over amalgamated free product factors has been given by Yamagami in [67]. Both these works, and more generally the research in the direction of constructing operator algebras out of finite-dimensional combinatorial data, have their roots in the work of Popa on the construction of subfactors associated to standard lattices [48], i.e., on the reverse of the machinery which associates to a subfactor its standard invariant (the system of its higher relative commutants). Moreover, the powerful deformation-rigidity theory developed by Popa [49], [50], [51] makes it possible to completely determine the bimodule categories for certain classes of type II_1 factors (see for example [8], [18], [19], [65] for some explicit results on the calculation of bimodule categories). More recently, Brothier, Hartglass and Penneys proved in [4] that every countably generated rigid C^*-tensor category can be realized as bimodules of free group factors (see also [3], [22], [23], [26], [32], [37]).

The purpose of the present work is to re-interpret the construction in [4] via Tomita bimodules (defined in the paper, see Definition 1.14), to generalize it in order to obtain different types of factors (possibly III_λ, $\lambda \in (0,1]$), and to prove the universality of free group factors for rigid C^*-tensor categories with uncountable spectrum (see Section 3).

Unlike [4], we do not use the language of Jones' planar algebras [31] (a planar diagrammatic axiomatization of Popa's standard lattices). Instead, we prefer to work with the tensor category itself in order to exploit its flexibility: our main trick is to double the spectrum of the given category and consider the set of all "letters" corresponding to inequivalent simple objects and to their conjugate objects. This allows us to define Tomita structures (see Section 2) on certain pre-Hilbert bimodules H without having to cope with ambiguities arising from the choice of solutions of the conjugate equations in the case of self-conjugate objects (depending on their Frobenius–Schur indicator, i.e., depending on the reality or pseudo-reality of self-conjugate objects, in the terminology of [42]). The algebra $\Phi(H)''$ associated to the Tomita bimodule H (via Fock space construction, see below) is of type II or III_λ, $\lambda \in (0,1]$, depending on the choice of Tomita structure, both in the finite and infinite spectrum case. Even if we choose the *trivial* Tomita structure, our construction is different from the one of [4] in the sense that for finitely generated categories we obtain different algebras, e.g., the trivial category with only one simple object $\mathbb{1}$ produces the free group factor with two generators $L(F_2)$.

The paper is organized as follows. In Section 1, we consider pre-Hilbert C^*-bimodules, a natural non-complete generalization of Hilbert C^*-bimodules, which we use to treat finite-dimensional purely algebraic issues before passing to the norm or Hilbert space completions. Moreover, we define Tomita structures on a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule (hence we define Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodules), where $\mathfrak{A}$ is a von Neumann algebra (Definition 1.14). The terminology is due to the fact that every Tomita algebra is a Tomita $\mathbb{C}$-$\mathbb{C}$ bimodule. We derive the main properties of Tomita bimodules only in the case of semifinite von Neumann algebras $\mathfrak{A}$ and choosing a reference normal semifinite faithful tracial weight τ on $\mathfrak{A}$. We recall the definition of (full) Fock space $\mathscr{F}(H)$ (crucial in Voiculescu's free probability theory [66]) here associated to a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule H and express the Tomita–Takesaki's modular objects of the von Neumann algebras $\Phi(H)''$, generated by $\mathfrak{A}$ and by the creation and annihilation operators, in terms of the Tomita structure of H and of the chosen tracial weight on $\mathfrak{A}$ (Theorem 1.28). This is the main result of the section.

In Section 2, which is the main part of this work, we associate to every rigid C^*-tensor category, with simple unit, an abelian von Neumann algebra $\mathfrak{A}$ and a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule H (Proposition 2.2). The Tomita structure is determined by an arbitrary choice of strictly positive numbers λ_α for every $\alpha \in \mathscr{S}$, where $\mathscr{S}$ is a representative set of simple objects in the category (i.e., $\mathscr{S}$ labels the spectrum of the category). The algebra $\mathfrak{A}$ is endowed with a "canonical" weight induced by the quantum trace on the category (the standard left and right inverses of [42] are indeed tracial). The associated von Neumann algebra $\Phi(H)''$ on Fock space (or better, on its Hilbert space completion with respect to the canonical weight) turns out to be a factor (Theorem 2.7), and we study the type of $\Phi(H)''$ depending on the size of the spectrum and on the chosen Tomita structure on H (Proposition 2.10).

In Section 3, we study $\Phi(H)''$ in the case of categories with infinite spectrum $\mathscr{S}$ and assuming $\lambda_\alpha = 1$ for every $\alpha \in \mathscr{S}$ (trivial Tomita structure). We prove that the free group factor $L(F_{\mathscr{S}})$, either with countably many or with uncountably many generators, sits in a corner of $\Phi(H)''$ (Theorem 3.7). Easy examples of rigid C^*-tensor categories which are not amenable and have uncountable spectrum come from the algebraic group algebras of uncountable discrete and non-amenable groups, see Example 2.2 in [27]. E.g., consider the pointed discrete categories with fusion ring equal to $\mathbb{C}[\mathbb{R} \times F_2]$ or $\mathbb{C}[F_\infty]$, where $\mathbb{R}$ is endowed with the discrete topology and F_∞ is the free group with uncountably many generators, or the category of G-graded finite-dimensional Hilbert spaces, where G is an uncountable and non-amenable group. Obtaining a realization result for such categories, which were previously not covered in the literature, was the original motivation of our work.

In Section 4, again in the infinite spectrum case, we interpret the algebra $\Phi(H)''$ as a corner of a bigger auxiliary algebra $\Phi(\mathscr{C})$. We associate to every object W of $W(\mathscr{C})$ (where $W(\mathscr{C})$ is a strict C^*-tensor category equivalent to $\mathscr{C}$, see Section 2 for its definition) a non-unital endomorphism of $\Phi(\mathscr{C})$, and we cut it down to an endomorphism of $\Phi(H)''$ denoted by $F(W)$. We conclude by showing that F is a fully faithful unitary tensor functor from $W(\mathscr{C})$ into $\mathrm{End}(\Phi(H)'')$, hence a realization of the category $W(\mathscr{C})$, thus $\mathscr{C}$, as endomorphisms with finite index of a factor (Theorem 4.14) which can be chosen to be either of type II_∞ or of type III_λ, $\lambda \in (0,1]$. Moreover, the realization always happens on a σ-finite factor (e.g., $L(F_\infty)$ if we choose trivial Tomita structure) by composing F with a suitable equivalence of bimodule (or endomorphism) categories coming from the amplification (Theorem 4.15).

In Appendix A, as we could not find a reference, we define and study the amalgamated free product of arbitrary von Neumann algebras (not necessarily σ-finite), as we need in our construction when dealing with uncountably generated categories.

In Appendix B, we generalize some results concerning the Jones projection and the structure of amalgamated free products, well-known in the σ-finite case, to the case of arbitrary von Neumann algebras.

1. PRELIMINARIES

Let $\mathcal{N}$ be a unital C^*-algebra. A *pre-Hilbert $\mathcal{N}$-$\mathcal{N}$ bimodule* is an $\mathcal{N}$-$\mathcal{N}$ bimodule H with a sesquilinear $\mathcal{N}$-valued inner product $\langle \cdot \,|\, \cdot \rangle_{\mathcal{N}} : H \times H \to \mathcal{N}$, fulfilling

(i) $\langle \xi_1 \,|\, B \cdot \xi_2 \cdot A \rangle_{\mathcal{N}} = \langle B^* \cdot \xi_1 \,|\, \xi_2 \rangle_{\mathcal{N}} A$, for every $\xi_1, \xi_2 \in H$ and $A, B \in \mathcal{N}$;
(ii) $\langle \xi \,|\, \xi \rangle_{\mathcal{N}} \geqslant 0$ and $\langle \xi \,|\, \xi \rangle_{\mathcal{N}} = 0$ implies $\xi = 0$;
(iii) $\|A \cdot \xi\|_H \leqslant \|A\| \|\xi\|_H$, where $\|\xi\|_H = \|\langle \xi \,|\, \xi \rangle_{\mathcal{N}}\|^{1/2}$.

Note that (ii) implies that $\langle \xi_1 \,|\, \xi_2 \rangle_{\mathcal{N}} = \langle \xi_2 \,|\, \xi_1 \rangle_{\mathcal{N}}^*$. A pre-Hilbert $\mathcal{N}$-$\mathcal{N}$ bimodule H which is complete in the norm $\|\cdot\|_H$ is called a *Hilbert $\mathcal{N}$-$\mathcal{N}$ bimodule*. Let $\mathcal{L}(H)$

and $\mathcal{B}(H)$ be respectively the set of adjointable and bounded adjointable linear mappings from H to H. Condition (iii) implies that the left action of $\mathcal{N}$ on H is a $*$-homomorphism from $\mathcal{N}$ into $\mathcal{B}(H)$. If H is a Hilbert $\mathcal{N}$-$\mathcal{N}$ bimodule, then $\mathcal{B}(H) = \mathcal{L}(H)$ (see [39]).

REMARK 1.1. Let H be a pre-Hilbert $\mathcal{N}$-$\mathcal{N}$ bimodule and $T \in \mathcal{B}(H)$. Note that

$$\|T^*\xi\|_H^2 = \|\langle T^*\xi \mid T^*\xi\rangle_{\mathcal{N}}\| = \|\langle \xi \mid TT^*\xi\rangle_{\mathcal{N}}\| \leqslant \|TT^*\xi\|_H\|\xi\|_H \leqslant \|T\|\|T^*\xi\|_H\|\xi\|_H.$$

Therefore $T^* \in \mathcal{B}(H)$. Thus if $\overline{H}$ is the completion of H, then each $T \in \mathcal{B}(H)$ extends uniquely to an element in $\mathcal{B}(\overline{H})$. But $\mathcal{B}(H)$ is not a norm dense subalgebra of $\mathcal{B}(\overline{H})$ in general.

EXAMPLE 1.2. Let $\mathcal{N} = \mathbb{C}$, $H = l^1(\mathbb{N})$ and $\overline{H} = l^2(\mathbb{N})$. Note that $\mathbb{N} = \bigcup\limits_{n=0}^{\infty} \{k : 2^n \leqslant k < 2^{n+1} - 1\}$. Let $\xi_j = 1/\sqrt{2^n} \sum\limits_{i=0}^{2^n-1} e_{2^n+i}$ where $\{e_1, e_2, \dots\}$ is the canonical orthonormal basis of $l^2(\mathbb{N})$. Define a partial isometry $V \in \mathcal{B}(l^2(\mathbb{N}))$ by $Ve_j = \xi_j$.

Assume there exists $T \in \mathcal{B}(l^1(\mathbb{N}))$ and $\|T - V\| < 1/2$. Note that $\|Te_n - \xi_n\| < 1/2$, thus

$$\sum_{i=0}^{2^n-1} |\langle e_{2^n+i} \mid Te_n\rangle| \geqslant \sqrt{2^n} - \sum_{i=0}^{2^n-1} \left|\langle e_{2^n+i} \mid Te_n\rangle - \frac{1}{\sqrt{2^n}}\right| \geqslant \sqrt{2^n}(1 - \|Te_n - \xi_n\|) \geqslant \frac{\sqrt{2^n}}{2}.$$

We now choose inductively two subsequences of non-negative numbers $1 = n_0 < n_1 < n_2 < \cdots$ and $n(1) < n(2) < \cdots$ such that $n(k) \geqslant k^2$ and

$$\sum_{i \notin [n_{l-1}, n_l - 1]} |\langle e_i \mid Te_{n(l)}\rangle| < 1, \quad l = 1, 2, \dots.$$

Let $n(1) = 1$. Since $Te_1 \in l^1(\mathbb{N})$, we can choose $n_1 \in \mathbb{N}$ that satisfies the above condition. Assume that $\{n_1, \dots, n_k\}$ and $\{n(1), \dots, n(k)\}$ are chosen. Recall that T^* also maps $l^1(\mathbb{N})$ into $l^1(\mathbb{N})$ and $\{T^*e_i\}_{i=1}^{n_k-1} \subset l^1(\mathbb{N})$. We may choose $n(k+1) > \max\{(k+1)^2, n_k, n(k)\}$ such that $\sum\limits_{i \in [1, n_k - 1]} |\langle e_i \mid Te_{n(k+1)}\rangle| < 1/2$. Now it is clear that we can choose $n_{k+1} > n_k$ such that $\sum\limits_{i \notin [n_k, n_{k+1}-1]} |\langle e_i \mid Te_{n(k+1)}\rangle| < 1$. Let $\beta = \sum\limits_{k=1}^{\infty} 1/k^2 e_{n(k)} \in l^1(\mathbb{N})$. Then

$$\|T\beta\|_1 = \sum_{l=1}^{\infty} \sum_{i \in [n_{l-1}, n_l - 1]} \left|\left\langle e_i \mid \sum_{k=1}^{\infty} \frac{1}{k^2} Te_{n(k)}\right\rangle\right| \geqslant \sum_{l=1}^{\infty} \frac{\sqrt{2^{n(l)}}}{2l^2} - \sum_{l=1}^{\infty} \frac{1}{l^2} = +\infty.$$

Thus T is not in $\mathcal{B}(l^1(\mathbb{N}))$, and $\mathcal{B}(l^1(\mathbb{N}))$ is not dense in $\mathcal{B}(l^2(\mathbb{N}))$.

In the following, we shall also consider the Hilbert space completion H_φ of a pre-Hilbert $\mathcal{N}$-$\mathcal{N}$ bimodule H, associated to a choice of weight on $\mathcal{N}$.

NOTATION 1.3. Let H be a pre-Hilbert $\mathcal{N}$-$\mathcal{N}$ bimodule, φ a faithful weight on the C^*-algebra $\mathcal{N}$ and denote $\mathfrak{N}(H,\varphi) = \{\xi \in H : \varphi(\langle \xi \mid \xi \rangle_{\mathcal{N}}) < +\infty\}$. The formula

$$\langle \xi_1 \mid \xi_2 \rangle := \varphi(\langle \xi_1 \mid \xi_2 \rangle_{\mathcal{N}}), \quad \xi_1, \xi_2 \in \mathfrak{N}(H,\varphi) \tag{1.1}$$

defines a positive definite inner product on $\mathfrak{N}(H,\varphi)$. We denote the norm associated with this inner product by $\|\cdot\|_2$, i.e., $\|\xi\|_2 = \varphi(\langle \xi \mid \xi \rangle_{\mathcal{N}})^{1/2}$, and the completion of $\mathfrak{N}(H,\varphi)$ relative to this norm by H_φ.

By Proposition 1.2 in [39], each $T \in \mathcal{B}(H)$ corresponds to a bounded operator $\pi_\varphi(T)$ on H_φ such that $\pi_\varphi(T)\xi = T\xi$, $\xi \in \mathfrak{N}(H,\varphi)$, and $T \mapsto \pi_\varphi(T)$ is a $*$-representation of $\mathcal{B}(H)$ on H_φ.

LEMMA 1.4. *Let $\mathfrak{A}$ be a von Neumann algebra and H a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule. If φ is a normal semifinite faithful* (n.s.f.) *weight on $\mathfrak{A}$, then* $\operatorname{Ker}(\pi_\varphi) = \{0\}$.

Proof. If $0 \neq T \in \mathcal{B}(H)$, then there exists $\xi \in H$ such that $\langle T\xi \mid T\xi \rangle_{\mathfrak{A}} \neq 0$. Since φ is semifinite and faithful, we can choose a self-adjoint operator $A \in \mathfrak{A}$ such that $0 < \varphi(A^2) < \infty$ and $A\langle T\xi \mid T\xi \rangle_{\mathfrak{A}} A \neq 0$. Note that $\xi \cdot A \in \mathfrak{N}(H,\varphi)$, this implies that $\pi_\varphi(T) \neq 0$. ∎

Let $\mathfrak{A}$ be a von Neumann algebra and φ a n.s.f. weight on $\mathfrak{A}$. For the rest of this section, H is a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule with a distinguished "vacuum vector" $\Omega \in H$ such that $\langle \Omega \mid \Omega \rangle_{\mathfrak{A}} = I$ and $A \cdot \Omega = \Omega \cdot A$ for all $A \in \mathfrak{A}$ (as we shall be equipped with in the Fock space construction at the end of this section). Let

$$e_{\mathfrak{A}}(\xi) := \langle \Omega \mid \xi \rangle_{\mathfrak{A}} \Omega, \quad \xi \in H. \tag{1.2}$$

It is easy to check that $e_{\mathfrak{A}}(A \cdot \xi \cdot B) = A \cdot e_{\mathfrak{A}}(\xi) \cdot B$, where $A, B \in \mathfrak{A}$.

PROPOSITION 1.5. *The operator $e_{\mathfrak{A}}$ is a projection in $\mathcal{B}(H)$, whose range is $\mathfrak{A} \cdot \Omega$.*

Proof. Since $\langle \xi \mid \Omega \rangle_{\mathfrak{A}} \langle \Omega \mid \xi \rangle_{\mathfrak{A}} \leqslant \langle \xi \mid \xi \rangle_{\mathfrak{A}}$, we have $\|e_{\mathfrak{A}}(\xi)\|_H \leqslant \|\xi\|_H$. It is also clear that $\langle \beta \mid e_{\mathfrak{A}}(\xi) \rangle_{\mathfrak{A}} = \langle \beta \mid \Omega \rangle_{\mathfrak{A}} \langle \Omega \mid \xi \rangle_{\mathfrak{A}} = \langle e_{\mathfrak{A}}(\beta) \mid \xi \rangle_{\mathfrak{A}}$, thus $e_{\mathfrak{A}} = e_{\mathfrak{A}}^*$. For any $A \in \mathfrak{A}$, we have $e_{\mathfrak{A}}(A \cdot \Omega) = A \cdot \Omega$. ∎

For the rest of this section, $\mathcal{M} \subset \mathcal{B}(H)$ is a $*$-algebra containing $\mathfrak{A}$ (which we regard as represented on H via its left action). Then

$$E_0(T) := \langle \Omega \mid T \cdot \Omega \rangle_{\mathfrak{A}}, \quad T \in \mathcal{M} \tag{1.3}$$

is a conditional expectation from $\mathcal{M}$ onto $\mathfrak{A}$ (see Theorem 4.6.15 in [59]). It is easy to check that $e_{\mathfrak{A}} T e_{\mathfrak{A}} = E_0(T) e_{\mathfrak{A}}$, $T \in \mathcal{M}$. Also note that $\mathcal{M}\Omega = \operatorname{span}\{T \cdot \Omega : T \in \mathcal{M}\}$ is a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule and that $L^2(\mathfrak{A},\phi)$ is canonically embedded into H_ϕ for any n.s.f. weight ϕ on $\mathfrak{A}$ via $A \in \mathfrak{A} \mapsto A \cdot \Omega$.

LEMMA 1.6. *The operator $e_{\mathfrak{A}}$ defined in equation* (1.2) *extends to an orthogonal projection, still denoted by $e_{\mathfrak{A}}$, from H_φ onto the subspace $L^2(\mathfrak{A},\varphi)$. Furthermore, we have that $e_{\mathfrak{A}} \pi_\varphi(T) e_{\mathfrak{A}} = \pi_\varphi(E_0(T)) e_{\mathfrak{A}}$, for every $T \in \mathcal{M}$.*

Proof. $\langle \beta \mid e_{\mathfrak{A}} \pi_\varphi(T) e_{\mathfrak{A}} \xi \rangle = \varphi(\langle \beta \mid \Omega \rangle_{\mathfrak{A}} E_0(T) \langle \Omega \mid \xi \rangle_{\mathfrak{A}}) = \langle \beta \mid \pi_\varphi(E_0(T)) e_{\mathfrak{A}} \xi \rangle$, for all $\xi, \beta \in \mathfrak{N}(H, \varphi)$. Thus $e_{\mathfrak{A}} \pi_\varphi(T) e_{\mathfrak{A}} = \pi_\varphi(E_0(T)) e_{\mathfrak{A}}$. ∎

Note that $A \in \mathfrak{A} \mapsto \pi_\varphi(A) e_{\mathfrak{A}}$ is a $*$-isomorphism, namely the Gelfand–Naimark–Segal (GNS) representation of $\mathfrak{A}$ associated with φ. By Lemma 1.6, $e_{\mathfrak{A}} \pi_\varphi(\mathcal{M})'' e_{\mathfrak{A}} = \pi_\varphi(\mathfrak{A}) e_{\mathfrak{A}}$. For any $B \in \pi_\varphi(\mathcal{M})''$, let $E_\varphi(B)$ be the unique element in $\mathfrak{A}$ satisfying $\pi_\varphi(E_\varphi(B)) e_{\mathfrak{A}} = e_{\mathfrak{A}} B e_{\mathfrak{A}}$.

It is not hard to check that $B \mapsto E_\varphi(B)$ is a normal completely positive $\mathfrak{A}$-$\mathfrak{A}$ bimodule map, i.e., $E_\varphi(\pi_\varphi(A_1) B \pi_\varphi(A_2)) = A_1 E_\varphi(B) A_2$ for $A_1, A_2 \in \mathfrak{A}$. By Lemma 1.6 $E_\varphi(\pi_\varphi(T)) = E_0(T)$, $T \in \mathcal{M}$.

LEMMA 1.7. *For any other* n.s.f. *weight ϕ on $\mathfrak{A}$, $\widetilde{\phi} = \phi \circ E_\varphi$ is a normal semifinite weight on $\pi_\varphi(\mathcal{M})''$ such that $\widetilde{\phi}(\pi_\varphi(T)) = \phi(E_0(T))$, $T \in \mathcal{M}$. If $\mathcal{M}\Omega$ is dense in H (with respect to the norm $\|\cdot\|_H$), then there exists a unitary $U : H_\phi \to L^2(\pi_\varphi(\mathcal{M})'', \widetilde{\phi})$ such that $U^* \pi_{\widetilde{\phi}}(\pi_\varphi(T)) U = \pi_\phi(T)$ for every $T \in \mathcal{M}$, where $\pi_{\widetilde{\phi}}$ is the* GNS *representation of $\pi_\varphi(\mathcal{M})''$ associated with $\widetilde{\phi}$.*

Proof. We claim that $\mathcal{M}\Omega \cap \mathfrak{N}(H, \phi)$ is dense in H_ϕ with respect to the Hilbert space norm. Indeed, let $\xi \in \mathfrak{N}(H, \phi)$ and $K = \langle \xi \mid \xi \rangle_{\mathfrak{A}}^{1/2}$. If $T_m = mK(I + mK)^{-1}$, then $KT_1 \leqslant KT_2 \leqslant KT_3 \leqslant \cdots$ and $KT_m \to K$ in norm. Note that $\phi(T_m^2) < \infty$. Since ϕ is normal, for any $\varepsilon > 0$, we can choose m_0 such that

$$\|\xi - \xi \cdot T_{m_0}\|_2 = |\phi(\langle \xi \cdot (I - T_{m_0}) \mid \xi \cdot (I - T_{m_0}) \rangle_{\mathfrak{A}})|^{1/2} < \varepsilon.$$

Let $C = \phi(T_{m_0}^2)$ and $\beta \in \mathcal{M}\Omega$ such that $\|\langle \beta - \xi \mid \beta - \xi \rangle_{\mathfrak{A}}\| < \varepsilon^2 / C$. Therefore

$$\|\beta \cdot T_{m_0} - \xi \cdot T_{m_0}\|_2 = \phi(T_{m_0} \langle \beta - \xi \mid \beta - \xi \rangle_{\mathfrak{A}} T_{m_0})^{1/2} < \varepsilon.$$

Thus $\mathcal{M}\Omega \cap \mathfrak{N}(H, \phi)$ is a dense subspace of H_ϕ.

Similarly, for $B \in \pi_\varphi(\mathcal{M})''$ satisfying $\phi(E_\varphi(B^* B)) < \infty$, let $K = E_\varphi(B^* B)^{1/2}$. Repeating the argument above, for any $\varepsilon > 0$, we can find a positive operator $T_{m_0} \in \mathfrak{A}$ such that $\phi(T_{m_0}^2) < \infty$ and $\phi((I - T_{m_0}) E_\varphi(B^* B)(I - T_{m_0}))^{1/2} < \varepsilon$. Since $\phi(T_{m_0} E_\varphi(\cdot) T_{m_0})$ is a normal functional on $\pi_\varphi(\mathcal{M})''$, we can find $A \in \mathcal{M}$ such that $\phi(T_{m_0} E_\varphi([\pi_\varphi(A) - B]^* [\pi_\varphi(A) - B]) T_{m_0})^{1/2} \leqslant \varepsilon$. Thus $\{\pi_\varphi(T) : \phi(E_0(T^* T)) < \infty, T \in \mathcal{M}\}$ is dense in $L^2(\pi_\varphi(\mathcal{M})'', \widetilde{\phi})$.

Since $\langle A\Omega \mid B\Omega \rangle = \phi(E_0(A^* B)) = \phi(E_\varphi(\pi_\varphi(A^* B))) = \langle \pi_\varphi(A) \mid \pi_\varphi(B) \rangle$, for $A\Omega, B\Omega \in \mathcal{M}\Omega \cap \mathfrak{N}(H, \phi)$. The map $A\Omega \mapsto \pi_\varphi(A) \in L^2(\pi_\varphi(\mathcal{M})'', \widetilde{\phi})$ can be extended to a unitary from H_ϕ to $L^2(\pi_\varphi(\mathcal{M})'', \widetilde{\phi})$. Finally, note that

$$\pi_{\widetilde{\phi}}(\pi_\varphi(T)) U A\Omega = \pi_\varphi(TA) = U \pi_\phi(T) A\Omega, \quad A\Omega \in \mathcal{M}\Omega \cap \mathfrak{N}(H, \phi). \quad ∎$$

THEOREM 1.8. *Let φ and ϕ be two* n.s.f. *weights on $\mathfrak{A}$. If $\mathcal{M}\Omega$ is dense in H (with respect to the norm $\|\cdot\|_H$), then the map $\pi_\varphi(T) \mapsto \pi_\phi(T)$, $T \in \mathcal{M}$, extends to a $*$-isomorphism between the two von Neumann algebras $\pi_\varphi(\mathcal{M})''$ and $\pi_\phi(\mathcal{M})''$.*

Proof. By Lemma 1.7, we have normal $*$-homomorphisms $\rho_1 : \pi_\varphi(\mathcal{M})'' \to \pi_\phi(\mathcal{M})''$ and $\rho_2 : \pi_\phi(\mathcal{M})'' \to \pi_\varphi(\mathcal{M})''$ such that $\rho_1(\pi_\varphi(T)) = \pi_\phi(T)$ and $\rho_2(\pi_\phi(T)) = \pi_\varphi(T)$ for every $T \in \mathcal{M}$. Thus ρ_1 is a $*$-isomorphism and $\rho_2 = \rho_1^{-1}$. ∎

1.1. PRE-HILBERT BIMODULES WITH NORMAL LEFT ACTION.

DEFINITION 1.9. Let H be a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule. We say that the *left action is normal* if the map $A \in \mathfrak{A} \mapsto \langle \xi \,|\, A \cdot \xi \rangle_{\mathfrak{A}}$ is normal for each $\xi \in H$, i.e., $A \mapsto \langle \xi \,|\, A \cdot \xi \rangle_{\mathfrak{A}}$ is continuous from $(\mathfrak{A})_1$ to $\mathfrak{A}$ both in their weak operator topology, where $(\mathfrak{A})_1$ denotes the unit ball of $\mathfrak{A}$.

REMARK 1.10. (i) In general, $\pi_\varphi(\mathfrak{A})'' \neq \pi_\varphi(\mathfrak{A})$. However, if the left action is normal, then $\pi_\varphi(\mathfrak{A})'' = \pi_\varphi(\mathfrak{A})$ (see Proposition 7.1.15 in [34]).

(ii) For every $\beta, \xi \in H$, the map $A \in \mathfrak{A} \mapsto \langle \beta \,|\, A \cdot \xi \rangle_{\mathfrak{A}}$ is continuous from $(\mathfrak{A})_1$ to $\mathfrak{A}$ both in their weak operator topology.

(iii) For every $\xi \in H$, the map $A \in \mathfrak{A} \mapsto \langle \xi \,|\, A \cdot \xi \rangle_{\mathfrak{A}}$ is completely positive. Indeed, let $(K_{ij})_{i,j}$ be a positive element in $M_n(\mathfrak{A})$, then we only need to check that $\sum_{l=1}^{n} (\langle K_{li} \cdot \xi \,|\, K_{lj} \cdot \xi \rangle_{\mathfrak{A}})_{i,j} \geqslant 0$, and this is true by Lemma 4.2 in [39].

The following easy fact is used implicitly in the paper and the proof is left as an exercise for the reader.

PROPOSITION 1.11. *Let $\mathcal{D}$ be a subset of H. Assume that $\mathfrak{A} \cdot \mathcal{D} \cdot \mathfrak{A} = \operatorname{span}\{A \cdot \beta \cdot B : A, B \in \mathfrak{A}, \beta \in \mathcal{D}\}$ is dense in H. Then the left action is normal if and only if $A \in \mathfrak{A} \mapsto \langle \beta \,|\, A \cdot \beta \rangle_{\mathfrak{A}}$ is normal for each $\beta \in \mathcal{D}$.*

PROPOSITION 1.12. *Let H be a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule. If the left action is normal, then $A \mapsto \langle \xi \,|\, A \cdot \xi \rangle_{\mathfrak{A}}$ is continuous from $(\mathfrak{A})_1$ to $\mathfrak{A}$ both in their strong operator topology.*

Proof. Assume that A_α tends to 0 in the strong operator topology, then $A_\alpha^* A_\alpha$ converges to 0 in the weak operator topology. Therefore $\langle A_\alpha \cdot \xi \,|\, A_\alpha \cdot \xi \rangle_{\mathfrak{A}}$ also converges to 0 in the weak operator topology. From the inequality $\langle A_\alpha \cdot \xi \,|\, A_\alpha \cdot \xi \rangle_{\mathfrak{A}} \geqslant 1/\|\xi\|_H^2 \langle A_\alpha \cdot \xi \,|\, \xi \rangle_{\mathfrak{A}} \langle \xi \,|\, A_\alpha \cdot \xi \rangle_{\mathfrak{A}}$, by Lemma 5.3 in [39], we have that $\langle \xi \,|\, A_\alpha \cdot \xi \rangle_{\mathfrak{A}}$ converges to 0 in the strong operator topology. ∎

PROPOSITION 1.13. *Let H_1 and H_2 be two pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodules. If the left action on H_2 is normal, then $H_1 \otimes_{\mathfrak{A}} H_2$, the algebraic tensor product over $\mathfrak{A}$, is a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule with $\mathfrak{A}$-valued inner product given on simple tensors by*

$$\langle \xi_1 \otimes \beta_1 \,|\, \xi_2 \otimes \beta_2 \rangle_{\mathfrak{A}} := \langle \beta_1 \,|\, \langle \xi_1 \,|\, \xi_2 \rangle_{\mathfrak{A}} \cdot \beta_2 \rangle_{\mathfrak{A}}, \quad \xi_1, \xi_2 \in H_1,\ \beta_1 \beta_2 \in H_2.$$

Proof. We only need to show that if $\zeta = \sum_i \xi_i \otimes \beta_i$ satisfying $\langle \zeta \,|\, \zeta \rangle_{\mathfrak{A}} = 0$, then $\zeta = 0$ in $H_1 \otimes_{\mathfrak{A}} H_2$.

Since $\langle \beta \mid T \cdot \beta \rangle_{\mathfrak{A}} = 0$ where $\beta = (\beta_1, \dots, \beta_n) \in H_2^n$ and $T = (\langle \xi_i \mid \xi_j \rangle_{\mathfrak{A}})_{i,j} \in M_n(\mathfrak{A})$, $T^{1/2} \cdot \beta = 0$. Thus $\langle \beta \mid T^{1/2} \cdot \beta \rangle_{\mathfrak{A}} = 0$ and this implies $T^{1/4} \cdot \beta = 0$. By induction, we have $T^{1/2^n} \cdot \beta = 0$, $n = 1, 2, \dots$. Since $T^{1/2^n}$ tends to the range projection $P = (p_{ij})_{i,j}$ of T in the strong operator topology, we have that $\langle \beta \mid T^{1/2^n} \cdot \beta \rangle_{\mathfrak{A}}$ converges to $\langle \beta \mid P \cdot \beta \rangle_{\mathfrak{A}}$ in the strong operator topology. Thus $P \cdot \beta = 0$ and $\sum\limits_{k=1}^{n} p_{i,k} \cdot \beta_k = 0$, $i = 1, \dots, n$.

Note that

$$0 = (I - P)(\langle \xi_k \mid \xi_l \rangle_{\mathfrak{A}})_{k,l}(I - P) = \Big(\Big\langle \xi_i - \sum_k \xi_k \cdot p_{k,i} \mid \xi_j - \sum_l \xi_l \cdot p_{l,j} \Big\rangle_{\mathfrak{A}} \Big)_{i,j}.$$

Therefore $\xi_i = \sum\limits_k \xi_k \cdot p_{k,i}$, $i = 1, \dots, n$. Then it is clear that

$$\zeta = \sum_i \xi_i \otimes \beta_i = \sum_{i,k} \xi_k \cdot p_{k,i} \otimes \beta_i = \sum_{i,k} \xi_k \otimes p_{k,i} \cdot \beta_i = 0. \qquad \blacksquare$$

1.2. TOMITA PRE-HILBERT BIMODULES AND FOCK SPACE CONSTRUCTION. In the remainder of this section, $\mathfrak{A}$ is a semifinite von Neumann algebra and τ is a n.s.f. tracial weight on $\mathfrak{A}$. Recall that $\mathfrak{N}(\mathfrak{A}, \tau) = \{A \in \mathfrak{A} : \tau(A^*A) < \infty\}$ is an ideal. Let H be a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule H with normal left action of $\mathfrak{A}$. Note that

$$\langle \xi \mid \beta \cdot A \rangle = \tau(\langle \xi \mid \beta \rangle_{\mathfrak{A}} A) = \tau((\langle \beta \mid \xi \rangle_{\mathfrak{A}} A^*)^*) = \langle \xi \cdot A^* \mid \beta \rangle.$$

Thus the right action $\xi \mapsto \xi \cdot B$, $\xi \in \mathfrak{N}(H, \tau)$, gives a (normal) $*$-representation of the opposite algebra $\mathfrak{A}^{\mathrm{op}}$ on the Hilbert space H_τ (see Notation 1.3). Moreover, H_τ is an $\mathfrak{A}$-$\mathfrak{A}$ bimodule with scalar-valued inner product (also called correspondence in [7], [41], [56]).

DEFINITION 1.14. Let H be a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule with normal left action of $\mathfrak{A}$. We say that H is a *Tomita* $\mathfrak{A}$-$\mathfrak{A}$ *bimodule* if

(i) $\mathfrak{N}(H, \tau) = H$, and in this case we regard H as a dense subspace of H_τ;

(ii) H admits an involution S such that $S(A \cdot \xi \cdot B) = B^* \cdot S(\xi) \cdot A^*$ and $S^2(\xi) = \xi$, for every $\xi \in H$ and $A, B \in \mathfrak{A}$;

(iii) H admits a complex one-parameter group $\{U(\alpha) : \alpha \in \mathbb{C}\}$ of linear isomorphisms satisfying the following properties:

(a) for every ξ, $\beta \in H$, the map $\alpha \mapsto U(\alpha)\beta$ is continuous with respect to $\|\cdot\|_H$ and the function $\alpha \mapsto \langle \xi \mid U(\alpha)\beta \rangle$ is entire;

(b) $S(U(\alpha)\xi) = U(\overline{\alpha})S(\xi)$;

(c) $\langle \xi \mid U(\alpha)\beta \rangle = \langle U(-\overline{\alpha})\xi \mid \beta \rangle$;

(d) $\langle S(\xi) \mid S(\beta) \rangle = \langle \beta \mid U(-\mathrm{i})\xi \rangle$.

REMARK 1.15. It is clear that every Tomita algebra is a Tomita $\mathbb{C}$-$\mathbb{C}$ bimodule. The same definition of Tomita bimodule could be given for $\mathfrak{A}$ not necessarily semifinite and for τ not necessarily tracial. In order to derive the following properties, and for the purpose of this paper, we stick to the semifinite case.

Note that $\|U(\alpha)\beta\|_2^2 = \langle\beta \,|\, U(2\mathrm{i}\,\mathrm{Im}\,\alpha)\beta\rangle$ is a continuous function of α for each $\beta \in H$, hence it is locally bounded. By Appendix A.1 in [62], $\alpha \in \mathbb{C} \mapsto U(\alpha)\beta \in H_\tau$ is entire in the (Hilbert space) norm for every $\beta \in H$. Furthermore, we have the following fact.

LEMMA 1.16. *Let H be a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule. For any ξ, $\beta \in H$, the map $\alpha \mapsto \langle\xi \,|\, U(\alpha)\beta\rangle_{\mathfrak{A}} \in \mathfrak{A}$ is entire in the (operator) norm.*

Proof. By Definition 1.14(iii)(a), the map is bounded on any compact subset of $\mathbb{C}$. For every $A, B \in \mathfrak{N}(\mathfrak{A}, \tau)$, $\tau(A^*\langle\xi \,|\, U(\alpha)\beta\rangle_{\mathfrak{A}}B) = \langle\xi \cdot AB^* \,|\, U(\alpha)\beta\rangle$ is entire. By Appendix A.1 in [62] we have the statement. ∎

REMARK 1.17. Let $\overline{H}$ be the closure of H with respect to $\|\cdot\|_H$. Using a similar argument as in the proof of Lemma 1.16, we can show that the map $\alpha \mapsto U(\alpha)\xi \in \overline{H}$ is entire for every $\xi \in H$.

Now, $U(\alpha)$ and S can be viewed as densely defined operators with domain $H \subset H_\tau$, and we have the following result.

PROPOSITION 1.18. *Let H be a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule. Then S is preclosed and we use the same symbol S to denote the closure. If $J\Delta^{1/2}$ is the polar decomposition of S, then $J^2 = I$, $J\Delta J = \Delta^{-1}$, $U(\alpha) = \Delta^{\mathrm{i}\alpha}|_H$ and H is a core for each $\Delta^{\mathrm{i}\alpha}$, $\alpha \in \mathbb{C}$.*

Proof. By Definition 1.14(iii)(d), H is in the domain of S^*. Thus S^* is densely defined, S is preclosed and $U(-\mathrm{i}) \subset \Delta$. Hence Lemma 1.5 from Chapter VI in [62] implies $J^2 = I$, $J\Delta J = \Delta^{-1}$. By the same argument used in the proof of Theorem 2.2(ii) form Chapter VI in [62], it can be shown that the closure of $U(\mathrm{i}t)$ is self-adjoint for every $t \in \mathbb{R}$, $U(\alpha) = \Delta^{\mathrm{i}\alpha}|_H$ and H is a core for each $\Delta^{\mathrm{i}\alpha}$.

For the convenience of the reader, we sketch the proof. By condition (iii)(c), $\{U(t)\}_{t\in\mathbb{R}}$ can be extended to a one-parameter unitary group on H_τ. By Stone's theorem, there exists a non-singular (unbounded) self-adjoint positive operator K such that $K^{\mathrm{i}t}|_H = U(t)$. By Lemma 2.3 from Chapter VI in [62], $H \subset \mathcal{D}(K^{\mathrm{i}\alpha})$ for every $\alpha \in \mathbb{C}$ and $K^{\mathrm{i}\alpha}|_H = U(\alpha)$. Thus $K|_H = \Delta|_H$. By Lemma 1.21 from Chapter VI in [62], H is a core for K^α, $\alpha \in \mathbb{C}$. Thus $K \subseteq \Delta$. Since $\Delta^* = \Delta$ and $K^* = K$, we have $K = \Delta$. ∎

REMARK 1.19. By Proposition 1.18, $S|_H = JU(-\mathrm{i}/2)$, thus $JH = H$.

LEMMA 1.20. *Let H be a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule. Then*

$$S(A \cdot \xi \cdot B) = B^* \cdot S(\xi) \cdot A^*, \quad S^*(A \cdot \beta \cdot B) = B^* \cdot S^*(\beta) \cdot A^*,$$
$$\Delta^{\mathrm{i}\alpha}(A \cdot \zeta \cdot B) = A \cdot \Delta^{\mathrm{i}\alpha}(\zeta) \cdot B, \quad J(A \cdot \zeta \cdot B) = B^* \cdot J(\zeta) \cdot A^*.$$

where ξ, β, ζ are in the domains of S, S^* and $\Delta^{\mathrm{i}\alpha}$, respectively and A, $B \in \mathfrak{A}$.

Proof. By Proposition 1.18, we only need to show that the equations hold for $\xi \in H$. Note that $\langle A \cdot \beta \cdot B \,|\, S(\xi)\rangle = \tau(\langle\beta \,|\, A^* \cdot S(\xi) \cdot B^*\rangle_{\mathfrak{A}}) = \langle\xi \,|\, B^* \cdot S^*(\beta) \cdot A^*\rangle$.

This implies $\Delta^{i\alpha}(A \cdot \xi \cdot B) = A \cdot \Delta^{i\alpha}(\xi) \cdot B$. Thus $J(A \cdot \xi \cdot B) = \Delta^{1/2} S(A \cdot \xi \cdot B) = B^* \cdot J(\xi) \cdot A^*$. ∎

LEMMA 1.21. *Let H be a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule. Then for any $\xi, \beta \in H$, we have*

$$\langle \xi \,|\, U(\alpha)\beta \rangle_{\mathfrak{A}} = \langle U(-\overline{\alpha})\xi \,|\, \beta \rangle_{\mathfrak{A}}.$$

Proof. By Lemma 1.20, we have

$$\tau(\langle \xi \,|\, U(\alpha)\beta \rangle_{\mathfrak{A}} A) = \tau(\langle \xi \,|\, U(\alpha)(\beta \cdot A) \rangle_{\mathfrak{A}}) = \tau(\langle U(-\overline{\alpha})\xi \,|\, \beta \rangle_{\mathfrak{A}} A), \quad \forall A \in \mathfrak{A}. \quad ∎$$

LEMMA 1.22. *Suppose that H_1 and H_2 are two Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodules. Then*

$$S(\xi_1 \otimes \xi_2) := S_2(\xi_2) \otimes S_1(\xi_1), \quad U(\alpha)(\xi_1 \otimes \xi_2) := U_1(\alpha)(\xi_1) \otimes U_2(\alpha)(\xi_2),$$

define a conjugate linear map S and a one-parameter group $\{U(\alpha)\}_{\alpha \in \mathbb{C}}$, where S_i and $U_i(\alpha)$ are the involution and one-parameter group for H_i, $i = 1, 2$, and $\xi_1 \otimes \xi_2 \in H_1 \otimes_{\mathfrak{A}} H_2$. Furthermore $\{U(\alpha)\}$ satisfies the conditions (iii)(a) *and* (iii)(d) *in Definition* 1.14.

Proof. By Proposition 1.13, Lemma 1.20, S and $U(\alpha)$ are well-defined. It is easy to see that $\alpha \mapsto U(\alpha)(\xi_1 \otimes \xi_2)$ is continuous with respect to $\|\cdot\|_{H_1 \otimes_{\mathfrak{A}} H_2}$. By Lemma 1.20, $\langle \xi_1 \otimes \xi_2 \,|\, U_1(\alpha)\beta_1 \otimes U_2(\alpha)\beta_2 \rangle = \langle U_2(-\overline{\alpha})\xi_2 \,|\, \langle \xi_1 \,|\, U_1(\alpha)\beta_1 \rangle_{\mathfrak{A}} \beta_2 \rangle$. Then Lemma 1.16 implies that $\{U(\alpha)\}$ satisfies the condition (iii)(a) in Definition 1.14.

For ξ_i and $\beta_i \in H_i$, $i = 1, 2$, we have

$$\begin{aligned} \langle S_2(\xi_2) \otimes S_1(\xi_1) \,|\, S_2(\beta_2) \otimes S_1(\beta_1) \rangle &= \tau(\langle S_2(\xi_2) \,|\, S_2(\beta_2) \rangle_{\mathfrak{A}} \langle \beta_1 \,|\, U_1(-\mathrm{i})\xi_1 \rangle_{\mathfrak{A}}) \\ &= \tau(\langle \beta_2 \,|\, \langle \beta_1 \,|\, U_1(-\mathrm{i})\xi_1 \rangle_{\mathfrak{A}} U_2(-\mathrm{i})\xi_2 \rangle_{\mathfrak{A}}) \\ &= \langle \beta_1 \otimes \beta_2 \,|\, U_1(-\mathrm{i})\xi_1 \otimes U_2(-\mathrm{i})\xi_2 \rangle, \end{aligned}$$

i.e.,

$$\langle S(\xi_1 \otimes \xi_2) \,|\, S(\beta_1 \otimes \beta_2) \rangle = \langle \beta_1 \otimes \beta_2 \,|\, U(-\mathrm{i})(\xi_1 \otimes \xi_2) \rangle. \quad ∎$$

PROPOSITION 1.23. *Given a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule H, then $H^{\otimes_{\mathfrak{A}}^n}$, $n \geqslant 1$, is also a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule with the $\mathfrak{A}$-valued inner product given by linear extension of*

$$\langle \xi_1 \otimes \cdots \otimes \xi_n \,|\, \beta_1 \otimes \cdots \otimes \beta_n \rangle_{\mathfrak{A}} := \langle \xi_n \,|\, \langle \cdots \langle \xi_2 \,|\, \langle \xi_1 \,|\, \beta_1 \rangle_{\mathfrak{A}} \beta_2 \rangle_{\mathfrak{A}} \cdots \rangle_{\mathfrak{A}} \beta_n \rangle_{\mathfrak{A}},$$

$\xi_i, \beta_i \in H$, and the involution S_n and one-parameter group $\{U_n(\alpha)\}_{\alpha \in \mathbb{C}}$ are respectively given by $S_n(\xi_1 \otimes \cdots \otimes \xi_n) := S(\xi_n) \otimes \cdots \otimes S(\xi_1)$ and $U_n(\alpha)(\xi_1 \otimes \cdots \otimes \xi_n) := U(\alpha)(\xi_1) \otimes \cdots \otimes U(\alpha)(\xi_n)$.

Proof. By Proposition 1.13, $H^{\otimes_{\mathfrak{A}}^n}$ is a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule with normal left action. Conditions (i), (ii), (iii)(b) in the definition of Tomita bimodule are trivial to check. By Lemma 1.20 and Lemma 1.21, (iii)(c) is also clear. By Lemma 1.22 and induction on n, we get (iii)(a) and (iii)(d). ∎

Let $S_n = J_n \Delta_n^{1/2}$ be the polar decomposition of S_n, $n \geqslant 1$. By Proposition 1.18, $H^{\otimes_{\mathfrak{A}}^n}$ is a common core for Δ_n^t, $t \in \mathbb{R}$, and $\Delta_n^t(\xi_1 \otimes \cdots \otimes \xi_n) = \Delta^t(\xi_1) \otimes \cdots \otimes \Delta^t(\xi_n)$. Therefore $J_n(\xi_1 \otimes \cdots \otimes \xi_n) = \Delta_n^{1/2} S_n(\xi_1 \otimes \cdots \otimes \xi_n) =$

$J(\xi_n) \otimes \cdots \otimes J(\xi_1)$. We recall the following definition, see, e.g., Section 4.6 in [59] for the same definition given for Hilbert bimodules over a C^*-algebra.

DEFINITION 1.24. Given a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule H, the associated *Fock space*, denoted by $\mathscr{F}(H)$, is the algebraic direct sum of $H^{\otimes_{\mathfrak{A}}^n}$, i.e., $\mathscr{F}(H) := \bigoplus_{n\geqslant 0} H^{\otimes_{\mathfrak{A}}^n}$, where $H^{\otimes_{\mathfrak{A}}^0} := \mathfrak{A}$ with the $\mathfrak{A}$-valued inner product $\langle A_1 \mid A_2\rangle_{\mathfrak{A}} := A_1^* A_2$ for A_1, $A_2 \in \mathfrak{A}$.

By Proposition 1.13, $\mathscr{F}(H)$ is a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule. It is clear that the identity I of $\mathfrak{A}$ is a "vacuum vector" for $\mathscr{F}(H)$, and we will use Ω to denote this vector as in the beginning of this section.

For $\xi \in H$, the creation and annihilation operators $L(\xi)$, $L^*(\xi)$ are defined by

$$\begin{aligned} &L(\xi)A := \xi \cdot A, \quad &&L(\xi)\beta_1 \otimes \cdots \otimes \beta_n := \xi \otimes \beta_1 \otimes \cdots \otimes \beta_n,\\ &L^*(\xi)A := 0, &&L^*(\xi)\beta_1 \otimes \cdots \otimes \beta_n := \langle \xi \mid \beta_1\rangle_{\mathfrak{A}} \cdot \beta_2 \otimes \cdots \otimes \beta_n, \end{aligned}$$

where $A \in \mathfrak{A}$ and $\beta_i \in H$, $i = 1, \ldots, n$.

DEFINITION 1.25. Let $\Phi(H)$ be the $*$-subalgebra of $\mathcal{B}(\mathscr{F}(H))$ generated by $\mathfrak{A}$ (acting on $\mathscr{F}(H)$ from the left) and $\{\Gamma(\xi) := L(\xi) + L^*(S(\xi)) : \xi \in H\}$.

Let $\Phi(H)''$ be the von Neumann algebra generated by $\pi_\tau(\Phi(H))$ on the Hilbert space completion $\mathscr{F}(H)_\tau$ (see Notation 1.3) and let $\Phi(H)'$ be the commutant of $\Phi(H)''$.

PROPOSITION 1.26. *With the above notation, we have that $\Phi(H)\Omega = \mathscr{F}(H)$.*

Proof. By induction, it is not hard to check that $\mathfrak{A} \cup \bigcup_{n\geqslant 1} \{\xi_1 \otimes \cdots \otimes \xi_n : \xi_i \in H, i = 1, \ldots, n\} \subset \Phi(H)\Omega$. ∎

The GNS space $L^2(\mathfrak{A}, \tau)$ can be identified with the closure of the subspace spanned by $\mathfrak{N}(\mathfrak{A}, \tau)$ in $\mathscr{F}(H)_\tau$. Let $e_{\mathfrak{A}}$ be the orthogonal projection from $\mathscr{F}(H)_\tau$ onto $L^2(\mathfrak{A}, \tau)$. By Lemma 1.6, $T \in \Phi(H)'' \mapsto e_{\mathfrak{A}} T e_{\mathfrak{A}}|_{L^2(\mathfrak{A},\tau)} \in \pi_\tau(\mathfrak{A}) e_{\mathfrak{A}}|_{L^2(\mathfrak{A},\tau)}$, induces a normal completely positive map E_τ from $\Phi(H)''$ onto $\mathfrak{A}$. Since $A \in \mathfrak{A} \mapsto \pi_\tau(A)$ is a $*$-isomorphism, $E = \pi_\tau \circ E_\tau$ is a conditional expectation from $\Phi(H)''$ onto $\pi_\tau(\mathfrak{A})$.

By Proposition 1.23, $\mathfrak{N}(\mathscr{F}(H), \tau) = \mathfrak{N}(\mathfrak{A}, \tau) \oplus \bigoplus_{n\geqslant 1} H^{\otimes_{\mathfrak{A}}^n}$ is a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule with involution $S_0 \oplus \bigoplus_{n\geqslant 1} S_n$ and one-parameter group $\mathrm{Id} \oplus \bigoplus_{n\geqslant 1} U_n(\alpha)$, where $S_0 = J_0$ is the modular conjugation associated with τ on the Hilbert space $L^2(\mathfrak{A}, \tau)$ and Id is the identity map on $\mathfrak{A}$. Let $\mathcal{J} := J_0 \oplus \bigoplus_{n\geqslant 1} J_n$. By Proposition 1.23, the polar decomposition of the closure of $S_0 \oplus \bigoplus_{n\geqslant 1} S_n$ (which we shall denote again by $S_0 \oplus \bigoplus_{n\geqslant 1} S_n$) is $\mathcal{J}\left(I \oplus \bigoplus_{n\geqslant 1} \Delta_n^{1/2}\right)$.

PROPOSITION 1.27. *For each* $\xi \in H$, $A \in \mathfrak{A}$, *we have that* $\mathcal{J}A\mathcal{J}B = BA^*$, $\mathcal{J}A\mathcal{J}\beta_1 \otimes \cdots \otimes \beta_n = \beta_1 \otimes \cdots \otimes (\beta_n \cdot A^*)$, $\mathcal{J}\Gamma(\xi)\mathcal{J}B = B \cdot J(\xi)$ *and*

$$\mathcal{J}\Gamma(\xi)\mathcal{J}\beta_1\otimes\cdots\otimes\beta_n = \beta_1\otimes\cdots\otimes\beta_n\otimes J(\xi) + \beta_1\otimes\cdots\otimes\beta_{n-1}\cdot\langle J(\beta_n) \mid S(\xi)\rangle_{\mathfrak{A}}, \tag{1.4}$$

where $B \in \mathfrak{N}(\mathfrak{A},\tau)$. *Thus* $\mathcal{J}\Phi(H)\mathcal{J} \subset \Phi(H)'$ *and the following conditional expectation is faithful:*

$$E := \pi_\tau \circ E_\tau : \Phi(H)'' \to \pi_\tau(\mathfrak{A}).$$

Proof. All the equations can be checked by easy computation. Let $\beta \in H$ and $B \in \mathfrak{N}(\mathfrak{A},\tau)$. Note that $S|_H = JU(-\mathrm{i}/2)$, Lemma 1.21 implies

$$\begin{aligned}[\Gamma(\beta)\mathcal{J}\Gamma(\xi)\mathcal{J}]B &= \beta \otimes B \cdot J(\xi) + \langle S(\beta) \mid B \cdot J(\xi)\rangle_{\mathfrak{A}} \\ &= \beta \cdot B \otimes J(\xi) + \langle J(\beta \cdot B) \mid S(\xi)\rangle_{\mathfrak{A}} = [\mathcal{J}\Gamma(\xi)\mathcal{J}\Gamma(\beta)]B.\end{aligned}$$

A similar computation shows that $\mathcal{J}\Phi(H)\mathcal{J} \subset \Phi(H)'$.

Note that Proposition 1.26 and equation (1.4) imply that $\Phi(H)'L^2(\mathfrak{A},\tau) = \{T\zeta : \zeta \in L^2(\mathfrak{A},\tau), T \in \Phi(H)'\}$ is dense in $\mathscr{F}(H)_\tau$. Since $E_\tau(T^*T) = 0$, $T \in \Phi(H)''$, if and only if $Te_{\mathfrak{A}} = 0$, then E_τ and consequently E are faithful. ∎

Since E_τ is faithful, $\widetilde{\tau} := \tau \circ E_\tau$ is a n.s.f. weight on $\Phi(H)''$. By Lemma 1.7, the map $A\Omega \in \mathfrak{N}(\mathscr{F}(H),\tau) \mapsto \pi_\tau(A) \in L^2(\Phi(H)'',\widetilde{\tau})$ extends to a unitary U such that $U^*\pi_{\widetilde{\tau}}(\pi_\tau(T))U = \pi_\tau(T)$ for every $T \in \Phi(H)$, where $\pi_{\widetilde{\tau}}$ is the GNS representation of $\Phi(H)''$ associated with $\widetilde{\tau}$.

Let $\mathcal{S}$ be the involution in the Tomita theory associated with $\widetilde{\tau}$. We claim that $U\mathfrak{N}(\mathscr{F}(H),\tau)$ is a core for $\mathcal{S}$. Indeed, let T be a self-adjoint operator in $\Phi(H)''$ such that $\tau(E_\tau(T^2)) < \infty$. Let Q be the projection onto the kernel of $E_\tau(T^2)$. Since E_τ is faithful, we have $T\pi_\tau(Q) = 0$. Thus, for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\tau(E_\tau(T^2)(I-P)) < \varepsilon, \quad \tau(E_\tau(T(I-\pi_\tau(P))T)) < \varepsilon$$

where P is the spectral projection of $E_\tau(T^2)$ corresponding to $[\delta, \|E_\tau(T^2)\|]$. Note that $P \in \mathfrak{N}(\mathfrak{A},\tau)$, and $\pi_\tau(P\Phi(H)P)$ is dense in $\pi_\tau(P)\Phi(H)''\pi_\tau(P)$. Therefore there is a self-adjoint operator $K \in \pi_\tau(P\Phi(H)P)$ such that

$$\begin{aligned}&\tau(E_\tau((K - T\pi_\tau(P))^*(K - T\pi_\tau(P)))) < \varepsilon,\\ &\tau(E_\tau((K - \pi_\tau(P)T)^*(K - \pi_\tau(P)T))) < \varepsilon.\end{aligned}$$

Note that $P\Phi(H)P\Omega \subset \mathfrak{N}(\mathscr{F}(H),\tau)$. We have that the graph of $\mathcal{S}|_{U\mathfrak{N}(\mathscr{F}(H),\tau)}$ is dense in the graph of $\mathcal{S}$.

By the definition of $\mathcal{S}$, it is clear that $U^*\mathcal{S}U(B) = B^*$ and $U^*\mathcal{S}U(\xi) = \Gamma(\xi)^*\Omega = S(\xi)$, where $B \in \mathfrak{N}(\mathfrak{A},\tau)$ and $\xi \in H$. Note that

$$\xi_1 \otimes \xi_2 = \Gamma(\xi_1)\Gamma(\xi_2)\Omega - \langle S(\xi_1) \mid \xi_2\rangle_{\mathfrak{A}}.$$

Since $U^*\mathcal{S}U(\Gamma(\xi_1)\Gamma(\xi_2)\Omega) = \Gamma(\xi_2)^*\Gamma(\xi_1)^*\Omega = S(\xi_2) \otimes S(\xi_1) + \langle \xi_2 \mid S(\xi_1)\rangle_{\mathfrak{A}}$, we have $U^*\mathcal{S}U(\xi_1 \otimes \xi_2) = S_2(\xi_1 \otimes \xi_2)$. A similar computation and induction on the degree of the tensor, it is not hard to check that $U^*\mathcal{S}U(\xi_1 \otimes \cdots \otimes \xi_n) =$

$S_n(\xi_1 \otimes \cdots \otimes \xi_n)$, for any $\xi_i \in H$, $i = 1, \ldots, n$. Thus $S_0 \oplus \bigoplus_{n \geqslant 1} S_n = U^* \mathcal{S} U$, where $S_0 = J_0$.

Therefore we have the following result which shows how the Tomita structure of H determines the modular objects of the associated von Neumann algebra $\Phi(H)''$. This is the main result of this section and it is crucial for the determination of the type of factors constructed in the next section.

THEOREM 1.28. *The commutant of $\Phi(H)''$ is $\mathcal{J}\Phi(H)''\mathcal{J}$. For each $\xi \in H$, we have that $\sigma_t^{\tau \circ E_\tau}(\Gamma(\xi)) = \Gamma(U(t)\xi)$ for every $t \in \mathbb{R}$.*

Proof. For every $t \in \mathbb{R}$, $A \in \mathfrak{N}(\mathfrak{A}, \tau)$ and $\beta_1 \otimes \cdots \otimes \beta_n \in H^{\otimes_{\mathfrak{A}}^n}$, we have

$$\begin{aligned}\sigma_t^{\tau \circ E_\tau}(\Gamma(\xi))\beta_1 \otimes \cdots \otimes \beta_n &= \Big(I \oplus \bigoplus_{n \geqslant 1} \Delta_n\Big)^{it} \Gamma(\xi) \Big(I \oplus \bigoplus_{n \geqslant 1} \Delta_n\Big)^{-it} \beta_1 \otimes \cdots \otimes \beta_n \\ &= U(t)\xi \otimes \beta_1 \otimes \cdots \otimes \beta_n + \langle S(U(t)\xi) \mid \beta_1 \rangle_{\mathfrak{A}} \beta_2 \otimes \cdots \otimes \beta_n,\end{aligned}$$

and $\sigma_t^{\tau \circ E_\tau}(\Gamma(\xi))A = U(t)\xi \cdot A$. ∎

1.3. FOCK SPACE AND AMALGAMATED FREE PRODUCTS. We now consider a family $\{(H_i, S_i, \{U_i(\alpha)\}_{\alpha \in \mathbb{C}})\}_{i \in I}$ of Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodules and define $H := \bigoplus_{i \in I} H_i = \{(\xi_i)_{i \in I} : \xi_i \in H_i, \xi_i$ is zero for all but for a finite number of indices$\}$. It is obvious that H admits an involution $S := \bigoplus_i S_i$ and a one-parameter group $\Big\{U(\alpha) := \bigoplus_i U_i(\alpha)\Big\}_{\alpha \in \mathbb{C}}$. If we define the $\mathfrak{A}$-valued inner product by setting $\langle (\xi_i) \mid (\beta_i) \rangle_{\mathfrak{A}} := \sum_i \langle \xi_i \mid \beta_i \rangle_{\mathfrak{A}}$, then H is a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule with left and right action given by $A \cdot (\xi_i) \cdot B := (A \cdot \xi_i \cdot B)$.

$\mathscr{F}(H_i)$ can be canonically embedded into $\mathscr{F}(H)$ as an $\mathfrak{A}$-$\mathfrak{A}$ sub-bimodule. We view $\mathscr{F}(H_i)_\tau$ as a Hilbert subspace of $\mathscr{F}(H)_\tau$. Let $e_{\mathfrak{A}}$ and e_i be the projection from $\mathscr{F}(H)_\tau$ onto $L^2(\mathfrak{A}, \tau)$ and $\mathscr{F}(H_i)$, respectively. It is clear that, for every $i \in I$, $e_i \in (\mathfrak{A} \cup \{\Gamma(\xi) : \xi \in H_i\})' \cap \mathcal{B}(\mathscr{F}(H)_\tau)$. By Proposition 1.27, the central carrier of $e_i \in (\mathfrak{A} \cup \{\Gamma(\xi) : \xi \in H_i\})'$ is the identity. Therefore

$$\begin{aligned}(\mathfrak{A} \cup \{\Gamma(\xi) : \xi \in H_i\})''(\subset \mathcal{B}(\mathscr{F}(H)_\tau)) &\cong (\mathfrak{A} \cup \{\Gamma(\xi) : \xi \in H_i\})'' e_i \\ &\cong \Phi(H_i)''(\subset \mathcal{B}(\mathscr{F}(H_i)_\tau)).\end{aligned}$$

Therefore, $\Phi(H_i)''$ can be identified with a von Neumann subalgebra of $\Phi(H)''$.

Let E be the faithful normal conditional expectation from $\Phi(H)''$ onto $\pi_\tau(\mathfrak{A})$ defined in Proposition 1.27 such that $E(T)e_{\mathfrak{A}} = e_{\mathfrak{A}} T e_{\mathfrak{A}}$. Consider $T_1, \ldots, T_l \in \Phi(H)$ such that $T_k \in \Phi(H_{i(k)})$, $i(1) \neq i(2) \neq \cdots \neq i(l)$ and $E(T_k) = 0$ for all $k = 1, \ldots, l$. Then each T_k can be written as a finite sum of elements of the form $L(\xi_1) \cdots L(\xi_n) L^*(\beta_1) \cdots L^*(\beta_m)$, where ξ_j, $\beta_j \in H_{i(k)}$ and $n + m > 0$. It is not hard to check that $e_{\mathfrak{A}} \pi_\tau(T_1 \cdots T_l) e_{\mathfrak{A}} = 0$ (see Theorem 4.6.15 in [59]) and $e_s \pi_\tau(T_1 \cdots T_l) e_s \neq 0$ only if $l = 1$ and $s = i(1)$. Therefore $e_s \Phi(H)'' e_s = \Phi(H_s)'' e_s$ and let $E_s(T)$ be the unique element in $\Phi(H_s)''$ such that $e_s T e_s = E_s(T) e_s$, for each $T \in \Phi(H)''$. By definition of E_s, we have $E_s(\pi_\tau(A) T \pi_\tau(B)) = \pi_\tau(A) E_s(T) \pi_\tau(B)$ and

$$E \circ E_s(T) e_{\mathfrak{A}} = e_{\mathfrak{A}} e_s T e_s e_{\mathfrak{A}} = E(T) e_{\mathfrak{A}},$$

where A, $B \in \Phi(H_s)$ and $T \in \Phi(H)''$. Thus E_s is a faithful normal conditional expectation from $\Phi(H)''$ onto $\Phi(H_s)''$ satisfying $E \circ E_s = E$.

Summarizing the above discussion, we have the following proposition (cf. Theorem 4.6.15 in [59] and Appendix A for the definition of amalgamated free product among arbitrary von Neumann algebras).

PROPOSITION 1.29. *Let $\{(H_i, S_i, \{U_i(\alpha)\}_{\alpha\in\mathbb{C}})\}_{i\in I}$ be a family of Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodules. Then $\Big(\Phi\Big(\bigoplus_i H_i\Big)'', E\Big) = *_{\pi_\tau(\mathfrak{A})}(\Phi(H_i)'', E|_{\Phi(H_i)''})$.*

2. FROM RIGID C^*-TENSOR CATEGORIES TO OPERATOR ALGEBRAS VIA TOMITA BIMODULES

Since every C^*-tensor category is equivalent to a strict one (see [43]), we shall exclusively work with strict C^*-tensor categories. Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a (small) strict rigid C^*-tensor category with simple (i.e., irreducible) unit $\mathbb{1}$, finite direct sums and subobjects, see [2], [16], [42]. For objects X in $\mathscr{C}$ we write with abuse of notation $X \in \mathscr{C}$, we denote by $t \in \mathrm{Hom}(X, Y)$ the arrows in $\mathscr{C}$ between $X, Y \in \mathscr{C}$ and by I_X the identity arrow in $\mathrm{Hom}(X, X)$. We use $\overline{X}$ to denote a *conjugate object* of X (also called dual object) equipped with unit and counit

$$\eta_X \in \mathrm{Hom}(\mathbb{1}, \overline{X} \otimes X), \quad \varepsilon_X \in \mathrm{Hom}(X \otimes \overline{X}, \mathbb{1})$$

satisfying the so-called conjugate equations, namely

$$(\varepsilon_X \otimes I_X)(I_X \otimes \eta_X) = I_X, \quad (I_{\overline{X}} \otimes \varepsilon_X)(\eta_X \otimes I_{\overline{X}}) = I_{\overline{X}}.$$

Hence conjugation is specified by four-tuples $(X, \overline{X}, \eta_X, \varepsilon_X)$. Furthermore, we always assume that $(\eta_X, \varepsilon_X^*)$ is a *standard* solution of the conjugate equations, see Section 3 in [42]. Important consequences of these assumptions on $\mathscr{C}$, for which we refer to [42], are semisimplicity (every object is completely reducible into a finite direct sum of simple ones), the existence of an additive and multiplicative dimension function on objects $d_X := \eta_X^* \eta_X = \varepsilon_X \varepsilon_X^* \in \mathbb{R}$, $d_X \geqslant 1$, and a left (= right) trace defined by

$$\eta_X^*(I_{\overline{X}} \otimes t)\eta_X = \varepsilon_X(t \otimes I_{\overline{X}})\varepsilon_X^*, \quad \forall t \in \mathrm{Hom}(X, X) \tag{2.1}$$

on the finite-dimensional C^*-algebra $\mathrm{Hom}(X, X)$. Moreover, for every X, $Y \in \mathscr{C}$ $\mathrm{Hom}(X, Y)$ is a finite-dimensional Hilbert space with inner product given by

$$\langle \xi_1 \mid \xi_2 \rangle := \eta_X^*[I_{\overline{X}} \otimes (\xi_1^* \xi_2)]\eta_X, \quad \xi_1, \xi_2 \in \mathrm{Hom}(X, Y). \tag{2.2}$$

In the following, we need to keep track of the distinction between $X \otimes Y$ and $X' \otimes Y'$, even if $X \otimes Y = X' \otimes Y'$ in $\mathscr{C}$ and $X \neq X'$ or $Y \neq Y'$.

Thus we introduce the C^*-tensor category $W(\mathscr{C})$ considered in the proof of Mac Lane's coherence theorem (Theorem XI.3.1 in [43]). The objects of $W(\mathscr{C})$ are the "words" constructed from the objects of $\mathscr{C}$ (finite strings of objects in $\mathscr{C}$). The tensor product of two words W_1 and W_2 is defined by the concatenation of strings and it is denoted by W_1W_2. Two words are equal if they have the same length and they are letterwise equal. Note that, e.g., $\mathbb{1}\mathbb{1}$ and $\mathbb{1}\otimes\mathbb{1} = \mathbb{1}$ are different in $W(\mathscr{C})$, since $\mathbb{1}\otimes\mathbb{1}$ is a one-letter word and $\mathbb{1}\mathbb{1}$ is a two-letter word. Let ι be the map

$$\iota : \varnothing \mapsto \mathbb{1}, \quad X_1X_2\cdots X_n \mapsto X_1\otimes X_2\otimes\cdots\otimes X_n, \tag{2.3}$$

where $X_i \in \mathscr{C}$, $i = 1,\dots,n$, $n \geqslant 1$, and $\varnothing$ denotes the empty word. The arrows between words W_1 and W_2, denoted by $\mathrm{Hom}(W_1, W_2)$, are $\mathrm{Hom}(\iota(W_1), \iota(W_2))$ with the composition and tensor product defined just as in $\mathscr{C}$. It is clear that $W(\mathscr{C})$ is a strict C^*-tensor category which is equivalent to $\mathscr{C}$. In particular, $W(\mathscr{C})$ has finite direct sums and subobjects. With abuse of notation we write the tensor product of arrows in $W(\mathscr{C})$ as $f\otimes g$ in $\mathrm{Hom}(W_1W_2, Y_1Y_2)$ for $f \in \mathrm{Hom}(W_1, Y_1)$ and $g \in \mathrm{Hom}(W_2, Y_2)$, where W_i, Y_i, $i = 1, 2$, are objects in $W(\mathscr{C})$.

NOTATION 2.1. (i) Let $\mathscr{S}$ be a representative set of simple objects in $\mathscr{C}$ such that $\mathbb{1} \in \mathscr{S}$. Let $\Lambda := \mathscr{S} \cup \overline{\mathscr{S}}$ (disjoint union) where $\overline{\mathscr{S}} := \{\overline{\alpha} : \alpha \in \mathscr{S}\}$. We shall always consider $\alpha \in \mathscr{S}$ and $\overline{\alpha} \in \overline{\mathscr{S}}$ as distinct elements ("letters") in Λ, even if $\alpha = \overline{\alpha}$ as objects in $\mathscr{C}$. In particular $\mathbb{1} \neq \overline{\mathbb{1}}$ in Λ.

(ii) We label the fundamental "fusion" Hom-spaces of $\mathscr{C}$ by triples in $\mathscr{S}\times\mathscr{S}\times\Lambda$, namely

$$(\beta_1, \beta_2, \alpha) \in \mathscr{S}\times\mathscr{S}\times\Lambda \mapsto \begin{bmatrix}\beta_2\alpha\\ \beta_1\end{bmatrix} := \mathrm{Hom}(\beta_1, \beta_2\otimes\alpha),$$

where $\beta_2\otimes\alpha$ is viewed as an element of $\mathscr{C}$.

(iii) For every $\alpha \in \Lambda$, let $r_\alpha := \eta_\alpha \in \mathrm{Hom}(\mathbb{1}, \overline{\alpha}\otimes\alpha)$ if $\alpha \in \mathscr{S}$, and $r_{\overline{\alpha}} := \varepsilon_\alpha^* \in \mathrm{Hom}(\mathbb{1}, \alpha\otimes\overline{\alpha})$ for the corresponding $\overline{\alpha} \in \overline{\mathscr{S}}$, where $(\eta_\alpha, \varepsilon_\alpha^*)$ is a standard solution of the conjugate equations as above.

(iv) For every $\alpha \in \mathscr{S}$, regarded as an element of Λ, let $\overline{\overline{\alpha}} := \alpha$. Thus $\alpha \in \Lambda \mapsto \overline{\alpha} \in \Lambda$ is an involution on Λ.

In the following, we use $\alpha, \beta, \gamma, \dots$ to denote elements in Λ, or in $\mathscr{S}$, and $W, X, Y, Z, \dots$ to denote objects in $W(\mathscr{C})$. For a set S, we denote its cardinality by $|S|$.

We define an abelian von Neumann algebra $\mathfrak{A}$ by

$$\mathfrak{A} := \bigoplus_{\beta\in\mathscr{S}} \mathrm{Hom}(\beta, \beta) = \bigoplus_{\beta\in\mathscr{S}} \mathbb{C}I_\beta \tag{2.4}$$

with a (non-normalized, semifinite) weight τ given by equation (2.1), i.e., the value on minimal projections equals the categorical dimension $\tau(I_\beta) = d_\beta$.

We are now ready to construct a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule H. Let

$$H := \bigoplus_{\alpha\in\Lambda,\beta_1,\beta_2\in\mathscr{S}} \begin{bmatrix}\beta_2\alpha\\ \beta_1\end{bmatrix}$$

be the algebraic direct sum of the $[{}^{\beta_2\alpha}_{\beta_1}]$. It is clear that H is an $\mathfrak{A}$-$\mathfrak{A}$ bimodule with left and right action given by

$$I_{\gamma_2}\cdot\xi\cdot I_{\gamma_1} := \delta_{\gamma_2,\beta_2}\delta_{\gamma_1,\beta_1}\xi, \quad \xi\in\begin{bmatrix}\beta_2\alpha\\ \beta_1\end{bmatrix}, \alpha\in\Lambda, \beta_i,\gamma_i\in\mathscr{S}, i=1,2.$$

We define an $\mathfrak{A}$-valued inner product on H by

$$\langle\xi_2\mid\xi_1\rangle_{\mathfrak{A}} := \delta_{\alpha_1,\alpha_2}\delta_{\omega_1,\omega_2}\xi_2^*\xi_1\in\mathrm{Hom}(\beta_1,\beta_2), \quad \xi_i\in\begin{bmatrix}\omega_i\alpha_i\\ \beta_i\end{bmatrix}, i=1,2.$$

It is easy to check that H is a pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule with normal left action, as we defined in Section 1.

Let $\alpha\in\Lambda$ and $\beta_1,\beta_2\in\mathscr{S}$. By Frobenius reciprocity (see Lemma 2.1 in [42]), we can define a bijection

$$\xi\in\begin{bmatrix}\beta_2\alpha\\ \beta_1\end{bmatrix}\mapsto\overline{\xi} := (\xi^*\otimes I_{\overline{\alpha}})(I_{\beta_2}\otimes r_{\overline{\alpha}})\in\begin{bmatrix}\beta_1\overline{\alpha}\\ \beta_2\end{bmatrix}. \tag{2.5}$$

It is clear that $\overline{\overline{\xi}}=\xi$.

We now define an involution S on H and a complex one-parameter group $\{U(z): z\in\mathbb{C}\}$ of isomorphisms that endow H with the structure of a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule (see Definition 1.14).

For each $\alpha\in\mathscr{S}$, we choose a real number $\lambda_\alpha>0$. For the corresponding $\overline{\alpha}\in\overline{\mathscr{S}}$, let $\lambda_{\overline{\alpha}} := 1/\lambda_\alpha$. Recall that $\alpha\neq\overline{\alpha}$ as elements in Λ. We define the action of S and $U(z)$ on every $\xi\in[{}^{\beta_2\alpha}_{\beta_1}]$, $\alpha\in\Lambda$, $\beta_1,\beta_2\in\mathscr{S}$ by

$$S:\xi\mapsto\sqrt{\lambda_\alpha}\,\overline{\xi}, \quad U(z):\xi\mapsto\lambda_\alpha^{\mathrm{i}z}\,\xi, \quad z\in\mathbb{C}, \tag{2.6}$$

where $\overline{\xi}$ is defined by equation (2.5). Summing up, we have the following proposition.

PROPOSITION 2.2. *The pre-Hilbert $\mathfrak{A}$-$\mathfrak{A}$ bimodule H is a Tomita $\mathfrak{A}$-$\mathfrak{A}$ bimodule, for every choice of λ_α and $\lambda_{\overline{\alpha}}=1/\lambda_\alpha$ associated with the elements $\alpha\in\Lambda$.*

In the following, we choose and write $\mathcal{O}^{\beta_2\alpha}_{\beta_1}=\{\xi_i\}_{i=1}^n$ for a fixed orthonormal basis of isometries in $[{}^{\beta_2\alpha}_{\beta_1}]$, i.e., $\xi_i^*\xi_j=\delta_{i,j}I_{\beta_1}$, where $n=\dim[{}^{\beta_2\alpha}_{\beta_1}]$, for every $\beta_1,\beta_2\in\mathscr{S}$ and $\alpha\in\Lambda$. Note that $\mathcal{O}^{\beta_2\alpha}_{\beta_1}=\emptyset$ if $n=0$. In the case $\alpha=\mathbb{1}$, we let $\mathcal{O}^{\beta\mathbb{1}}_{\beta}=\{I_\beta\}$. Thus we have that

$$H=\bigoplus_{\alpha,\beta_1,\beta_2\in\mathscr{S}}\ \bigoplus_{\xi\in\mathcal{O}^{\beta_2\alpha}_{\beta_1}}\{a\xi+b\overline{\xi}: a,b\in\mathbb{C}\}. \tag{2.7}$$

From now on, we identify $\mathfrak{A}$ with the von Neumann subalgebra $\pi_\tau(\mathfrak{A})$ of $\Phi(H)''$ acting on the Fock (Hilbert) space $\mathscr{F}(H)_\tau$ (see Definition 1.24 and Definition 1.25), and we denote by E the normal faithful conditional expectation from $\Phi(H)''$ onto $\mathfrak{A}$ such that

$$E(\Gamma(\xi_1)\cdots\Gamma(\xi_n)) = \langle \Omega \,|\, \Gamma(\xi_1)\cdots\Gamma(\xi_n)\Omega\rangle_{\mathfrak{A}}, \tag{2.8}$$

where $\Omega = I$ is the vacuum vector in $\mathscr{F}(H)$.

LEMMA 2.3. *Suppose that $\mathcal{O}_\beta^{\beta\alpha} \neq \emptyset$ and let $\xi \in \mathcal{O}_\beta^{\beta\alpha}$, where $\beta \in \mathscr{S}, \alpha \in \Lambda$. If $\lambda_\alpha \leqslant 1$ set $\eta = \xi$. Otherwise $\lambda_{\overline{\alpha}} = 1/\lambda_\alpha \leqslant 1$ and set $\eta = \overline{\xi}$. Let $\mathcal{N}_\beta$ be the von Neumann algebra generated by $\Gamma(\eta)$, then*

$$\Phi(\{a\xi + b\overline{\xi} : a, b \in \mathbb{C}\})'' \cong \mathcal{N}_\beta \oplus \bigoplus_{\gamma \in \mathscr{S}\setminus\{\beta\}} \mathbb{C}I_\gamma \quad \textit{and}$$

$$\mathcal{N}_\beta \cong \begin{cases} L(F_2) & \textit{if } \lambda_\alpha = 1, \\ \textit{free Araki–Woods type } \mathrm{III}_\lambda \textit{ factor} & \textit{otherwise}, \end{cases}$$

where $\lambda = \min\{\lambda_\alpha, \lambda_{\overline{\alpha}}\}$. Furthermore, in the polar decomposition $\Gamma(\eta) = VK$, we have $\mathrm{Ker}\, K = \{0\}$ *and V, K are $*$-free with respect to E. The distribution of $K^2 = \Gamma(\eta)^*\Gamma(\eta)$ with respect to the state $d_\beta^{-1}\tau \circ E|_{\mathcal{N}_\beta}$ is*

$$\frac{\sqrt{4\lambda - (t-(1+\lambda))^2}}{2\pi\lambda t}\mathrm{d}t, \quad t \in ((1-\sqrt{\lambda})^2, (1+\sqrt{\lambda})^2).$$

Proof. By exchanging the roles of ξ and $\overline{\xi}$, hence of α and $\overline{\alpha}$, we can assume $\lambda_\alpha \leqslant 1$. It is clear that $\Phi(\{a\xi + b\overline{\xi}\})''$ is generated by $\mathfrak{A}$ and $\Gamma(\eta) = L(\xi) + \sqrt{\lambda_\alpha}L^*(\overline{\xi})$. Since I_β is a minimal projection in $\mathfrak{A}$ and $I_\beta\Gamma(\eta) = \Gamma(\eta) = \Gamma(\eta)I_\beta$, we have that $E(T) = d_\beta^{-1}\tau(E(T))I_\beta$ for any $T \in \mathcal{N}_\beta$. Note that $\langle \xi \,|\, \overline{\xi}\rangle_{\mathfrak{A}} = 0$. Then Remark 4.4, Theorem 4.8 and Theorem 6.1 in [57] imply the result (see also Example 2.6.2 in [28] for the case $\lambda_\alpha = 1$). ∎

REMARK 2.4. With the notation of Lemma 2.3, we have that $\sigma_t^{\tau\circ E}(\Gamma(\xi)) = \lambda_\alpha^{it}\Gamma(\xi)$ for every $t \in \mathbb{R}$ by Theorem 1.28. Therefore $K = (\Gamma(\xi)^*\Gamma(\xi))^{1/2}$ is in the centralizer of $\tau \circ E$, denoted by $\Phi(H)''_{\tau\circ E}$.

In the following, we adopt the notation used in [12] to specify the (non-normalized) trace on a direct sum of algebras. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two finite von Neumann algebra with two given states ω_1 and ω_2, respectively. We use $\underset{t_1}{\overset{p}{\mathfrak{A}_1}} \oplus \underset{t_2}{\overset{q}{\mathfrak{A}_2}}$ to denote the direct sum algebra with distinguished positive linear functional $t_1\omega_1(a) + t_2\omega_2(b)$, $a \in \mathfrak{A}_1$, $b \in \mathfrak{A}_2$, where p and q are projections corresponding to the identity elements of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. A special case of this is when ω_1, ω_2 and $t_1\omega_1(\cdot) + t_2\omega_2(\cdot)$ are tracial.

LEMMA 2.5. *Let $\xi \in \mathcal{O}_{\beta_1}^{\beta_2\alpha}$ for $\alpha \in \Lambda$ and $\beta_1, \beta_2 \in \mathscr{S}$, $\beta_1 \neq \beta_2$. If $\lambda = \lambda_\alpha d_{\beta_1}/d_{\beta_2} \leqslant 1$, then*

$$\Phi(\{a\xi + b\overline{\xi} : a, b \in \mathbb{C}\})'' \cong \left(\underset{d_{\beta_1}+\lambda_\alpha d_{\beta_1}}{\overset{I_{\beta_1}+p_{\beta_2}}{L^\infty([0,1]) \otimes M_2(\mathbb{C})}} \right) \oplus \underset{d_{\beta_2}-\lambda_\alpha d_{\beta_1}}{\overset{q_{\beta_2}}{\mathbb{C}}} \oplus \bigoplus_{\gamma \in \mathscr{S}\setminus\{\beta_1,\beta_2\}} \mathbb{C} I_\gamma,$$

where p_{β_2} and q_{β_2} are two projections such that $p_{\beta_2} + q_{\beta_2} = I_{\beta_2}$ and $\tau \circ E(p_{\beta_2}) = \lambda_\alpha d_{\beta_1}$, $\tau \circ E(q_{\beta_2}) = d_{\beta_2} - \lambda_\alpha d_{\beta_1}$.

Proof. As in the previous lemma, $\Phi(\{a\xi + b\overline{\xi}\})''$ is the von Neumann subalgebra of $\Phi(H)''$ generated by $\mathfrak{A}$ and $\Gamma(\xi)$. By Lemma 4.3 and Remark 4.4 in [57], we have that the distributions of $\Gamma(\xi)^*\Gamma(\xi)$ and $\Gamma(\xi)\Gamma(\xi)^*$ with respect to $d_{\beta_1}^{-1}\tau \circ E$ and $d_{\beta_2}^{-1}\tau \circ E$ are

$$\frac{\sqrt{4\lambda - (t-(1+\lambda))^2}}{2\pi\lambda t}\mathrm{d}t \quad \text{and} \quad \frac{\sqrt{4\lambda - (t-(1+\lambda))^2}}{2\pi t}\mathrm{d}t + (1-\lambda)\delta_0,$$

where δ_0 is the Dirac measure concentrated in 0.

Let $\xi_1 = \xi$ and $\xi_2 = \sqrt{1/\lambda}\, S(\xi) \in [{}^{\beta_1\overline{\alpha}}_{\beta_2}]$. Then $\xi_2^*\xi_2 = I_{\beta_2}$, because $\tau(\xi_2^*\xi_2) = d_{\beta_2}$, and the operator $\Gamma(\xi) = L(\xi_1) + \sqrt{\lambda}L(\xi_2)^*$ fulfills $I_{\beta_2}\Gamma(\xi) = \Gamma(\xi) = \Gamma(\xi)I_{\beta_1}$. Let $\Gamma(\xi) = UK$ be the polar decomposition of $\Gamma(\xi)$. By the proof of Lemma 4.3 in [57], we know that $K \in I_{\beta_1}\Phi(H)''I_{\beta_1}$ and $\operatorname{Ker} K = \{0\}$ (even when $\lambda = 1$), and $U^*U = I_{\beta_1}$. Since $\sigma_t^{\tau\circ E}(U) = \lambda_\alpha^{\mathrm{i}t}U$, $t \in \mathbb{R}$, we have

$$d_{\beta_1} = \tau \circ E(U^*U) = \frac{1}{\lambda_\alpha}\tau \circ E(UU^*),$$

thus $\tau \circ E(UU^*) = \lambda_\alpha d_{\beta_1} \leqslant d_{\beta_2}$. Since $\tau \circ E$ is faithful, we know that UU^* is a subprojection of I_{β_2}. Let $p_{\beta_2} = UU^*$ and $q_{\beta_2} = I_{\beta_2} - UU^*$. Note that $p_{\beta_2} = I_{\beta_2}$ if and only if $\lambda = 1$. The fact that the distribution of K^2 is non-atomic implies that the von Neumann algebra generated by K is isomorphic to $L^\infty([0,1])$. Thus the von Neumann algebra generated by I_{β_1}, I_{β_2} and $\Gamma(\xi)$ is isomorphic to

$$\left(\underset{d_{\beta_1}+\lambda_\alpha d_{\beta_1}}{\overset{I_{\beta_1}+p_{\beta_2}}{L^\infty([0,1]) \otimes M_2(\mathbb{C})}} \right) \oplus \underset{d_{\beta_2}-\lambda_\alpha d_{\beta_1}}{\overset{q_{\beta_2}}{\mathbb{C}}}. \qquad \blacksquare$$

Now consider the sets of isometries $\mathcal{O}_\beta^{\beta\alpha}$ in the special case $\alpha = \mathbb{1}$. For every $\beta \in \mathscr{S}$, let $\xi_\beta = I_\beta$ be the only element in $\mathcal{O}_\beta^{\beta\mathbb{1}}$. By Lemma 2.3, we have that

$$\Phi\Big(\bigoplus_{\beta \in \mathscr{S}} \{a\xi_\beta + b\overline{\xi}_\beta : a, b \in \mathbb{C}\} \Big)'' \cong \bigoplus_{\beta \in \mathscr{S}} \mathcal{N}_\beta \tag{2.9}$$

where $\mathcal{N}_\beta$ is $L(F_2)$ or the free Araki–Woods type III_λ factor, $\lambda = \min\{\lambda_{\mathbb{1}}, \lambda_{\overline{\mathbb{1}}}\}$, and $\mathcal{N}_\beta$ is generated by a partial isometry V_β and a positive operator K_β in the centralizer of $\tau \circ E$ (see Remark 2.4). If we let $U_\beta = f(K_\beta^2)$ where $f(x) = \mathrm{e}^{2\pi\mathrm{i}h(x)}$

and $h(x) = \int\limits_{(1-\sqrt{\lambda})^2}^{x} \sqrt{4\lambda - (t-(1+\lambda))^2}/(2\pi\lambda t)\mathrm{d}t$, we get a Haar unitary, i.e., $E(U_\beta^n) = 0$ for every $n \neq 0$, which generates $\{K_\beta\}''$. Moreover, U_β and V_β are $*$-free with respect to E.

Denote by $\mathcal{M}_1$ the l.h.s. of equation (2.9) and by $\widetilde{\mathfrak{M}}_1$ the strongly dense $*$-subalgebra of $\mathcal{M}_1$ generated by $\mathfrak{A}$ and $\{U_\beta, V_\beta : \beta \in \mathscr{S}\}$. By Proposition 1.29 and equation (2.7) we have an amalgamated free product decomposition of $\Phi(H)''$, namely

$$(\Phi(H)'', E) = (\mathcal{M}_1, E|_{\mathcal{M}_1}) *_{\mathfrak{A}} (\mathcal{M}_2, E|_{\mathcal{M}_2}) \tag{2.10}$$

where $\mathcal{M}_2 := \Phi\Big(\bigoplus_{\alpha,\beta_1,\beta_2\in\mathscr{S},(\beta_1=\beta_2,\alpha\neq\mathbb{1} \text{ or } \beta_1\neq\beta_2)} \bigoplus_{\xi\in\mathcal{O}_{\beta_1}^{\beta_2\alpha}} \{a\xi + b\overline{\xi} : a, b \in \mathbb{C}\}\Big)''$.

LEMMA 2.6. *With the notations above, there is a unitary $U \in \Phi(H)''_{\tau\circ E} \cap \mathcal{M}_1$ such that $E(t_1 U^n t_2) \to 0$ for $n \to +\infty$ in the strong operator topology for every $t_1, t_2 \in \widetilde{\mathfrak{M}}_1$.*

Proof. By Lemma 2.3 and the discussion above, $U = \sum\limits_{\beta\in\mathscr{S}} U_\beta$ satisfies the conditions. ∎

THEOREM 2.7. *The von Neumann algebra $\Phi(H)''$ is a factor.*

Proof. Let U be the unitary in Lemma 2.6. If $T \in \Phi(H)'' \cap \Phi(H)'$, then $T \in \{U\}' \cap \mathcal{M}_1$ by Proposition B.3. Thus Lemma 2.5 implies $\Phi(H)'' \cap \Phi(H)' = \Phi(H)' \cap \mathfrak{A} = \mathbb{C}I$. ∎

LEMMA 2.8. *Let $\mathcal{Z}(\Phi(H)''_{\tau\circ E})$ be the center of $\Phi(H)''_{\tau\circ E}$. Then $\mathcal{Z}(\Phi(H)''_{\tau\circ E}) \subseteq \mathfrak{A}$.*

Proof. Proceeding as in the proof of Theorem 2.7, we have $\mathcal{Z}(\Phi(H)''_{\tau\circ E}) \subset \mathcal{M}_1$. If $\lambda_{\mathbb{1}} = 1$, then $\mathcal{M}_1 \subset \Phi(H)''_{\tau\circ E}$ and we have $\mathcal{Z}(\Phi(H)''_{\tau\circ E}) \subseteq \mathfrak{A}$. By Corollary 4 in [1], Corollary 5.5 in [57] and Lemma 2.3, it is also clear that $\mathcal{Z}(\Phi(H)''_{\tau\circ E}) \subseteq \mathfrak{A}$ if $\lambda_{\mathbb{1}} \neq 1$. ∎

Recall that the Connes' invariant $S(\mathcal{M})$ of a factor $\mathcal{M}$ is given by $\cap\{\mathrm{Sp}(\Delta_\varphi) : \varphi$ is a n.s.f. weight on $\mathcal{M}\}$ and that $\mathbb{R}_*^+ \cap S(\mathcal{M}) = \mathbb{R}_*^+ \cap \{\mathrm{Sp}(\Delta_{\varphi_e}) : 0 \neq e \in \mathrm{Proj}(\mathcal{Z}(\mathcal{M}_\varphi)\}$ is a closed subgroup of $\mathbb{R}_*^+ = \{t \in \mathbb{R}, t > 0\}$, where $\varphi_e = \varphi|_{e\mathcal{M}e}$ and $\mathcal{Z}(\mathcal{M}_\varphi)$ is the center of $\mathcal{M}_\varphi$. See Chapter III in [5], Chapter VI in [60]. Here $\mathrm{Proj}(\cdot)$ denotes the set of orthogonal projections of a von Neumann algebra and $\mathrm{Sp}(\cdot)$ the spectrum of an operator.

LEMMA 2.9. *Let G be the closed subgroup of $\mathbb{R}_*^+$ generated by $\{\lambda_\alpha\lambda_{\beta_2}/\lambda_{\beta_1} : \alpha \in \Lambda, \beta_1, \beta_2 \in \mathscr{S}$ such that $\left[\begin{smallmatrix}\beta_2\alpha\\ \beta_1\end{smallmatrix}\right] \neq \{0\}\}$. Then $\mathbb{R}_*^+ \cap S(\Phi(H)'') = G$.*

Proof. We use φ to denote $\tau \circ E$. By Lemma 2.8 and Section 15.4 in [60], we know that $\mathbb{R}_*^+ \cap S(\Phi(H)'') = \mathbb{R}_*^+ \cap \{\mathrm{Sp}(\Delta_{\varphi_{I_\beta}}) : \beta \in \mathscr{S}\}$. Furthermore, note that $(I_\beta\Phi(H)''I_\beta)_{\varphi_{I_\beta}} = I_\beta(\Phi(H)'')_\varphi I_\beta$. Thus the center of $(I_\beta\Phi(H)''I_\beta)_{\varphi_{I_\beta}}$ is trivial by

Proposition 5.5.6 in [34]. Then Corollary 16.2.1, Section 16.1 in [60] imply $\mathbb{R}_*^+ \cap S(\Phi(H)'') = \mathbb{R}_*^+ \cap \mathrm{Sp}(\Delta_{\varphi_{I_{\mathbb{1}}}})$.

Let $\xi_1 \in [{}^{\beta_1\alpha_1}_{\mathbb{1}}], \xi_2 \in [{}^{\beta_2\alpha_2}_{\beta_1}], \dots, \xi_n \in [{}^{\mathbb{1}\alpha_n}_{\beta_{n-1}}]$ and $\alpha_i \in \Lambda, \beta_i \in \mathscr{S}$, $n = 1, 2, \dots$. By Theorem 1.28, for every $t \in \mathbb{R}$, we have

$$\sigma_t^{\varphi}(\Gamma(\xi_n)\cdots\Gamma(\xi_1)) = \Big(\frac{\lambda_{\mathbb{1}}\lambda_{\alpha_n}}{\lambda_{\beta_{n-1}}}\Big)^{\mathrm{i}t}\cdots\Big(\frac{\lambda_{\beta_2}\lambda_{\alpha_2}}{\lambda_{\beta_1}}\Big)^{\mathrm{i}t}\Big(\frac{\lambda_{\beta_1}\lambda_{\alpha_1}}{\lambda_{\mathbb{1}}}\Big)^{\mathrm{i}t}\Gamma(\xi_n)\cdots\Gamma(\xi_1).$$

Also note that $\sigma_t^{\varphi}(\Gamma(\zeta_2)\Gamma(\xi)\Gamma(\zeta_1)) = \big(\frac{\lambda_{\beta_2}\lambda_\alpha}{\lambda_{\beta_1}}\big)^{\mathrm{i}t}\Gamma(\zeta_2)\Gamma(\xi)\Gamma(\zeta_1)$, where $\zeta_1 \in [{}^{\beta_1\overline{\beta}_1}_{\mathbb{1}}]$, $\xi \in [{}^{\beta_2\alpha}_{\beta_1}]$, $\zeta_2 \in [{}^{\mathbb{1}\beta_2}_{\beta_2}]$. Since the linear span of $\{I_{\mathbb{1}}\} \cup \{\Gamma(\xi_n)\cdots\Gamma(\xi_1)\}$ as above is dense in the space $L^2(I_{\mathbb{1}}\Phi(H)''I_{\mathbb{1}}, \varphi_{I_{\mathbb{1}}})$, we have $\mathrm{Sp}(\Delta_{\varphi_{I_{\mathbb{1}}}}) \setminus \{0\} = G$. ∎

PROPOSITION 2.10. *The von Neumann algebra $\Phi(H)''$ is semifinite if and only if $\lambda_{\beta_1} = \lambda_{\beta_2}\lambda_\alpha$ for every $\alpha \in \Lambda$ and $\beta_1, \beta_2 \in \mathscr{S}$ such that $[{}^{\beta_2\alpha}_{\beta_1}] \neq \{0\}$. If $\Phi(H)''$ is not semifinite, then it is a type III_λ factor for $\lambda \in (0,1]$, where λ depends on the choice of (non-trivial) Tomita structure.*

Proof. By Proposition 28.2 in [60], the algebra $\Phi(H)''$ is semifinite if and only if $S(\Phi(H)'') = \{1\}$. If $\Phi(H)''$ is semifinite, Lemma 2.9 implies that $\lambda_{\beta_1} = \lambda_{\beta_2}\lambda_\alpha$ for every $\alpha \in \Lambda$ and $\beta_1, \beta_2 \in \mathscr{S}$ such that $[{}^{\beta_2\alpha}_{\beta_1}] \neq \{0\}$.

Conversely, if $\lambda_{\beta_1} = \lambda_{\beta_2}\lambda_\alpha$, by Theorem 2.11 from Chapter VIII in [62], we have

$$\sigma_t^{\tau_K\circ E}(\Gamma(\xi)) = K^{\mathrm{i}t}\sigma_t^{\tau\circ E}(\Gamma(\xi))K^{-\mathrm{i}t} = \Gamma(\xi),$$

for every $\xi \in [{}^{\beta_2\alpha}_{\beta_1}]$, $\alpha \in \Lambda$, $\beta_1, \beta_2 \in \mathscr{S}$, where $K = \sum_{\beta\in\mathscr{S}} \lambda_\beta I_\beta$ and $\tau_K(\cdot) = \tau(K^{1/2} \cdot K^{1/2})$. Thus $\Phi(H)''$ is semifinite. ∎

REMARK 2.11. If $\Phi(H)''$ is semifinite, then $\lambda_{\mathbb{1}}^2 = \lambda_{\mathbb{1}}$ implies $\lambda_{\mathbb{1}} = 1$. If $\alpha \in \mathscr{S}$ is self-conjugate, i.e., $\alpha \cong \overline{\alpha}$ in $\mathscr{C}$, then $\lambda_\alpha^2 = 1$, which implies $\lambda_\alpha = 1$. If α is not self-conjugate, then there is $\beta \in \mathscr{S}$ which is a conjugate of α and we have $\lambda_\alpha\lambda_\beta = 1$.

If the spectrum $\mathscr{S}$ is finite, then $\Phi(H)''$ is a type II_1 factor. If $\mathscr{S}$ is infinite, then $\tau_K \circ E(I) = \sum_{\beta\in\mathscr{S}} \lambda_\beta d_\beta = +\infty$, where τ_K is defined in the proof of Proposition 2.10. Thus $\Phi(H)''$ is a II_∞ factor. Observe that $\tau_K \circ E$ is a trace (tracial weight), whereas $\tau \circ E$ is a trace if and only if $\lambda_\alpha = 1$ for every $\alpha \in \Lambda$.

3. THE CASE $\lambda_\alpha = 1$

In this section, we assume that the spectrum $\mathscr{S}$ of the category $\mathscr{C}$ is infinite (not necessarily denumerable) and study the algebra $\Phi(H)''$ constructed before, in the special case $\lambda_\alpha = 1$ for every $\alpha \in \mathscr{S}$ (hence for every $\alpha \in \Lambda$), i.e., when the $\mathfrak{A}$-$\mathfrak{A}$ pre-Hilbert bimodule H has trivial Tomita bimodule structure. The main result is that, in the trivial Tomita structure case, the free group factor $L(F_{\mathscr{S}})$ (with countably infinite or uncountably many generators) sits in a corner of $\Phi(H)''$, namely $I_{\mathbb{1}}\Phi(H)''I_{\mathbb{1}} \cong L(F_{\mathscr{S}})$ (Theorem 3.7).

All von Neumann algebras in this section are semifinite and all conditional expectations are trace-preserving. Since the conditional expectation onto a von Neumann subalgebra with respect to a fixed tracial weight is unique (see Theorem 4.2 from Chapter IX in [62]), we sometimes omit the conditional expectations from the expressions of amalgamated free products to simplify the notation, when there is no ambiguity.

Recall that $\mathfrak{A} = \bigoplus_{\beta\in\mathscr{S}} \mathbb{C}I_\beta$ is the abelian von Neumann algebra with tracial weight τ such that $\tau(I_\beta) = d_\beta$, where I_β is the identity of $\mathrm{Hom}(\beta,\beta)$, and d_β is the categorical dimension of $\beta \in \mathscr{S}$. By Theorem 2.7 and Proposition 2.10,

$$(\Phi(H)'', E) = *_{\mathfrak{A},\alpha,\beta_1,\beta_2\in\mathscr{S},\xi\in\mathcal{O}_{\beta_1}^{\beta_2\alpha}} (\Phi(\{a\xi + b\overline{\xi} : a, b \in \mathbb{C}\})'', E)$$

is a II_∞ factor (semifiniteness is due to the assumption $\lambda_\alpha = 1, \alpha \in \mathscr{S}$).

We use $L(F_s)$, $1 < s < \infty$, to denote the interpolated free group factor [14], [54]. In general, for every non-empty set S, let $L(F_S)$ be the free group factor of the free group over S with the canonical tracial state τ_S (see Section 6 in [47]).

We let $\mathcal{M}(F_S) = \bigoplus_{\beta\in\mathscr{S}} L(F_S)$ and identify $\mathfrak{A}$ with the subalgebra $\bigoplus_{\beta\in\mathscr{S}} \mathbb{C}$ of $\mathcal{M}(F_S)$. Let E_S be the faithful normal conditional expectation from $\mathcal{M}(F_S)$ onto $\mathfrak{A}$ such that $E_S\Big(\bigoplus_{\beta\in\mathscr{S}} a_\beta\Big) = \sum_{\beta\in\mathscr{S}} \tau_S(a_\beta)I_\beta$, where $a_\beta \in L(F_S)$. By Lemma 2.3, we have

$$(\mathcal{M}(F_2), E_2) = *_{\mathfrak{A},\beta\in\mathscr{S},\xi\in\mathcal{O}_{\beta}^{\beta\mathbb{1}}} (\Phi(\{a\xi + b\overline{\xi} : a, b \in \mathbb{C}\})'', E).$$

The following fact is well known at least when S is denumerable [53] and it shows that the fundamental group of an uncountably generated free group factor is $\mathbb{R}_*^+ = \{t \in \mathbb{R}, t > 0\}$ as well.

LEMMA 3.1. *Let S be an infinite set (not necessarily denumerable). Then $L(F_S) \cong pL(F_S)p$ for every non-zero projection $p \in L(F_S)$.*

Proof. Let $S = \bigcup_{i\in\Theta} S_i$ be a partition of S such that $|S_i| = |\mathbb{N}|$ and $|\Theta| = |S|$. Let $(\mathcal{M}, \tau)$ be a W^*-probability space with a normal faithful tracial state τ, containing a copy of the hyperfinite II_1-factor R and a semicircular family $\{X_{ij} : i \in \Theta, j \in S_i\}$ (see Definition 5.1.1 in [66]) such that $\{X_{ij} : i \in \Theta, j \in S_i\}$ and R are free. Consider the von Neumann algebra $\mathcal{N} = (R \cup \{qX_{ij}q : i \in \Theta, j \in S_i\})''$, where $q \in R$ is a projection with the same trace value as p.

Let $\mathcal{N}_i = (R \cup \{qX_{ij}q : j \in S_i\})''$. For each $i \in \Theta$, by Proposition 2.2 in [14], there exists a semicircular family $\{Y_{ij} : j \in S_i\} \subset \mathcal{N}_i$ such that $\mathcal{N}_i = (R \cup \{Y_{ij} : j \in S_i\})''$ and $\{Y_{ij} : j \in S_i\}$ and R are free.

Note that $\{Y_{i_0 j} : j \in S_{i_0}\}$ and $R \cup \{X_{ij} : i \in \Theta \setminus \{i_0\}, j \in S_i\}$ are free for every $i_0 \in \Theta$. Thus $\{Y_{ij} : i \in \Theta, j \in S_i\}$ is a semicircular family which is free with R.

By Theorem 4.1 in [13], $L(F_S) \cong (R \cup \{Y_{ij} : i \in \Theta, j \in S_i\})'' = \mathcal{N}$. Since $q\mathcal{N}q = (qRq \cup \{qX_{ij}q : i \in \Theta, j \in S_i\})''$ and $qRq \cong R$, we have $L(F_S) \cong pL(F_S)p$ by Theorem 1.3 in [14]. ∎

LEMMA 3.2. *For every set S such that $1 \leqslant |S| \leqslant |\mathscr{S}|$, let*

$$(\mathcal{M}, \mathcal{E}) := (\mathcal{M}(F_S), E_S) *_{\mathfrak{A}, \alpha \in \mathscr{S} \setminus \{\mathbb{1}\}, \xi \in \mathcal{O}_\alpha^{\mathbb{1}\alpha}} (\Phi(\{a\xi + b\bar{\xi} : a, b \in \mathbb{C}\})'', E).$$

Then $\mathcal{M}$ is a II_∞ factor with a tracial weight $\tau \circ \mathcal{E}$ and $I_{\mathbb{1}} \mathcal{M} I_{\mathbb{1}} \cong L(F_{\mathscr{S}})$.

Proof. Proposition A.6 in the appendix implies that $\tau \circ \mathcal{E}$ is a tracial weight. By Lemma 2.5, we have

$$\mathcal{M} = \left(\underset{d_{\mathbb{1}}}{\overset{I_{\mathbb{1}}}{L(F_S)}} \bigoplus_{\alpha \in \mathscr{S} \setminus \{\mathbb{1}\}} \underset{d_\alpha}{\overset{I_\alpha}{\mathbb{C}}} \right) *_{\mathfrak{A}, \beta \in \mathscr{S} \setminus \{\mathbb{1}\}} \mathcal{M}_\beta,$$

where $\mathcal{M}_\beta$ is

$$\mathcal{M}_\beta = \left(\underset{d_\beta}{\overset{I_\beta}{L(F_S)}} \bigoplus_{\alpha \in \mathscr{S} \setminus \{\beta\}} \underset{d_\alpha}{\overset{I_\alpha}{\mathbb{C}}} \right) *_{\mathfrak{A}} \left(\left(L(F_1) \underset{2}{\overset{I_{\mathbb{1}} + p_\beta}{\otimes}} M_2(\mathbb{C}) \right) \oplus \underset{d_\beta - 1}{\overset{q_\beta}{\mathbb{C}}} \bigoplus_{\gamma \in \mathscr{S} \setminus \{\mathbb{1}, \beta\}} \underset{d_\gamma}{\overset{I_\gamma}{\mathbb{C}}} \right),$$

and $p_\beta + q_\beta = I_\beta$. It is clear that the central carrier of I_β in $\mathcal{M}_\beta$ is $I_{\mathbb{1}} + I_\beta$. By Lemma 4.3 in [15],

$$I_\beta \mathcal{M}_\beta I_\beta = \underset{d_\beta}{\overset{I_\beta}{L(F_S)}} * \left(\underset{1}{\overset{p_\beta}{L(F_1)}} \oplus \underset{d_\beta - 1}{\overset{q_\beta}{\mathbb{C}}} \right).$$

By Proposition 1.7(i) in [12], $I_\beta \mathcal{M}_\beta I_\beta \cong L(F_{t'_\beta})$ where t'_β is a real number bigger than 1 if S is a finite set (see [14]) or $t_\beta = S$ if S is an infinite set. Then Theorem 2.4 in [14] and Lemma 3.1 imply $I_{\mathbb{1}} \mathcal{M}_\beta I_{\mathbb{1}} \cong L(F_{t_\beta})$ where t_β is a real number bigger than 1 if S is a finite set or $t_\beta = S$ if S is an infinite set.

Note that $I_{\mathbb{1}} \mathcal{M} I_{\mathbb{1}} = L(F_S) *_{\beta \in \mathscr{S} \setminus \{\mathbb{1}\}} I_{\mathbb{1}} \mathcal{M}_\beta I_{\mathbb{1}}$. Let $\mathscr{S} = \bigcup_{j \in \Theta} S_j$ where Θ is a set such that $|\Theta| = |\mathscr{S}|$ and $\{S_j\}_{j \in \Theta}$ are disjoint countably infinite subsets of $\mathscr{S}$. If S is a finite set, by Proposition 4.3 in [12], $I_{\mathbb{1}} \mathcal{M} I_{\mathbb{1}} \cong *_{j \in \Theta} L(F_{\mathbb{N}}) \cong L(F_{\mathscr{S}})$. Otherwise, $I_{\mathbb{1}} \mathcal{M} I_{\mathbb{1}} \cong *_{j \in \Theta} L(F_S) \cong L(F_{\mathscr{S}})$. Finally, $\mathcal{M}$ is a type II_∞ factor because the central carrier of $I_{\mathbb{1}}$ is I. ∎

LEMMA 3.3. *For every S such that $1 \leqslant |S| \leqslant |\mathscr{S}|$, let*

$$(\mathcal{M}, \mathcal{E}) := (\mathcal{M}(F_S), E_S) *_{\mathfrak{A}, \alpha \in \mathscr{S} \setminus \{\mathbb{1}\}, \xi \in \mathcal{O}_\alpha^{\mathbb{1}\alpha}} \left(\underset{2}{\overset{I_{\mathbb{1}} + p_\alpha}{M_2(\mathbb{C})}} \oplus \underset{d_\alpha - 1}{\overset{q_\alpha}{\mathbb{C}}} \bigoplus_{\beta \in \mathscr{S} \setminus \{\mathbb{1}, \alpha\}} \underset{d_\beta}{\overset{I_\beta}{\mathbb{C}}} \right),$$

where $p_\alpha + q_\alpha = I_\alpha$. Then $\mathcal{M}$ is a II_∞ factor with a tracial weight $\tau \circ \mathcal{E}$ and $I_{\mathbb{1}} \mathcal{M} I_{\mathbb{1}} \cong L(F_{\mathscr{S}})$.

For the proof invoke Proposition 2.4 in [12] instead of Proposition 1.7(i) in [12], and the lemma can be proved by the same argument used in the proof of Lemma 3.2.

The following fact is well known; we sketch the proof for the reader's convenience.

LEMMA 3.4. *Let* $0 < t < 1$ *and* $\mathcal{M} := *_{i\in\mathbb{N}}\Big(\underset{t}{\mathbb{C}} \oplus \underset{1-t}{\mathbb{C}}\Big)$. *Then* $\mathcal{M} \cong L(F_{\mathbb{N}})$.

Proof. We assume $t \geqslant 1/2$. Then by Theorem 1.1 in [12],

$$\Big(\underset{t}{\mathbb{C}} \oplus \underset{1-t}{\mathbb{C}}\Big) * \Big(\underset{t}{\mathbb{C}} \oplus \underset{1-t}{\mathbb{C}}\Big) \cong \Big(\underset{2-2t}{L(F_1) \otimes M_2(\mathbb{C})}\Big) \oplus \underset{\max\{2t-1,0\}}{\mathbb{C}}.$$

By Theorem 4.6 in [12],

$$\begin{aligned}\Big(\Big(\underset{2-2t}{L(F_1) \otimes M_2(\mathbb{C})}\Big) \oplus \underset{\max\{2t-1,0\}}{\mathbb{C}}\Big) * \Big(\Big(\underset{2-2t}{L(F_1) \otimes M_2(\mathbb{C})}\Big) \oplus \underset{\max\{2t-1,0\}}{\mathbb{C}}\Big)\\ \cong \underset{1-\max\{4t-3,0\}}{L(F_{s_1})} \oplus \underset{\max\{4t-3,0\}}{\mathbb{C}},\end{aligned}$$

where $s_1 \geqslant 1$. By Proposition 2.4 in [12], there exists $n \in \mathbb{N}$ such that

$$*_{i=1\ldots n}\Big(\underset{t}{\mathbb{C}} \oplus \underset{1-t}{\mathbb{C}}\Big) \cong L(F_{s_2}),$$

where $s_2 \geqslant 1$. Then Propositions 4.3 and 4.4 in [12] imply the result. ∎

LEMMA 3.5. *Let* $(\mathcal{M}, \tau)$ *be a* W^**-noncommutative probability space with normal faithful tracial state* τ *and let*

$$\mathfrak{A}_1 := \underset{2/(1+d)}{\overset{e_{11}+e_{22}}{M_2(\mathbb{C})}} \oplus \underset{(d-1)/(1+d)}{\overset{q}{\mathbb{C}}}$$

be a unital subalgebra of $\mathcal{M}$, *and* $d > 1$. *We use* $\{e_{ij}\}_{i,j}$ *to denote the canonical matrix units of* $M_2(\mathbb{C})$ *in* $\mathfrak{A}_1$. *Let* $\mathfrak{A}_0$ *be the abelian subalgebra generated by* e_{11} *and* $P = e_{22} + q$ *and* E *be the trace-preserving conditional expectation from* $\mathcal{M}$ *onto* $\mathfrak{A}_0$.

Assume that $\{e_{22}\} \cup \{p_s\}$ *is a* $*$*-free family of projections in* $(e_{22}+q)\mathcal{M}(e_{22}+q)$ *such that* $\tau(p_s) = 1/(1+d)$. *We denote by* $\mathcal{N}$ *the von Neumann subalgebra of* $(e_{22} + q)\mathcal{M}(e_{22}+q)$ *generated by* $\{e_{22}\} \cup \{p_s\}_{s\in S}$. *Then there exist partial isometries* v_s *in* $\mathcal{N}$ *such that* $v_s^*v_s = p_s$ *and* $v_sv_s^* = e_{22}$. *Suppose that* $\{w_s\}_{s\in S}$ *is a free family of Haar unitaries in* $e_{22}\mathcal{M}e_{22}$ *which is* $*$*-free with* $e_{22}\mathcal{N}e_{22}$. *Then*

$$\Big(\Big[\mathfrak{A}_1 \cup \Big(\bigcup_{s\in S}\{e_{11}, e_{12}w_sv_s, v_s^*w_s^*e_{21}, p_s, P\}\Big)\Big]'', E\Big) \cong (\mathfrak{A}_1, E) *_{\mathfrak{A}_0} (*_{\mathfrak{A}_0, s\in S}(\mathfrak{A}_1, E)).$$

Proof. The existence of v_s is implied by Theorem 1.1 in [12]. It is clear that

$$\{e_{11}, e_{12}w_sv_s, v_s^*w_s^*e_{21}, p_s, P\}'' \cong \mathfrak{A}_1.$$

We could assume that $0 \notin S$ and denote by $Z_0 = \{e_{12}, e_{21}, de_{22} - P\}$ and by $Z_s = \{e_{12}w_sv_s, v_s^*w_s^*e_{21}, dp_s - P\}$ for every $s \in S$. Note that $E(a) = (1+d)\tau(e_{11}ae_{11})e_{11}$

$+ [(1+d)/d]\tau(PaP)P$. We only need to show that $E(t_1,\dots,t_n)=0$, where $t_i \in Z_{k(i)}, k(i)\in\{0\}\cup S$ and $k(1)\neq k(2)\neq\cdots\neq k(n), n\geqslant 1$.

Note that $E(t_1\cdots t_n)=0$ if $t_1\cdots t_n \in e_{11}\mathcal{M}P$ or $P\mathcal{M}e_{11}$. Therefore we assume that $0\neq t_1\cdots t_n$ is in $e_{11}\mathcal{M}e_{11}$ or in $P\mathcal{M}P$ and show that $\tau(t_1\cdots t_n)=0$.

Case 1. $t_i \notin \bigcup_{s\in S}\{e_{12}w_sv_s, v_s^*w_s^*e_{21}\}$. If e_{12} and e_{21} are not in $\{t_1,\dots,t_n\}$, then it is clear that $\tau(t_1\cdots t_n)=0$. If e_{12} or e_{21} is in $\{t_1,\dots,t_n\}$, then $t_1=e_{12}$, $t_n=e_{21}$. Note that $t_i=de_{22}-P$ or $t_i\in\{dp_s-P\}_{s\in S}$, $i=2,\dots,n-1$ and $t_{n-1}\neq de_{22}-P$, $i=2,\dots,n-1$. Since $\{p_s\}_{s\in S}$ and e_{22} are free, then

$$\tau(t_1\cdots t_n)=\tau(t_2\cdots t_{n-1}e_{22})=0.$$

Case 2. If the number of the elements in $\left\{i: 1\leqslant i\leqslant n, t_i\in \bigcup_{s\in S}\{e_{12}w_sv_s\}\right\}$ and $\left\{i: 1\leqslant i\leqslant n, t_i\in \bigcup_{s\in S}\{v_s^*w_s^*e_{21}\}\right\}$ are different, then $\tau(t_1\cdots t_n)=0$ since $\{w_s\}_{s\in S}$ and $e_{22}\mathcal{N}e_{22}=e_{22}(\mathfrak{A}_1\cup\mathcal{N})''e_{22}$ are free.

Therefore, if $\tau(t_1\cdots t_n)\neq 0$, there must exist $i_1<i_2$ such that

$$t_{i_1}=v_s^*w_s^*e_{21},\quad t_{i_2}=e_{12}w_sv_s \quad\text{or}\quad t_{i_1}=e_{12}w_sv_s,\quad t_{i_2}=v_s^*w_s^*e_{21}.$$

and $t_i\notin\bigcup_{s\in S}\{e_{12}w_sv_s, v_s^*w_s^*e_{21}\}$, $i_1<i<i_2$.

Assume that $t_{i_1}=v_s^*w_s^*e_{21}, t_{i_2}=e_{12}w_sv_s$. Since $t_{i_1}\cdots t_{i_2}\neq 0$, $t_{(i_1+1)}=e_{12}$, $t_{(i_2-1)}=e_{21}$, $t_i\in\{dp_s-P\}_{s\in S}\cup\{de_{22}-P\}$ $i_1+1<i<i_2-1$ and $t_{(i_1+2)}\neq de_{22}-P$, $t_{(i_1-2)}\neq de_{22}-P$, then by Case 1, we have

$$\tau(e_{22}t_{(i_1+2)}\cdots t_{(i_2-2)}e_{22})=0.$$

Now assume that $t_{i_1}=e_{12}w_sv_s, t_{i_2}=v_s^*w_s^*e_{21}$. Since $t_{i_1}\cdots t_{i_2}\neq 0$, $t_j\notin\{e_{12},e_{21}\}$, $i_1\leqslant j\leqslant i_2$. Then by Case 1,

$$\tau(v_st_{(i_1+2)}\cdots t_{(i_2-2)}v_s^*)=\tau(p_st_{(i_1+2)}\cdots t_{(i_2-2)})=0.$$

This, however, implies that $\tau(t_1\cdots t_n)=0$. ∎

LEMMA 3.6. *Suppose that S is a set such that $|S|=|\mathscr{S}|$, let*

$$\mathcal{M}:=*_{\mathfrak{A}_0,s\in S}\left(\underset{2}{\overset{I_1+p}{M_2(\mathbb{C})}}\oplus\underset{d-1}{\overset{q}{\mathbb{C}}}\right)$$

where $\mathfrak{A}_0=\underset{1}{\overset{I_1}{\mathbb{C}}}\oplus\underset{d}{\overset{I_2}{\mathbb{C}}}$ *and* $I_2=p+q$, $d\geqslant 1$. *Then* $\mathcal{M}$ *is a* II_1 *factor and* $I_1\mathcal{M}I_1\cong L(F_{\mathscr{S}})$.

Proof. Let $s_0\in S$.

Case 1. $d=1$. Then a similar argument as in the proof of Lemma 3.5 implies that $I_1\mathcal{M}I_1$ is generated by a free family of Haar unitaries $\{u_s\}_{s\in S\setminus\{s_0\}}$. Thus $I_1\mathcal{M}I_1\cong L(F_{\mathscr{S}})$.

Case 2. $d>1$. By Lemma 3.5, $I_2\mathcal{M}I_2$ is generated by a free family of projections $\{p_s\}$ and a family of Haar unitaries $\{w_s\}\subset p_{s_0}\mathcal{M}p_{s_0}$ such that $\tau(p_s)=1$, $\{w_s\}$ and $p_{s_0}\{p_s\}_{s\in S}''p_{s_0}$ are free. By Lemma 3.4, $\{p_s\}_{s\in S}''\cong L(F_{\mathscr{S}})$. Then

Lemma 3.1 implies $p_{s_0}\mathcal{M}p_{s_0} \cong L(F_{\mathscr{S}}) * L(F_{\mathscr{S}}) \cong L(F_{\mathscr{S}})$. Note that the central carrier of p_{s_0} is $I_1 + I_2$, thus $\mathcal{M}$ is a factor and $I_1\mathcal{M}I_1 \cong L(F_{\mathscr{S}})$. ∎

THEOREM 3.7. *We have a $*$-isomorphism of factors $I_{\mathbb{1}}\Phi(H)''I_{\mathbb{1}} \cong L(F_{\mathscr{S}})$.*

Proof. By the axiom of choice, $|\mathscr{S} \times \mathscr{S}| = |\mathscr{S}|$. Let

$$(\mathcal{M},\mathcal{E}) = (\mathcal{M}(F_{\mathscr{S}}),E_{\mathscr{S}}) *_{\mathfrak{A},\,\alpha\in\mathscr{S}\setminus\{\mathbb{1}\},\xi\in\mathcal{O}_\alpha^{\mathbb{1}\alpha}} \left(\underset{2}{\overset{I_{\mathbb{1}}+p_\alpha}{M_2(\mathbb{C})}} \oplus \underset{d_\alpha-1}{\overset{q_\alpha}{\mathbb{C}}} \bigoplus_{\beta\in\mathscr{S}\setminus\{\mathbb{1},\alpha\}} \mathbb{C}, E \right),$$

where $p_\alpha + q_\alpha = I_\alpha$. Note that

$$(\mathcal{M},\mathcal{E}) = (\mathcal{M}(F_{\mathscr{S}}),E_{\mathscr{S}}) *_{\mathfrak{A},\,\alpha\in\mathscr{S}\setminus\{\mathbb{1}\}} \mathcal{M}_\alpha,$$

where

$$\mathcal{M}_\alpha = \left(\underset{d_\alpha}{\overset{I_\alpha}{L(F_{\mathscr{S}})}} \bigoplus_{\beta\in\mathscr{S}\setminus\{\alpha\}} \underset{d_\beta}{\overset{I_\beta}{\mathbb{C}}} \right) *_{\mathfrak{A}} \left(\left(\underset{2}{\overset{I_{\mathbb{1}}+p_\alpha}{M_2(\mathbb{C})}} \oplus \underset{d_\alpha-1}{\overset{q_\alpha}{\mathbb{C}}} \right) \bigoplus_{\gamma\in\mathscr{S}\setminus\{\mathbb{1},\alpha\}} \underset{d_\gamma}{\overset{I_\gamma}{\mathbb{C}}} \right).$$

Note that $\mathcal{M}_\alpha \cong \left(\underset{d_{\mathbb{1}}+d_\alpha}{\overset{I_{\mathbb{1}}+I_\alpha}{L(F_{\mathscr{S}})}} \bigoplus_{\beta\in\mathscr{S}\setminus\{\mathbb{1},\alpha\}} \underset{d_\beta}{\overset{I_\beta}{\mathbb{C}}} \right)$ by Lemma 4.3 in [15]. By Lemma 3.6 and Proposition B.5, we have

$$(\mathcal{M},\mathcal{E}) \cong (\mathcal{M}(F_{\mathscr{S}}),E_{\mathscr{S}}) *_{\mathfrak{A},\,\alpha\in\mathscr{S}\setminus\{\mathbb{1}\},s\in S} \left(\underset{2}{\overset{I_{\mathbb{1}}+p_\alpha}{M_2(\mathbb{C})}} \oplus \underset{d_\alpha-1}{\overset{q_\alpha}{\mathbb{C}}} \bigoplus_{\beta\in\mathscr{S}\setminus\{\mathbb{1},\alpha\}} \mathbb{C}, E \right),$$

where S is an index set such that $|S| = |\mathscr{S}|$. Thus $(\mathcal{M},\mathcal{E}) \cong *_{\mathfrak{A},\,s\in S}(\mathcal{M},\mathcal{E})$.

By Lemma 2.3, Lemma 2.5, Lemma 3.2, Lemma 3.3 and Proposition B.5, we have

$$(\Phi(H)'',E) \cong (\mathcal{M},\mathcal{E}) *_{\mathfrak{A},\,\alpha\in\mathscr{S},\,\beta_1\neq\beta_2\in\mathscr{S},\,\beta_2\neq\mathbb{1},\,\xi\in\mathcal{O}_{\beta_1}^{\beta_2\alpha}} (\Phi(\{a\xi + b\overline{\xi} : a,b\in\mathbb{C}\})'',E).$$

Thus $\Phi(H)'' \cong *_{\mathfrak{A},\,\alpha\in\mathscr{S},\beta_1\neq\beta_2\in\mathscr{S},\beta_2\neq\mathbb{1},\xi\in\mathcal{O}_{\beta_1}^{\beta_2\alpha}} \mathcal{M}_\xi$ where we set

$$\begin{aligned}\mathcal{M}_\xi = (\mathcal{M}(F_{\mathscr{S}}),E_{\mathscr{S}}) *_{\mathfrak{A},\,\beta\in\mathscr{S}\setminus\{\mathbb{1},\beta_2\}} &\left(\underset{2}{\overset{I_{\mathbb{1}}+p_\beta}{M_2(\mathbb{C})}} \oplus \underset{d_\beta-1}{\overset{q_\beta}{\mathbb{C}}} \bigoplus_{\eta\in\mathscr{S}\setminus\{\mathbb{1},\beta\}} \mathbb{C}, E \right) \\ *_{\mathfrak{A}} &\left(\underset{2d_{\beta_1}}{\overset{I_{\beta_1}+p_{\beta_2}}{(L(F_1)\otimes M_2(\mathbb{C}))}} \oplus \underset{d_{\beta_2}-d_{\beta_1}}{\overset{q_{\beta_2}}{\mathbb{C}}} \bigoplus_{\gamma\in\mathscr{S}\setminus\{\beta_1,\beta_2\}} \mathbb{C}, E \right)\end{aligned}$$

if $d_{\beta_1} \leqslant d_{\beta_2}$, and $\mathcal{M}_\xi$ is similarly defined in the case $d_{\beta_2} \leqslant d_{\beta_1}$. Arguing as in the proof of Lemma 3.2, we have that $\mathcal{M}_\xi$ is a factor and $I_{\mathbb{1}}\mathcal{M}_\xi I_{\mathbb{1}} \cong L(F_{\mathscr{S}})$.

By Proposition B.5, $\Phi(H)'' \cong *_{\mathfrak{A},\,\alpha\in\mathscr{S},\beta_1\neq\beta_2\in\mathscr{S},\beta_2\neq\mathbb{1},\xi\in\mathcal{O}_{\beta_1}^{\beta_2\alpha}}(\mathcal{M},\mathcal{E}) \cong (\mathcal{M},\mathcal{E})$ and Lemma 3.3 implies the result. ∎

4. REALIZATION AS ENDOMORPHISMS OF $\Phi(H)''$

In this section, we assume that the spectrum $\mathscr{S}$ of $\mathscr{C}$ is infinite, and we show that $W(\mathscr{C})$ considered in Section 2, thus $\mathscr{C}$, can be realized as endomorphisms of the factor $\Phi(H)''$ constructed in Section 2. Recall that $\mathscr{C}$ and $W(\mathscr{C})$ are equivalent as strict C^*-tensor categories. We first introduce for convenience an auxiliary algebra living on a bigger Hilbert space.

From now on, we use Φ and $L^2(\Phi)$ to denote $\Phi(H)''$ and $L^2(\Phi(H)'', \tau \circ E)$, unless stated otherwise. Recall that $L^2(\Phi)$ is a Φ-Φ bimodule and $I_\beta \in \Phi$, $\beta \in \mathscr{S}$. Let

$$\mathcal{H}(\mathscr{C}) := \bigoplus_{W \in W(\mathscr{C})} \mathcal{H}^W, \quad \text{where } \mathcal{H}^W := \bigoplus_{\beta,\gamma \in \mathscr{S}} \mathrm{Hom}(\beta, W\gamma) \otimes (I_\beta L^2(\Phi)).$$

Here by $\oplus$ and $\otimes$ we mean the Hilbert space completions of the algebraic direct sums and tensor products. The inner product on $\mathcal{H}(\mathscr{C})$ is given by

$$\langle u_1 \otimes \xi_1 \mid u_2 \otimes \xi_2 \rangle := \frac{1}{d_\beta} \langle u_1 \mid u_2 \rangle \langle \xi_1 \mid \xi_2 \rangle = \langle \xi_1 \mid (u_1^* u_2) \xi_2 \rangle,$$

where $\xi_1, \xi_2 \in I_\beta L^2(\Phi)$, $u_1, u_2 \in \mathrm{Hom}(\beta, W\gamma)$ and $\langle u_1 \mid u_2 \rangle$ is given as in Section 2 by equation (2.2). To simplify the notation, we will use $u\xi$ to denote $u \otimes I_\beta \xi$ for every $u \in \mathrm{Hom}(\beta, W\gamma)$ and $\xi \in L^2(\Phi)$. Moreover, note that $L^2(\Phi)$ is canonically identified with $\mathcal{H}^\varnothing$.

Each $f \in \mathrm{Hom}(W_1\beta_1, W_2\beta_2)$, for $\beta_1, \beta_2 \in \mathscr{S}$, $W_1, W_2 \in W(\mathscr{C})$, acts on $\mathcal{H}(\mathscr{C})$ as follows. On the linear span of the $u\xi$'s as above let

$$f \cdot (u\xi) := \delta_{W_1\beta_1, W\gamma} (fu)\xi, \quad u \in \mathrm{Hom}(\beta, W\gamma).$$

It is routine to check that

$$\Big\langle \sum_i^n (fu_i)\xi_i \mid \sum_j^n (fu_j)\xi_j \Big\rangle \leqslant \|f\|^2 \sum_{i,j}^n \langle \xi_i \mid (u_i^* u_j) \xi_j \rangle,$$

since $(u_i^* f^* f u_j)_{i,j} \leqslant \|f\|^2 (u_i^* u_j)_{i,j}$. Thus the map extends by continuity to a bounded linear operator in $\mathcal{B}(\mathcal{H}(\mathscr{C}))$, again denoted by f.

Let $f_i \in \mathrm{Hom}(W_i\alpha_i, Z_i\beta_i)$, for $\alpha_i, \beta_i \in \mathscr{S}$, W_i, $Z_i \in W(\mathscr{C})$, $i = 1, 2$. By the above definition, for every $\zeta, \zeta_1, \zeta_2 \in \mathcal{H}(\mathscr{C})$, we have

$$f_1 \cdot (f_2 \cdot \zeta) = \delta_{W_1\alpha_1, Z_2\beta_2} (f_1 f_2) \cdot \zeta \quad \text{and} \quad \langle \zeta_1 \mid f_1 \cdot \zeta_2 \rangle = \langle f_1^* \cdot \zeta_1 \mid \zeta_2 \rangle.$$

In particular, for $u \in \mathrm{Hom}(\beta, W\gamma)$ and $\xi \in L^2(\Phi) = \mathcal{H}^\varnothing$, we have $u \cdot \xi = u\xi$. Therefore, we write $f \cdot \zeta$ as $f\zeta$ from now on.

It is also clear that

$$\mathcal{H}(\mathscr{C}) = \bigoplus_{W \in W(\mathscr{C}), \beta \in \mathscr{S}} \bigoplus_{u \in \mathcal{O}_\beta^{W\mathscr{S}}} uL^2(\Phi), \tag{4.1}$$

where $\mathcal{O}_\beta^{W\mathscr{S}} = \bigcup_{\gamma\in\mathscr{S}} \mathcal{O}_\beta^{W\gamma}$ (disjoint union) and $\mathcal{O}_\beta^{W\gamma}$ is a fixed orthonormal basis of isometries in $\mathrm{Hom}(\beta, W\gamma)$, for every $\beta,\gamma \in \mathscr{S}$, $W \in W(\mathscr{C})$, while $\mathcal{O}_\beta^{W\gamma} = \emptyset$ if $\dim \mathrm{Hom}(\beta, W\gamma) = 0$.

For every $\xi \in [{}^{\beta_2\alpha}_{\beta_1}] \subset H$, $\alpha \in \Lambda$, $\beta_1, \beta_2 \in \mathscr{S}$, the operator $\Gamma(\xi) \in \Phi$ acts from the left on $\mathcal{H}(\mathscr{C})$ in the following way

$$\begin{cases} \Gamma(\xi)(\zeta) = \Gamma(\xi)\zeta & \text{if } \zeta \in \mathcal{H}^\emptyset, \\ \Gamma(\xi)(u\zeta) = 0 & \text{if } u\zeta \notin \mathcal{H}^\emptyset, \end{cases}$$

and we denote again by $\Gamma(\xi)$ the operator acting on $\mathcal{H}(\mathscr{C})$.

DEFINITION 4.1. Let $\Phi(\mathscr{C})$ be the von Neumann subalgebra of $\mathcal{B}(\mathcal{H}(\mathscr{C}))$ generated by $\{f \in \mathrm{Hom}(W_1\beta_1, W_2\beta_2) : \beta_1, \beta_2 \in \mathscr{S}, W_1, W_2 \in W(\mathscr{C})\}$ and $\{\Gamma(\xi) : \xi \in [{}^{\beta_2\alpha}_{\beta_1}], \alpha \in \Lambda, \beta_1, \beta_2 \in \mathscr{S}\}$.

Let $P_W : \mathcal{H}(\mathscr{C}) \to \mathcal{H}^W$ be the orthogonal projection onto $\mathcal{H}^W$ for $W \in W(\mathscr{C})$. We have the following easy result and its proof is omitted.

LEMMA 4.2. *The algebra Φ constructed in Section 2 is a corner of $\Phi(\mathscr{C})$, namely $\Phi \cong P_\emptyset \Phi(\mathscr{C}) P_\emptyset$ and a canonical $*$-isomorphism is given by*

$$I_\beta \mapsto I_\beta|_{\mathcal{H}^\emptyset}, \quad \Gamma(\xi) \mapsto \Gamma(\xi)|_{\mathcal{H}^\emptyset},$$

for every $\xi \in [{}^{\beta_2\alpha}_{\beta_1}]$, where $\alpha \in \Lambda$, $\beta_1, \beta_2 \in \mathscr{S}$.

From now on, we identify $L^2(\Phi)$ and Φ with $\mathcal{H}^\emptyset$ and $P_\emptyset \Phi(\mathscr{C}) P_\emptyset$, respectively.

THEOREM 4.3. *For every word $W \in W(\mathscr{C})$, the projection P_W is equivalent to $P_\emptyset$ in $\Phi(\mathscr{C})$. Thus $P_W \Phi(\mathscr{C}) P_W \cong P_\emptyset \Phi(\mathscr{C}) P_\emptyset = \Phi$.*

Proof. Note that $P_\emptyset = \sum_{\beta\in\mathscr{S}} I_\beta$ and $P_W = \sum_{\beta\in\mathscr{S}} \sum_{u\in\mathcal{O}_\beta^{W\mathscr{S}}} uu^*$. Since the central carrier of $P_\emptyset$ is I, $\Phi(\mathscr{C})$ is either a type II_∞ or a type III factor by Proposition 5.5.6, Corollary 9.1.4 in [34].

Assume first that $\Phi(\mathscr{C})$ is a type III factor. By Corollary 6.3.5 in [34], I_β and $\sum_{u\in\mathcal{O}_\beta^{W\mathscr{S}}} uu^*$ are equivalent for every $\beta \in \mathscr{S}$, since they are both countably decomposable projections. Therefore P_W is equivalent to $P_\emptyset$.

Assume now that $\Phi(\mathscr{C})$ is type II_∞. Let $\mathscr{S} = \bigcup_{i\in\Theta} \mathscr{S}_i$ be a partition of $\mathscr{S}$ such that each $\mathscr{S}_i$ contains countably many elements and $|\mathscr{S}| = |\Theta|$. Note that each projection $\bigoplus_{u\in\mathcal{O}_\beta^{W\mathscr{S}}} uu^*$ is a finite projection and equivalent to a subprojection of $\bigoplus_{\gamma\in\mathscr{S}_{\theta(\beta)}} I_\gamma$, where θ is a one-to-one correspondence between $\mathscr{S}$ and Θ. This implies that P_W is equivalent to $P_\emptyset$. ∎

Let Δ be the modular operator of Φ with respect to $\tau \circ E$. Then we can define a one-parameter family of unitaries in $\mathcal{B}(\mathcal{H}(\mathscr{C}))$ as follows

$$U_t : u\zeta \to u(\Delta^{\mathrm{i}t}\zeta), \quad t \in \mathbb{R},$$

for every $\zeta \in L^2(\Phi), u \in \mathcal{O}_\beta^{W\mathscr{S}}$. It is easy to check that

$$U_t f U_t^* = f, \quad U_t \Gamma(\xi) U_t^* = \lambda_\alpha^{\mathrm{i}t} \Gamma(\xi), \quad t \in \mathbb{R},$$

where $f \in \mathrm{Hom}(W_1\beta_1, W_2\beta_2)$ and $\xi \in [{}^{\beta_2\alpha}_{\beta_1}]$, for $\alpha \in \Lambda$, $\beta_1, \beta_2 \in \mathscr{S}$, W_1, $W_2 \in W(\mathscr{C})$. Therefore $\mathrm{Ad}U_t \in \mathrm{Aut}(\Phi(\mathscr{C}))$.

Note that $\{uI_\beta : u \in \mathcal{O}_\beta^{W\mathscr{S}}, I_\beta \in L^2(\Phi), W \in W(\mathscr{C}), \beta \in \mathscr{S}\}$ is a total set of vectors for $\Phi(\mathscr{C})'$ (here I_β is the operator in $\mathfrak{A}$ and treated as a vector in $L^2(\Phi)$, see equation (2.4)). Thus

$$\varphi(A) := \sum_{\beta \in \mathscr{S}, W \in W(\mathscr{C})} \sum_{u \in \mathcal{O}_\beta^{W\mathscr{S}}} \langle uI_\beta \mid A(uI_\beta) \rangle, \quad A \in \Phi(\mathscr{C}),\ A \geqslant 0, \tag{4.2}$$

defines a n.s.f. weight on $\Phi(\mathscr{C})$. Moreover, $\varphi(U_t \cdot U_t^*) = \varphi(\cdot)$ for every $t \in \mathbb{R}$, since $U_t(uI_\beta) = uI_\beta$. Now we show that U_t, $t \in \mathbb{R}$, implement the modular group of $\Phi(\mathscr{C})$ with respect to φ.

LEMMA 4.4. *For every $A \in \Phi(\mathscr{C})$ we have $\sigma_t^\varphi(A) = U_t A U_t^*$, $t \in \mathbb{R}$.*

Proof. Note that $\varphi(uA) = \langle I_\beta \mid A(uI_\beta) \rangle = \langle I_\beta \mid (Au)I_\beta \rangle = \varphi(Au)$ for every $u \in \mathcal{O}_\beta^{W\mathscr{S}}$, $\beta \in \mathscr{S}$, $W \in W(\mathscr{C})$. Then Theorem 2.6 from Chapter VIII in [62] implies that the von Neumann subalgebra generated by $\{f \in \mathrm{Hom}(W_1\beta_1, W_2\beta_2) : \beta_1, \beta_2 \in \mathscr{S}, W_1, W_2 \in W(\mathscr{C})\}$ is contained in the centralizer $\Phi(\mathscr{C})_\varphi$ and $\sigma_t^\varphi(f) = f = U_t f U_t^*$.

Since $E(B) = \sum_{\beta \in \mathscr{S}} d_\beta^{-1} \langle I_\beta \mid BI_\beta \rangle I_\beta$, for every $B \in \Phi$, we have $\tau \circ E(B) = \varphi(B)$. Then Theorem 1.2 from Chapter VIII in [62] implies $\sigma_t^\varphi(\Gamma(\xi)) = U_t \Gamma(\xi) U_t^*$ for every $\xi \in [{}^{\beta_2\alpha}_{\beta_1}]$. By the definition of $\Phi(\mathscr{C})$, the lemma is proved. ∎

LEMMA 4.5. *For every word $W \in W(\mathscr{C})$ there exists a non-unital injective $*$-endomorphism ρ_W of $\Phi(\mathscr{C})$ such that*

$$\rho_W : f \in \mathrm{Hom}(W_1\beta_1, W_2\beta_2) \mapsto I_W \otimes f \in \mathrm{Hom}(WW_1\beta_1, WW_2\beta_2) \quad \textit{and}$$

$$\rho_W : \Gamma(\xi) \mapsto \sum_{\gamma_1, \gamma_2 \in \mathscr{S}} \sum_{u \in \mathcal{O}_{\gamma_2}^{W\beta_2}, v \in \mathcal{O}_{\gamma_1}^{W\beta_1}} u\Gamma([u^* \otimes I_\alpha][I_W \otimes \xi]v)v^*$$

for every $\xi \in [{}^{\beta_2\alpha}_{\beta_1}]$, where $\alpha \in \Lambda$, $\beta_1, \beta_2 \in \mathscr{S}$, $W_1, W_2 \in W(\mathscr{C})$.

Proof. Let $d_W = d_{\iota(W)}$. It is easy to check that $\varphi(f) = 1/d_W \varphi(I_W \otimes f)$. Let $\xi_i \in [{}^{\beta_{i+1}\alpha_i}_{\beta_i}]$ where $\beta_i \in \mathscr{S}$, $\alpha_i \in \Lambda$, $i = 1, \ldots, m+1$, $m \geqslant 1$. Note that $\varphi(\Gamma(\xi_m) \cdots \Gamma(\xi_1)) = 0 = d_W^{-1} \varphi(\rho_W(\Gamma(\xi_m)) \cdots \rho_W(\Gamma(\xi_1)))$ unless $m = 2n$ and $\beta_1 = \beta_{2n+1}$.

We claim that $\varphi(\Gamma(\xi_{2n})\cdots\Gamma(\xi_1)) = d_W^{-1}\varphi(\rho_W(\Gamma(\xi_{2n}))\cdots\rho_W(\Gamma(\xi_1)))$ if $\beta_1 = \beta_{2n+1}$. By the definition of τ and E (see equation (2.8)), we have

$$\tau\circ E(\Gamma(\xi_{2n})\cdots\Gamma(\xi_1)) = d_{\beta_1}\sum_{\substack{(\{j_1,i_1\},\dots,\{j_n,i_n\})\\ \in NC_2(2n), i_k<j_k}}\prod_{k=1}^{n}\frac{\sqrt{\lambda_{\alpha_{j_k}}}}{d_{\beta_{i_k}}}\langle\overline{\xi}_{j_k}\mid\xi_{i_k}\rangle,$$

where $NC_2(2n)$ stands for the set of non-crossing pairings of $\{1,\dots,2n\}$, i.e., there does not exist $1\leqslant i_k < i_l < j_k < j_l \leqslant 2n$ (see Notation 8.7 in [46] for more details). Thus we only need to show that, for every $(\{j_1,i_1\},\dots,\{j_n,i_n\}) \in NC_2(2n)$, $d_{\beta_1}d_W\prod_{k=1}^{n}(1/d_{\beta_{i_k}})\langle\overline{\xi}_{j_k}\mid\xi_{i_k}\rangle$ equals

$$\sum_{\gamma_1,\dots,\gamma_{2n}} d_{\gamma_1}\sum_{\substack{u_i\in\mathcal{O}^{W\beta_i}_{\gamma_i},\\ i=1,\dots,2n}}\prod_{k=1}^{n}\frac{1}{d_{\gamma_{i_k}}}\langle(u^*_{j_k}\otimes I_{\overline{\alpha}_{j_k}})(I_W\otimes\overline{\xi}_{j_k})u_{j_k+1}\mid(u^*_{i_k+1}\otimes I_{\alpha_{i_k}})(I_W\otimes\xi_{i_k})u_{i_k}\rangle,$$

here $u_{2n+1}=u_1$. We show that the two expressions are equal by induction on n.

If $n=1$, then

$$\sum_{\gamma_1,\gamma_2}\sum_{u_1\in\mathcal{O}^{W\beta_1}_{\gamma_1}, u_2\in\mathcal{O}^{W\beta_2}_{\gamma_2}}\langle(u_2^*\otimes I_{\overline{\alpha}_2})(I_W\otimes\overline{\xi}_2)u_1\mid(u_2^*\otimes I_{\alpha_1})(I_W\otimes\xi_1)u_1\rangle$$
$$=\sum_{\gamma_1}\sum_{u\in\mathcal{O}^{W\beta_1}_{\gamma_1}}\frac{d_{\gamma_1}}{d_{\beta_1}}\langle\overline{\xi}_2\mid\xi_1\rangle = d_W\langle\overline{\xi}_2\mid\xi_1\rangle.$$

Assume the claim holds for $n=2l$. For $n=2l+2$, let $k\in\{1,\dots,2l+2\}$ such that $j_k = i_k+1$. Note that

$$\sum_{\gamma_{i_k+1}}\sum_{u_{i_k+1}\in\mathcal{O}^{W\beta_{i_k+1}}_{\gamma_{i_k+1}}}\frac{1}{d_{\gamma_{i_k}}}\langle(u^*_{i_k+1}\otimes I_{\overline{\alpha}_{i_k+1}})(I_W\otimes\overline{\xi}_{i_k+1})u_{i_k+2}\mid(u^*_{i_k+1}\otimes I_{\alpha_{i_k}})(I_W\otimes\xi_{i_k})u_{i_k}\rangle$$
$$=\frac{1}{d_{\beta_{i_k}}}\delta_{\beta_{i_k},\beta_{i_k+2}}\delta u_{i_k+2}, u_{i_k}\langle\overline{\xi}_{j_k}\mid\xi_{i_k}\rangle.$$

Then the inductive hypothesis implies the claim.

Let Φ_0 be the $*$-subalgebra of $\Phi(\mathscr{C})$ generated by $\{f : \mathrm{Hom}(W_1\beta_1, W_2\beta_2), \beta_1,\beta_2\in\mathscr{S}, W_1,W_2\in W(\mathscr{C})\}$ and $\{\Gamma(\xi) : \xi\in[\begin{smallmatrix}\beta_2\alpha\\ \beta_1\end{smallmatrix}], \beta_1,\beta_2\in\mathscr{S},\alpha\in\Lambda\}$. Moreover, let $\rho_W(\Phi_0)$ be the $*$-subalgebra of $P\Phi(\mathscr{C})P$ generated by $\{\rho_W(f) : \mathrm{Hom}(W_1\beta_1, W_2\beta_2), \beta_1,\beta_2\in\mathscr{S}, W_1,W_2\in W(\mathscr{C})\}$ and $\{\rho_W(\Gamma(\xi)) : \xi\in[\begin{smallmatrix}\beta_2\alpha\\ \beta_1\end{smallmatrix}], \beta_1,\beta_2\in\mathscr{S},\alpha\in\Lambda\}$, where $P=\sum_{V\in W(\mathscr{C})}P_{WV}$.

By Lemma 4.4 and Lemma 2.1 in [29], Φ_0 and $\rho_W(\Phi_0)$ are respectively dense in $L^2(\Phi(\mathscr{C}),\varphi)$ and $L^2(\rho_W(\Phi_0)'', d_W^{-1}\varphi)$. By the discussion above,

$$\varphi(u\Gamma(\xi_n)\cdots\Gamma(\xi_1)v^*) = d_W^{-1}\varphi(\rho_W(u)\rho_W(\Gamma(\xi_n))\cdots\rho_W(\Gamma(\xi_1))\rho_W(v^*)),$$

where $\xi_i \in [\substack{\beta_{i+1}\alpha_i \\ \beta_i}]$, $u \in \mathcal{O}^{W_2\mathscr{S}}_{\beta_{n+1}}$ and $v \in \mathcal{O}^{W_1\mathscr{S}}_{\beta_1}$, $\beta_i \in \mathscr{S}$, $\alpha_i \in \Lambda$, $i = 1, \ldots, n$, $n \geqslant 1$, and $W_1, W_2 \in W(\mathscr{C})$.

Thus there exists a unitary U from $L^2(\Phi(\mathscr{C}), \varphi)$ to $L^2(\rho_W(\Phi_0)'', d_W^{-1}\varphi)$ such that $U^*\pi(\rho_W(f))U = f$ and $U^*\pi(\rho_W(\Gamma(\xi)))U = \Gamma(\xi)$ where π is the GNS representation of $\rho_W(\Phi_0)''$ with respect to $d_W^{-1}\varphi$. ∎

REMARK 4.6. With the notations used in Lemma 4.5, it is routine to check that

$$\sum_{u \in \mathcal{O}^{W\beta_2}_{\gamma_2}, v \in \mathcal{O}^{W\beta_1}_{\gamma_1}} u\Gamma([u^* \otimes I_\alpha][I_W \otimes \xi]v)v^*$$

does not depend on the choice of orthonormal basis of isometries in $\mathrm{Hom}(\gamma_i, W\beta_i)$ where the sum over u and v runs. Then it is not hard to check that $\rho_{W_2} \circ \rho_{W_1} = \rho_{W_2W_1}$ for every $W_1, W_2 \in W(\mathscr{C})$.

LEMMA 4.7. *For every pair of words $W_1, W_2 \in W(\mathscr{C})$ and $f \in \mathrm{Hom}(W_1, W_2)$, we denote*

$$f \otimes I := \sum_{\beta \in \mathscr{S}, W \in W(\mathscr{C})} f \otimes I_{W\beta} \in \Phi(\mathscr{C}),$$

where $f \otimes I_{W\beta} \in \mathrm{Hom}(W_1W\beta, W_2W\beta)$. Then the following intertwining relation in $\mathrm{End}(\Phi(\mathscr{C}))$ holds:

$$(f \otimes I)\rho_{W_1}(A) = \rho_{W_2}(A)(f \otimes I), \quad \forall A \in \Phi(\mathscr{C}).$$

Proof. We only need to show that $(f \otimes I)\rho_{W_1}(\Gamma(\xi)) = \rho_{W_2}(\Gamma(\xi))(f \otimes I)$ for every $\xi \in [\substack{\beta_2\alpha \\ \beta_1}]$, $\alpha \in \Lambda$, $\beta_i \in \mathscr{S}$, $i = 1, 2$. Without loss of generality, we assume that $f = u_2u_1^*$ where $u_i \in \mathcal{O}^{W_i}_\beta$, $\beta \in \mathscr{S}$, $i = 1, 2$. Note that $\widetilde{\mathcal{O}}^{W_i\beta_j}_\gamma = \{(u \otimes I_{\beta_j})v : u \in \mathcal{O}^{W_i}_\eta, v \in \mathcal{O}^{\eta\beta_j}_\gamma, \eta \in \mathscr{S}\}$ is an orthonormal basis of isometries in $\mathrm{Hom}(\gamma, W_i\beta_j)$, $i = 1, 2$, $j = 1, 2$. Hence by Remark 4.6 we have the desired statement. ∎

We choose and fix a partial isometry $V_W \in \Phi(\mathscr{C})$ such that $V_W^*V_W = P_W$ and $V_WV_W^* = P_\varnothing$ (whose existence is guaranteed by Theorem 4.3) for every object $W \in W(\mathscr{C})$ and $V_\varnothing = P_\varnothing$. Note that $\rho_W(P_\varnothing) = P_W$.

DEFINITION 4.8. For every object $W \in W(\mathscr{C})$, let $F(W)$ be the unital $*$-endomorphism of Φ $(= P_\varnothing\Phi(\mathscr{C})P_\varnothing)$ defined by

$$F(W)(A) := V_W\rho_W(A)V_W^*, \quad A \in \Phi.$$

In particular, $F(\varnothing) = \mathrm{Id}_\Phi$.

Let $\mathrm{End}(\Phi)$ be the (strict) C^*-tensor category of endomorphisms of Φ (normal faithful unital and $*$-preserving), with tensor structure given on objects by the composition of endomorphisms and C^*-norm on arrows given by the one of Φ. The tensor unit Id_Φ of $\mathrm{End}(\Phi)$ is simple because Φ is a factor.

By Lemma 4.7, for every $W, Y \in W(\mathscr{C})$ and $f \in \mathrm{Hom}(W, Y)$ we have

$$[V_Y(f \otimes I)V_W^*]\, F(W)(A) = F(Y)(A)\, [V_Y(f \otimes I)V_W^*], \quad \forall A \in \Phi.$$

Therefore

$$W \mapsto F(W), \quad f \mapsto F(f) := V_Y(f \otimes I)V_W^*$$

is a $*$-functor from $W(\mathscr{C})$ into $\mathrm{End}(\Phi)$. In particular, $F(I_W) = I_{F(W)}$ and $F(f^*) = F(f)^*$ hold.

We now show that F is a fully faithful tensor functor from $W(\mathscr{C})$ to $\mathrm{End}(\Phi)$, hence an equivalence of C^*-tensor categories onto its image.

The (unitary) tensorator of the functor

$$\begin{aligned} J_{W_1,W_2} : F(W_1) \otimes F(W_2) &= \mathrm{Ad}(V_{W_1}\rho_{W_1}(V_{W_2})) \circ \rho_{W_1}(\rho_{W_2}(\cdot)) \\ &\mapsto \mathrm{Ad}V_{W_1W_2} \circ \rho_{W_1W_2} = F(W_1W_2) \end{aligned}$$

is defined for every $W_1, W_2 \in W(\mathscr{C})$ by

$$J_{W_1,W_2} := V_{W_1W_2}\rho_{W_1}(V_{W_2}^*)V_{W_1}^* \in \Phi.$$

LEMMA 4.9. *The family of morphisms $\{J_{W_1,W_2}\}$ is natural in W_1 and W_2, i.e., the following diagram commutes in* $\mathrm{End}(\Phi)$ *for every $f_i \in \mathrm{Hom}(W_i, Y_i)$, $W_i, Y_i \in W(\mathscr{C})$, $i = 1, 2$:*

$$\begin{array}{ccc} F(W_1) \otimes F(W_2) & \xrightarrow{J_{W_1,W_2}} & F(W_1W_2) \\ \Big\downarrow{\scriptstyle F(f_1)\otimes F(f_2)} & & \Big\downarrow{\scriptstyle F(f_1\otimes f_2)} \\ F(Y_1) \otimes F(Y_2) & \xrightarrow{J_{Y_1,Y_2}} & F(Y_1Y_2). \end{array}$$

Proof. Since $\rho_W(P_\varnothing) = P_W$, $W \in W(\mathscr{C})$, Lemma 4.7 implies that

$$\begin{aligned} J_{Y_1,Y_2}(F(f_1) \otimes F(f_2)) &= V_{Y_1Y_2}\rho_{Y_1}(V_{Y_2}^*)(f_1 \otimes I)\rho_{W_1}(V_{Y_2})\rho_{W_1}(f_2 \otimes I)\rho_{W_1}(V_{W_2}^*)V_{W_1}^* \\ &= V_{Y_1Y_2}\Big[(f_1 \otimes I)\rho_{W_1}\Big(\sum_{\beta\in\mathscr{S}} f_2 \otimes I_\beta\Big)\Big]\rho_{W_1}(V_{W_2}^*)V_{W_1}^* \end{aligned}$$

and $F(f_1 \otimes f_2)J_{W_1,W_2} = V_{Y_1Y_2}[(f_1 \otimes f_2) \otimes I]\rho_{W_1}(V_{W_2}^*)V_{W_1}^*$. Note that

$$(f_1 \otimes I)\rho_{W_1}\Big(\sum_{\beta\in\mathscr{S}} f_2 \otimes I_\beta\Big) = \sum_{\beta\in\mathscr{S}} (f_1 \otimes f_2) \otimes I_\beta = [(f_1 \otimes f_2) \otimes I]P_{W_1W_2}.$$

Thus $\{J_{X_1,X_2}\}$ is natural. ∎

LEMMA 4.10. *The functor F defined above is a tensor functor.*

Proof. By Section 2.4 in [16], we only need to check that the following diagram commutes:

$$\begin{array}{ccccc} F(W)\otimes F(Y)\otimes F(Z) & \xrightarrow{J_{W,Y}\otimes I_{F(Z)}} & F(WY)\otimes F(Z) & \xrightarrow{J_{WY,Z}} & F(WYZ) \\ \downarrow & & & & \downarrow \\ F(W)\otimes F(Y)\otimes F(Z) & \xrightarrow{I_{F(W)}\otimes J_{Y,Z}} & F(W)\otimes F(YZ) & \xrightarrow{J_{W,YZ}} & F(WYZ). \end{array}$$

By Lemma 4.7 and by $\rho_W(\rho_Y(P_Z)) = P_{WYZ}$, we have

$$J_{WY,Z}(J_{W,Y}\otimes I_{F(Z)}) = V_{WYZ}\rho_W(\rho_Y(V_Z^*))\rho_W(V_Y^*)V_W^* = J_{W,YZ}(I_{F(W)}\otimes J_{Y,Z}). \quad \blacksquare$$

We denote by $\mathfrak{A}(\mathscr{C})$ the von Neumann subalgebra of $\Phi(\mathscr{C})$ generated by $\{f \in \mathrm{Hom}(\beta_1, W\beta_2) : \beta_1, \beta_2 \in \mathscr{S}, W \in W(\mathscr{C})\}$. By Theorem 4.2 from Chapter IX in [62] and Lemma 4.4, there exists a faithful normal conditional expectation $\widetilde{E} : \Phi(\mathscr{C}) \to \mathfrak{A}(\mathscr{C})$ with respect to φ defined by equation (4.2). Note that $\widetilde{E}|_\Phi = E : \Phi \to \mathfrak{A}$ defined in equation (2.8).

LEMMA 4.11. *For every $W \in W(\mathscr{C})$, let $\Phi(W) := P_W\Phi(\mathscr{C})P_W$ and $\mathfrak{A}(W) := \mathfrak{A}(\mathscr{C}) \cap \Phi(W)$. Then*

$$(\Phi(W), \widetilde{E}) = (\mathcal{M}_1(W), \widetilde{E}) *_{\mathfrak{A}(W)} (\mathcal{M}_2(W), \widetilde{E}),$$

where $\mathcal{M}_j(W)$, $j = 1, 2$, is the von Neumann subalgebra of $\Phi(W)$ generated by

$$\{u_2Au_1^* : A \in \mathcal{M}_j, u_i \in \mathrm{Hom}(\beta_i, W\gamma_i), \beta_i, \gamma_i \in \mathscr{S}, i = 1, 2\},$$

and $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined by equation (2.9) *and equation* (2.10) *respectively.*

Proof. By equation (2.10), it is not hard to check that $\Phi(W)$ is generated by $\mathcal{M}_1(W)$ and $\mathcal{M}_2(W)$. Since $\widetilde{E}(u_2Au_1^*) = u_2E(A)u_1^*$ and

$$\widetilde{E}[(u_1A_1v_1^*)(u_2A_2v_2^*)\cdots(u_nA_nv_n^*)] = u_1E[(A_1v_1^*u_2)(A_2v_2^*u_3)\cdots A_n]v_n^* = 0,$$

whenever $A_k \in \mathrm{Ker}\, E \cap \mathcal{M}_{i_k}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$, $u_i \in \mathrm{Hom}(\beta_i, W\gamma_i)$ and $v_i \in \mathrm{Hom}(\beta_i', W\gamma_i')$, the lemma is proved. $\blacksquare$

LEMMA 4.12. *The endomorphisms ρ_W of $\Phi(\mathscr{C})$ defined in Lemma* 4.5 *map Φ into $\Phi(W)$, moreover $\rho_W(\Phi)' \cap \Phi(W) \cong \mathrm{Hom}(W, W)$ for every $W \in W(\mathscr{C})$.*

Proof. With the notation as in the previous lemma, it is clear that

$$\mathcal{M}_1(W) = \{u_2Au_1^* : A \in \mathcal{N}_\gamma, u_i \in \mathrm{Hom}(\gamma, W\beta_i), \gamma, \beta_1, \beta_2 \in \mathscr{S}\}'',$$

where $\mathcal{N}_\gamma$ is the factor generated by $\Gamma(\xi_\gamma)$, $\xi_\gamma = I_\gamma \in \mathcal{O}_\gamma^{\gamma\mathbb{1}}$ (see equation (2.9)).

By the definition of ρ_W, we have

$$\rho_W(\Gamma(\xi_\beta)) = \sum_{\gamma\in\mathscr{S}, u\in\mathcal{O}_\gamma^{W\beta}} u\Gamma([u^*\otimes I_{\mathbb{1}}][I_W\otimes\xi_\beta]u)u^* = \sum_{\gamma\in\mathscr{S}, u\in\mathcal{O}_\gamma^{W\beta}} u\Gamma(\xi_\gamma)u^*.$$

Let $\{U_\gamma\}_{\gamma\in\mathscr{S}}$ be the unitaries defined before Lemma 2.6; it is clear that

$$\sum_{\beta\in\mathscr{S}} \sum_{\gamma\in\mathscr{S}, u\in\mathcal{O}_\gamma^{W\beta}} uU_\gamma u^* \in \rho_W(\Phi) \cap \mathcal{M}_1(W).$$

Let $\mathcal{B}\Big(\bigoplus_{\beta\in\mathscr{S}}\mathrm{Hom}(\gamma, W\beta)\Big)$ be the algebra of bounded linear operators acting on the Hilbert space $\bigoplus_{\beta\in\mathscr{S}}\mathrm{Hom}(\gamma, W\beta)$. Then it is easy to check that

$$u_2 A u_1^* \mapsto u_2 u_1^* \otimes A, \quad A \in \mathcal{N}_\gamma, u_i \in \mathrm{Hom}(\gamma, W\beta_i),$$

induces a $*$-isomorphism from $\mathcal{M}_1(W)$ onto $\bigoplus_{\gamma\in\mathscr{S}}\mathcal{B}\Big(\bigoplus_{\beta\in\mathscr{S}}\mathrm{Hom}(\gamma, W\beta)\Big)\otimes\mathcal{N}_\gamma$. Therefore Lemma 2.6 implies $\rho_W(\Phi)' \cap \Phi(W) \subset \mathcal{M}_1(W)$.

Since Φ is a factor, Proposition 2 from Chapter 2 in [9] implies that the reduction map by $I_{W\mathbb{1}} = \rho_W(I_{\mathbb{1}})$ is an isomorphism $\rho_W(\Phi)' \cap \mathcal{M}_1(W) \ni A \mapsto AI_{W\mathbb{1}} \in (I_{W\mathbb{1}}\rho_W(\Phi)I_{W\mathbb{1}})' \cap I_{W\mathbb{1}}\mathcal{M}_1(W)I_{W\mathbb{1}}$.

Note that

$$I_{W\mathbb{1}}\mathcal{M}_1(W)I_{W\mathbb{1}} \cong \bigoplus_{\gamma\in\mathscr{S}}\mathcal{B}(\mathrm{Hom}(\gamma, W\mathbb{1}))\otimes\mathcal{N}_\gamma,$$

and $\rho_W(\Gamma(\xi_{\mathbb{1}})) = \sum_{\gamma\in\mathscr{S}, u\in\mathcal{O}_\gamma^{W\mathbb{1}}} u\Gamma(\xi_\gamma)u^*$. We have

$$(I_{W\mathbb{1}}\rho_W(\Phi)I_{W\mathbb{1}})' \cap I_{W\mathbb{1}}\mathcal{M}_1(W)I_{W\mathbb{1}} \subseteq \bigoplus_{\gamma\in\mathscr{S}}\mathcal{B}(\mathrm{Hom}(\gamma, W\mathbb{1}))\otimes I_{\mathcal{N}_\gamma} \cong \mathrm{Hom}(W,W),$$

and the claim is proved by Lemma 4.7. ∎

LEMMA 4.13. *For every $W_1, W_2 \in W(\mathscr{C})$, we have that* $\mathrm{Hom}(F(W_1), F(W_2)) \cong \mathrm{Hom}(W_1, W_2)$.

Proof. Let $f_i \in \mathrm{Hom}(W_i, W_1 \oplus W_2)$ be isometries such that $f_i^* f_i = I_{W_i}$ and $e_1 + e_2 = I_{W_1\oplus W_2}$, where $e_i = f_i f_i^*$, $i = 1, 2$. Here $W_1 \oplus W_2 \in W(\mathscr{C})$ denotes the direct sum of objects in $W(\mathscr{C})$. Then $T \in \mathrm{Hom}(F(W_1), F(W_2))$ if and only if

$$(f_2 \otimes I)V_{W_2}^* T V_{W_1}\rho_{W_1}(A)(f_1^* \otimes I) = (f_2 \otimes I)\rho_{W_2}(A)V_{W_2}^* T V_{W_1}(f_1^* \otimes I), \quad \forall A \in \Phi.$$

Thus $(f_2 \otimes I)V_{W_2}^* T V_{W_1}(f_1^* \otimes I) \in (e_2 \otimes I)[\rho_{W_1\oplus W_2}(\Phi)' \cap \Phi(W_1 \oplus W_2)](e_1 \otimes I)$ and Lemma 4.12 implies the result. ∎

Denoted by $\mathrm{End}_0(\Phi)$ the subcategory of finite-dimensional (i.e., finite index) endomorphisms of Φ, by Lemma 4.9, Lemma 4.10 and Lemma 4.13 we can conclude the following theorem.

THEOREM 4.14. *The functor F is a fully faithful tensor $*$-functor (non-strict but unital), hence it gives an equivalence of $W(\mathscr{C})$ with (the repletion of) its image $F(W(\mathscr{C}))$ $\subset \mathrm{End}_0(\Phi)$ as C^*-tensor categories.*

Recall that a von Neumann algebra is called *σ-finite* (or *countably decomposable*) if every decomposition of unit by means of non-zero orthogonal projections is countable, and that σ-finiteness is equivalent to the existence of normal faithful states. In the case of categories with infinite and non-denumerable spectrum $\mathscr{S}$, the factor $\Phi \cong \Phi(H)''$ is not σ-finite (and it is either of type II_∞ or

III_λ, $\lambda \in (0,1]$). Indeed, $\pi_\tau(\mathfrak{A}) \subset \Phi(H)''$ and $\pi_\tau(I_\beta) \in \pi_\tau(\mathfrak{A})$, $\beta \in \mathscr{S}$, are uncountably many mutually orthogonal projections summing up to the identity. However, $\Phi \cong \widetilde{\Phi} \otimes \mathcal{B}(\mathcal{H})$ where $\widetilde{\Phi}$ is a σ-finite factor and $\mathcal{H}$ is a non-separable Hilbert space. Moreover, $\mathrm{Bimod}_0(\widetilde{\Phi} \otimes \mathcal{B}(\mathcal{H})) \simeq \mathrm{Bimod}_0(\widetilde{\Phi})$ where, e.g., $\mathrm{Bimod}_0(\widetilde{\Phi})$ denotes the category of faithful normal $\widetilde{\Phi}$-$\widetilde{\Phi}$ bimodules (in the sense of correspondences) with finite dimension (i.e., with finite index) and $\simeq$ denotes an equivalence of C^*-tensor categories. In the type III case, we also have $\mathrm{Bimod}_0(\widetilde{\Phi}) \simeq \mathrm{End}_0(\widetilde{\Phi})$. In [21] we give a proof of these last facts, that we could not find stated in the literature, cf. Corollary 8.6 in [55], [44]. Recalling Theorem 3.7 and summing up the previous discussion, we can conclude the following theorem.

THEOREM 4.15. *Any rigid C^*-tensor category with simple unit and infinite spectrum $\mathscr{S}$ (not necessarily denumerable) can be realized as finite index endomorphisms of a σ-finite type* III *factor, or as bimodules of a σ-finite type* II *factor, such as the free group factor $L(F_{\mathscr{S}})$ that arises by choosing the trivial Tomita structure.*

We leave open the questions on whether every countably generated rigid C^*-tensor category with simple unit can be realized on a hyperfinite factor, and on whether the free Araki–Woods factors defined by Shlyakhtenko in [57] are universal as well for rigid C^*-tensor categories with simple unit, as our construction (see Lemma 2.3, Lemma 2.9 and Theorem 3.7, the latter in the trivial Tomita structure case) might suggest.

Appendix A. AMALGAMATED FREE PRODUCT

In [63], the amalgamated free product of σ-finite von Neumann algebras is constructed. The purpose of this appendix is to demonstrate that the method used in Section 2 of [63] can be applied to construct the amalgamated free product of arbitrary von Neumann algebras.

DEFINITION A.1. Let $\{\mathcal{M}_s\}_{s\in S}$ be a family of von Neumann algebras having a common von Neumann subalgebra $\mathcal{N}$ such that each inclusion $I \in \mathcal{N} \subset \mathcal{M}_s$ has a normal faithful conditional expectation $E_s : \mathcal{M}_s \to \mathcal{N}$. The *amalgamated free product* $(\mathcal{M}, E) = *_{\mathcal{N},s\in S}(\mathcal{M}_s, E_s)$ of the family $(\mathcal{M}_s, E_s)$ is a von Neumann algebra $\mathcal{M}$ with a conditional expectation E satisfying:

(i) there exist normal $*$-isomorphisms π_s from $\mathcal{M}_s$ into $\mathcal{M}$ and $\pi_s|_{\mathcal{N}} = \pi_{s'}|_{\mathcal{N}}$, $s, s' \in S$; let $\pi = \pi_s|_{\mathcal{N}}$;

(ii) $\mathcal{M}$ is generated by $\pi_s(\mathcal{M}_s)$, $s \in S$;

(iii) E is a faithful normal conditional expectation onto $\pi(\mathcal{N})$ such that

$$E(\pi_s(m)) = \pi(E_s(m)),$$

and the family $\{\pi_s(\mathcal{M}_s)\}$ is free in $(\mathcal{M}, E)$, i.e.,

$$E(\pi_{s_1}(m_1)\cdots\pi_{s_k}(m_k)) = 0$$

if $s_1 \neq s_2 \neq \cdots \neq s_k$ and $m_i \in \operatorname{Ker} E_{s_i}$, $i = 1, \ldots, k$, where $k \geqslant 1$.

LEMMA A.2. *Let $\{(\mathcal{M}_s, E_s)\}_{s\in S}$ and $\{(\widetilde{\mathcal{M}}_s, \widetilde{E}_s)\}_{s\in S}$ be two families of von Neumann algebras having common unital von Neumann subalgebras $\mathcal{N}$ and $\widetilde{\mathcal{N}}$, respectively, and $E_s : \mathcal{M}_s \to \mathcal{N}$ and $\widetilde{E}_s : \widetilde{\mathcal{M}}_s \to \widetilde{\mathcal{N}}$ normal faithful conditional expectations.*

Let $(\mathcal{M}, E) := *_{\mathcal{N}, s\in S}(\mathcal{M}_s, E_s)$ and $(\widetilde{\mathcal{M}}, \widetilde{E}) := *_{\widetilde{\mathcal{N}}, s\in S}(\widetilde{\mathcal{M}}_s, \widetilde{E}_s)$ with normal $*$-isomorphisms π_s from $\mathcal{M}_s$ into $\mathcal{M}$ and $\widetilde{\pi}_s$ from $\mathcal{M}_s$ into $\widetilde{\mathcal{M}}$. If there is a family of $*$-isomorphisms $\rho_s : \mathcal{M}_s \to \widetilde{\mathcal{M}}_s$ such that $\rho_s(E_s(m)) = \widetilde{E}_s(\rho_s(m))$ and $\rho_s|_{\mathcal{N}} = \rho_{s'}|_{\mathcal{N}}$, then there exists a unique $*$-isomorphism $\rho : \mathcal{M} \to \widetilde{\mathcal{M}}$ such that $\widetilde{E} \circ \rho = \rho \circ E$ and $\rho : \pi_s(m) \mapsto \widetilde{\pi}_s(\rho_s(m))$ for every $m \in \mathcal{M}_s$, $s \in S$.

Proof. Without loss of generality, we can assume that $\mathcal{M}_s \subseteq \mathcal{M}$ and $\widetilde{\mathcal{M}}_s \subseteq \widetilde{\mathcal{M}}$ and $\pi_s(m) = m$, $\widetilde{\pi}_s(\widetilde{m}) = \widetilde{m}$ for every $s \in S$, $m \in \mathcal{M}_s$ and $\widetilde{m} \in \widetilde{\mathcal{M}}_s$.

Let φ be a n.s.f. weight on $\mathcal{N}$ and $\mathfrak{A}_0$ be the $*$-subalgebra of $\mathcal{M}$ generated by $\bigcup_{s\in S}(\mathfrak{N}(\mathcal{M}_s, \varphi\circ E_s) \cap \mathfrak{N}(\mathcal{M}_s, \varphi\circ E_s)^*)$. For each $m \in \mathfrak{N}(\mathcal{M}, \varphi\circ E)$, we use $m\varphi^{1/2}$ to denote the vector given by the canonical injection $m \in \mathfrak{N}(\mathcal{M}, \varphi\circ E) \mapsto L^2(\mathcal{M}, \varphi\circ E)$. Note that $\mathfrak{A}_0 \subset \mathfrak{N}(\mathcal{M}, \varphi\circ E) \cap \mathfrak{N}^*(\mathcal{M}, \varphi\circ E)$. We claim that $\mathfrak{A}_0\varphi^{1/2}$ is dense in $L^2(\mathcal{M}, \varphi\circ E)$.

For $a \in \mathfrak{N}(\mathcal{M}, \varphi\circ E) \cap \mathfrak{N}^*(\mathcal{M}, \varphi\circ E)$ and $\varepsilon > 0$, there is a spectral projection p of $E(a^*a)$ such that $\|(a - ap)\varphi^{1/2}\| < \varepsilon$ and $\varphi(p) < \infty$. Since $\mathfrak{A}_0$ is a dense $*$-subalgebra of $\mathcal{M}$ in the strong operator topology, there exists $b \in \mathfrak{A}_0$ such that $\|(b-a)p\varphi^{1/2}\| < \varepsilon$. Thus $\mathfrak{A}_0$ is dense in $L^2(\mathcal{M}, \varphi\circ E)$.

Note that $\widetilde{\varphi} = \varphi \circ \rho_s^{-1}$ is a n.s.f. weight on $\widetilde{\mathcal{N}}$. Similarly, let $\widetilde{\mathfrak{A}}_0$ be the $*$-subalgebra generated by $\bigcup_{s\in S}(\mathfrak{N}(\widetilde{\mathcal{M}}_s, \widetilde{\varphi}\circ \widetilde{E}_s) \cap \mathfrak{N}(\widetilde{\mathcal{M}}_s, \widetilde{\varphi}\circ \widetilde{E}_s)^*)$, then $\widetilde{\mathfrak{A}}_0\widetilde{\varphi}^{1/2}$ is dense in $L^2(\widetilde{\mathcal{M}}, \widetilde{\varphi}\circ\widetilde{E})$.

Let U be the linear map given by linear extension of

$$Um_1m_2\cdots m_k\varphi^{1/2} = \rho_{s(1)}(m_1)\rho_{s(2)}(m_2)\cdots\rho_{s(k)}(m_k)\widetilde{\varphi}^{1/2}$$

where $m_i \in \mathfrak{N}(\mathcal{M}_{s(i)}, \varphi\circ E_{s(i)}) \cap \mathfrak{N}(\mathcal{M}_{s(i)}, \varphi\circ E_{s(i)})^*$, $i(1) \neq i(2) \neq \cdots \neq i(k)$, $k \geqslant 1$. By Definition A.1(iii), it is not hard to check that U can be extended to a unitary from $L^2(\mathcal{M}, \varphi\circ E)$ onto $L^2(\widetilde{\mathcal{M}}, \widetilde{\varphi}\circ\widetilde{E})$ such that

$$U\pi_\varphi(m)U^* = \pi_{\widetilde{\varphi}} \circ \rho_s(m), \quad \forall m \in \mathcal{M}_s, s \in S,$$

where π_φ and $\pi_{\widetilde{\varphi}}$ are GNS representations of $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ with respect to $\varphi\circ E$ and $\widetilde{\varphi}\circ\widetilde{E}$. Then $\rho(\cdot) = \pi_{\widetilde{\varphi}}^{-1}(U\pi_\varphi(\cdot)U^*)$ satisfies the conditions in the lemma. ∎

To construct the amalgamated free product, we first fix a n.s.f. weight φ on $\mathcal{N}$ and regard $\mathcal{M}_s$ as a concrete von Neumann algebra acting on $\mathcal{H}_s = L^2(\mathcal{M}_s, \varphi\circ E_s)$ for every $s \in S$. Let $\mathcal{M}_s^\circ = \operatorname{Ker} E_s$ and $\mathfrak{N}(\mathcal{M}_s, \varphi\circ E_s) = \{m \in \mathcal{M}_s :$

$\varphi \circ E_s(m^*m) < \infty\}$. For each $m \in \mathfrak{N}(\mathcal{M}_s, \varphi \circ E_s)$, we use $m\varphi^{1/2}$ to denote the vector given by the canonical injection $\mathfrak{N}(\mathcal{M}_s, \varphi \circ E_s) \to \mathcal{H}_s$. Let J_s, Δ_s be the modular operators and $\{\sigma_t^{\varphi\circ E_s}\}_{t\in\mathbb{R}}$ be the modular automorphism group of $\mathcal{M}_s$ with respect to $\varphi \circ E_s$. Recall that $\mathcal{H}_s$ is a $\mathcal{M}_s$-$\mathcal{M}_s$ bimodule with left and right actions given by

$$m_1(a\varphi^{1/2})m_2 = J_s m_2^* J_s m_1 a\varphi^{1/2}, \quad m_1, m_2 \in \mathcal{M}_s, a \in \mathfrak{N}(\mathcal{M}_s, \varphi \circ E_s).$$

By Lemma 3.18 from Chapter VIII in [62], $m_1(a\varphi^{1/2})m_2 = m_1 a\sigma_{-\mathrm{i}/2}^{\varphi\circ E_s}(m_2)\varphi^{1/2}$ if $m_2 \in \mathcal{D}(\sigma_{-\mathrm{i}/2}^{\varphi\circ E_s})$. In the following, let $\mathfrak{A}_s$ be the maximal Tomita algebra of the left Hilbert algebra $\mathfrak{N}(\mathcal{M}_s, \varphi \circ E_s) \cap \mathfrak{N}(\mathcal{M}_s, \varphi \circ E_s)^*$, i.e.,

$$\mathfrak{A}_s=\{m\in\mathcal{M}_s : m\in\mathcal{D}(\sigma_z^{\varphi\circ E_s}) \text{ and } \sigma_z^{\varphi\circ E_s}(m)\in\mathfrak{N}(\mathcal{M}_s, \varphi\circ E_s)\cap\mathfrak{N}(\mathcal{M}_s, \varphi\circ E_s)^*, \forall z\in\mathbb{C}\},$$

and $\mathfrak{A}_{\mathcal{N}} = \mathcal{N} \cap \mathfrak{A}_s = E_s(\mathfrak{A}_s)$. For m_1, $m_2 \in \mathfrak{A}_s$, we use $m_1\varphi^{1/2}m_2$ to denote $m_1\sigma_{-\mathrm{i}/2}^{\varphi\circ E_s}(m_2)\varphi^{1/2}$. In particular, $\varphi^{1/2}m_2 = \sigma_{-\mathrm{i}/2}^{\varphi\circ E_s}(m_2)\varphi^{1/2}$ and $\langle \varphi^{1/2}m_1 \mid \varphi^{1/2}m_2\rangle = \varphi \circ E_s(m_2 m_1^*)$ (see Appendix B in [7], Section 2 in [17], Chapter IX in [62]).

Note that $n \in \mathfrak{N}(\mathcal{N}, \varphi) \mapsto n\varphi^{1/2} \in \mathcal{H}_s$ gives the natural embedding of $L^2(\mathcal{N}, \varphi)$ into $L^2(\mathcal{M}_s, \varphi \circ E_s)$. By [61], we can identify $L^2(\mathcal{N}, \varphi)$ as an $\mathcal{N}$-$\mathcal{N}$ sub-bimodule of $L^2(\mathcal{M}_s, \varphi \circ E_s)$, moreover J_s and Δ_s can be decomposed as

$$J_s = \begin{pmatrix} J_{\mathcal{N}} & \\ & J_s^\circ \end{pmatrix}, \quad \Delta_s = \begin{pmatrix} \Delta_{\mathcal{N}} & \\ & \Delta_s^\circ \end{pmatrix} \quad \text{on} \begin{pmatrix} L^2(\mathcal{N}, \varphi) \\ \mathcal{H}_s^\circ \end{pmatrix},$$

where $\mathcal{H}_s^\circ = \mathcal{H}_s \ominus L^2(\mathcal{N}, \varphi)$.

Define $\mathcal{H} = L^2(\mathcal{N}, \varphi) \oplus \bigoplus_{k=1}^{\infty} \Big(\bigoplus_{s_1 \neq \cdots \neq s_k} \mathcal{H}_{s_1}^\circ \otimes_\varphi \cdots \otimes_\varphi \mathcal{H}_{s_k}^\circ \Big)$, where $\otimes_\varphi$ is the relative tensor product ([56]). By Remarque 2.2(a) in [56],

$$\begin{aligned} \mathcal{H}_0 &= \mathfrak{A}_{\mathcal{N}}\varphi^{1/2} \oplus \bigoplus_{k=1}^{\infty} \Big(\bigoplus_{s_1\neq\cdots\neq s_k} \operatorname{span}\{a_1\varphi^{1/2} \otimes \cdots \otimes a_k\varphi^{1/2} : a_i \in \mathfrak{A}_{s_i}^\circ\} \Big) \\ &= \mathfrak{A}_{\mathcal{N}}\varphi^{1/2} \oplus \bigoplus_{k=1}^{\infty} \Big(\bigoplus_{s_1\neq\cdots\neq s_k} \operatorname{span}\{\varphi^{1/2}a_1 \otimes \cdots \otimes \varphi^{1/2}a_k : a_i \in \mathfrak{A}_{s_i}^\circ\} \Big) \end{aligned}$$

is dense in $\mathcal{H}$, where $\mathfrak{A}_s^\circ = \mathfrak{A}_s \cap \mathcal{M}_s^\circ$. By the definition of relative tensor product and Lemme 1.5(c) in [56] (or Proposition 3.1 in [17]), we have

$$\begin{aligned} &\langle a_1\varphi^{1/2} \otimes \cdots \otimes a_k\varphi^{1/2} \mid b_1\varphi^{1/2} \otimes \cdots \otimes b_k\varphi^{1/2}\rangle \\ &\qquad = \varphi(E_{s_k}(a_k^* E_{s_{k-1}}(\cdots a_2^* E_{s_1}(a_1^* b_1)b_2 \cdots)b_k)), \\ &\langle \varphi^{1/2}a_1 \otimes \cdots \otimes \varphi^{1/2}a_k \mid \varphi^{1/2}b_1 \otimes \cdots \otimes \varphi^{1/2}b_k\rangle \\ &\qquad = \varphi(E_{s_1}(b_1 E_{s_2}(\cdots b_{k-1} E_{s_k}(b_k a_k^*) a_{k-1}^* \cdots) a_1^*)). \end{aligned}$$

For each $s \in S$, let $\mathcal{H}(s,l) = L^2(\mathcal{N},\varphi) \oplus \bigoplus_{k=1}^{\infty} \Big(\bigoplus_{s_1 \neq \cdots \neq s_k, s_1 \neq s} \mathcal{H}^\circ_{s_1} \otimes_\varphi \cdots \otimes_\varphi \mathcal{H}^\circ_{s_k} \Big)$. By Proposition 3.5 in [17] (or Section 2.4 in [56]), we have unitaries

$$U_s : \mathcal{H}_s \otimes_\varphi \mathcal{H}(s,l) = (L^2(\mathcal{N},\varphi) \oplus \mathcal{H}^\circ_s) \otimes_\varphi \mathcal{H}(s,l) \to \mathcal{H}.$$

For $m \in \mathcal{M}_s$, let $\pi_s(m) = U_s m \otimes_\varphi I_{\mathcal{H}(s,l)} U_s^*$ (see Corollary 3.4(ii) in [17]). If $m_s \in \mathfrak{A}_s$, it is not hard to check that $\pi_s(m_s) m\varphi^{1/2} = E_s(m_s m)\varphi^{1/2} + (m_s m - E_s(m_s m))\varphi^{1/2}$ and that

$$\begin{aligned} &\pi_s(m_s) a_1\varphi^{1/2} \otimes \cdots \otimes a_k\varphi^{1/2} \\ &\qquad = \begin{cases} (m_s a_1 - E_s(m_s a_1))\varphi^{1/2} \otimes a_2\varphi^{1/2} \cdots \otimes a_k\varphi^{1/2} \\ \qquad + E_s(m_s a_1) a_2 \varphi^{1/2} \otimes \cdots \otimes a_k \varphi^{1/2}, & s_1 = s, \\ (m_s - E_s(m_s))\varphi^{1/2} \otimes a_1\varphi^{1/2} \otimes \cdots \otimes a_k\varphi^{1/2} \\ \qquad + E_s(m_s) a_1 \otimes \cdots \otimes a_k \varphi^{1/2}, & s_1 \neq s, \end{cases} \end{aligned} \tag{A.1}$$

where $m \in \mathfrak{A}_s$ and $a_i \in \mathfrak{A}^\circ_{s_i}$. Note that $L^2(\mathcal{N},\varphi) \oplus \mathcal{H}^\circ_s \cong L^2(\mathcal{M}_s, \varphi \circ E_s)$ and π_s is a $*$-isomorphism. If $n \in \mathcal{N}$, then it is clear that $\pi_s(n) = \pi_{s'}(n)$, $s, s' \in S$. And we use π to denote $\pi_s|_{\mathcal{N}}$. Define $\mathcal{M}$ to be the von Neumann algebra generated by $\{\pi_s(\mathcal{M}_s) : s \in S\}$.

Let J be the conjugate-linear map defined on $\mathcal{H}_0$ by $Jn\varphi^{1/2} = J_{\mathcal{N}} n \varphi^{1/2}$, $n \in \mathfrak{A}_{\mathcal{N}}$ and $J(a_1\varphi^{1/2} \otimes \cdots \otimes a_k\varphi^{1/2}) = J_{s_k}(a_k\varphi^{1/2}) \otimes \cdots \otimes J_{s_1}(a_1\varphi^{1/2}) = \varphi^{1/2}a_k^* \otimes \cdots \otimes \varphi^{1/2} a_1^*$, where $a_i \in \mathcal{H}^\circ_{s_i}$. Note that

$$\begin{aligned} &\langle \varphi^{1/2}a_k^* \otimes \cdots \otimes \varphi^{1/2}a_1^* \mid \varphi^{1/2}b_k^* \otimes \cdots \otimes \varphi^{1/2}b_1^*)\rangle \\ &\qquad = \langle b_1\varphi^{1/2} \otimes \cdots \otimes b_k\varphi^{1/2} \mid a_1\varphi^{1/2} \otimes \cdots \otimes a_k\varphi^{1/2}\rangle. \end{aligned} \tag{A.2}$$

Thus J can be extended to a conjugate-linear involutive unitary and we still use J to denote its extension.

The following fact is well known, at least when the von Neumann algebras are σ-finite (see Lemma 2.2 in [63]).

PROPOSITION A.3. *For every $m_s \in \mathcal{M}_s, m_{s'} \in \mathcal{M}_{s'}$, $s, s' \in S$, the operators $\pi_s(m_s)$ and $J\pi_{s'}(m_{s'}^*)J$ commute. Furthermore, the linear subspaces $\mathcal{M}L^2(\mathcal{N},\varphi)$ and $J\mathcal{M}JL^2(\mathcal{N},\varphi)$ are both dense in $\mathcal{H}$.*

Let $e_{\mathcal{N}}$ and e_s be the projections from $\mathcal{H}$ onto $L^2(\mathcal{N},\varphi)$ and $L^2(\mathcal{N},\varphi) \oplus \mathcal{H}^\circ_s$, respectively.

PROPOSITION A.4. *Let $s \in S$, then $e_s\pi_{s_1}(m_1)\pi_{s_2}(m_2)\cdots\pi_{s_k}(m_k)e_s = 0$ if $m_i \in \mathcal{M}^\circ_{s_i}$, $s_1 \neq s_2 \neq \cdots \neq s_k$ and $k > 1$. Moreover, $e_s\pi_{s_1}(m_1)e_s = \delta_{s,s_1}\pi_s(m_1)e_s$ and $e_{\mathcal{N}}\pi_{s_1}(m_1)e_{\mathcal{N}} = 0$. If $n \in \mathcal{N}$, then $e_s\pi(n)e_s = \pi(n)e_s$ and $e_{\mathcal{N}}\pi(n)e_{\mathcal{N}} = \pi(n)e_{\mathcal{N}}$.*

Proof. By equation (A.1), an easy calculation verifies all the equations when $m_{s_i} \in \mathfrak{A}^\circ_{s_i}$ and $n \in \mathfrak{A}_{\mathcal{N}}$. Since $\mathfrak{A}_s$ is dense in the strong operator topology in $\mathcal{M}_s$, the assertion is proved. ∎

By Proposition A.4, it is clear that $e_s \in \pi_s(\mathcal{M}_s)'$ ($e_\mathcal{N} \in \pi(\mathcal{N})'$) and $e_s\mathcal{M}e_s = \pi_s(\mathcal{M}_s)e_s$ ($e_\mathcal{N}\mathcal{M}e_\mathcal{N} = \pi(\mathcal{N})e_\mathcal{N}$). Let $E_{\mathcal{M}_s}$ and E be the normal conditional expectations from $\mathcal{M}$ onto $\pi_s(\mathcal{M}_s)$ and $\pi(\mathcal{N})$, respectively defined by

$$e_sAe_s = E_{\mathcal{M}_s}(A)e_s, \quad e_\mathcal{N}Ae_\mathcal{N} = E(A)e_\mathcal{N} \quad A \in \mathcal{M}.$$

By Proposition A.3, E and $E_{\mathcal{M}_s}$ are faithful. Also note that $E(E_{\mathcal{M}_s}(A)) = E(A)$ and $E(\pi_s(m)) = \pi(E_s(m))$. Therefore $(\mathcal{M}, E)$ is an amalgamated free product of the family $(\mathcal{M}_s, E_s)$ as in Definition A.1.

PROPOSITION A.5. *Let $(\mathcal{M}, E) = *_{\mathcal{N},s\in S}(\mathcal{M}_s, E_s)$. Then there exist normal conditional expectations $E_{\mathcal{M}_s} : \mathcal{M} \to \pi_s(\mathcal{M}_s)$ such that $E \circ E_{\mathcal{M}_s} = E$. Furthermore, for every* n.s.f. *weight ϕ on $\mathcal{N}$, we have $\sigma_t^{\widetilde{\phi}\circ E}(\pi_s(m)) = \pi_s(\sigma_t^{\phi\circ E_s}(m))$, $m \in \mathcal{M}_s$, where $\widetilde{\phi} = \phi \circ \pi^{-1}$.*

Proof. Note that $\widetilde{\phi} \circ E = (\widetilde{\phi} \circ E|_{\pi_s(\mathcal{M}_s)}) \circ E_{\mathcal{M}_s}$ and $\widetilde{\phi} \circ E(\pi_s(m)) = \phi \circ E_s(m)$. Then [61] implies the result. ∎

By Proposition A.5 and Lemma 2.1 in [29], we can identity $L^2(\mathcal{M}, \varphi \circ \pi^{-1} \circ E)$ with $\mathcal{H}$ by using the unitary given in Lemma A.2. Let S be the involution on the left Hilbert algebra $\mathfrak{N}(\mathcal{M}, \varphi\circ\pi^{-1}\circ E) \cap \mathfrak{N}(\mathcal{M}, \varphi\circ\pi^{-1}\circ E)^*$, i.e., $S : m \mapsto m^*$.

PROPOSITION A.6. *The left Hilbert algebra $\mathcal{H}_0$ is a Tomita algebra with involution $S_0 = S|_{\mathcal{H}_0}$ and complex one-parameter group $\{U(z) : z \in \mathbb{C}\}$ given by*

$$U(z)(a_1\varphi^{1/2} \otimes \cdots \otimes a_k\varphi^{1/2}) := \sigma_z^{\varphi\circ E_{s_1}}(a_1)\varphi^{1/2} \otimes \cdots \otimes \sigma_z^{\varphi\circ E_{s_k}}(a_k)\varphi^{1/2},$$

and $U(z)(n\varphi^{1/2}) := \sigma_z^{\varphi}(n)\varphi^{1/2}$, where $a_i \in \mathfrak{A}_s^\circ$ and $n \in \mathfrak{A}_\mathcal{N}$. In particular, the modular operator Δ associated with $\varphi\circ\pi^{-1}\circ E$ is the closure of $U(-\mathrm{i})$ and the modular conjugation is J.

Proof. For a_i, $b_i \in \mathfrak{A}_{s_i}^\circ$, $s_1 \neq \cdots \neq s_k$, let $\xi = a_1\varphi^{1/2} \otimes \cdots \otimes a_k\varphi^{1/2}$ and $\beta = b_1\varphi^{1/2} \otimes \cdots \otimes b_k\varphi^{1/2}$. Note that $S_0\xi = a_k^*\varphi^{1/2} \otimes \cdots \otimes a_1^*\varphi^{1/2}$. Thus we have

$$\begin{aligned}\langle \xi \mid U(z)\beta\rangle &= \varphi(E_{s_k}(a_k^* E_{s_{k-1}}(\cdots a_2^* E_{s_1}(a_1^*\sigma_z^{\varphi\circ E_{s_1}}(b_1))\sigma_z^{\varphi\circ E_{s_2}}(b_2)\cdots)\sigma_z^{\varphi\circ E_{s_k}}(b_k)))\\ &= \varphi(E_{s_k}(\sigma_{-\overline{z}}^{\varphi\circ E_{s_k}}(a_k)^* E_{s_{k-1}}(\cdots \sigma_{-\overline{z}}^{\varphi\circ E_{s_1}}(a_2)^* E_{s_1}(\sigma_{-\overline{z}}^{\varphi\circ E_{s_1}}(a_1)^* b_1)b_2\cdots)b_k))\\ &= \langle U(-\overline{z})\xi \mid \beta\rangle.\end{aligned}$$

By the definition of J and equation (A.2), we have

$$\langle S_0\xi \mid S_0\beta\rangle = \Big\langle JU\Big(-\frac{\mathrm{i}}{2}\Big)\xi \mid JU\Big(-\frac{\mathrm{i}}{2}\Big)\beta\Big\rangle = \Big\langle U\Big(-\frac{\mathrm{i}}{2}\Big)\beta \mid U\Big(-\frac{\mathrm{i}}{2}\Big)\xi\Big\rangle = \langle U(-\mathrm{i})\beta \mid \xi\rangle.$$

The other conditions in the definition of Tomita algebra can also be checked easily. Let $\overline{S}_0$ be the closure of S_0. By Theorem 2.2 from Chapter VI in [62], $\mathcal{H}_0$ is a core for both $\overline{S}_0$ and S_0^*. Note that $\mathcal{H}_0$ is contained in the maximal Tomita algebra in $\mathfrak{N}(\mathcal{M}, \varphi\circ\pi^{-1}\circ E) \cap \mathfrak{N}(\mathcal{M}, \varphi\circ\pi^{-1}\circ E)^*$. Therefore $\mathcal{H}_0 \subset \mathcal{D}(S) \cap \mathcal{D}(S^*)$ and by Theorem 2.2 from Chapter VI in [62], Δ and $\Delta^{1/2}$ are the closure of $U(-\mathrm{i})$ and $U(-\mathrm{i}/2)$, respectively. Since $\mathcal{H}_0$ is dense in $\mathcal{H}$, the modular conjugation is J. ∎

Appendix B. TECHNICAL RESULTS ON NON-σ-FINITE ALGEBRAS

The results of this appendix are well known in the case of *σ-finite* von Neumann algebras (i.e., von Neumann algebras that admit normal faithful states), but we could not find them stated in the literature in more general situations. In the following we generalize them to the case of arbitrary von Neumann algebras, as we shall need in our construction (when considering categories with uncountably many inequivalent simple objects).

Let $\mathcal{N}$ be a von Neumann subalgebra of a von Neumann algebra $\mathcal{M}$ and let E be a normal, faithful conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. Then $\mathcal{M}$ is a pre-Hilbert $\mathcal{N}$-$\mathcal{N}$ bimodule, as defined in Section 1, with the $\mathcal{N}$-valued inner product $\langle m_1 \,|\, m_2\rangle_{\mathcal{N}} := E(m_1^* m_2)$. The left action of $\mathcal{N}$ on $\mathcal{M}$ is clearly normal. For a n.s.f. weight φ of $\mathcal{N}$, the Hilbert space H_φ defined in Section 1 (see Notation 1.3), with $H = \mathcal{M}$, coincides with the GNS Hilbert space $L^2(\mathcal{M}, \varphi \circ E)$. The inner product $\langle \cdot \,|\, \cdot \rangle$ associated with H and φ is the GNS inner product $\langle \cdot \,|\, \cdot\rangle_{\varphi\circ E}$. We regard $\mathcal{M}$ and $\mathcal{N}$ as a concrete von Neumann algebras acting on $L^2(\mathcal{M}, \varphi \circ E)$. Recall that $e_{\mathcal{N}} \in \mathcal{N}'$ is the Jones projection from $L^2(\mathcal{M}, \varphi \circ E)$ onto the subspace $L^2(\mathcal{N}, \varphi)$, and $e_{\mathcal{N}} T e_{\mathcal{N}} = E(T) e_{\mathcal{N}}$ for every $T \in \mathcal{M}$. Moreover, there exists a unique n.s.f. operator-valued weight E^{-1} from $\mathcal{N}'$ to $\mathcal{M}'$ characterized by

$$\frac{\mathrm{d}(\varphi \circ E)}{\mathrm{d}\phi} = \frac{\mathrm{d}\varphi}{\mathrm{d}(\phi \circ E^{-1})},$$

where φ and ϕ are n.s.f. weights on $\mathcal{N}$ and $\mathcal{M}'$, respectively (see [24], [25], [38]).

The following generalizes Lemma 3.1 in [38] to the non-σ-finite case.

LEMMA B.1. *With the above notations $E^{-1}(e_{\mathcal{N}}) = I$.*

Proof. Let $\xi \in L^2(\mathcal{N}, \varphi)$ be a φ-bounded vector (see [6]) regarded as an element of $L^2(\mathcal{M}, \varphi \circ E)$, namely such that the map $R^\varphi(\xi) : n \in \mathfrak{N}(\mathcal{N}, \varphi) \mapsto n\xi \in L^2(\mathcal{M}, \varphi \circ E)$ extends to a bounded linear operator from $L^2(\mathcal{N}, \varphi)$ to $L^2(\mathcal{M}, \varphi \circ E)$. Note that

$$\|m\xi\|_2 = \|E(m^*m)^{1/2}\xi\|_2 \leqslant \|R^\varphi(\xi)\| \|m\|_2, \quad m \in \mathfrak{N}(\mathcal{M}, \varphi \circ E).$$

Thus ξ is a $\varphi \circ E$-bounded vector and the map $R^{\varphi\circ E}(\xi) : m \mapsto m\xi$ is a bounded operator in $\mathcal{M}'$.

It is clear that $(I - e_{\mathcal{N}}) R^{\varphi\circ E}(\xi) e_{\mathcal{N}} = 0$ for each φ-bounded vector $\xi \in L^2(\mathcal{N}, \varphi)$. Let $m \in \mathfrak{N}(\mathcal{M}, \varphi \circ E)$ such that $E(m) = 0$. Then $\langle \beta \,|\, R^{\varphi\circ E}(\xi) m\rangle_{\varphi\circ E} = \langle \beta \,|\, E(m)\xi\rangle_{\varphi\circ E} = 0$, for every $\beta \in L^2(\mathcal{N}, \varphi)$. Therefore, we have $R^{\varphi\circ E}(\xi) e_{\mathcal{N}} = e_{\mathcal{N}} R^{\varphi\circ E}(\xi)$ and $R^{\varphi\circ E}(\xi) e_{\mathcal{N}}|_{L^2(\mathcal{N},\varphi)} = R^\varphi(\xi)$.

By Proposition 3 in [6], there exists a family $\{\xi_\alpha\}_{\alpha\in S}$ of φ-bounded vectors in $L^2(\mathcal{N}, \varphi)$ such that

$$e_{\mathcal{N}} = \sum_\alpha R^\varphi(\xi_\alpha) R^\varphi(\xi_\alpha)^* = \sum_\alpha R^{\varphi\circ E}(\xi_\alpha) R^{\varphi\circ E}(\xi_\alpha)^* e_{\mathcal{N}}.$$

Since $\mathcal{M}e_{\mathcal{N}}L^2(\mathcal{M},\varphi\circ E) = \operatorname{span}\{m\xi : \xi \in L^2(\mathcal{N},\varphi), m \in \mathcal{M}\}$ is dense in $L^2(\mathcal{M},\varphi\circ E)$, we have

$$\sum_\alpha R^{\varphi\circ E}(\xi_\alpha)R^{\varphi\circ E}(\xi_\alpha)^* = I.$$

Therefore $E^{-1}(e_{\mathcal{N}}) = \sum_\alpha E^{-1}(R^{\varphi}(\xi_\alpha)R^{\varphi}(\xi_\alpha)^*) = \sum_\alpha R^{\varphi\circ E}(\xi_\alpha)R^{\varphi\circ E}(\xi_\alpha)^* = I$ (see Lemma 3.1 in [38] for more details on this part). ∎

Let J be the modular conjugation associated with $\varphi\circ E$ on the Hilbert space $L^2(\mathcal{M},\varphi\circ E)$. By Theorem 4.2 from Chapter IX in [62] (or see [61]), $Je_{\mathcal{N}}J = e_{\mathcal{N}}$. The basic extension of $\mathcal{M}$ by $e_{\mathcal{N}}$ is the von Neumann algebra generated by $\mathcal{M}$ and $e_{\mathcal{N}}$ ([30]), which coincides with $J\mathcal{N}'J$. For $A\in\mathcal{B}(L^2(\mathcal{M},\varphi\circ E))$, let $j(A) = JA^*J$. Then $\widehat{E} : A \in J\mathcal{N}'J \mapsto j\circ E^{-1}\circ j(A) \in \mathcal{M}$ is a n.s.f. operator-valued weight. By Lemma B.1, $\widehat{E}(e_{\mathcal{N}}) = I$. Moreover, the same arguments as in the proof of Proposition 2.2 in [29] lead us to the following lemma.

LEMMA B.2. *The projection $e_{\mathcal{N}}$ is in the centralizer of $\phi := \varphi\circ E\circ\widehat{E}$, namely $\sigma_t^{\phi}(e_{\mathcal{N}}) = e_{\mathcal{N}}$, $t\in\mathbb{R}$.*

Proof. By Lemma 1.3 in [38], $\widehat{E} = (j\circ E\circ j)^{-1}$. Thus

$$\frac{\mathrm{d}\phi}{\mathrm{d}\varphi\circ j} = \frac{\mathrm{d}\varphi\circ E}{\mathrm{d}(\varphi\circ E\circ j)} = \Delta_{\varphi\circ E}.$$

By Theorem 9 in [6] and Takesaki's theorem (Theorem 4.2 Chapter IX in [62]) we conclude $\sigma_t^{\phi}(e_{\mathcal{N}}) = \Delta_{\varphi\circ E}^{\mathrm{i}t}e_{\mathcal{N}}\Delta_{\varphi\circ E}^{-\mathrm{i}t} = e_{\mathcal{N}}$. ∎

Along the same line of arguments used in the proof of Proposition 3.3 in [64], we have the following result on the structure of amalgamated free products in the non-finite, non-σ-finite setting (see Appendix A for their definition).

PROPOSITION B.3. *Suppose that $\mathcal{M}_1 \supseteq \mathcal{N} \subseteq \mathcal{M}_2$ are von Neumann algebras with normal faithful conditional expectations $E_i : \mathcal{M}_i \to \mathcal{N}$, $i = 1,2$. Let*

$$(\mathcal{M},E) := (\mathcal{M}_1,E_1) *_{\mathcal{N}} (\mathcal{M}_2,E_2)$$

be their amalgamated free product (see Definition A.1*) and φ be a* n.s.f. *weight on $\mathcal{N}$.*

For $i = 1,2$, we use $\mathfrak{M}_i \subset \mathcal{M}_i$ to denote the set of analytic elements $A\in\mathcal{M}_i$ such that $\sigma_z^{\varphi\circ E}(A) \in \mathfrak{N}(\mathcal{M}_i,\varphi\circ E_i)\cap\mathfrak{N}(\mathcal{M}_i,\varphi\circ E_i)^*$, $z\in\mathbb{C}$.

Let $\widetilde{\mathfrak{M}}_1$ be a strongly dense $*$-subalgebra of $\mathfrak{M}_1$ which is globally invariant under the modular automorphism group $\sigma_t^{\varphi\circ E_1}$ and such that $E_1(\widetilde{\mathfrak{M}}_1)\subset\widetilde{\mathfrak{M}}_1$. Let $\mathfrak{A}$ be a unital von Neumann subalgebra of the centralizer $(\mathcal{M}_1)_{\varphi\circ E_1}$ such that:

(i) *$\varphi\circ E_1|_{\mathfrak{A}'\cap\mathcal{M}_1}$ is semifinite;*

(ii) *there is a net U_α of unitaries in $\mathfrak{A}$ satisfying $E_1(AU_\alpha B)\to 0$ in the strong operator topology for all A, $B\in\{I\}\cup\widetilde{\mathfrak{M}}_1$.*

Then any unitary $V\in\mathcal{M}$ satisfying $V\mathfrak{A}V^*\subset\mathcal{M}_1$ must be contained in $\mathcal{M}_1$. In particular $\mathfrak{A}'\cap\mathcal{M} = \mathfrak{A}'\cap\mathcal{M}_1$.

Analogous statements hold replacing $\mathcal{M}_1$, $\mathfrak{M}_1$, $\widetilde{\mathfrak{M}}_1$ with $\mathcal{M}_2$, $\mathfrak{M}_2$, $\widetilde{\mathfrak{M}}_2$.

Proof. We assume that $\mathcal{M}$ acts on $L^2(\mathcal{M}, \varphi \circ E)$ and take V as above. Let J be the modular conjugation associated with $\varphi \circ E$ and $\widehat{E} = j \circ E_{\mathcal{M}_1}^{-1} \circ j$ be the dual operator-valued weight from $J\mathcal{M}_1'J$ onto $\mathcal{M}$, where $E_{\mathcal{M}_1}$ is the conditional expectation from $\mathcal{M}$ onto $\mathcal{M}_1$ defined as in the discussion after Proposition A.4. We use $e_{\mathcal{M}_1}$ to denote the Jones projection from $L^2(\mathcal{M}, \varphi \circ E)$ onto $L^2(\mathcal{M}_1, \varphi \circ E_1)$. Note that $e_{\mathcal{M}_1} T e_{\mathcal{M}_1} = E_{\mathcal{M}_1}(T) e_{\mathcal{M}_1}$, $T \in \mathcal{M}$, and $L^2(\mathcal{M}_1, \varphi \circ E_1)$ is a separating set in $L^2(\mathcal{M}, \varphi \circ E)$ (see Proposition A.3 and Lemma A.2). Thus we only need to show that $(I - e_{\mathcal{M}_1}) V^* e_{\mathcal{M}_1} V (I - e_{\mathcal{M}_1}) = 0$.

Note that $V\mathfrak{A}V^* \subset \mathcal{M}_1$, $V^* e_{\mathcal{M}_1} V \in \mathfrak{A}' \cap J\mathcal{M}_1'J$ and $\widehat{E}(V e_{\mathcal{M}_1} V^*) = 1$ thanks to Lemma B.1. By taking spectral projections, it is sufficient to show that if F is a projection in $\mathfrak{A}' \cap J\mathcal{M}_1'J$ such that $F \leqslant I - e_{\mathcal{M}_1}$ and $\|\widehat{E}(F)\| < +\infty$, then $F = 0$.

Let $\mathfrak{N} = \widetilde{\mathfrak{M}}_1 \cap \mathcal{N}$, $\widetilde{\mathfrak{M}}_1^\circ = \operatorname{Ker} E_1|_{\widetilde{\mathfrak{M}}_1}$, $\mathfrak{M}_2^\circ = \operatorname{Ker} E_2|_{\mathfrak{M}_2}$ and $\mathfrak{M} = \mathfrak{N} + \operatorname{span}\Lambda(\widetilde{\mathfrak{M}}_1^\circ, \mathfrak{M}_2^\circ)$, where $\Lambda(\widetilde{\mathfrak{M}}_1^\circ, \mathfrak{M}_2^\circ)$ is the set of all alternating words in $\widetilde{\mathfrak{M}}_1^\circ$ and $\mathfrak{M}_2^\circ$. By Lemma 2.1 in [29], $\mathfrak{M}$ is dense as a subspace of $L^2(\mathcal{M}, \varphi \circ E)$. Let $H_0 = \operatorname{span}\{A e_{\mathcal{M}_1} B : A, B \in \mathfrak{M}\}$. By Lemma B.2 and Theorem 4.7 in [24], H_0 is globally invariant under σ_t^ϕ, where $\phi = \varphi \circ E \circ \widehat{E}$ is a n.s.f. weight on $J\mathcal{M}_1'J$. Therefore H_0 is dense as a subspace of $L^2(J\mathcal{M}_1'J, \phi)$ by Lemma 2.1 in [29].

Let $P \in \mathfrak{A}' \cap \mathcal{M}_1$ be a projection such that $0 < \gamma^2 = \phi(PFP)$. Let $T = \sum_{i=1}^n A_i e_{\mathcal{M}_1} B_i \in H_0$ satisfying $\phi(T^*T)^{1/2} \leqslant 3\gamma/2$ and $\phi((T - (PFP)^{1/2})^*(T - (PFP)^{1/2}))^{1/2} \leqslant \gamma/5$. Moreover, we can assume that $(I - e_{\mathcal{M}_1})T = T = T(I - e_{\mathcal{M}_1})$ by Lemma B.2 and Lemma 3.18(i) from Chapter VIII in [62]. Therefore $E_{\mathcal{M}_1}(A_i) = 0 = E_{\mathcal{M}_1}(B_i)$, $i = 1, \ldots, n$, and we could assume that A_i, $B_i \in \operatorname{span}(\Lambda(\widetilde{\mathfrak{M}}_1^\circ, \mathfrak{M}_2^\circ) \setminus \widetilde{\mathfrak{M}}_1^\circ)$.

For any U_α as in our assumptions, we have

$$\gamma^2 - |\phi(T^* U_\alpha T U_\alpha{}^*)| \leqslant |\phi((PFP)^{1/2} U_\alpha (PFP)^{1/2} U_\alpha^*) - \phi(T^* U_\alpha T U_\alpha^*)| \leqslant \frac{\gamma^2}{2}.$$

Thus

$$\begin{aligned}
\gamma^2 &\leqslant 2 \sum_{i,j=1}^n |\phi(B_i^* e_{\mathcal{M}_1} A_i^* U_\alpha A_j e_{\mathcal{M}_1} B_j U_\alpha^*)| \\
&= 2 \sum_{i,j=1}^n |\varphi \circ E(\sigma_{i}^{\varphi\circ E}(B_j) U_\alpha^* B_i^* E_{\mathcal{M}_1}(A_i^* U_\alpha A_j))| \\
&\leqslant 2 \Big(\max_{i,j} \|B_i\| \|\sigma_{-\mathrm{i}}^{\varphi\circ E}(B_j^*)\|_2 \Big) \Big(\sum_{i,j} \|E_{\mathcal{M}_1}(A_i^* U_\alpha A_j)\|_2 \Big),
\end{aligned}$$

the second equality is due to Proposition 2.17 in [60]. Note that each operator in $\Lambda(\widetilde{\mathfrak{M}}_1^\circ, \mathfrak{M}_2^\circ) \setminus \widetilde{\mathfrak{M}}_1^\circ$ can be written as $C\delta$ where $C \in \{1\} \cup \widetilde{\mathfrak{M}}_1^\circ$ and δ is a reduced word that starts with an operator in $\mathfrak{M}_2^\circ$. Let $C_1\delta_1$ and $C_2\delta_2$ be two such words.

By the fact $\varphi \circ E(\delta_i^* \delta_i) < +\infty$ and Proposition A.4, we have

$$\begin{aligned}\|E_{\mathcal{M}_1}(\delta_1^* C_1^* U_\alpha C_2 \delta_2)\|_2 &= \|E_{\mathcal{M}_1}(\delta_1^* E_1(C_1^* U_\alpha C_2)\delta_2)\|_2 \\ &\leqslant \|\delta_1^*\| \|E_1(C_1^* U_\alpha C_2)\delta_2\|_2 \to 0.\end{aligned}$$

Recall that $A_i \in \operatorname{span}(\Lambda(\widetilde{\mathfrak{M}}_1^\circ, \mathfrak{M}_2^\circ) \setminus \widetilde{\mathfrak{M}}_1^\circ)$. The above estimation implies that $\gamma = 0$ and $F = 0$. ∎

We conclude with two facts that are used in Section 3.

LEMMA B.4. *Let φ_i be a n.s.f. weight on a von Neumann algebra $\mathcal{M}_i$, and let $\mathcal{N}_i$ be a von Neumann subalgebra of $\mathcal{M}_i$ such that $\varphi_i|_{\mathcal{N}_i}$ is semifinite, $i = 1, 2$. Let $E_i : \mathcal{M}_i \to \mathcal{N}_i$ be the normal faithful conditional expectation satisfying $\varphi_i \circ E_i = \varphi_i$. If $\rho : \mathcal{M}_1 \to \mathcal{M}_2$ is a $*$-isomorphism such that $\rho(\mathcal{N}_1) = \mathcal{N}_2$ and $\varphi_1 = \varphi_2 \circ \rho$, then $\rho \circ E_1 = E_2 \circ \rho$.*

Proof. Let $m \in \mathfrak{N}(\mathcal{M}_1, \varphi_1)$. For every $n \in \mathfrak{N}(\mathcal{N}_1, \varphi_1|_{\mathcal{N}_1})$, we have

$$\varphi_2(\rho \circ E_1(n^* m)) = \varphi_1(n^* E_1(m)) = \varphi_1(n^* m) = \varphi_2(\rho(n)^* E_2(\rho(m))).$$

Thus $E_2(\rho(m)) = \rho(E_1(m))$. Since $\mathfrak{N}(\mathcal{M}_1, \varphi_1)$ is dense in $\mathcal{M}_1$, $\rho \circ E_1 = E_2 \circ \rho$. ∎

PROPOSITION B.5. *Let $\mathcal{M}_1 \supseteq \mathcal{N} \subseteq \mathcal{M}_2$ be von Neumann algebras. Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are semifinite factors and that $\mathcal{N}$ is a discrete abelian von Neumann algebra. Let $\{P_s\}_{s \in S}$ be the set of minimal projections of $\mathcal{N}$. Assume that $\tau_1(P_s) = \tau_2(P_s) < \infty$ for every P_s, where τ_1, τ_2 are tracial weights on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. If $P_{s_0}\mathcal{M}_1 P_{s_0} \cong P_{s_0}\mathcal{M}_2 P_{s_0}$, then there exists a $*$-isomorphism $\rho : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\rho(P_s) = P_s$ and $\rho \circ E_1 = E_2 \circ \rho$, where E_1 and E_2 are the normal conditional expectations satisfying $\tau_1 \circ E_1 = \tau_1$ and $\tau_2 \circ E_2 = \tau_2$.*

Proof. Let $Q_i \in \mathcal{M}_i$, $i = 1, 2$, be a subprojection of P_{s_0} such that $\tau_1(Q_1) = \tau_2(Q_2)$ and $\mathcal{M}_i \cong Q_i \mathcal{M}_i Q_i \otimes \mathcal{B}(\mathcal{H})$. Since $Q_1 \mathcal{M}_1 Q_1 \cong Q_2 \mathcal{M}_2 Q_2$, $\rho = \rho_0 \otimes \mathrm{Id}$ is a $*$-isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$, where ρ_0 is a $*$-isomorphism from $Q_1 \mathcal{M}_1 Q_1$ onto $Q_2 \mathcal{M}_2 Q_2$. By the uniqueness of the tracial state on $Q_1 \mathcal{M}_1 Q_1$, we have $\tau_1|_{Q_1 \mathcal{M}_1 Q_1} = \tau_2|_{Q_2 \mathcal{M}_2 Q_2} \circ \rho$. Note that $\tau_i = \tau_i|_{Q_i \mathcal{M}_i Q_i} \otimes \mathrm{Tr}$. Thus $\tau_1 = \tau_2 \circ \rho$. Therefore there exists a unitary $U \in \mathcal{M}_2$ such that $U\rho(P_s)U^* = P_s$. Thus $\mathrm{Ad}U \circ \rho$ satisfies the conditions in Lemma B.4. ∎

Acknowledgements. We would like to thank Yemon Choi for letting us including Example 1.2 in the paper, and Florin Rădulescu for suggesting a reference and informing us about an alternative proof of the computation of the fundamental group of an uncountably generated free group factor contained in Lemma 3.1. We are also indebted to Arnaud Brothier and Roberto Longo for comments and questions on earlier versions of the manuscript (some of which are reported at the end of Section 4), and to Rui Liu and Chi-Keung Ng for constructive discussions. Luca Giorgetti acknowledges the hospitality of the Academy of Mathematics and Systems Science (AMSS), Chinese Academy of Sciences (CAS) during a one-month invitation to Beijing in January 2017 that made this research possible. Research by the authors is supported by the Youth Innovation Promotion Association, CAS and the NSFC under grant numbers 11871303, 11871127, by the ERC Advanced Grant QUEST 669240, and by the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006. We also want to thank the anonymous referee for constructive criticisms and valuable comments.

REFERENCES

[1] L. BARNETT, Free product von Neumann algebras of type III, *Proc. Amer. Math. Soc.* **123**(1995), 543–553.

[2] M. BISCHOFF, Y. KAWAHIGASHI, R. LONGO, K.-H. REHREN, *Tensor Categories and Endomorphisms of von Neumann Algebras with Applications to Quantum Field Theory*, Springer Briefs in Math. Phy., vol. 3, Springer, Berlin 2015.

[3] A. BROTHIER, Unshaded planar algebras and their assocated II_1 factors, *J. Funct. Anal.* **262**(2012), 3839–3871.

[4] A. BROTHIER, M. HARTGLASS, D. PENNEYS, Rigid C^*-tensor categories of bimodules over interpolated free group factors, *J. Math. Phys.* **53**(2012), 123525.

[5] A. CONNES, Une classification des facteurs de type III, *Ann. Scient. École Norm. Sup.* **6**(1973), 133–252.

[6] A. CONNES, Spatial theory of von Neumann algebras, *J. Funct. Anal.* **35**(1980), 153–164.

[7] A. CONNES, *Noncommutative Geometry*, Academic Press, San Diego, CA 1994.

[8] S. DEPREZ, S. VAES, A classification of all finite index subfactors for a class of group-measure space II_1 factors, *J. Noncommut. Geom.* **5**(2011), 523–545.

[9] J. DIXMIER, *Von Neumann Algebras*, North-Holland Math. Library, vol. 27, North-Holland Publ. Co., Amsterdam-New York-Oxford 1981.

[10] S. DOPLICHER, R. HAAG, J.E. ROBERTS, Local observables and particle statistics. I, *Comm. Math. Phys.* **23**(1971), 199–230.

[11] S. DOPLICHER, J.E. ROBERTS, A new duality theory for compact groups, *Invent. Math.* **98**(1989), 157–218.

[12] K. DYKEMA, Free products of hyperfinite von Neumann algebras and free dimension, *Duke Math. J.* **69**(1993), 97–119.

[13] K. DYKEMA, On certain free product factors via an extended matrix model, *J. Funct. Anal.* **112**(1993), 31–60.

[14] K. DYKEMA, Interpolated free group factors, *Pacific J. Math.* **163**(1994), 123–135.

[15] K. DYKEMA, Amalgamated free products of multi-matrix algebras and a construction of subfactors of a free group factor, *Amer. J. Math.* **117**(1995), 1555–1602.

[16] P. ETINGOF, S. GELAKI, D. NIKSHYCH, V. OSTRIK, *Tensor Categories*, Math. Surveys Monogr., vol. 205, Amer. Math. Soc., Providence, RI 2015.

[17] T. FALCONE, L^2-von Neumann modules, their relative tensor products and the spatial derivative, *Illinois J. Math.* **44**(2000), 407–437.

[18] S. FALGUIÈRES, S. RAUM, Tensor C^*-categories arising as bimodule categories of II_1 factors, *Adv. Math.* **237**(2013), 331–359.

[19] S. FALGUIÈRES, S. VAES, The representation category of any compact group is the bimodule category of a II_1 factor, *J. Reine Angew. Math.* **643**(2010), 171–199.

[20] P. GHEZ, R. LIMA, J.E. ROBERTS, W^*-Categories, *Pacific J. Math.* **120**(1985), 79–109.

[21] L. GIORGETTI, L. WANG, W. YUAN, title, in preparation.

[22] A. GUIONNET, V.F.R. JONES, D. SHLYAKHTENKO, Random matrices, free probability, planar algebras and subfactors, *Clay Math. Proc.* **11**(2010), 201–239.

[23] A. GUIONNET, V.F.R. JONES, D. SHLYAKHTENKO, A semi-finite algebra associated to a subfactor planar algebra, *J. Funct. Anal.* **261**(2011), 1345–1360.

[24] U. HAAGERUP, Operator valued weights in von Neumann algebras. I, *J. Funct. Anal.* **32**(1979), 175–206.

[25] U. HAAGERUP, Operator valued weights in von Neumann algebras. II, *J. Funct. Anal.* **33**(1979), 339–361.

[26] M. HARTGLASS, Free product von Neumann algebras associated to graphs and Guionnet, Jones, Shlyakhtenko subfactors in infinite depth, *J. Funct. Anal.* **265**(2013), 3305–3324.

[27] T. HAYASHI, S. YAMAGAMI, Amenable tensor categories and their realizations as AFD bimodules, *J. Funct. Anal.* **172**(2000), 19–75.

[28] F. HIAI, D. PETZ, *The Semicircle Law, Free Random Variables and Entropy*, Math. Surveys Monogr., vol. 77, Amer. Math. Soc., Providence, RI 2006.

[29] M. IZUMI, R. LONGO, S. POPA, A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras, *J. Funct. Anal.* **155**(1998), 1451–1468.

[30] V.F.R. JONES, Index for subfactors, *Invent. Math.* **72**(1983), 1–25.

[31] V.F.R. JONES, Planar algebras. I, arXiv math.QA/9909027, 1999.

[32] V.F.R. JONES, D. SHLYAKHTENKO, K. WALKER, An orthogonal approach to the subfactor of a planar algebra, *Pacific J. Math.* **246**(2010), 187–197.

[33] V.F.R. JONES, V.S. SUNDER, *Introduction to Subfactors*, London Math. Soc. Lecture Note Ser., vol. 234, Cambridge Univ. Press, Cambridge 1997.

[34] R.V. KADISON, J.R. RINGROSE, *Fundamentals of the Theory of Operator Algebras*, Vol. I, II, Grad. Stud. Math., vol. 15, 21, Amer. Math. Soc., Providence, RI 1997.

[35] T. KAJIWARA, C. PINZARI, Y. WATATANI, Jones index theory for Hilbert C^*-bimodules and its equivalence with conjugation theory, *J. Funct. Anal.* **215**(2004), 1–49.

[36] T. KAJIWARA, Y. WATATANI, Jones index theory by Hilbert C^*-bimodules and K-theory, *Trans. Amer. Math. Soc.* **352**(2000), 3429–3472.

[37] V. KODIYALAM, V.S. SUNDER, Guionnet–Jones–Shlyakhtenko subfactors associated to finite-dimensional Kac algebras, *J. Funct. Anal.* **257**(2009), 3930–3948.

[38] H. KOSAKI, Extension of Jones' theory on index to arbitrary factors, *J. Funct. Anal.* **66**(1986), 123–140.

[39] E.C. LANCE, *Hilbert C^*-Modules. A Toolkit for Operator Algebraists*, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge Univ. Press, Cambridge 1995.

[40] R. LONGO, Index of subfactors and statistics of quantum fields. I, *Comm. Math. Phys.* **126**(1989), 145–155.

[41] R. LONGO, Index of subfactors and statistics of quantum fields. II. Correspondences, braid group statistics and Jones polynomial, *Comm. Math. Phys.* **130**(1990), 285–309.

[42] R. LONGO, J.E. ROBERTS, A theory of dimension, *K-Theory* **11**(1997), 103–159.

[43] S. MACLANE, *Categories for the Working Mathematician*, Grad. Texts in Math., vol. 5, Springer-Verlag, New York 1998.

[44] R. MEYER, Morita equivalence in algebra and geometry, preprint 1997.

[45] M. MÜGER, Tensor categories: a selective guided tour, *Rev. Un. Mat. Argentina* **51**(2010), 95–163.

[46] A. NICA, R. SPEICHER, *Lectures on the Combinatorics of Free Probability*, London Math. Soc. Lecture Note Ser., vol. 335, Cambridge Univ. Press, Cambridge 2006.

[47] S. POPA, Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras, *J. Operator Theory* **9**(1983), 253–268.

[48] S. POPA, An axiomatization of the lattice of higher relative commutants of a subfactor, *Invent. Math.* **120**(1995), 427–445.

[49] S. POPA, On a class of type II_1 factors with Betti numbers invariants, *Ann. of Math.* **163**(2006), 809–899.

[50] S. POPA, Strong rigidity of II_1 factors arising from malleable actions of ω-rigid groups, Part I, *Invent. Math.* **165**(2006), 369–408.

[51] S. POPA, Strong rigidity of II_1 factors arising from malleable actions of ω-rigid groups, Part II, *Invent. Math.* **165**(2006), 409–451.

[52] S. POPA, D. SHLYAKHTENKO, Universal properties of $L(F_\infty)$ in subfactor theory, *Acta Math.* **191**(2003), 225–257.

[53] F. RĂDULESCU, The fundamental group of the von Neumann algebra of a free group with infinitely many generators is $\mathbb{R}_+ \setminus \{0\}$, *J. Amer. Math. Soc.* **5**(1992), 517–532.

[54] F. RĂDULESCU, Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index, *Invent. Math.* **115**(1994), 347–389.

[55] M.A. RIEFFEL, Morita equivalence for C^*-algebras and W^*-algebras, *J. Pure Appl. Algebra* **5**(1974), 51–96.

[56] J.-L. SAUVAGEOT, Sur le produit tensoriel relatif d'espaces de Hilbert, *J. Operator Theory* **9**(1983), 237–252.

[57] D. SHLYAKHTENKO, Free quasi-free states, *Pacific J. Math.* **177**(1997), 329–367.

[58] D. SHLYAKHTENKO, A-valued semicircular systems, *J. Funct. Anal.* **166**(1999), 1–47.

[59] R. SPEICHER, Combinatorial theory of the free product with amalgamation and operator-valued free probability theory, *Mem. Amer. Math. Soc.* **132**(1998), 1–88.

[60] S. STRĂTILĂ, *Modular Theory in Operator Algebras*, Editura Academiei and Abacus Press, Bucureşti and Tunbridge Wells, Kent 1981.

[61] M. TAKESAKI, Conditional expectations in von Neumann algebras, *J. Funct. Anal.* **9**(1972), 306–321.

[62] M. TAKESAKI, *Theory of Operator Algebras*. II, Encyclopaedia Math. Sci., vol. 125, Springer, Berlin 2003.

[63] Y. UEDA, Amalgamated free product over Cartan subalgebras, *Pacific J. Math.* **191**(1999), 359–392.

[64] Y. UEDA, Some analysis of amalgamated free products of von Neumann algebras in the non-tracial setting, *J. London Math. Soc.* **88**(2013), 25–48.

[65] S. VAES, Explicit computations of all finite index bimodules for a family of II_1 factors, *Ann. Sci. École Norm. Sup.* **41**(2008), 743–788.

[66] D. VOICULESCU, K. DYKEMA, A. NICA, *Free Random Variables. A noncommutative Probability Approach to Free Products with Applications to Random Matrices, Operator Algebras and Harmonic Analysis on Free Groups*, CRM Monograph Ser., vol. 1, Amer. Math. Soc., Providence, RI 1992.

[67] S. YAMAGAMI, C^*-tensor categories and free product bimodules, *J. Funct. Anal.* **197**(2003), 323–346.

[68] S. YAMAGAMI, Frobenius duality in C^*-tensor categories, *J. Operator Theory* **52**(2004), 3–20.

LUCA GIORGETTI, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA TOR VERGATA, VIA DELLA RICERCA SCIENTIFICA, 1, ROMA, I-00133, ITALY *and* DIPARTIMENTO DI MATEMATICA "GUIDO CASTELNUOVO", SAPIENZA UNIVERSITÀ DI ROMA, PIAZZALE ALDO MORO, 5, ROMA, I-00185, ITALY

E-mail address: giorgett@mat.uniroma2.it

WEI YUAN, INSTITUTE OF MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, CHINA *and* SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, BEIJING, 100049, CHINA

E-mail address: wyuan@math.ac.cn

Received March 8, 2018; revised October 2, 2018 and January 3, 2019.