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FUNCTORIAL PROPERTIES OF $\mathrm{Ext}_{\mathrm{u}}(\cdot, \mathcal{B})$ WHEN $\mathcal{B}$ IS SIMPLE WITH CONTINUOUS SCALE

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ABSTRACT. In this note we define two functors Ext and $\mathrm{Ext}_{\mathrm{u}}$ which capture unitary equivalence classes of extensions in a manner which is finer than KK^1.

We prove that for every separable nuclear C^*-algebra $\mathcal{A}$, and for every σ-unital nonunital simple continuous scale C^*-algebra $\mathcal{B}$, $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ is an abelian group. We have a similar result for $\mathrm{Ext}_{\mathrm{u}}$. We study some functorial properties of the covariant functor $X \mapsto \mathrm{Ext}_{\mathrm{u}}(C(X), \mathcal{B})$, where X ranges over the category of compact metric spaces.

KEYWORDS: *K-theory, extension theory, Brown–Douglas–Fillmore theory, real rank zero.*

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1. INTRODUCTION

Motivated by the goal of classifying essentially normal operators using Fredholm indices, Brown, Douglas and Fillmore (BDF) classified all extensions of the form

$$0 \to \mathcal{K} \to \mathcal{E} \to C(X) \to 0$$

where X is a compact subset of the plane ([3]; see also [4]).

Perhaps one reason for the success of their theory is that the Calkin algebra $\mathbb{B}(l_2)/\mathcal{K}$ has particularly nice structure. Among other things, $\mathbb{B}(l_2)/\mathcal{K}$ is a simple purely infinite C^*-algebra, and, for example, the BDF–Voiculescu result ([1], [3], [28]), which roughly says that every essential extension is absorbing, would not be true if $\mathbb{B}(l_2)/\mathcal{K}$ were not simple.

In fact, comparison theory and structure theory for multiplier algebras and corona algebras have dotted the landscape of this subject even from the very beginning — though oftentimes these considerations were only present implicitly. (See, for example, [1], [3], [4], [6], [11], [12], [16], [20], [26], [28].) For example, it is by now clear, that for the classical theory of absorbing extensions, "niceness" of the extension theory corresponds to "niceness" of the corona algebra structure. (E.g., see [7].)

Recall that a σ-unital simple C^*-algebra $\mathcal{B}$ is said to have *continuous scale* if $\mathcal{B}$ has an approximate unit $\{e_n\}_{n=1}^{\infty}$ such that (i) $e_{n+1}e_n = e_n$ for all n, and (ii) for all $b \in \mathcal{B}_+ - \{0\}$, there exists $N \geqslant 1$ such that for all $m > n \geqslant N$,

$$e_m - e_n \preceq b.$$

Lin proved that for every σ-unital simple C^*-algebra $\mathcal{B}$, $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple if and only if $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is simple purely infinite if and only if either $\mathcal{B} \cong \mathcal{K}$ or $\mathcal{B}$ has continuous scale. (See [14], [21].) The class of simple continuous scale C^*-algebras is an ideal first context for a generalization of BDF theory, and such a theory was first definitively constructed in [16] (see also [17] and [18]). We note that in this context, $\mathcal{M}(\mathcal{B})$ can be too small to admit infinite repeats, and certain portions of the classical theory of absorbing extensions are no longer available. Thus, one needs to develop a type of nonstable absorption theory, where the fine structure of the additional K-theory is taken into account. This is part of the interesting new technical challenges which were first addressed in [16] and related papers.

In this paper, we continue the work of previous authors. Our first result shows that $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ is always an abelian group when $\mathcal{A}$ is separable nuclear and $\mathcal{B}$ is simple continuous scale. We also have a result for the unital case. We note that similar, but independent, results were proven in [23] for the case where $\mathcal{A}$ is unital commutative but $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is purely infinite without the requirement of simplicity. Moreover, the argument of [23] gives an explicit construction of the neutral element, whereas the proof of the present paper only gives existence. Functorial properties are then proven for the unital case when $\mathcal{A}$ is commutative. The proof techniques for the latter, which are similar in spirit to those in the original BDF paper ([3]), extensively involve real rank zero, and seem to also to play a role for classifying extensions in the setting of nonsimple purely infinite corona algebras.

This paper is part of a series of papers. In [10], we characterize (not necessarily simple) purely infinite corona algebras. I.e., under mild conditions on a simple separable nonunital finite C^*-algebra $\mathcal{B}$, we show that the following statements are equivalent:

(i) $\mathcal{B}$ has quasicontinuous scale;
(ii) $\mathcal{M}(\mathcal{B})$ has strict comparison;
(iii) $\mathcal{M}(\mathcal{B})/\mathcal{B}$ is purely infinite;
(iv) $\mathcal{M}(\mathcal{B})/I_{\mathrm{min}}$ is purely infinite;
(v) $\mathcal{M}(\mathcal{B})$ has finitely many ideals;
(vi) $I_{\mathrm{min}} = I_{\mathrm{fin}}$;
(vii) $V(\mathcal{M}(\mathcal{B}))$ has finitely many order ideals.

Moreover, under the conditions of the previous paragraph, $\mathcal{M}(\mathcal{B})/\mathcal{B}$ has real rank zero ([25]).

In a future paper, we will use the ideas from this paper to classify all extensions of the form

$$0 \to \mathcal{B} \to \mathcal{E} \to C(X) \to 0$$

where $\mathcal{B}$ is a nonunital simple real rank zero continuous scale C^*-algebra and X is a compact subset of the plane. We note that extensions of the above form have already been classified for the case where X is a finite CW complex but with no restriction on the topological dimension ([24]).

2. WHEN Ext IS A GROUP

Good references for extension theory, continuous scale algebras and some of the notation here used are [2], [6], [9], [14], [15], [16], [19], and [21]. Though KK-theory is not explicitly used, it nonetheless is spiritually present, and we refer the reader to the good references [2], [12] and [19].

We begin by fixing some notation. For a C^*-algebra $\mathcal{A}$, we let $\mathcal{A}^\sim$ denote the unitization of $\mathcal{A}$ if $\mathcal{A}$ is nonunital, and $\mathcal{A} \oplus \mathbb{C}$ if $\mathcal{A}$ is unital. If $\mathcal{A}$ is unital then $U(\mathcal{A})$ denotes the unitary group of $\mathcal{A}$ and $U_0(\mathcal{A})$ the elements of $U(\mathcal{A})$ that are in the connected component of the identity. $\mathrm{Proj}(\mathcal{A})$ denotes the projections in $\mathcal{A}$. For a nonunital simple C^*-algebra $\mathcal{B}$, we let $T(\mathcal{B})$ be the collection of all (norm) lower semicontinuous densely defined traces on $\mathcal{B}$ which are normalized on a fixed nonzero element of $\mathrm{Ped}(\mathcal{B})_+$ ($\mathrm{Ped}(\mathcal{B})$ is the Pedersen ideal of $\mathcal{B}$). For stably finite $\mathcal{B}$, we will always assume that all quasitraces are traces. We let $\mathrm{Aff}(T(\mathcal{B}))$ denote the affine continuous functions $T(\mathcal{B}) \to \mathbb{R}$, and $\mathrm{Aff}(T(\mathcal{B}))_{++}$ denote the strictly positive elements of $\mathrm{Aff}(T(\mathcal{B}))$. $\mathcal{M}(\mathcal{B})$ and $\mathcal{C}(\mathcal{B})$ denote the multiplier algebra and corona algebra of $\mathcal{B}$, respectively. For $A \in \mathcal{M}(\mathcal{B})_+$, $\widehat{A} : T(\mathcal{A}) \to [0, \infty]$ is the affine lower semicontinuous function given by $\widehat{A}(\tau) =_{\mathrm{df}} \tau(A)$ for all $\tau \in T(\mathcal{B})$. We let $\chi : K_0(\mathcal{B}) \to \mathrm{Aff}(T(\mathcal{B}))$ denote the map induced by $\chi([p]) =_{\mathrm{df}} \widehat{p}$ for all $p \in \mathrm{Proj}(\mathcal{B} \otimes \mathcal{K})$.

For a continuous map $\rho : X \to Y$ between compact Hausdorff topological spaces, we let $\phi_\rho : C(Y) \to C(X)$ denote the corresponding $*$-homomorphism (i.e., $\phi_\rho(f) =_{\mathrm{df}} f \circ \rho$ for all $f \in C(X)$).

For $A \in \mathcal{M}(\mathcal{B})_+$, we let $\mathrm{Her}(A) =_{\mathrm{df}} \overline{A\mathcal{M}(\mathcal{B})A}$ and $\mathrm{her}(A) =_{\mathrm{df}} \overline{A\mathcal{B}A}$, the hereditary subalgebras of $\mathcal{M}(\mathcal{B})$ and $\mathcal{B}$, respectively, generated by A.

Recall that to each extension $0 \to \mathcal{B} \to \mathcal{E} \to \mathcal{A} \to 0$, we can associate a $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ called the *Busby invariant* of the extension ([2], 15.2). Recall that ϕ determines the extension up to "strong isomorphism" (in the terminology of Blackadar; [2], 15.4). Moreover, the extension is essential if and only if ϕ is injective. Throughout this paper, we will be identifying an extension with its Busby invariant.

Let $\phi, \psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ be two extensions. We say that ϕ and ψ are *weakly unitarily equivalent* if there exists a unitary $u \in \mathcal{C}(\mathcal{B})$ such that $\phi(a) = u\psi(a)u^*$

for all $a \in \mathcal{A}$. We say that ϕ and ψ are *unitarily equivalent* or *strongly unitarily equivalent* if there exists a unitary $U \in \mathcal{M}(\mathcal{B})$ such that $\phi(a) = \pi(U)\psi(a)\pi(U^*)$ for all $a \in \mathcal{A}$, where $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{C}(\mathcal{B})$ is the standard quotient map. We write $\phi \sim \psi$ to mean that both ϕ and ψ are unitarily equivalent.

If $\mathcal{A}, \mathcal{C}$ are C^*-algebras with $\mathcal{C}$ unital, then we say that a $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{C}$ is *nonunital* if $1_{\mathcal{C}} \notin \mathrm{Ran}(\phi)$.

PROPOSITION 2.1. *Let $\mathcal{A}$ be a σ-unital C^*-algebra, and let $\mathcal{B}$ be a σ-unital simple continuous scale C^*-algebra. Suppose that $\phi, \psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ are two nonunital $*$-homomorphisms. If ϕ and ψ are weakly unitarily equivalent then they are unitarily equivalent.*

Proof. Firstly, we prove that there is a nonzero projection $p \in \mathcal{C}(\mathcal{B})$ such that $p \perp \psi(\mathcal{A})$.

Since $\mathcal{A}$ is σ-unital, let c be a strictly positive element of $\mathcal{A}$. Since ψ is nonunital, $\mathrm{her}(\psi(c)) =_{\mathrm{df}} \overline{\psi(c)\mathcal{C}(\mathcal{B})\psi(c)}$ is a proper hereditary C^*-subalgebra of $\mathcal{C}(\mathcal{B})$. By Theorem 7.7 of [27] $\mathrm{her}(\psi(c))^{\perp\perp} = \mathrm{her}(\psi(c))$. Hence, $\mathrm{her}(\psi(c))^{\perp} \neq 0$. Hence, since $\mathcal{C}(\mathcal{B})$ has real rank zero, there exists a nonzero projection $p \in \mathrm{her}(\psi(c))^{\perp}$.

Since ϕ and ψ are weakly unitarily equivalent, let $u \in U(\mathcal{C}(\mathcal{B}))$ be such that $\phi(a) = u\psi(a)u^*$ for all $a \in \mathcal{A}$. Hence, since $C(\mathcal{B})$ is simple purely infinite, there exists a $v \in U(p\mathcal{C}(\mathcal{B})p)$ such that $u' =_{\mathrm{df}} u(v + 1 - p)$ is in the connected component of 1 in $U(\mathcal{C}(\mathcal{B}))$. Clearly, $\phi(a) = u'\psi(a)(u')^*$ for all $a \in \mathcal{A}$ and u' lifts to a unitary in $U(\mathcal{M}(\mathcal{B}))$. ∎

Under additional hypotheses, we will prove that Proposition 2.1 holds for unital extensions.

LEMMA 2.2. *Let $\mathcal{A}$ be a C^*-algebra with no one-dimensional hereditary C^*-subalgebras, and let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ be finitely many nonzero hereditary C^*-subalgebras of $\mathcal{A}$.*

Then for every $\varepsilon > 0$ and for every finite subset $S \subset \mathcal{A}$, there exist norm one positive elements $b_j \in \mathcal{B}_j$ ($1 \leqslant j \leqslant n$) such that

$$\|b_j x b_k\| < \varepsilon$$

for all $x \in S$ and for all $j \neq k$.

The proof follows from [5] and an induction argument.

LEMMA 2.3. *Let $\mathcal{D}$ be a unital simple purely infinite C^*-algebra.*

Then for every $\varepsilon > 0$, for every finitely many $a_1, a_2, \ldots, a_n \in \mathcal{D}_+$ with norm one, and for every finite subset $S \subset \mathcal{D}$, there exist contractions $r_1, r_2, \ldots, r_n \in \mathcal{D}$ such that

$$\|r_j a_j r_j^* - 1\| < \varepsilon \quad \textit{and} \quad \|r_j x r_k^*\| < \varepsilon$$

for all $j \neq k$ and for all $x \in S$.

Proof. Let $f : [0,1] \to [0,1]$ be the unique continuous function satisfying

$$f(t) = \begin{cases} 0 & t \leqslant 1 - \varepsilon/3, \\ 1 & t \geqslant 1 - \varepsilon/6, \\ \text{linear on} & [1-\varepsilon/3, 1-\varepsilon/6]. \end{cases}$$

For all j, let $\mathcal{B}_j =_{\mathrm{df}} \mathrm{her}(f(a_j))$. By Lemma 2.2, for all j, let $b_j \in (\mathcal{B}_j)_+$ with $\|b_j\| = 1$ be such that every element of $b_j S b_k$ has norm less than ε for all $j \neq k$.

For all j, since $1 - \varepsilon/3 \leqslant \|b_j a_j b_j\| \leqslant 1$ and since $\mathcal{D}$ is simple purely infinite, let $x_j \in \mathcal{D}$ with $\|x_j\| = 1$ be such that $\|x_j b_j a_j b_j x_j^* - 1\| < \varepsilon$.

For all j, let $r_j =_{\mathrm{df}} x_j b_j$. ∎

Versions of the next two results were proven in [23] for the case where $\mathcal{A}$ is commutative, but $\mathcal{C}(\mathcal{B})$ can be a nonsimple purely infinite corona algebra.

THEOREM 2.4. *Let $\mathcal{B}$ be a nonunital simple σ-unital continuous scale C^*-algebra, $\mathcal{A}$ be a unital separable nuclear C^*-algebra, and $\phi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ be a unital $*$-monomorphism.*

Suppose that $\psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ is a c.p.c. map. ("c.p.c." abbreviates "completely positive contractive").

Then there exists a $v \in \mathcal{C}(\mathcal{B})$ such that

$$\psi(a) = v^* \phi(a) v$$

for all $a \in \mathcal{A}$.

Proof. The proof is exactly the same as that of Theorem 3.4 in [13] except that in the argument of Lemma 3.3 in [13] we replace Propositions 2.10 (and 2.9) in [13] with Lemma 2.3 in our paper. (This allows us to remove the "no one-dimensional hereditary C^*-subalgebras" condition from the hypothesis.) We further note that, in the arguments, separability of $\mathcal{B}$ can be replaced with σ-unitality. ∎

Recall that for a unital C^*-algebra $\mathcal{A}$, the "unitization" $\mathcal{A}^\sim =_{\mathrm{df}} \mathcal{A} \oplus \mathbb{C}$. Suppose that $\mathcal{A}, \mathcal{C}$ are unital C^*-algebras and $\psi : \mathcal{A} \to \mathcal{C}$ is a unital map. By the "unitization" map $\widetilde{\psi} : \mathcal{A}^\sim \to \mathcal{C}$, we mean the unique $*$-homomorphism such that $\widetilde{\psi}|_{\mathcal{A}} = \psi$ and $\widetilde{\psi}(\mathbb{C}) = 0$. When ψ or $\mathcal{A}$ is not unital, we have the usual definitions of "unitization".

COROLLARY 2.5. *Let $\mathcal{B}$ be a nonunital simple σ-unital continuous scale C^*-algebra, $\mathcal{A}$ be a separable nuclear C^*-algebra, and $\phi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ be a nonunital $*$-monomorphism.*

Suppose that $\psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ is a $*$-homomorphism.

Then there exists an isometry $v \in \mathcal{C}(\mathcal{B})$ such that

$$\psi(a) = v^* \phi(a) v$$

for all $a \in \mathcal{A}$.

Proof. Let $\widetilde{\phi}, \widetilde{\psi} : \mathcal{A}^{\sim} \to \mathcal{C}(\mathcal{B})$ be the unitizations of ϕ and ψ, respectively. By the argument of Proposition 2.1, there exists a nonzero projection orthogonal to $\phi(\mathcal{A})$. Then $\widetilde{\phi}$ is also a $*$-monomorphism. The result then follows from Theorem 2.4. We note that v is an isometry since $\widetilde{\phi}$ and $\widetilde{\psi}$ are unital. ∎

PROPOSITION 2.6. *Let $\mathcal{B}$ be a nonunital simple σ-unital continuous scale C^*-algebra, with $[1_{\mathcal{C}(\mathcal{B})}] = 0$ in $K_0(\mathcal{C}(\mathcal{B}))$. Let $\mathcal{A}$ be a unital separable nuclear C^*-algebra, $\phi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ a unital $*$-monomorphism, and $\psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ is a unital $*$-homomorphism.*

Then there exists a $*$-homomorphism $\sigma : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ and isometries $S, T \in \mathcal{C}(\mathcal{B})$ with $SS^* + TT^* = 1$ such that

$$\phi(a) = S\psi(a)S^* + T\sigma(a)T^*$$

for all $a \in \mathcal{A}$. Moreover, we can choose σ to be injective.

Proof. Since $[1]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$, let $X, Y \in \mathcal{C}(\mathcal{B})$ be isometries such that $XX^* + YY^* = 1$. Consider the unital $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ given by $\rho(a) =_{\mathrm{df}} X\phi(a)X^* + Y\psi(a)Y^*$ for all $a \in \mathcal{A}$.

By Theorem 2.4, let $v \in \mathcal{C}(\mathcal{B})$ be such that $v^*\phi(a)v = \rho(a)$ for all $a \in \mathcal{A}$. Since ϕ and ρ are unital, v is an isometry.

Now for all $a \in \mathcal{A}_+$, $v^*\phi(a)vv^*\phi(a)v = \rho(a)^2 = \rho(a^2) = v^*\phi(a)^2 v$. So for all $a \in \mathcal{A}_+$, $vv^*\phi(a)(1 - vv^*)\phi(a)vv^* = 0$. Hence, the projection vv^* commutes with every element of $\phi(\mathcal{A})$.

Hence, for all $a \in \mathcal{A}$, $\phi(a) = \phi(a)(1 - vv^*) + v\rho(a)v^* = \phi(a)(1 - vv^*) + vX\phi(a)X^*v^* + vY\psi(a)Y^*v^*$.

Let $T \in \mathcal{C}(\mathcal{B})$ be an isometry such that $TT^* = (1 - vv^*) + vXX^*v^*$, let $\sigma(a) =_{\mathrm{df}} T^*(\phi(a)(1 - vv^*) + vX\phi(a)X^*v^*)T$ for all $a \in \mathcal{A}$, and let $S =_{\mathrm{df}} vY$. ∎

PROPOSITION 2.7. *Let $\mathcal{B}$ be a nonunital simple σ-unital continuous scale C^*-algebra, $\mathcal{A}$ be a separable nuclear C^*-algebra, $\phi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ a nonunital $*$-monomorphism, and $\psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ a $*$-homomorphism.*

Then there exists a $*$-homomorphism $\sigma : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ and isometries $S, T \in \mathcal{C}(\mathcal{B})$ with $SS^* + TT^* \leqslant 1$ such that

$$\phi(a) = S\psi(a)S^* + T\sigma(a)T^*$$

for all $a \in \mathcal{A}$. Moreover, we can choose σ to be injective and nonunital.

Proof. The proof is a modification of the proof of Proposition 2.6. Let $X, Y \in \mathcal{C}(\mathcal{B})$ be isometries with $XX^* + YY^* \leqslant 1$. Consider the nonunital $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ given by $\rho(a) =_{\mathrm{df}} X\phi(a)X^* + Y\psi(a)Y^*$ for all $a \in \mathcal{A}$.

By Corollary 2.5, let $v \in \mathcal{C}(\mathcal{B})$ be an isometry such that $v^*\phi(a)v = \rho(a)$ for all $a \in \mathcal{A}$.

By the same argument as that of Proposition 2.6, the projection vv^* commutes with every element of $\phi(\mathcal{A})$.

So as in the proof of Propositon 2.6, for all $a \in \mathcal{A}$, $\phi(a) = \phi(a)(1 - vv^*) + vX\phi(a)X^*v^* + vY\psi(a)Y^*v^*$.

By the argument of Proposition 2.1, there is a nonzero subprojection of vXX^*v^* which is orthogonal to $vX\phi(\mathcal{A})X^*v^*$. Hence, there exists a projection $p \in \mathcal{C}(\mathcal{B})$ with $p \sim 1$, $p \perp vYY^*v^*$, and p is strictly bigger than some local unit for $\phi(\mathcal{A})(1 - vv^*) + vX\phi(\mathcal{A})X^*v^*$. Let $T \in \mathcal{C}(\mathcal{B})$ be an isometry with $TT^* = p$, let $\sigma(a) =_{\mathrm{df}} T^*(\phi(a)(1 - vv^*) + vX\phi(a)X^*v^*)T$ for all $a \in \mathcal{A}$, and let $S =_{\mathrm{df}} vY$. ∎

DEFINITION 2.8. Let $\mathcal{A}$ be a separable C^*-algebra, and let $\mathcal{B}$ be a nonunital σ-unital simple continuous scale C^*-algebra.

Let $\mathrm{Ext}(\mathcal{A},\mathcal{B})$ denote the collection of all unitary equivalence classes of nonunital essential extensions $\phi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$.

If, in addition $\mathcal{A}$ is unital, let $\mathrm{Ext_u}(\mathcal{A},\mathcal{B})$ denote the collection of all unitary equivalence classes of unital essential extensions $\phi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$.

To simplify notation, for a compact metric space X, we will often write "$\mathrm{Ext_u}(X,\mathcal{B})$" for $\mathrm{Ext_u}(C(X),\mathcal{B})$.

Let $\mathcal{A}$ be a separable C^*-algebra, and let $\mathcal{B}$ be a nonunital σ-unital continuous scale C^*-algebra. Let $S, T \in \mathcal{C}(\mathcal{B})$ be isometries such that $SS^* + TT^* \leqslant 1$, and let $\phi, \psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ be nonunital essential extensions. We define

$$\phi(a) \oplus \psi(a) =_{\mathrm{df}} S\phi(a)S^* + T\psi(a)T^*$$

for all $a \in \mathcal{A}$. This sum is well-defined up to weak unitary equivalence. Thus, by Proposition 2.1, this sum is well-defined up to unitary equivalence. Thus we have the following proposition.

PROPOSITION 2.9. *Let $\mathcal{A}$ be a separable nuclear C^*-algebra, and let $\mathcal{B}$ be a nonunital σ-unital continuous scale C^*-algebra.*

Then $\mathrm{Ext}(\mathcal{A},\mathcal{B})$ *is an abelian semigroup.*

THEOREM 2.10. *Let $\mathcal{A}$ be a separable nuclear C^*-algebra, and let $\mathcal{B}$ be a nonunital σ-unital continuous scale C^*-algebra.*

Then $\mathrm{Ext}(\mathcal{A},\mathcal{B})$ *is an abelian group.*

Proof. Firstly, we prove the existence of a neutral element. Let $x \in \mathrm{Ext}(\mathcal{A},\mathcal{B})$. By Proposition 2.7, there exists a $y \in \mathrm{Ext}(\mathcal{A},\mathcal{B})$ such that $x = x + y$.

We claim that y is the identity for $\mathrm{Ext}(\mathcal{A},\mathcal{B})$. Let $x' \in \mathrm{Ext}(\mathcal{A},\mathcal{B})$ be arbitrary. By Proposition 2.7, there exists $x'' \in \mathrm{Ext}(\mathcal{A},\mathcal{B})$ such that $x' = x'' + x$. Hence

$$x' + y = x'' + x + y = x'' + x = x'.$$

Since x' was arbitrary, y is the identity for $\mathrm{Ext}(\mathcal{A},\mathcal{B})$.

Finally, let $x' \in \mathrm{Ext}(\mathcal{A},\mathcal{B})$ again be arbitrary. By Proposition 2.7, there exists $x'' \in \mathrm{Ext}(\mathcal{A},\mathcal{B})$ such that $y = x' + x''$. Thus x'' is the inverse for x'. ∎

PROPOSITION 2.11. *Let $\mathcal{A}$ be a unital separable nuclear C^*-algebra such that $\mathcal{A}$ has a one-dimensional $*$-representation. Let $\mathcal{B}$ be a nonunital σ-unital simple continuous scale C^*-algebra with $[1_{\mathcal{C}(\mathcal{B})}] = 0$ in $K_0(\mathcal{C}(\mathcal{B}))$.*

Suppose that $\phi, \psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ are two unital $*$-monomorphisms.

If ϕ and ψ are weakly unitarily equivalent then they are unitarily equivalent.

Proof. Let $u \in U(\mathcal{C}(\mathcal{B}))$ be such that $\phi(a) = u\psi(a)u^*$ for all $a \in \mathcal{A}$. Let $\sigma : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ be a unital $*$-homomorphism with one-dimensional range. By Proposition 2.6, let $\rho : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ be a unital $*$-homomorphism, and let $S, T \in \mathcal{C}(\mathcal{B})$ be isometries with $SS^* + TT^* = 1$ such that

$$\psi(a) = S\sigma(a)S^* + T\rho(a)T^*$$

for all $a \in \mathcal{A}$.

Let $p =_{\text{df}} SS^*$. Since $\mathcal{C}(\mathcal{B})$ is simple purely infinite, we can find a unitary $v \in U(p\mathcal{C}(\mathcal{B})p)$ such that $w =_{\text{df}} u(v + (1-p))$ is in the connected component of the identity of $U(\mathcal{C}(\mathcal{B}))$.

Then for all $a \in \mathcal{A}$, $w\psi(a)w^* = u\psi(a)u^* = \phi(a)$. ∎

We can now define an addition on Ext_{u} in a manner similar to that on Ext. Let $\mathcal{A}$ be a unital separable nuclear C^*-algebra such that $\mathcal{A}$ has a one-dimensional $*$-representation. Let $\mathcal{B}$ be a nonunital σ-unital continuous scale C^*-algebra with $[1_{\mathcal{C}(\mathcal{B})}] = 0$ in $K_0(\mathcal{C}(\mathcal{B}))$. Let $S, T \in \mathcal{C}(\mathcal{B})$ be isometries such that $SS^* + TT^* = 1$, and let $\phi, \psi : \mathcal{A} \to \mathcal{C}(\mathcal{B})$ be unital essential extensions. We define

$$\phi(a) \oplus \psi(a) =_{\text{df}} S\phi(a)S^* + T\psi(a)T^*$$

for all $a \in \mathcal{A}$. This sum is well-defined up to weak unitary equivalence, and thus, by Proposition 2.11, this sum is well-defined up to unitary equivalence. Thus, we have the following proposition.

PROPOSITION 2.12. *Let $\mathcal{A}$ be a unital separable nuclear C^*-algebra such that $\mathcal{A}$ has a one-dimensional $*$-representation. Let $\mathcal{B}$ be a nonunital σ-unital continuous scale C^*-algebra with $[1_{\mathcal{C}(\mathcal{B})}] = 0$ in $K_0(\mathcal{C}(\mathcal{B}))$.*

Then $\text{Ext}_{\text{u}}(\mathcal{A}, \mathcal{B})$ *is an abelian semigroup.*

THEOREM 2.13. *Let $\mathcal{A}$ be a unital separable nuclear C^*-algebra such that $\mathcal{A}$ has a one-dimensional $*$-representation. Let $\mathcal{B}$ be a nonunital σ-unital continuous scale C^*-algebra with $[1_{\mathcal{C}(\mathcal{B})}] = 0$ in $K_0(\mathcal{C}(\mathcal{B}))$.*

Then $\text{Ext}_{\text{u}}(\mathcal{A}, \mathcal{B})$ *is an abelian group.*

The proof is exactly the same as that of Theorem 2.10, except that we replace every occurrence of Proposition 2.7 with Proposition 2.6.

COROLLARY 2.14. *Let X be a locally compact second countable topological space, and let $\mathcal{B}$ be a nonunital σ-unital continuous scale C^*-algebra.*

Then $\text{Ext}(C_0(X), \mathcal{B})$ *is an abelian group.*

If, in addition, X is compact and $[1]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$, then $\mathrm{Ext_u}(C(X),\mathcal{B})$ *is an abelian group.*

3. NULL EXTENSIONS

The notion of a null extension (defined in Theorem 3.1) is due to Lin (e.g., see [16]), with precursors in the original BDF work ([3]). Some of what follows, in this and the next section, have the flavour of operator theory. (E.g., see Halmos' proof of the Weyl–von Neumann–Berg theorem; Theorem 5.3 of [3] and [8].)

THEOREM 3.1. *Let X be a compact metric space and let $\mathcal{B}$ be a nonunital σ-unital simple continuous scale* C^**-algebra such that $[1_{\mathcal{C}(\mathcal{B})}] = 0$ in $K_0(\mathcal{C}(\mathcal{B}))$. Then there exists a null unital essential extension $\phi : C(X) \to \mathcal{C}(\mathcal{B})$. I.e., there exists a unital essential extension $\phi : C(X) \to \mathcal{C}(\mathcal{B})$ and a unital commutative* AF*-subalgebra $\mathcal{C} \subset \mathcal{C}(\mathcal{B})$ such that* $\mathrm{Ran}(\phi) \subseteq \mathcal{C}$ *and $[p] = 0$ in $K_0(\mathcal{C}(\mathcal{B}))$ for every projection $p \in \mathcal{C}$.*

Proof. Since $\mathcal{C}(\mathcal{B})$ is simple purely infinite with $[1]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$, this follows immediately from standard results. (E.g., $C(X)$ can easily be unitally embedded into a commutative unital AF-algebra, and any commutative unital AF-algebra can easily be unitally embedded into O_2.) ∎

THEOREM 3.2. *Let $\mathcal{B}$ be a nonunital simple σ-unital continuous scale C^*-algebra with $[1_{\mathcal{C}(\mathcal{B})}] = 0$ in $K_0(\mathcal{C}(\mathcal{B}))$. Let X be a compact metric space and let $\phi : C(X) \to \mathcal{C}(\mathcal{B})$ be a unital essential extension. Then the following statements are equivalent:*

(i) *ϕ is null;*

(ii) *ϕ is self-absorbing, i.e., $\phi \oplus \phi \sim \phi$, i.e., $[\phi] + [\phi] = [\phi]$ in* $\mathrm{Ext_u}(C(X),\mathcal{B})$.

Proof. The proof is contained in Theorem 3.5 of [23]. Since this paper is not yet published, for the convenience of the reader, we here sketch the argument of the proof.

(i) $\Rightarrow$ (ii) The argument for this direction of the proof was pointed out to us by Professor Huaxin Lin.

Since ϕ is null, let $\mathcal{C} \subseteq \mathcal{C}(\mathcal{B})$ be a commutative unital AF-subalgebra such that $\phi(C(X)) \subseteq \mathcal{C}$ and $[p]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$ for all $p \in \mathrm{Proj}(\mathcal{C})$.

Let $\{p_{i_1,i_2,\ldots,i_n}\}_{1\leqslant n<\infty, 1\leqslant i_j \leqslant m_j}$ be a collection of projections in $\mathcal{C}$ such that

(1) $1 = \sum\limits_{j=1}^{m_1} p_j$,

(2) for all $i_1,\ldots,i_n$, $p_{i_1,\ldots,i_n} = \sum\limits_{j=1}^{m_{n+1}} p_{i_1,\ldots,i_n,j}$, and

(3) the linear span of $\{p_{i_1,i_2,\ldots,i_n}\}_{1\leqslant n<\infty, 1\leqslant i_j \leqslant m_j}$ is norm dense in $\mathcal{C}$.

Let $b \in \mathcal{B}$ be a strictly positive element. Let $S, T \in \mathcal{C}(\mathcal{B})$ be isometries such that $SS^* + TT^* = 1$. For all $i_1,\ldots,i_n$, let $q_{i_1,\ldots,i_n} =_{\mathrm{df}} Sp_{i_1,\ldots,i_n}S^* + Tp_{i_1,\ldots,i_n}T^*$.

For each $i_1, \ldots, i_n$, we construct contractive positive elements $A_{i_1,\ldots,i_n}, A'_{i_1,\ldots,i_n}, B_{i_1,\ldots,i_n}, B'_{i_1,\ldots,i_n} \in \mathcal{M}(\mathcal{B})$ and a unitary $U_n \in U(\mathcal{M}(\mathcal{B}))$. The construction is by induction on n.

Let $\{A_1, \ldots, A_{m_1}\}$ and $\{A'_1, \ldots, A'_{m_1}\}$ be two collections of pairwise orthogonal contractive positive elements of $\mathcal{M}(\mathcal{B})$ such that $\pi(A_j) = \pi(A'_j) = p_j$ and $A'_j A_j = A_j$ for all j. Let $V_1 \in U(\mathcal{C}(\mathcal{B}))$ such that $V_1 p_j V_1^* = q_j$. Moreover, since $\mathcal{C}(\mathcal{B})$ is simple purely infinite, we can choose V_1 to be path-connected to 1. Hence, there exists $U_1 \in U(\mathcal{M}(\mathcal{B}))$ for which $\pi(U_1) = V_1$. Let $B_j =_{\mathrm{df}} U_1 A_j U_1^*$ and $B'_j =_{\mathrm{df}} U_1 A'_j U_1^*$, for all j.

Suppose that $U_n, A_{i_1,\ldots,i_n}, A'_{i_1,\ldots,i_n}, B_{i_1,\ldots,i_n}$ and $B'_{i_1,\ldots,i_n}$ have been constructed for $1 \leqslant i_j \leqslant m_j$. We now construct $U_{n+1}, A_{i_1,\ldots,i_{n+1}}, A'_{i_1,\ldots,i_{n+1}}, B_{i_1,\ldots,i_{n+1}}$ and $B'_{i_1,\ldots,i_{n+1}}$, for $1 \leqslant i_j \leqslant m_j$.

For all $i_1, i_2, \ldots, i_n$, let $A_{i_1,\ldots,i_n,j}$ and $A'_{i_1,\ldots,i_n,j}$, $1 \leqslant j \leqslant m_{n+1}$, be two collection of contractive, pairwise orthogonal, positive elements of $\mathrm{Her}(A_{i_1,\ldots,i_n})$ such that $\pi(A_{i_1,\ldots,i_n,j}) = \pi(A'_{i_1,\ldots,i_n,j}) = p_{i_1,\ldots,i_n,j}$ and $A'_{i_1,\ldots,i_n,j} A_{i_1,\ldots,i_n,j} = A_{i_1,\ldots,i_n,j}$ for all j. Let $W =_{\mathrm{df}} U_n \cdots U_1$. By the induction hypothesis, $W A_{i_1,\ldots,i_n} W^* \in \mathrm{Her}(B_{i_1,\ldots,i_n})$, $W A'_{i_1,\ldots,i_n} W^* \in \mathrm{Her}(B'_{i_1,\ldots,i_n})$, $A'_{i_1,\ldots,i_n} A_{i_1,\ldots,i_n} = A_{i_1,\ldots,i_n}$, $B'_{i_1,\ldots,i_n} B_{i_1,\ldots,i_n} = B_{i_1,\ldots,i_n}$, and

$$\begin{aligned}\pi(W)\pi(A_{i_1,\ldots,i_n})\pi(W^*) &= \pi(W)\pi(A'_{i_1,\ldots,i_n})\pi(W^*) = \pi(W) p_{i_1,\ldots,i_n} \pi(W^*) \\ &= \pi(B_{i_1,\ldots,i_n}) = \pi(B'_{i_1,\ldots,i_n}) = q_{i_1,\ldots,i_n}.\end{aligned}$$

Hence, $W A'_{i_1,\ldots,i_n,j} W^* \in \mathrm{Her}(B_{i_1,\ldots,i_n})$ for all j. (Recall that $A'_{i_1,\ldots,i_n,j} \in \mathrm{Her}(A_{i_1,\ldots,i_n})$.)

For all $i_1, \ldots, i_n$, let $V_{i_1,\ldots,i_n} \in U(q_{i_1,\ldots,i_n} \mathcal{C}(\mathcal{B}) q_{i_1,\ldots,i_n})$ be such that

$$V_{i_1,\ldots,i_n} \pi(W) p_{i_1,\ldots,i_n,j} \pi(W^*) V^*_{i_1,\ldots,i_n} = q_{i_1,\ldots,i_n,j}$$

for all j. Since $q_{i_1,\ldots,i_n} \mathcal{C}(\mathcal{B}) q_{i_1,\ldots,i_n}$ is simple purely infinite, we may choose $V_{i_1,\ldots,i_n}$ to be homotopic to $q_{i_1,\ldots,i_n}$. Hence, since $\mathcal{C}(\mathcal{B})$ is simple purely infinite, there exist self-adjoint elements $C_{i_1,\ldots,i_n}, C'_{i_1,\ldots,i_n} \in q_{i_1,\ldots,i_n} \mathcal{C}(\mathcal{B}) q_{i_1,\ldots,i_n}$, with norm at most 2π, such that $V_{i_1,\ldots,i_n} = \exp(\mathrm{i} C_{i_1,\ldots,i_n}) \exp(\mathrm{i} C'_{i_1,\ldots,i_n})$.

Let $D_{i_1,\ldots,i_n}, D'_{i_1,\ldots,i_n} \in \mathrm{Her}(B_{i_1,\ldots,i_n})$ be self-adjoint elements, with norm at most 2π, such that $\pi(D_{i_1,\ldots,i_n}) = C_{i_1,\ldots,i_n}$ and $\pi(D'_{i_1,\ldots,i_n}) = C'_{i_1,\ldots,i_n}$. Let $U_{i_1,\ldots,i_n} \in U(\mathrm{Her}(B_{i_1,\ldots,i_n}))^{\sim}$ be given by $U_{i_1,\ldots,i_n} =_{\mathrm{df}} \exp(\mathrm{i} D_{i_1,\ldots,i_n}) \exp(\mathrm{i} D'_{i_1,\ldots,i_n})$. For each $i_1, \ldots, i_n$, by replacing $D_{i_1,\ldots,i_n}$ and $D'_{i_1,\ldots,i_n}$ with $(1-c) D_{i_1,\ldots,i_n} (1-c)$ and $(1-c) D'_{i_1,\ldots,i_n} (1-c)$, respectively, where $c \in \mathrm{her}(B_{i_1,\ldots,i_n})_+$ is an appropriate element, we may assume that $\|U_{i_1,\ldots,i_n} W b - W b\| < 1/2^n 2^{m_1+\cdots+m_n}$ and $\|b U_{i_1,\ldots,i_n} - b\| < 1/2^n 2^{m_1+\cdots+m_n}$.

Finally, define

$$\begin{aligned}B_{i_1,\ldots,i_n,j} &=_{\mathrm{df}} U_{i_1,\ldots,i_n} W A_{i_1,\ldots,i_n,j} W^* U^*_{i_1,\ldots,i_n} \quad \text{and} \\ B'_{i_1,\ldots,i_n,j} &=_{\mathrm{df}} U_{i_1,\ldots,i_n} W A'_{i_1,\ldots,i_n,j} W^* U^*_{i_1,\ldots,i_n}\end{aligned}$$

for all j. (Note that $\pi(B_{i_1,\ldots,i_n,j}) = \pi(B'_{i_1,\ldots,i_n,j}) = q_{i_1,\ldots,i_n,j}$ for all j.)

Let $U_{n+1} =_{\mathrm{df}} \prod_{1 \leqslant i_j \leqslant m_j} U_{i_1,\ldots,i_n}$.

Then the sequence of products $\{U_n \cdots U_1\}_{n=1}^{\infty}$ converges in the strict topology to a unitary $U \in \mathcal{M}(\mathcal{B})$. Moreover, for all $f \in C(X)$, $U\phi(f)U^* = S\phi(f)S^* + T\phi(f)T^*$.

(ii) $\Rightarrow$ (i) From the previous direction ((i) $\Rightarrow$ (ii)) and from Theorem 3.1, we have proven that unital null essential extensions exist and are self-absorbing. Hence, it suffices to prove that any two unital self-absorbing essential extensions are unitarily equivalent.

Let $\phi_1, \phi_2 : C(X) \to \mathcal{C}(\mathcal{B})$ be two unital essential self-absorbing extensions. By Theorem 2.13, let $\psi : C(X) \to C(\mathcal{B})$ be a unital extension such that $\phi_1 \sim \psi \oplus \phi_2$. Hence, $\phi_1 \oplus \phi_2 \sim \psi \oplus \phi_2 \oplus \phi_2 \sim \psi \oplus \phi_2 \sim \phi_1$.

By a similar argument, $\phi_1 \oplus \phi_2 \sim \phi_2$. Hence, $\phi_1 \sim \phi_2$. ∎

Let $\mathcal{B}$ be a σ-unital nonunital simple continuous scale C^*-algebra such that $[1_{\mathcal{C}(\mathcal{B})}] = 0$ in $K_0(\mathcal{C}(\mathcal{B}))$.

Let X, Y be compact metric spaces and let $\rho : X \to Y$ be a continuous map. Then ρ induces a group homomorphism $\rho_* : \mathrm{Ext_u}(X, \mathcal{B}) \to \mathrm{Ext_u}(Y, \mathcal{B})$ in the following manner: say that $\phi : C(X) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is a unital essential extension. Then $\rho_*([\phi]) =_{\mathrm{df}} [(\phi \circ \phi_\rho) \oplus \psi]$ where $\psi : C(Y) \to \mathcal{M}(\mathcal{B})/\mathcal{B}$ is a null unital essential extension. It is straightforward to check that the map ρ_* is a well-defined group homomorphism. (Recall that we often denote $\mathrm{Ext_u}(C(X), \mathcal{B})$ by $\mathrm{Ext_u}(X, \mathcal{B})$ etc.)

PROPOSITION 3.3. $\mathrm{Ext_u}(\cdot, \mathcal{B})$ *is a covariant functor from the category of compact metrizable spaces to the category of abelian groups.*

4. FUNCTORIAL PROPERTIES

In this section, we follow closely the ideas of [3] and [16].

LEMMA 4.1. *Let $\mathcal{B}$ be a nonunital σ-unital simple real rank zero continuous scale stably finite C^*-algebra. Suppose that $\{P_n\}$ is a sequence of nonzero pairwise orthogonal projections in $\mathcal{M}(\mathcal{B})$ such that $\sum P_n$ converges strictly in $\mathcal{M}(\mathcal{B})$.*

Then there exists a sequence $\{P'_n\}$ of nonzero pairwise orthogonal projections in $\mathcal{M}(\mathcal{B})$ such that:

(i) *$P'_n \leqslant P_n$ for all n;*

(ii) *$P_n - P'_n \in \mathcal{B}$ for all n; and*

(iii) *for all $L, M \geqslant 1$, there exists $N \geqslant 1$ such that $M\tau(P_n) < \tau(P_l - P'_l)$ for all $n \geqslant N$, for all $l \leqslant L$ and for all $\tau \in T(\mathcal{B})$.*

Proof. This follows immediately from the facts that $\mathcal{B}$ has real rank zero, and that since $\mathcal{B}$ has continuous scale, for all $A \in \mathcal{M}(\mathcal{B})_+$, $\widehat{A}$ is a continuous function on $T(\mathcal{B})$. ∎

LEMMA 4.2. *Let $\mathcal{B}$ be a nonunital σ-unital simple real rank zero continuous scale C^*-algebra with strict comparison.*

If $p \in \mathrm{Proj}(\mathcal{C}(\mathcal{B}))$ and $[p]_{K_0(\mathcal{B})} = 0$ then there exists a $P \in \mathrm{Proj}(\mathcal{M}(\mathcal{B}))$ with $\widehat{P} \in \chi(K_0(\mathcal{B}))$ such that $\pi(P) = p$.

Proof. Suppose that p is a nonzero proper subprojection of $1_{\mathcal{C}(\mathcal{B})}$. Let $R \in \mathrm{Proj}(\mathcal{M}(\mathcal{B})) - \{0\}$ be such that $1 - R \notin \mathcal{B}$ and $\widehat{R} \in \chi(K_0(\mathcal{B})_+)$. Then $[\pi(R)]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$ and there exists a unitary $u \in U_0(\mathcal{C}(\mathcal{B}))$ such that $u\pi(R)u^* = p$. Let $U \in U(\mathcal{M}(\mathcal{B}))$ be such that $\pi(U) = u$. Then $\pi(URU^*) = p$. ∎

LEMMA 4.3. *Let $\mathcal{B}$ be a nonunital σ-unital simple real rank zero continuous scale C^*-algebra with strict comparison. If $\{p_n\}$ is a sequence of nonzero pairwise orthogonal projections in $\mathcal{C}(\mathcal{B})$ such that $[p_n]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$ for all n, then there exists a sequence $\{P_n\}$ of pairwise orthogonal projections in $\mathcal{M}(\mathcal{B})$ with $\widehat{P}_n \in \chi(K_0(\mathcal{B}))$ such that $\pi(P_n) = p_n$ for all n and $\sum P_n$ converges strictly to $1_{\mathcal{M}(\mathcal{B}))}$.*

Proof. The proof is a variation on Lemma 1.1 of [18]. Let $b \in \mathcal{B}$ be a strictly positive element. Suppose, for induction, that pairwise orthogonal $P_1, P_2, \ldots, P_n$ have already been constructed. Let $P =_{\mathrm{df}} \sum_{j=1}^{n} P_j$. Then $(1-P)\mathcal{B}(1-P)$ has continuous scale and $[p_{n+1}]_{K_0(\mathcal{C}((1-P)\mathcal{B}(1-P)))} = 0$. So by Lemma 4.2, we can find $P_{n+1} \in (1-P)\mathcal{M}(\mathcal{B})(1-P)$ such that $\pi(P_{n+1}) = p_{n+1}$. Since $\mathcal{B}$ has real rank zero, by adding a projection in $(1 - P_{n+1} - P)\mathcal{B}(1 - P_{n+1} - P)$ to P_{n+1} if necessary, we may assume that

$$\|(P_{n+1} + P)b - b\| < \frac{1}{n+1}. \quad ∎$$

The next lemma is an example of how real rank zero had a persistent (though implicit) presence in the BDF arguments.

LEMMA 4.4. *Let $\mathcal{B}$ be a nonunital σ-unital simple real rank zero continuous scale C^*-algebra with strict comparison and cancellation. Let X be a compact metric space and $F \subseteq X$ a closed subset such that X is the disjoint union of F and clopen subsets $X_1, X_2, \ldots$.*

Let $\phi : C(X) \to \mathcal{C}(\mathcal{B})$ be a $*$-homomorphism for which $[\phi(\chi_n)]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$ for all n. (Here, $\chi_n =_{\mathrm{df}} \chi_{X_n}$, the characteristic function of X_n, $\forall n$.)

Then there exists a sequence $\{P_n\}_{n=0}^{\infty}$ of pairwise orthogonal projections in $\mathcal{M}(\mathcal{B})$ with $\sum_{n=0}^{\infty} P_n$ converging strictly to 1 in $\mathcal{M}(\mathcal{B})$ such that $\pi(P_n) = \phi(\chi_n)$ for all n and for all $g \in C(F)$,

$$\phi(g \circ r) = P_0\phi(g \circ r)P_0 + \pi\Big(\sum_{n=1}^{\infty} g(x_n)P_n\Big)$$

where $r : X \twoheadrightarrow F$ is a retraction which brings each X_n to a nearest point $x_n \in F$.

Moreover, we can find a unital commutative AF*-subalgebra* $\mathcal{C} \subset (1-P_0)\mathcal{C}(\mathcal{B})(1-P_0)$ *for which* $[p]=0$ *in* $K_0(\mathcal{C}(\mathcal{B}))$ *for all* $p \in \mathrm{Proj}(\mathcal{C})$ *and*

$$\phi(C_0(X-F)) \cup \Big\{\pi\Big(\sum_{n\geqslant 1} g(x_n)P_n\Big) : g \in C(F)\Big\} \subseteq \mathcal{C}.$$

Proof. Let $\{g_l\}_{l=1}^{\infty}$ be a dense sequence in the closed unit ball of $C(F)_+$. Let $\{\widetilde{g}_l\}_{l=1}^{\infty}$ be elements of the closed unit ball of $\mathcal{M}(\mathcal{B})_+$ such that $\pi(\widetilde{g}_l) = \phi(g_l \circ r)$ for all l.

Step 1. We will first construct a sequence $\{Q_n\}$ in $\mathrm{Proj}(\mathcal{M}(\mathcal{B}))$ which satisfies all the conditions except the statement about $\mathcal{C}$. Moreover, we will additionally have that $\|Q_n\widetilde{g}_l - g_l(x_n)Q_n\| < 1/10^n$ and $Q_n\widetilde{g}_l - g_l(x_n)Q_n \in \mathcal{B}$, for all $l \leqslant n$.

By Lemma 4.3, let $\{Q_n\}$ be a sequence of pairwise orthogonal projections in $\mathcal{M}(\mathcal{B})$ with $\widehat{Q}_n \in \chi(K_0(\mathcal{B}))$ such that $\pi(Q_n) = \phi(\chi_n)$ for all n and $\sum Q_n$ converges strictly. Note that for all n, l, $\phi((g_l \circ r)\chi_n) = g_l(x_n)\pi(Q_n)$. Hence, since $\mathcal{B}$ has real rank zero, replacing each Q_n by a subprojection (which differs by a projection in $\mathcal{B}$) if necessary, we may assume that $\|Q_n\widetilde{g}_l - g_l(x_n)Q_n\| < 1/10^n$ and $Q_n\widetilde{g}_l - g_l(x_n)Q_n \in \mathcal{B}$, for all $l \leqslant n$. Let $Q_0 =_{\mathrm{df}} 1 - \sum Q_n$. Hence, for all l,

$$\widetilde{g}_l = \widetilde{g}_l Q_0 + \sum \widetilde{g}_l Q_n = \widetilde{g}_l Q_0 + \sum g(x_n)Q_n \ (\mathrm{mod}\ \mathcal{B}).$$

So for all l,

$$\phi(g_l \circ r) = \phi(g_l \circ r)\pi(Q_0) + \pi\Big(\sum g(x_n)Q_n\Big).$$

This completes Step 1.

Apply Lemma 4.1 to $\{Q_n\}_{n=1}^{\infty}$ to get a sequence of subprojections $\{Q'_{n,1}\}_{n=1}^{\infty}$.

Now let $\{F_{l_1,l_2,\ldots,l_k}\}_{1\leqslant l_j\leqslant m_j, 1\leqslant j\leqslant k, 1\leqslant k<\infty}$ be a collection of subsets of F such that the following are true:

(i) $F_l \cap F_{l'} = \emptyset$ for all $l \neq l'$;

(ii) for all $l_1,\ldots,l_{k-1}$, for all $l \neq l'$, $F_{l_1,\ldots,l_{k-1},l} \cap F_{l_1,\ldots,l_{k-1},l'} = \emptyset$;

(iii) $F = \bigcup_{l=1}^{m_1} F_l$;

(iv) for all $l_1,\ldots,l_{k-1}$; $F_{l_1,\ldots,l_{k-1}} = \bigcup_{l=1}^{m_k} F_{l_1,\ldots,l_{k-1},l}$;

(v) for all $l_1,\ldots,l_{k-1}$, for all $1 \leqslant l \leqslant m_k$, $\mathrm{diam}(F_{l_1,\ldots,l_{k-1},l}) < 1/10^k$.

For all $l_1,\ldots,l_k$, let $S_{l_1,\ldots,l_k} =_{\mathrm{df}} \{n \in \mathbb{Z}^+ : x_n \in F_{l_1,\ldots,l_k}\}$.

For all k, we will now construct a sequence $\{Q'_{n,k}\}_{n=1}^{\infty}$ in $\mathrm{Proj}(\mathcal{M}(\mathcal{B}))$ such that:

(1) $Q'_{n,k} \leqslant Q_n$ for all n,k;

(2) $Q_n - Q'_{n,k} \in \mathcal{B}$ for all n,k;

(3) $Q'_{n,k} = Q'_{n,l}$ for all $k,l \geqslant n$; and

(4) for all $j \geqslant k$, for all $l_1,\ldots,l_k$, $\sum_{n\in S_{l_1,\ldots,l_k}} \widehat{Q}'_{n,j} = \sum_{n\in S_{l_1,\ldots,l_k}} \widehat{Q}'_{n,k} \in \chi(K_0(\mathcal{B})_+)$.

The construction is by induction on k.

Say that $\{Q'_{n,k}\}_{n=1}^{\infty}$ has been constructed. Let $\{\overrightarrow{l_1}, \ldots, \overrightarrow{l_L}\} = \{(l_1, \ldots, l_k) : 1 \leqslant l_j \leqslant m_j, 1 \leqslant j \leqslant k\}$.

Fix $1 \leqslant j \leqslant L$.

If for all $1 \leqslant l \leqslant m_{k+1}$,

$$\sum_{n \in S_{(\overrightarrow{l_j}, l)}} \widehat{Q}'_{n,k} \in \chi(K_0(\mathcal{B})_+)$$

then define, for all $n \in S_{(\overrightarrow{l_j}, l)}$ and for all $1 \leqslant l \leqslant m_{k+1}$,

$$Q'_{n,k+1} =_{\mathrm{df}} Q'_{n,k}.$$

Suppose that there exists $1 \leqslant l \leqslant m_{k+1}$ such that

$$\sum_{n \in S_{(\overrightarrow{l_j}, l)}} \widehat{Q}'_{n,k} \notin \chi(K_0(\mathcal{B})_+).$$

For simplicity, let us assume that for all $1 \leqslant l \leqslant m_{k+1}$, $S_{(\overrightarrow{l_j}, l)}$ is infinite. (The proofs for the other cases are similar.)

For all $1 \leqslant l \leqslant m_{k+1}$, let $\{n_{(l,s)}\}_{s=1}^{\infty}$ be a subsequence of $\mathbb{Z}^+$ such that the following are true:

(a) $n_{(l,1)} \geqslant 10^k$;

(b) $n_{(l,s)} \in S_{(\overrightarrow{l_j}, l)}$ for all s;

(c) if $l \leqslant m_{k+1} - 1$ then for all s, for all $\tau \in T(\mathcal{B})$, $10^{k+1}\tau(Q'_{n_{(l+1,s)},k}) \leqslant \tau(Q_{n_{(l,s)}} - Q'_{n_{(l,s)},k})$.

By an inductive construction, for $2 \leqslant l \leqslant m_{k+1}$, there exists a sequence $\{p'_{(l,s)}\}_{s=1}^{\infty}$ in $\mathrm{Proj}(\mathcal{B})$, and for $1 \leqslant l \leqslant m_{k+1} - 1$, there exists a sequence $\{p_{(l,s)}\}_{s=1}^{\infty}$ in $\mathrm{Proj}(\mathcal{B})$ such that the following are true:

(i) for all l, s, $\bigoplus^{10^{k+1}} p_{(l,s)} \leqslant Q_{n_{(l,s)}} - Q'_{n_{(l,s)},k}$ and $\bigoplus^{10^{k+1}} p'_{(l,s)} \leqslant Q'_{n_{(l,s)},k}$;

(ii) for all $1 \leqslant l \leqslant m_{k+1} - 1$, $p_{(l,s)} \sim p'_{(l+1,s)}$;

(iii) $\sum_{s=1}^{\infty} (Q'_{n_{(m_{k+1},s)},k} - p'_{(m_{k+1},s)})\widehat{\ } + \sum_{n \in S_{\overrightarrow{l_j}, m_{k+1}}, n \neq n_{(m_{k+1},s)} \forall s} \widehat{Q}'_{n,k} \in \chi(K_0(\mathcal{B})_+)$;

(iv) $\sum_{s=1}^{\infty} (Q'_{n(1,s),k} \oplus p_{(1,s)})\widehat{\ } + \sum_{n \in S_{\overrightarrow{l_j},1}, n \neq n(1,s) \forall s} \widehat{Q}'_{n,k} \in \chi(K_0(\mathcal{B})_+)$;

(v) for all $2 \leqslant l \leqslant m_{k+1} - 1$, $\sum_{s=1}^{\infty} ((Q'_{n(l,s),k} - p'_{(l,s)}) \oplus p_{(l,s)})\widehat{\ } + \sum_{n \in S_{\overrightarrow{l_j},l}, n \neq n_{(l,s)} \forall s} \widehat{Q}'_{n,k}$ $\in \chi(K_0(\mathcal{B})_+)$.

For all s, define

$$Q'_{n(m_{k+1},s),k+1} =_{\mathrm{df}} Q'_{n(m_{k+1},s),k} - p'_{m_{k+1},s}, \quad Q'_{n(1,s),k+1} =_{\mathrm{df}} Q'_{n(1,s),k} \oplus p_{1,s},$$

and for all $2 \leqslant l \leqslant m_{k+1} - 1$,

$$Q'_{n(l,s),k+1} =_{\mathrm{df}} (Q'_{n(l,s),k} - p'_{(l,s)}) \oplus p_{(l,s)}.$$

Finally, for all $1 \leqslant l \leqslant m_{k+1}$, for all $n \in S_{\overrightarrow{l_j},l}$, if

$$n \neq n(l,s)$$

for all s, then define

$$Q'_{n,k+1} =_{\mathrm{df}} Q'_{n,k}.$$

This completes the inductive construction.

For all n, let

$$P_n =_{\mathrm{df}} Q'_{n,k}$$

for all $k \geqslant n$. It follows that

(i.) for all $l_1, \ldots, l_k$, $\sum_{n \in S_{(l_1,\ldots,l_k)}} \widehat{P}_n \in \chi(K_0(\mathcal{B})_+)$;

(ii.) $P_n \leqslant Q_n$ and $Q_n - P_n \in \mathcal{B}$ for all n; and

(iii.) $\|P_n \widetilde{g}_l - g_l(x_n)P_n\| < 1/10^n$ for all n and for all $l \leqslant n$, and $P_n \widetilde{g}_l - g_l(x_n)P_n \in \mathcal{B}$ for all n and for all l, n.

As a consequence, for all l,

$$\pi\Big(\sum_{n=1}^{\infty} P_n\Big)\phi(g_l \circ r) = \pi\Big(\sum_{n=1}^{\infty} g_l(x_n)P_n\Big).$$

Let $P_0 =_{\mathrm{df}} 1 - \sum_{n=1}^{\infty} P_n$. Then for all $g \in C(F)$,

$$\phi(g \circ r) = \pi(P_0)\phi(g \circ r) + \pi\Big(\sum_{n=1}^{\infty} g(x_n)P_n\Big).$$

For all $l_1, \ldots, l_k$, let $P_{S(l_1,\ldots,l_k)} =_{\mathrm{df}} \sum_{n \in S(l_1,\ldots,l_k)} P_n$. Then $P_{S(l_1,\ldots,l_k)} \in \mathrm{Proj}(\mathcal{M}(\mathcal{B}))$ and $\widehat{P}_{S(l_1,\ldots,l_k)} \in \chi(K_0(\mathcal{B})_+)$.

For all k, let $\widetilde{\mathcal{C}}_k$ be given by $\widetilde{\mathcal{C}}_k =_{\mathrm{df}} \bigoplus_{1 \leqslant l_j \leqslant m_j, 1 \leqslant j \leqslant k} \mathbb{C}P_{S(l_1,\ldots,l_k)}$.

Then $\mathcal{C}_k =_{\mathrm{df}} \pi(\widetilde{\mathcal{C}}_k)$ is a finite dimensional C^*-subalgebra of $\mathcal{C}(\mathcal{B})$ and $[p]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$ for all $p \in \mathrm{Proj}(\mathcal{C}_k)$.

Note that for all k, we have a unital inclusion $\mathcal{C}_k \subseteq \mathcal{C}_{k+1}$. Then

$$\mathcal{C} =_{\mathrm{df}} C^*\Big(\phi\Big(C_0(X - F) \cup \bigcup_k \mathcal{C}_k\Big)\Big)$$

is a unital commutative AF C^*-subalgebra of $\mathcal{C}(\mathcal{B})$ for which $[p]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$ for all $p \in \mathrm{Proj}(\mathcal{C})$, and moreover,

$$\phi(C_0(X - F)) \cup \Big\{\pi\Big(\sum_{n=1}^{\infty} g(x_n)P_n\Big) : g \in C(F)\Big\} \subseteq \mathcal{C}. \quad \blacksquare$$

We note that there is a slight gap in the proof of Theorem 5.2 in [16] (half exactness of the functor). In particular, in Lemma 5.1 of [16] one cannot conclude (from the argument of 3.2 in [16]) that the extension τ_X is null (in the notation of Lemma 5.2 in [16]). This is remedied by our Lemma 4.4 (in particular, the last paragraph in the statement of our Lemma 4.4).

THEOREM 4.5. *Let X be a compact metric space and $F \subseteq X$ a closed subset. Let $\mathcal{B}$ be a nonunital simple σ-unital real rank zero continuous scale C^*-algebra with strict comparison and cancellation such that $[1_{\mathcal{C}(\mathcal{B})}]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$.*

Then

$$\mathrm{Ext_u}(F, \mathcal{B}) \to \mathrm{Ext_u}(X, \mathcal{B}) \to \mathrm{Ext_u}(X/F, \mathcal{B})$$

is exact.

Proof. The proof is exactly the same as Theorem 5.2 of [16], except that we replace Lemma 5.1 of [16] with Lemma 4.4 in our paper. Also, we use the existence and characterization of null extension from Theorems 3.1 and 3.2 in our paper. ∎

THEOREM 4.6. *Let $\mathcal{B}$ be a nonunital σ-unital simple real rank zero continuous scale C^*-algebra with strict comparison and cancellation such that $[1_{\mathcal{C}(\mathcal{B})}]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$. For all n, let $\rho_n : X_{n+1} \to X_n$ be a continuous map between compact metrizable spaces. Suppose that X is a compact metrizable space which can be expressed as an inverse limit*

$$X = \lim_{\leftarrow}(X_n, \rho_n).$$

Then the map $\Phi : \mathrm{Ext_u}(X, \mathcal{B}) \to \lim\limits_{\leftarrow} \mathrm{Ext_u}(X_n, \mathcal{B})$ is surjective.

Proof. The proof is exactly the same as that of Theorem 5.3 in [16] (which, in turn, is essentially the same as Theorem 8.4 of [3]). The main (minor) differences is that we replace Theorem 5.2 of [16] with Theorem 4.5 in our paper, and we use the existence and characterization of null extensions from Theorems 3.1 and 3.2 in our paper. ∎

COROLLARY 4.7. *Assume the same hypotheses and notation as Theorem 4.6. Assume, in addition, that each $\rho_n : X_{n+1} \to X_n$ is surjective and each X_n has only finitely many points.*

Then Φ is an isomorphism.

The (short) proof is exactly the same as that of Corollary 5.4 in [16].

Following the argument of [16], we have that the above results imply that $\mathrm{Ext_u}(\cdot, \mathcal{B})$ is homotopy invariant.

THEOREM 4.8. *Let $\mathcal{B}$ be a nonunital σ-unital simple real rank zero continuous scale C^*-algebra with strict comparison and cancellation such that $[1_{\mathcal{C}(\mathcal{B})}]_{K_0(\mathcal{C}(\mathcal{B}))} = 0$.*

Then $\mathrm{Ext_u}(\cdot, \mathcal{B})$ is homotopy-invariant.

The proof follows from the beginning of Section 2 of [4] and Theorem 4.5, Theorem 4.6 and Corollary 4.7 in our paper.

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