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PURELY INFINITE CORONA ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a simple, $\sigma-$unital, non-unital C*-algebra, with metrizable tracial simplex $\mathcal{T}(\mathcal{A})$, projection surjectivity and injectivity, and strict comparison of positive elements by traces. Then the following are equivalent:

(i) $\mathcal{A}$ has quasicontinuous scale;
(ii) $\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements by traces;
(iii) $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is purely infinite;
(iii′) $\mathcal{M}(\mathcal{A})/I_{\min}$ is purely infinite;
(iv) $\mathcal{M}(\mathcal{A})$ has finitely many ideals;
(v) $I_{\min} = I_{\mathrm{fin}}$.

If furthermore $M_n(\mathcal{A})$ has projection surjectivity and injectivity for every n, then the above conditions are equivalent to:

(vi) $V(\mathcal{M}(\mathcal{A}))$ has finitely many order ideals.

KEYWORDS: *Multiplier algebras, ideals in multiplier algebras, corona algebras, strict comparison.*

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INTRODUCTION

In the study of multiplier algebras of C^*-algebras, an important role is played by the associated corona algebras. In the case of the algebra $\mathcal{K}$ of compact operators on a separable Hilbert space H, $\mathcal{M}(\mathcal{K}) = B(H)$ and the corona algebra $\mathcal{M}(\mathcal{K})/\mathcal{K}$ is the Calkin algebra which is well known to be both simple and purely infinite.

Perhaps one reason for the success of the Brown–Douglas–Fillmore theory ([5], [6]) is that in their context, the multiplier algebra $\mathcal{M}(\mathcal{K})$ and the corona algebra $\mathcal{M}(\mathcal{K})/\mathcal{K}$ have particularly nice structure. For example, the BDF–Voiculescu result which, roughly says that every essential extension is absorbing, would not be true if the corona algebra $\mathcal{M}(\mathcal{K})/\mathcal{K}$ were not simple ([1], [5], [59]). It is by now clear that, in the classical theory of absorbing extensions, "nice" extension theory corresponds to suitable corona algebra structure (e.g., [14], [24], [25], [36], [38], [46], [64]). Furthermore, progress in the study of extensions of stable C^*-algebras beyond the known cases will require new knowledge of extensions of nonstable algebras ([37]) which in turn requires knowledge of the structure and comparison theory for multiplier algebras and corona algebras in the non stable case.

If $\mathcal{A}$ is a simple, σ-unital but not unital, and non-elementary C^*-algebra, Lin showed in [34] and [39] that $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is simple if and only if $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is simple and purely infinite, if and only if $\mathcal{A}$ has continuous scale. Simple continuous scale algebras are one of the most interesting classes in generalizations of BDF theory.

However, the continuity of the scale is not necessary for $\mathcal{M}(\mathcal{A})/\mathcal{A}$ to be purely infinite. Indeed, Kucerovsky and Perera [27] proved that a weaker condition, which they called quasicontinuity of the scale (see Definition 3.1), is both necessary and sufficient in the case when $\mathcal{A}$ is simple, separable, non-unital, with real rank zero, stable rank one, strict comparison of positive elements and finitely many infinite extremal traces.

In the case of stable algebras with a nonzero projection (where the scale is quasicontinous if and only if the algebra has finitely many extremal traces) their work was then extended in Theorem A of [29] to simple, separable C^*-algebras that are either exact and $\mathcal{Z}$-stable or are AH-algebras with slow dimension growth.

The main goal of this paper is to extend these two results to a wider class of stable and of non stable algebras and to add further properties equivalent to the pure infiniteness of the corona algebra (see Abstract or Theorem 6.11).

The regularity conditions that we will require for the C^*-algebras studied, which we call projection surjectivity and injectivity, have been used implicitly in various forms in the study of multiplier algebras (see Definitions 4.1). Roughly speaking, they permit to identify the projections in $\mathcal{M}(\mathcal{A})$ that are not in $\mathcal{A}$ with certain strictly positive lower semicontinuous affine functions on the tracial simplex $\mathcal{T}(\mathcal{A})$ (see Definitions 4.1). Real rank zero simple σ-unital nonunital nonelementary algebras with stable rank one and strict comparison of projections (e.g., see [48]) are projection surjective and injective. We can drop the real rank zero condition for stable separable algebras (see Corollary 4.11). This class includes a wide class of algebras (see list after Corollary 4.11).

A key role in our study is played by three distinguished ideals of $\mathcal{M}(\mathcal{A})$, I_{min}, I_{cont}, and I_{fin}. I_{min} (see (1.12)) is the intersection of all the ideals of $\mathcal{M}(\mathcal{A})$ that properly contain $\mathcal{A}$, and was studied in [31], [34], [39], [43], [48], and more recently in [23]. If $\mathcal{A}$ is separable or if $\mathcal{A}$ has strict comparison, then $\mathcal{A} \neq I_{\text{min}}$ and then $\frac{I_{\text{min}}}{\mathcal{A}}$ is purely infinite and simple ([23], Corollary 3.15, Theorem 4.8). I_{cont} is the ideal generated by the elements with continuous evaluation functions and it coincides with I_{min} if $\mathcal{A}$ has strict comparison. Without strict comparison we have an example where the two are different ([23], Theorem 5.6, Theorem 7.8). I_{fin} (see [48] and Definition 1.6) is the ideal generated by the elements with evaluation function finite on the extremal boundary $\partial_e(\mathcal{T}(\mathcal{A}))$ of the tracial simplex $\mathcal{T}(\mathcal{A})$. Several of the properties in Theorem 6.11 are proven to be equivalent to the condition $I_{\text{min}} = I_{\text{fin}}$.

Another key role in our study is played by the notion of strict comparison in the multiplier algebra. In Theorem 6.6 of [22] we had proven that if $\mathcal{A}$ is simple, σ-unital, has strict comparison of positive elements by traces, and has quasicontinuous scale, then $\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements by traces. Here we prove that the converse holds if the algebra is projection surjective and injective and then use this property to prove that the multiplier algebra has finitely many ideals and that $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is purely infinite.

The paper is organized as follows: in Section 2, we present some background material on the tracial simplex $\mathcal{T}(\mathcal{A})$ and on ideals of $\mathcal{M}(\mathcal{A})$. In Section 3, we present some technical lemmas on lower semicontinuous affine functions. In Section 4, we introduce quasicontinuity of the scale and its relation to the finiteness of the ideal lattice of $\mathcal{M}(\mathcal{A})$. In Section 5, we introduce and discuss the notion of projection surjectivity and injectivity. In Section 6 we apply projection surjectivity and injectivity to study ideals in multiplier algebras. Finally, in Section 7 we prove our main result (Theorem 6.11) linking all the various equivalent conditions. Then, under additional assumptions on the tracial simplex, we derive further properties of the ideal lattice of $\mathcal{M}(\mathcal{A})$; in particular, the existence of infinite chains of principal ideals.

1. NOTATIONS AND PRELIMINARIES

1.1. THE TRACIAL SIMPLEX AND STRICT COMPARISON. In this paper $\mathcal{A}$ will always denote a simple σ-unital (possibly unital) C*-algebra. If e is a nonzero positive element in the Pedersen ideal $\mathrm{Ped}(\mathcal{A})$ of $\mathcal{A}$, denote by $\mathcal{T}(\mathcal{A})$ the collection of the (norm) lower semicontinuous densely defined tracial weights τ on $\mathcal{A}_+$, that are normalized on e. Explicitly, a trace $\tau \in \mathcal{T}(\mathcal{A})$ is an additive and homogeneous map from $\mathcal{A}_+$ into $[0,\infty]$ (a weight); satisfies the trace condition $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{A}$; is densely defined (also called densely finite, or semifinite), i.e., the positive cone $\{x \in \mathcal{A}_+ : \tau(x) < \infty\}$ is dense in $\mathcal{A}_+$; satisfies the lower semicontinuity condition $\tau(x) \leqslant \underline{\lim}\, \tau(x_n)$ for $x, x_n \in \mathcal{A}_+$ with $\|x_n - x\| \to 0$, or equivalently, $\tau(x) = \lim_n \tau(x_n)$ for $0 \leqslant x_n \uparrow x$ in norm; and is normalized on e, i.e., $\tau(e) = 1$. We will mostly assume that $\mathcal{T}(\mathcal{A}) \neq \emptyset$ and hence that $\mathcal{A}$ is stably finite.

When equipped with the topology of pointwise convergence on $\mathrm{Ped}(\mathcal{A})$, $\mathcal{T}(\mathcal{A})$ is a Choquet simplex (e.g., see Proposition 3.4 of [58] and [15]). In particular, $\mathcal{T}(\mathcal{A})$ is a compact convex subset of a locally convex linear topological Hausdorff space, *compact convex space* for short. The collection of the extreme points of $\mathcal{T}(\mathcal{A})$ is denoted by $\partial_e(\mathcal{T}(\mathcal{A}))$ and is called the *extremal boundary* of $\mathcal{T}(\mathcal{A})$. For simplicity's sake we call the elements of $\mathcal{T}(\mathcal{A})$ (respectively, $\partial_e(\mathcal{T}(\mathcal{A}))$) traces (respectively, extremal traces). Tracial simplexes $\mathcal{T}(\mathcal{A})$ arising from different nonzero positive elements in $\mathrm{Ped}(\mathcal{A})$ are homeomorphic; so we will not reference explicitly which element e is used for the normalization. A trace τ on $\mathcal{A}$

extends naturally to a trace on $\mathcal{A} \otimes \mathcal{K}$ (explicitly to the trace $\tau \otimes \mathrm{Tr}$), and so we can identify $\mathcal{T}(\mathcal{A} \otimes \mathcal{K})$ with $\mathcal{T}(\mathcal{A})$. By the work of F. Combes ([8], Proposition 4.1, Proposition 4.4) and Ortega, Rørdam, and Thiel ([45], Proposition 5.2) every trace $\tau \in \mathcal{T}(\mathcal{A})$ has a unique extension to a lower semicontinuous (i.e., normal) tracial weight (trace for short) on the enveloping von Neumann algebra $\mathcal{A}^{**}$, and hence to a trace on the multiplier algebra $\mathcal{M}(\mathcal{A})$. We will still denote that extension by τ. Finally, we recall that the tracial simplex $\mathcal{T}(\mathcal{A})$ is studied also for non simple algebras, but then the positive element $e \in \mathrm{Ped}(\mathcal{A})$ used to normalize the traces is chosen to be full. For more details, see [15], [58] and also [20] and [22].

Although we will use the following notions mainly for the case when K is a Choquet simplex, it is customary (and more convenient) to formulate them for compact convex spaces.

DEFINITION 1.1. Given a compact convex space K,

(i) $\mathrm{Aff}(\mathcal{K})$ denotes the *Banach space of the continuous real-valued affine functions* on K with the uniform norm;

(ii) $\mathrm{LAff}(\mathcal{K})$ denotes the *collection of the lower semicontinuous affine functions* on K with values in $\mathbb{R} \cup \{+\infty\}$;

(iii) $\mathrm{Aff}(\mathcal{K})_+$ (respectively, $\mathrm{LAff}(\mathcal{K})_+$) denotes the *cone of the positive functions* (i.e., $f(x) \geqslant 0$ for all $x \in K$) in $\mathrm{Aff}(\mathcal{K})$ (respectively, in $\mathrm{LAff}(\mathcal{K})$).

(iv) $\mathrm{Aff}(\mathcal{K})_{++}$ (respectively, $\mathrm{LAff}(\mathcal{K})_{++}$) denotes the *cone of the strictly positive functions* (i.e., $f(x) > 0$ for all $x \in K$) in $\mathrm{Aff}(\mathcal{K})$ (respectively, in $\mathrm{LAff}(\mathcal{K})$);

(v) $\mathrm{LAff}_\sigma(\mathcal{K})$, (respectively, $\mathrm{LAff}_\sigma(\mathcal{K})_+$, $\mathrm{LAff}_\sigma(\mathcal{K})_{++}$) denotes the *collection of functions* in $\mathrm{LAff}(\mathcal{K})$ (respectively, $\mathrm{LAff}(\mathcal{K})_+$, $\mathrm{LAff}(\mathcal{K})_{++}$) that are the increasing pointwise limit of a *sequence* of functions in $\mathrm{Aff}(\mathcal{K})$;

(vi) given $\mathcal{S} \in \mathrm{LAff}_\sigma(\mathcal{K})_{++}$, an element $f \in \mathrm{LAff}_\sigma(\mathcal{K})_{++}$ is said to be *complemented* under $\mathcal{S}$ if there is a $g \in \mathrm{LAff}_\sigma(\mathcal{K})_{++} \sqcup \{0\}$ such that $f + g = \mathcal{S}$.

For every $A \in \mathcal{M}(\mathcal{A})_+$, $\widehat{A}$ denotes the evaluation map and $\widehat{[A]}$ the dimension function of A:

$$\mathcal{T}(\mathcal{A}) \ni \tau \mapsto \widehat{A}(\tau) := \tau(A) \in [0, \infty]; \tag{1.1}$$

$$\mathcal{T}(\mathcal{A}) \ni \tau \mapsto \widehat{[A]}(\tau) := d_\tau(A) = \lim_n \tau(A^{1/n}) \in [0, \infty]. \tag{1.2}$$

As shown in Remark 5.3 of [45],

$$d_\tau(A) = \tau(R_A) \quad \text{where } R_A \in \mathcal{A}^{**} \text{ is the range projection of } A. \tag{1.3}$$

It is well known that both $\widehat{A} \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_+$ and $\widehat{[A]} \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_+$ for every $A \neq 0$. In fact, $\widehat{A} \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++}$ and $\widehat{[A]} \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++}$ due to the standing assumption that $\mathcal{A}$ is simple. In particular the *scale* $\mathcal{S}$ of $\mathcal{A}$ is defined as $\mathcal{S} := \widehat{1_{\mathcal{M}(\mathcal{A})}}$ and thus

$$\mathcal{S} \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++}. \tag{1.4}$$

We will use frequently the following well known facts. If $A, B \in \mathcal{M}(\mathcal{A})_+$, and $\tau \in \mathcal{T}(\mathcal{A})$ then

$$A \leqslant B \Rightarrow \widehat{A}(\tau) \leqslant \widehat{B}(\tau), \tag{1.5}$$

$$A \preceq B \Rightarrow d_\tau(A) \leqslant d_\tau(B), \tag{1.6}$$

where "$\preceq$" denotes Cuntz subequivalence, that is the existence of a sequence $X_n \in \mathcal{M}(\mathcal{A})$ such that $\|X_n B X_n^* - A\| \to 0$;

$$AB = 0 \Rightarrow d_\tau(A+B) = d_\tau(A) + d_\tau(B), \tag{1.7}$$

$$\tau(A) \leqslant \|A\| d_\tau(A), \tag{1.8}$$

$$d_\tau((A-\delta)_+) < \frac{1}{\delta}\widehat{A}(\tau) \quad \forall\, \delta > 0, \tag{1.9}$$

and by Lemma 2.4(iii) of [22],

$$d_\tau((A+B-\delta_1-\delta_2)_+) \leqslant d_\tau((A-\delta_1)_+) + d_\tau((B-\delta_2)_+) \quad \forall \delta_1 \geqslant 0,\ \delta_2 \geqslant 0. \tag{1.10}$$

By the definition of the topology on $\mathcal{T}(\mathcal{A})$, if $a \in \mathrm{Ped}(\mathcal{A})$, then $\widehat{a} \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))$. Notice that $\widehat{[a]}$ is not necessarily continuous. We will use the fact that $\widehat{[a]}$ is bounded.

LEMMA 1.2. *Let $\mathcal{A}$ be a simple, σ-unital C^*-algebra with non empty tracial simplex $\mathcal{T}(\mathcal{A})$ and let $a \in \mathrm{Ped}(\mathcal{A})_+$. Then* $\sup_{\tau \in \mathcal{T}(\mathcal{A})} d_\tau(a) < \infty$.

Proof. Since $a \leqslant \sum_{j=1}^{n} a_j$ for some $n \in \mathbb{N}$ elements $a_j \in \mathcal{A}_+$, with *local unit*, i.e., such that $b_j a_j = a_j$ for some $b_j \in \mathcal{A}_+$, and since $d_\tau(a) \leqslant \sum_{j=1}^{n} d_\tau(a_j)$, it is enough to verify the claim for an $a \in \mathcal{A}_+$ that has a local unit $b \in \mathcal{A}_+$ (i.e., $ba = a$). Assume without loss of generality that $\|a\| = 1$. Since a and b commute, we can identify them as continuous functions on a compact space X, i.e., $a = a(x)$ and $b = b(x)$. Then for all $x \in X$ such that $a(x) \neq 0$ we have $b(x) = 1$ and hence $f_{1/2}(b(x)) = 1$, where for every $\varepsilon > 0$, the function $f_\varepsilon(t)$ is defined as follows:

$$f_\varepsilon(t) := \begin{cases} 0 & \text{for } t \in [0, \varepsilon], \\ \frac{t-\varepsilon}{\varepsilon} & \text{for } t \in (\varepsilon, 2\varepsilon], \\ 1 & \text{for } t \in (2\varepsilon, \infty). \end{cases} \tag{1.11}$$

But then $f_{1/2}(b)a = a$ and $R_a \leqslant f_{1/2}(b)$. Thus $f_{1/2}(b)$ is also a local unit for a and since itself belongs to $\mathrm{Ped}(\mathcal{A})_+$ as $f_{1/4}(b) f_{1/2}(b) = f_{1/2}(b)$, its evaluation function $\widehat{f_{1/2}(b)}$ is continuous on $\mathcal{T}(\mathcal{A})$. Thus $d_\tau(a) = \tau(R_a) \leqslant \tau(f_{1/2}(b))$ for every $\tau \in \mathcal{T}(\mathcal{A})$ and hence $\sup_{\tau \in \mathcal{T}(\mathcal{A})} d_\tau(a) < \infty$. ∎

The same result was obtained in Lemma 1.6 of [29] under the additional conditions that $\mathcal{A}$ is the stabilization of a simple, unital, exact algebra with strict comparison.

The notion of strict comparison has played an important role in the theory of C^*-algebras especially after [2].

DEFINITION 1.3. Let $\mathcal{A}$ be a simple, σ-unital C^*-algebra with non empty tracial simplex $\mathcal{T}(\mathcal{A})$. Then we say that:

(i) $\mathcal{A}$ has strict comparison of positive elements by traces if $a \preceq b$ whenever $a, b \in \mathcal{A}_+$ and $d_\tau(a) < d_\tau(b)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(b) < \infty$;

(ii) $\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements by traces if $A \preceq B$ whenever $A, B \in \mathcal{M}(\mathcal{A})_+$, A belongs to the ideal $I(B)$ generated by B, and $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$.

Strict comparison is often defined in terms of 2-quasitraces. In Theorem 2.9 of [20] we proved that if a simple, unital C^*-algebra of real rank zero and stable rank one has strict comparison of positive elements by traces (equivalently, of projections, due to real rank zero) then all 2-quasitraces are traces. Recently it was shown the same conclusion holds without the real rank zero and stable rank one hypotheses ([44], Theorem 3.6).

Notice that in (ii), the condition that $A \in I(B)$ (which is obviously necessary for $A \preceq B$) does not follow in general from the condition that $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$. Indeed if there is an element $B \in \mathcal{A}_+$ with $d_\tau(B) = \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$ (and this is certainly the case when $\mathcal{A}$ is stable) then the condition $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$ is trivially satisfied for every $A \in \mathcal{M}(\mathcal{A})_+ \setminus \mathcal{A}$ and yet $A \not\preceq B$.

1.2. IDEALS AND TRACES.

We first recall the following well-known result.

LEMMA 1.4. *Let $\mathcal{B}$ be a C^*-algebra and let $A, T \in \mathcal{B}_+$. Then $A \in I(T)$ (the principal ideal generated by T) if and only if for every $\varepsilon > 0$ there is some $m \in \mathbb{N}$ such that $(A - \varepsilon)_+ \preceq \bigoplus_{k=1}^{m} T$ in $M_m(\mathcal{B})$. In particular, if P is a projection, then $P \in I(T)$ if and only if there is an $m \in \mathbb{N}$ such that $P \preceq \bigoplus_{k=1}^{m} T$ in $M_m(\mathcal{B})$.*

We will focus on the ideals of the multiplier algebra $\mathcal{M}(\mathcal{A})$ of a simple, σ-unital, non-unital C^*-algebra $\mathcal{A}$.

The ideal

$$I_{\min} := \bigcap \{\mathcal{J} \lhd \mathcal{M}(\mathcal{A}) : \mathcal{A} \subsetneq \mathcal{J}\} \tag{1.12}$$

is called the minimal ideal of $\mathcal{M}(\mathcal{A})$ and $A \subset I_{\min}$. We do not know in general whether $\mathcal{A} \neq I_{\min}$ although by Corollary 3.15, Proposition 5.4, Theorem 5.6 of [23] this conclusion holds when $\mathcal{A}$ is also non-elementary and with any of the following properties:

(i) $\mathcal{A}$ is separable;

(ii) the Cuntz semigroup of $\mathcal{A}$ is order separable;

(iii) $\mathcal{A}$ has the (SP) property and its dimension semigroup $D(\mathcal{A})$ of Murray–von Neumann equivalence classes of projections is order separable;

(iv) $\mathcal{A}$ has strict comparison of positive elements by traces.

The conclusion $\mathcal{A} \neq I_{\min}$ holds also if $\mathcal{A}$ has continuous scale (in particular, if $\mathcal{A}$ is purely infinite), because then $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is simple ([39], Theorem 2.8) and hence $I_{\min} = \mathcal{M}(\mathcal{A})$.

Following Lin's approach, but not using his notations ([32]), one can characterize $I_{\min}$ in terms of approximate identities of $\mathcal{A}$. Given any approximate identity $\{e_n\}$, which henceforth we will always assume to satisfy the condition $e_{n+1}e_n = e_n$, the ideal $I_{\min}$ is shown (see [23] and [32]) to coincide with the norm closure of the linear span of

$$K_o(\{e_n\}) := \{X \in \mathcal{M}(\mathcal{A})_+ : \forall\, 0 \neq a \in \mathcal{A}_+ \;\exists\, N \in \mathbb{N} \\ \ni\; m > n \geqslant N \Rightarrow (e_m - e_n)X(e_m - e_n) \preceq a\}.$$

When $\mathcal{A}$ is a simple, σ-unital, non-unital, non-elementary C^*-algebra with non-empty tracial simplex, then another natural ideal is I_{cont}.

DEFINITION 1.5. ([23], Definition 5.1, Proposition 5.2) I_{cont} is the norm closure of the linear span of $K_c := \{X \in \mathcal{M}(\mathcal{A})_+ : \widehat{X} \in \operatorname{Aff}(\mathcal{T}(\mathcal{A}))\}$.

An immediate consequence of the definition (see Proposition 5.2 of [23]) is that if $0 \neq X \in \mathcal{M}(\mathcal{A})_+$ and $0 \neq P \in \mathcal{M}(\mathcal{A})$ is a projection, then

$$X \in (I_{\text{cont}})_+ \quad \text{if and only if } \widehat{(X-\delta)}_+ \in \operatorname{Aff}(\mathcal{T}(\mathcal{A}))\ \forall\, \delta > 0; \tag{1.13}$$

$$P \in (I_{\text{cont}})_+ \quad \text{if and only if } \widehat{P} \in \operatorname{Aff}(\mathcal{T}(\mathcal{A}))_{++}. \tag{1.14}$$

There are simple, separable, non-unital C^*-algebras where $I_{\min} \neq I_{\text{cont}}$ ([23], Theorem 7.8), however $I_{\min} = I_{\text{cont}}$ when $\mathcal{A}$ has strict comparison of positive elements ([23], Theorem 5.6).

It is well known that every trace τ gives rise to a (not necessarily proper) ideal I_τ which is the norm closure of the linear span of the hereditary cone

$$\{X \in \mathcal{M}(\mathcal{A})_+ : \tau(X) < \infty\}.$$

As a consequence, if $0 \neq X \in \mathcal{M}(\mathcal{A})_+$ and $0 \neq P \in \mathcal{M}(\mathcal{A})$ is a projection, then

$$X \in (I_\tau)_+ \quad \text{if and only if } \tau((X-\delta)_+) < \infty\ \forall\, \delta > 0; \tag{1.15}$$

$$P \in (I_\tau)_+ \quad \text{if and only if } \tau(P) < \infty. \tag{1.16}$$

In this paper the following ideals will play an important role.

DEFINITION 1.6. Let $\mathcal{A}$ be a σ-unital, non-unital, non-elementary C^*-algebra with non-empty tracial simplex.

(i) $I_{\text{fin}} := \bigcap_{\tau \in \partial_e(\mathcal{T}(\mathcal{A}))} I_\tau$;

(ii) $I_{\mathrm{b}} := \bigcap_{\tau \in \mathcal{T}(\mathcal{A})} I_\tau$.

Perera introduced in a different way the ideal I_{fin}, which he called the *finite ideal*, for σ-unital, non-unital, non-elementary C^*-algebras with real rank zero, stable rank one, and weakly unperforated K_0 group ([48], Proposition 6.1). The following inclusions are obvious:

$$I_{\mathrm{cont}} \subset I_{\mathrm{b}} \subset I_{\mathrm{fin}}. \tag{1.17}$$

Also an immediate consequence of the definition and of (1.15), (1.16) is the following lemma.

LEMMA 1.7. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with non-empty tracial simplex.*

(i) $(I_{\mathrm{fin}})_+ = \{X \in \mathcal{M}(\mathcal{A})_+ : \tau((X-\delta)_+) < \infty \; \forall \delta > 0, \tau \in \partial_e(\mathcal{T}(\mathcal{A}))\}$;

(ii) $(I_{\mathrm{b}})_+ = \{X \in \mathcal{M}(\mathcal{A})_+ : \tau((X-\delta)_+) < \infty \; \forall \delta > 0, \tau \in \mathcal{T}(\mathcal{A})\}$.

In particular, if P is a projection, then

(iii) $P \in I_{\mathrm{fin}} \;\Leftrightarrow\; \widehat{P}(\tau) < \infty$ *for all* $\tau \in \partial_e(\mathcal{T}(\mathcal{A}))$.

To explain the notation of I_{b}, we need the following elementary observation.

LEMMA 1.8. *Let K be a compact convex space and let $f \in \mathrm{LAff}(\mathcal{K})_+$. Then*

(i) $\sup_{x \in \partial_e(\mathcal{K})} f(x) = \sup_{x \in K} f(x)$;

(ii) $\sup_{x \in K} f(x) < \infty$ *if and only if* $f(x) < \infty$ *for all* $x \in K$.

Proof. (i) It is obvious that $\sup_{x \in \partial_e(\mathcal{K})} f(x) = \sup_{x \in \mathrm{co}\partial_e(\mathcal{K})} f(x)$. Then the conclusion follows from the density of $\mathrm{co}\partial_e(\mathcal{K})$ in K (the Krein–Millman theorem) and the lower semicontinuity of f.

(ii) The necessity being trivial, assume that $\sup_{x \in K} f(x) = \infty$ and choose a sequence $x_k \in \partial_e(\mathcal{K})$ for which $f(x_k) \geqslant 2^k$ for all k. If $f(x_k) = \infty$ for some k, then we are done, thus assume that $f(x_k) < \infty$ for all k. Let μ_k be the Dirac measure on x_k and $\mu := \sum_{k=1}^{\infty} \frac{\mu_k}{2^k}$. Then μ is a probability measure on $\partial_e(\mathcal{K})$. Let $x \in K$ be the corresponding element, i.e.,

$$g(x) = \int_{\partial_e(\mathcal{K})} g(y) \mathrm{d}\mu(y) = \sum_{k=1}^{\infty} \frac{g(x_k)}{2^k}$$

for all $g \in \mathrm{Aff}(\mathcal{K})$ and hence also $f(x) = \sum_{k=1}^{\infty} \frac{f(x_k)}{2^k} = \infty$. ∎

The argument in (ii) is similar to the one in Lemma 4.4 of [48].

COROLLARY 1.9. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with non-empty tracial simplex, and let $0 \neq X \in \mathcal{M}(\mathcal{A})_+$ and $0 \neq P \in \mathcal{M}(\mathcal{A})$ be a projection. Then*

(i) $X \in I_{\mathrm{b}}$ *if and only if* $\tau \in \partial_e(\mathcal{T}(\mathcal{A})) \sup\limits_{\tau \in \partial_e(\mathcal{T}(\mathcal{A}))} \tau((X-\delta)_+) < \infty \; \forall \delta > 0$;

(ii) $P \in I_{\mathrm{b}}$ *if and only if* $\sup\limits_{\tau \in \partial_e(\mathcal{T}(\mathcal{A}))} \tau(P) < \infty$.

COROLLARY 1.10. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra, with non-empty $\mathcal{T}(\mathcal{A})$, and with $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$. Then $I_{\mathrm{cont}} = I_{\mathrm{b}} = I_{\mathrm{fin}}$.*

Finally we list our notations for order ideals. If $\mathcal{B}$ is a C^*algebra, denote by $V(\mathcal{B})$ the semigroup of Murray von–Neumann equivalence classes of projections in $M_\infty(\mathcal{B})$, where $[P] + [Q] := [P \oplus Q]$ for $P, Q \in M_\infty(\mathcal{B})$. An order ideal H of $V(\mathcal{B})$ is a hereditary sub-semigroup of $V(\mathcal{B})$. When $[P] \in V(\mathcal{B})$, denote the principal order ideal generated by $[P]$ by

$$I([P]) := \{[R] \in V(\mathcal{B}) : [R] \leqslant n[P] \text{ for some } n \in \mathbb{N}\}. \tag{1.18}$$

The connection between principal ideals and principal order ideals generated by a projection of $\mathcal{B}$ is an immediate consequence of Lemma 1.4.

LEMMA 1.11. *If P, Q are projections in $\mathcal{B}$, then the following are equivalent:*

(i) $I(P) \subsetneq I(Q)$;

(ii) $\exists\, n$ *such that* $[P] \leqslant n[Q]$, $\nexists\, n$ *such that* $[Q] \leqslant n[P]$;

(iii) $I([P]) \subsetneq I([Q])$.

2. PRELIMINARIES ON LOWER SEMICONTINUOUS AFFINE FUNCTIONS

Our paper makes use of some technical results on lower semicontinuous affine functions on Choquet simplexes. We collect in this section some technical results on lower semicontinuous affine functions on Choquet simplexes. We start by listing for convenience of reference some results that will be used throughout the paper.

PROPOSITION 2.1. *Let K be a compact convex metrizable space.*

(i) $\mathrm{LAff}(\mathcal{K})_{++} = \mathrm{LAff}_\sigma(\mathcal{K})_{++}$ ([58], *Lemma* 4.2, *see also comments before Proposition* 4.10 *in* [48]). *In particular, for every $f \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ there is a decomposition $f = \sum\limits_{j=1}^{\infty} f_j$ (pointwise convergence) with $f_j \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))_{++}$.*

(ii) ([49], *Choquet Theorem, p.* 14) *For every $x \in K$ there exists a probability measure μ on $\partial_e(\mathcal{K})$ such that $f(x) = \int\limits_{\partial_e(\mathcal{K})} f(t)\mathrm{d}\mu(t)$ for all $f \in \mathrm{LAff}(\mathcal{K})_+$.*

(iii) *If $f, g \in \mathrm{LAff}(\mathcal{K})_+$ and $f(x) \geqslant g(x)$ (respectively, $f(x) > g(x)$), (respectively, $f(x) = g(x)$) for all $x \in \partial_e(\mathcal{K})$ then $f(x) \geqslant g(x)$, (respectively, $f(x) > g(x)$), (respectively, $f(x) = g(x)$), for all $x \in K$.*

(iv) *If furthermore K is a Choquet simplex, then the measure in* (ii) *is unique* ([49], *Choquet Theorem, p.* 60).

THEOREM 2.2. ([18], *Theorem* 11.14, *Corollary* 11.15) *Let K be a Choquet simplex, $X \subseteq \partial_e K$ a compact subset of the extremal boundary of K, $f : K \to \{-\infty\} \cup \mathbb{R}$ an upper semicontinuous convex function, $h : K \to \mathbb{R} \cup \{\infty\}$ a lower semicontinuous concave function, and $g_0 : X \to \mathbb{R}$ and continuous function, such that $f \leqslant h$ and $f|_X \leqslant g_0 \leqslant h|_X$. Then there exists a function $g \in \mathrm{Aff}(\mathcal{K})$ such that:*

(i) $f \leqslant g \leqslant h$, *and*

(ii) $g|_X = g_0$.

In particular, every function $g_0 \in C(X, \mathbb{R})$ has an extension $g \in \mathrm{Aff}(\mathcal{K})$ such that $\|g\| = \|g_0\|$.

The following is an elementary observation which we will use in a number of occasions.

LEMMA 2.3. *Assume that $g = G + F$ where G and F are finite and lower semicontinuous functions on a compact set K and that there is a sequence $K \ni x_n \to x \in K$ such that $g(x_n) \to g(x)$. Then $G(x_n) \to G(x)$. In particular, if $G, F \in \mathrm{LAff}(\mathcal{K})_{++}$ and $g := G + F \in \mathrm{Aff}(\mathcal{K})$, then $G, F \in \mathrm{Aff}(\mathcal{K})_{++}$.*

Now recall that if K is a simplex, then the complementary face F' of a face F is the union of all the faces of K that are disjoint from F. A face F of K is said to be *split* if $K = F \dot{+} F'$ where $\dot{+}$ denotes the direct convex sum. By Theorem 11.28 of [18],

(2.1) if K is a Choquet simplex then every closed face is split.

It is elementary and most likely well known that if K is a Choquet simplex and F is a split face, then every pair of affine nonnegative functions f on F and g on F' has unique extension to an affine function on K. We will need to use this fact and some refinements of it, collected in the following lemma.

LEMMA 2.4. *Let K be a Choquet simplex and F a split face. Assume that f and g are affine nonnegative extended real valued function on F and F', respectively and let $f \dot{+} g$ be the function defined on K as follows: if $k = tx + (1-t)y$ for some $x \in F$, $y \in F'$, and $0 \leqslant t \leqslant 1$ then*

$$(f \dot{+} g)(k) := \begin{cases} tf(x) + (1-t)g(y) & 0 < t < 1, \\ g(y) & t = 0, \\ f(x) & t = 1. \end{cases}$$

Then

(i) *$f \dot{+} g$ is the unique affine function that agrees with f on F and with g on F'.*

(ii) *Assume that F is closed, $f \in \mathrm{LAff}(F)_+$, $g \in \mathrm{LAff}(\mathcal{K})_+$, and $f(x) \leqslant g(x)$ for all $x \in F$. Then $f \dot{+} g|_{F'} \in \mathrm{LAff}(\mathcal{K})_+$ and $f \dot{+} g|_{F'} \leqslant g$.*

(iii) *Assume that F and F' are closed, $f \in \mathrm{LAff}(F)_+$, and $g \in \mathrm{LAff}(F')_+$. Then $f \dot{+} g \in \mathrm{LAff}(\mathcal{K})_+$.*

Proof. (i) Recall that a decomposition $k = tx + (1-t)y$ is unique but for the case when $k \in F$ and then $t = 1$, $x = k$, and y is arbitrary or $k \in F'$ and then $t = 0$, $y = k$, and x is arbitrary. Therefore the function $f \dot{+} g$ is well defined. Also, the definition given can be simplified by the convention that $0 \cdot \infty = 0$. A lengthy straightforward computation shows that $f \dot{+} g$ is indeed affine and that it is the unique affine function that agrees with f on F and with g on F'.

(ii) Assume that $k_\lambda \to k$ for some net $k_\lambda \in K$. Let $k_\lambda = t_\lambda x_\lambda + (1-t_\lambda)y_\lambda$ for some $x_\lambda \in F$, $y_\lambda \in F'$ and $t_\lambda \in [0,1]$, and let $k = \alpha x_o + (1-\alpha)y_o$ for some $x_o \in F$, $y_o \in F'$ and $\alpha \in [0,1]$. By passing if necessary to a subnet, we can assume that $(f \dot{+} g)(k_\lambda)$ converges to $\underline{\lim}_\lambda (f \dot{+} g)(k_\lambda)$. Then

$$(f \dot{+} g)(k_\lambda) = t_\lambda f(x_\lambda) + (1-t_\lambda)g(y_\lambda). \tag{2.2}$$

By the compactness of $[0,1]$, F, and K, and by passing if necessary to subnets of subnets, which will not affect neither $\underline{\lim}_\lambda (f \dot{+} g)(k_\lambda)$ nor $(f \dot{+} g)(k)$, we can assume that $t_\lambda \to t$, $x_\lambda \to x$, and $y_\lambda \to \beta x' + (1-\beta)y$ for some $x, x' \in F$, $y \in F'$ and $0 \leqslant \beta \leqslant 1$. (Notice that if F' is closed, then $x' = 0$.) Then

$$k = tx + \beta(1-t)x' + (1-\beta)(1-t)y. \tag{2.3}$$

We leave to the reader the simpler case when $t = \beta = 0$, i.e., $k = y$, and thus assume that t and β do not both vanish. Set

$$x'' := \frac{t}{t+\beta(1-t)}x + \frac{\beta(1-t)}{t+\beta(1-t)}x'. \tag{2.4}$$

Then $x'' \in F$ and $k = (t + \beta(1-t))x'' + (1-\beta)(1-t)y$ is the decomposition of k in $F \dot{+} F'$. Then

$$\begin{aligned}
(f \dot{+} g)(k) &= (t+\beta(1-t))f(x'') + (1-\beta)(1-t)g(y) \quad \text{(by definition of } f \dot{+} g)\\
&= tf(x) + \beta(1-t)f(x') + (1-\beta)(1-t)g(y) \quad \text{(by (2.4) as } f \text{ is affine)}\\
&\leqslant tf(x) + \beta(1-t)g(x') + (1-\beta)(1-t)g(y) \quad \text{(as } f \leqslant g \text{ on } F)\\
&= tf(x) + (1-t)(g(\beta x' + (1-\beta)y)) \quad (g \text{ is affine})\\
&\leqslant \underline{\lim}_\lambda\, t_\lambda f(x_\lambda) + \underline{\lim}_\lambda (1-t_\lambda)g(y_\lambda) \quad (f \text{ and } g \text{ are lsc})\\
&\leqslant \underline{\lim}_\lambda (t_\lambda f(x_\lambda) + (1-t_\lambda)g(y_\lambda))\\
&= \underline{\lim}_\lambda (f \dot{+} g)(k_\lambda) \quad \text{(by definition of } f \dot{+} g).
\end{aligned}$$

(iii) Follows from the same proof as in (ii). ∎

Notice that if both F and F' are closed and f and g are continuous, then the same computation shows that $f \dot{+} g$ is continuous, which of course is well known (e.g., see Corollary 11.23 of [18]).

Lemma 2.4 provides a generalization of Corollaries 4.11-13 of [48] and Proposition 4.10 to the case when F is closed but not necessarily finite dimensional, and without requiring the metrizability of K.

COROLLARY 2.5. *Let K be a metrizable Choquet simplex, $h \in \mathrm{LAff}(\mathcal{K})_{++}$, F is a closed face of K such that $h|_F = \infty$. Then*

(i) *$f \dot{+} \frac{h}{2} \in \mathrm{LAff}(K)_{++}$ is complemented under h for every $f \in \mathrm{LAff}(F)_{++}$.*

(ii) *If $f \in \mathrm{LAff}(F)_{++}$, $g \in \mathrm{Aff}(K)_{++}$, $f \leqslant g|_F$ and $g(x) < h(x)$ for all x, then $f \dot{+} g$ is complemented under h. In particular, if $\sup f < \min h$, then for every $\sup f < \gamma < \min h$, $f \dot{+} \gamma$ is complemented under h, where γ denotes the constant function $\gamma(x) = \gamma$.*

(iii) *If also F' is closed, $f \in \mathrm{LAff}(\mathcal{K})_{++}$, and f is continuous on F', then f is complemented under nh for some $n \in \mathbb{N}$.*

Proof. (i) $f \dot{+} \frac{h}{2} \in \mathrm{LAff}(K)_{++}$ by Lemma 2.4 as $f \leqslant h|_F$. Moreover

$$f \dot{+} \frac{h}{2} + \frac{h}{2} = \begin{cases} f(x) + \infty = h(x) & x \in F, \\ \frac{h(x)}{2} + \frac{h(x)}{2} = h(x) & x \in F', \end{cases}$$

and hence $f \dot{+} \frac{h}{2} + \frac{h}{2} = h$. Obviously, $\frac{h}{2} \in \mathrm{LAff}(K)_{++}$.

(ii) $f \dot{+} g \in \mathrm{LAff}(K)_{++}$ by Lemma 2.4, $h - g \in \mathrm{LAff}(K)_{++}$, and

$$(f \dot{+} g + h - g)(x) = \begin{cases} f(x) + h(x) - g(x) = \infty = h(x) & x \in F, \\ g(x) + h(x) - g(x) = h(x) & x \in F'. \end{cases}$$

Therefore $f \dot{+} g + h - g = h$, which concludes the proof.

(iii) h has a strictly positive minumum on the compact set F', hence we can find $n \in \mathbb{N}$ such that $f(x) < nh(x)$ for all $x \in F'$ and let $g := nh|_{F'} - f|_{F'}$. Then $h\,|_F \dot{+} g \in \mathrm{LAff}(\mathcal{K})_{++}$ by Lemma 2.4. Reasoning as above, $f + h\,|_F \dot{+} g = nh$. ∎

COROLLARY 2.6. *Let K be a Choquet simplex and $\{x_j\}_1^\infty \subset \partial_e(\mathcal{K})$ be a sequence with distinct terms. Then for every nondecreasing sequence of scalars $0 < \alpha_j < \infty$ there is a function $f \in \mathrm{LAff}_\sigma(\mathcal{K})_{++}$ such that $f(x_j) = \alpha_j$ for all j and $\alpha_1 \leqslant f \leqslant \sup_j \alpha_j$.*

Proof. Starting with $g_o = \alpha_1 > 0$ we construct an increasing sequence of functions $g_k \in \mathrm{Aff}(\mathcal{K})_{++}$ such that $g_k(x_j) = \alpha_j$ for all $1 \leqslant j \leqslant k$ and $g_k(x) \leqslant \alpha_k$ for all $x \in K$.

Assuming the construction up to $k-1$ for some $k \geqslant 1$, set $X_k := \{x_1, x_2, \ldots, x_k\}$ and $g_{k,o}(x_j) := \begin{cases} g_{k-1}(x_j) & 1 \leqslant j < k, \\ \alpha_k & j = k. \end{cases}$ Then $g_{k-1} \leqslant \alpha_{k-1} \leqslant \alpha_k$ and hence $g_{k-1}|_{X_k} \leqslant g_{k,o} \leqslant \alpha_k$. Let $g_k \in \mathrm{Aff}(\mathcal{K})$ be the extension of $g_{k,o}$ to K for which $g_{k-1} \leqslant g_k \leqslant \alpha_k$ that is given by Theorem 2.2. Then $f := \lim_k g_k$ satisfies the desired properties. ∎

Next we present two technical constructions of lower semicontinuous functions that will be needed in the study of principal ideals in $\mathcal{M}(\mathcal{A})$.

LEMMA 2.7. *Let K be a compact metrizable space and g be a non-negative, finite, lower semicontinuous function on K that is not continuous at some point $x_o \in K$. Then there is a decomposition $g = G + F$ into the sum of lower-semicontinuous non-negative functions G and F which are both discontinuous at x_o but for which there is a sequence $y_k \to x_o$ such that $G(y_k) \to G(x_o)$ but $g(y_k) \not\to g(x_o)$.*

If furthermore K is also convex and $g \in \mathrm{LAff}(\mathcal{K})_+$ (respectively, $g \in \mathrm{LAff}(\mathcal{K})_{++}$), then we can choose G, F to be in $\mathrm{LAff}(\mathcal{K})_+$ (respectively, in $\mathrm{LAff}(\mathcal{K})_{++}$).

Proof. Since g is lower semicontinuous and K is metrizable, we can decompose it into a sum $g = \sum_{k=1}^{\infty} g_k$ of functions $g_i \in C(K, \mathbb{R})_+$ (respectively, $g_i \in C(K, \mathbb{R})_{++}$ if g is strictly positive). Since g is not continuous at x_o, there is a sequence $x_j \to x_o$ and a number β such that $g(x_j) > \beta > g(x_o)$ for all j. Let $\delta := \frac{\beta - g(x_o)}{3}$. We construct inductively two sequences of positive integers $m_j \leqslant n_j < m_{j+1}$ starting with $m_1 = 1$ and two strictly increasing sequences of integers s_k and t_k such that if we set $G_k := \sum_{j=1}^{k} \sum_{i=m_j}^{n_j} g_i$, we have for all integers $k \geqslant 1$:

(i) $G_k(x_{s_k}) > G_k(x_o) + \delta$;

(ii) $|G_k(x_j) - G_k(x_o)| < \frac{\delta}{k}$ for all $j \geqslant t_k$;

(iii) $\sum_{i=m_{k+1}}^{\infty} g_i(x_{t_k}) < \frac{\delta}{k}$ and $\sum_{i=m_{k+1}}^{\infty} g_i(x_o) < \frac{\delta}{k}$.

We start the induction by setting $m_1 = 1$ and $s_1 = 1$. Since $g(x_{s_1}) > \beta$, choose an integer $n_1 \geqslant 1$ such that $\sum_{i=1}^{n_1} g_i(x_{s_1}) > \beta$. Thus

$$G_1(x_{s_1}) > \beta = g(x_o) + 3\delta \geqslant G_1(x_o) + 3\delta > G_1(x_o) + \delta,$$

which satisfies condition (i). By the continuity of G_1 we can find an index t_1 for which (ii) is satisfied. By the convergence of the series $\sum_1^{\infty} g_i(x)$ for every x, choose $m_2 > n_1$ so to satisfy (iii). Thus conditions (i)–(iii) are satisfied for $k = 1$.

Next assume the construction up to some integer k and notice that this includes the existence of $m_{k+1} > n_k$ that satisfies (iii). By the continuity of $\sum_{i=1}^{m_{k+1}-1} g_i$

and of $G_k(x)$, choose an integer $s_{k+1} > s_k$ such that for all $j \geqslant s_{k+1}$ we have

$$\Big|\sum_{i=1}^{m_{k+1}-1} g_i(x_j) - \sum_{i=1}^{m_{k+1}-1} g_i(x_o)\Big| < \delta \quad \text{and} \quad |G_k(x_j) - G_k(x_o)| < \delta. \tag{2.5}$$

Since $g(x_{s_{k+1}}) > \beta$, choose an integer $n_{k+1} \geqslant m_{k+1}$ so that

$$\sum_{i=1}^{n_{k+1}} g_i(x_{s_{k+1}}) > \beta. \tag{2.6}$$

Then

$$\begin{aligned}
&G_{k+1}(x_{s_{k+1}}) - G_{k+1}(x_o)\\
&= G_k(x_{s_{k+1}}) - G_k(x_o) + \sum_{i=m_{k+1}}^{n_{k+1}} g_i(x_{s_{k+1}}) - \sum_{i=m_{k+1}}^{n_{k+1}} g_i(x_o)\\
&= G_k(x_{s_{k+1}}) - G_k(x_o) + \sum_{i=1}^{n_{k+1}} g_i(x_{s_{k+1}}) - \sum_{i=1}^{n_{k+1}} g_i(x_o) - \sum_{i=1}^{m_{k+1}-1} g_i(x_{s_{k+1}}) + \sum_{i=1}^{m_{k+1}-1} g_i(x_o)\\
&> -\delta + \beta - g(x_o) - \delta = \delta
\end{aligned}$$

where

$$\begin{aligned}
&|G_k(x_{s_{k+1}}) - G_k(x_o)| < \delta \quad \text{(by (2.5))},\\
&\sum_{i=1}^{n_{k+1}} g_i(x_{s_{k+1}}) > \beta \quad \text{(by (2.6))},\\
&\sum_{i=1}^{n_{k+1}} g_i(x_o) \leqslant g(x_o) \quad \text{(by the definition of } g),\\
&\Big|\sum_{i=1}^{m_{k+1}-1} g_i(x_{s_{k+1}}) - \sum_{i=1}^{m_{k+1}-1} g_i(x_o)\Big| < \delta \quad \text{(by (2.5))}.
\end{aligned}$$

Thus condition (i) is satisfied for $k+1$. Since G_{k+1} is continuous, choose $t_{k+1} > t_k$ so to satisfy (ii). By the convergence of $\sum_{i=1}^{\infty} g_i(x)$ for all x, choose $m_{k+2} > n_{k+1}$ so to satisfy (iii).

Thus by induction we can continue the construction for all k and obtain the function $G := \lim G_k$. As a sum of nonnegative continuous functions, G is nonnegative lower semicontinuous. Then $F = g - G = \sum_{j=1}^{\infty} \sum_{i=n_j+1}^{m_{j+1}-1} g_i$, hence F too is nonnegative lower semicontinuous. Since $G(x_{s_k}) \geqslant G_k(x_{s_k}) > G_k(x_o) + \delta$ for all k, we have

$$\varliminf_k G(x_{s_k}) \geqslant \lim_k G_k(x_o) + \delta = G(x_o) + \delta.$$

Since $x_{s_k} \to x_o$, it follows that G is not continuous at x_o.

By (iii),

$$0 \leqslant G(x_{t_k}) - G_k(x_{t_k}) = \sum_{j=k+1}^{\infty} \sum_{i=m_j}^{n_j} g_i(x_{t_k}) \leqslant \sum_{i=m_{k+1}}^{\infty} g_i(x_{t_k}) < \frac{\delta}{k}$$

and similarly $0 \leqslant G(x_o) - G_k(x_o) < \frac{\delta}{k}$. Since by (ii), $|G_k(x_{t_k}) - G_k(x_o)| < \frac{\delta}{k}$, it follows that

$$|G(x_{t_k}) - G(x_o)| < \frac{3\delta}{k}$$

and hence $G(x_{t_k}) \to G(x_o)$. Then set $y_k := x_{t_k}$. Since $g(y_k) > \beta > g(x_o)$, it follows that $F(y_k) \not\to F(x_o)$.

Finally, if K is convex and $g \in \mathrm{LAff}(\mathcal{K})$, then by Proposition 11.8 of [18] and Lemma 4.2 of [58], g is the supremum of an increasing sequence of functions in $\mathrm{Aff}(\mathcal{K})$ and thus we can assume that $g_i \in \mathrm{Aff}(\mathcal{K})_{++}$. The rest of the conclusions are now immediate. ∎

LEMMA 2.8. *Let K be a metrizable Choquet simplex, $h \in \mathrm{LAff}(\mathcal{K})_{++}$, and assume there is a sequence $\{x_n\}_1^\infty \subset \partial_e(\mathcal{K})$ of distinct elements for which $\lim_n h(x_n) = \infty$. Then h can be decomposed into the sum of two functions F and $G \in \mathrm{LAff}(\mathcal{K})_{++}$ such that:*

(i) $G(x_n) < \infty$ *for every* n;

(ii) $\sup_n G(x_n) = \infty$;

(iii) $\sup_n \frac{h(x_n)}{G(x_n)} = \infty$.

Proof. By Proposition 2.1(i) there is an increasing sequence of functions $h_m \in \mathrm{Aff}(\mathcal{K})_{++}$ that converges pointwise to h. Start with integers $n_1 > 1$ such that $h(x_{n_1}) > 1$ and $m_1 \geqslant 1$ such that $h_{m_1}(x_{n_1}) \geqslant 1$. Then construct recursively two strictly increasing sequences of integers n_k and m_k such that

$$h_{m_k}(x_{n_k}) \geqslant k^2 + k\|h_{m_{k-1}}\|_\infty. \tag{2.7}$$

Let $\gamma := \min_{x \in K} h_{m_1}(x)$. Since h_{m_1} is strictly positive, it follows that $\gamma > 0$. Let $X_1 := \{x_1, x_2, \dots, x_{n_1}\}$ and define for every $x_j \in X_1$

$$g_{1,0}(x_j) := \begin{cases} \frac{1}{2}\gamma & 1 \leqslant j < n_1, \\ \frac{1}{2}h_{m_1}(x_{n_1}) & j = n_1. \end{cases}$$

We verify the conditions of Theorem 2.2: X_1 is a compact subset of $\partial_e(\mathcal{K})$, $g_{1,0} \in C(X_1, \mathbb{R})$, the constant function $\frac{1}{2}\gamma$ is continuous and convex on K, the function $\frac{1}{2}h_{m_1} \in \mathrm{Aff}(\mathcal{K})$ is continuous and concave on K and

$$\frac{1}{2}\gamma\ |_{X_1} \leqslant g_{1,0} \leqslant \frac{1}{2}h_{m_1}\ |_{X_1}\ .$$

Thus by Theorem 2.2 there is an extension $g_1 \in \mathrm{Aff}(\mathcal{K})$ of $g_{1,0}$ for which

$$\frac{1}{2}\gamma \leqslant g_1 \leqslant \frac{1}{2}h_{m_1}.$$

In particular, $g_1 \in \mathrm{Aff}(\mathcal{K})_{++}$. Let $f_1 := h_{m_1} - g_1$. Then $f_1 \geqslant \frac{1}{2}h_{m_1}$ and hence also $f_1 \in \mathrm{Aff}(\mathcal{K})_{++}$.

Now for every $k > 1$, set $X_k := \{x_1, x_2, \ldots, x_{n_k}\}$ and define for every $x_j \in X_k$

$$g_{k,0}(x_j) := \begin{cases} 0 & 1 \leqslant j < n_k, \\ \frac{1}{k}(h_{m_k} - h_{m_{k-1}})(x_{n_k}) & j = n_k. \end{cases}$$

Then $g_{k,0} \in C(X_k, \mathbb{R})$ and $0 \leqslant g_{k,0} \leqslant \frac{1}{k}(h_{m_k} - h_{m_{k-1}})|_{X_k}$ hence it has an extension $g_k \in \mathrm{Aff}(\mathcal{K})$ for which

$$0 \leqslant g_k \leqslant \frac{1}{k}(h_{m_k} - h_{m_{k-1}}) \quad \text{and} \quad g_k(x_{n_k}) = \frac{1}{k}(h_{m_k} - h_{m_{k-1}})(x_{n_k}). \tag{2.8}$$

Set $f_k := h_{m_k} - h_{m_{k-1}} - g_k$. Then $f_k \geqslant 0$ and $f_k \in \mathrm{Aff}(\mathcal{K})$.

Set $m_\mathrm{o} := 0$ and $h_{m_\mathrm{o}} = 0$. Then for all k

$$\sum_{j=1}^{k} f_j + \sum_{j=1}^{k} g_j = \sum_{j=1}^{k} (h_{m_j} - h_{m_{j-1}}) = h_{m_k}.$$

In particular,

$$\sum_{j=1}^{k} g_j < h_{m_k} \quad \forall k. \tag{2.9}$$

Let $F = \sum_{k=1}^{\infty} f_k$ and $G = \sum_{k=1}^{\infty} g_k$, then $F + G = h$ and F, $G \in \mathrm{LAff}(\mathcal{K})_{++}$, where the strict positivity of F and G follows from the strict positivity of f_1 and g_1.

To show that (i) holds, for every n, choose $n_k > n$. By definition, $g_{k'}(x_n) = 0$ for every $k' > k$ and hence by (2.9),

$$G(x_n) = \sum_{i=1}^{k} g_i(x_n) \leqslant h_{m_k}(x_n) < \infty.$$

Now

$$\begin{aligned} G(x_{n_k}) &\geqslant g_k(x_{n_k}) \\ &= \frac{1}{k}(h_{m_k} - h_{m_{k-1}})(x_{n_k}) \quad \text{(by (2.8))} \\ &\geqslant \frac{1}{k}h_{m_k}(x_{n_k}) - \frac{1}{k}\|h_{m_{k-1}}\|_\infty \\ &\geqslant \frac{1}{k}(k^2 + k\|h_{m_{k-1}}\|_\infty) - \frac{1}{k}\|h_{m_{k-1}}\|_\infty \quad \text{(by (2.7))} \\ &> k \end{aligned}$$

and hence (ii) holds. Finally, (iii) follows from the inequalities:

$$\begin{aligned}
G(x_{n_k}) &= \sum_{j=1}^{k-1} g_j(x_{n_k}) + g_k(x_{n_k}) \\
&\leqslant h_{m_{k-1}}(x_{n_k}) + \frac{1}{k}(h_{m_k} - h_{m_{k-1}})(x_{n_k}) \quad \text{(by (2.9) and (2.8))} \\
&\leqslant \|h_{m_{k-1}}\|_\infty + \frac{1}{k} h_{m_k}(x_{n_k}) \\
&\leqslant \frac{2}{k} h_{m_k}(x_{n_k}) \quad \text{(by (2.7))} \\
&\leqslant \frac{2}{k} h(x_{n_k}). \quad \blacksquare
\end{aligned}$$

3. QUASICONTINUOUS SCALE AND IDEALS IN $\mathcal{M}(\mathcal{A})$

Kucerovsky and Perera introduced in [27] the notion of quasicontinuity of the scale for simple, separable C^*-algebras of real rank zero in terms of quasitraces. In [22] we studied this notion in terms of traces.

DEFINITION 3.1 ([22], Definition 2.10). Let $\mathcal{A}$ be a simple, σ-unital C^*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$. The *scale* $\mathcal{S} := \widehat{1_{\mathcal{M}(\mathcal{A})}}$ of $\mathcal{A}$ is said to be *quasicontinuous* if:

(i) the set $F_\infty := \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) : \mathcal{S}(\tau) = \infty\}$ is finite (possibly empty) and hence the face $\mathrm{co}(F_\infty)$ is closed;

(ii) the complementary face F'_∞ of $\mathrm{co}(F_\infty)$ is closed (possibly empty);

(iii) the restriction $\mathcal{S}|_{F'_\infty} : F'_\infty \to (0, \infty]$ of the scale $\mathcal{S}$ to F'_∞ is continuous and hence finite-valued.

As we have remarked in Definition 2.10 of [22], while the scale function $\mathcal{S}$ depends on the normalization chosen for $\mathcal{T}(\mathcal{A})$, the quasicontinuity of $\mathcal{S}$ does not. Notice also that when $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$, the scale is necessarily quasicontinuous. If $\mathcal{A}$ is the stabilization of a unital algebra and hence $\mathcal{S}(\tau) = \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$, then $F_\infty = \partial_e(\mathcal{T}(\mathcal{A}))$ and thus the scale is quasicontinuous if and only if $|\partial_e(\mathcal{T}(\mathcal{A}))| < \infty$. Algebras with quasicontinuous scale have interesting regularity properties. Among them, and essential for the main result of this paper is the following theorem.

THEOREM 3.2 ([22], Theorem 6.6). *Let $\mathcal{A}$ be a simple, σ-unital C^*-algebra with quasicontinuous scale and with strict comparison of positive elements by traces. Then strict comparison of positive element by traces holds in $\mathcal{M}(\mathcal{A})$.*

Extending the work by Lin ([32], Theorem 2) on AF-algebras to simple, non-unital, non-elementary C^*-algebras that are the stabilization of a unital algebra, have strict comparison of positive elements by traces, and have a finite tracial extremal boundary, Rørdam ([55],Theorem 4.4) proved that their multiplier algebras have only finitely many ideals ($2^m - 1$ when $m = |\partial_e(\mathcal{T}(\mathcal{A}))|$). In a related result, Kucerovsky and Perera proved ([27], Corollary 3.5) for the case of simple, separable, non-unital, non-elementary C^*-algebras, with real rank zero, stable rank one, strict comparison of positive elements by quasitraces, and quasicontinuous scale, that there are finitely many ideals in $\mathcal{M}(\mathcal{A})$.

The techniques in [22] permit us to extend these results to algebras with quasicontinuous scale.

THEOREM 3.3. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra, with quasicontinuous scale and strict comparison of positive elements by traces. For every $B \in \mathcal{M}(\mathcal{A})_+ \setminus \mathcal{A}$, let $T(B) := \{\tau \in F_\infty : B \in I_\tau\}$ and $I(B)$ be the ideal generated by B. If $T(B) \neq \emptyset$, then $I(B) = \bigcap_{\tau \in T(B)} I_\tau$; if $T(B) = \emptyset$, then $I(B) = \mathcal{M}(\mathcal{A})$.*

To prove Theorem 3.3, we need the following theorem and two lemmas obtained in [22]. For the convenience of the readers and ease of reference we reproduce them here.

THEOREM 3.4 ([22], Theorem 4.2). *Let $\mathcal{B}$ be a σ-unital C^*-algebra and let $T \in \mathcal{M}(\mathcal{B})_+$. Then for every $\varepsilon > 0$ there exist a bi-diagonal series $\sum_1^\infty d_k$ with each $d_k \in \mathcal{B}_+$ and a selfadjoint element $t_\varepsilon \in \mathcal{B}$ with $\|t_\varepsilon\| < \varepsilon$ such that $T = \sum_1^\infty d_k + t_\varepsilon$. The elements d_k can be chosen in $\mathrm{Ped}(\mathcal{B})$.*

For every approximate identity $\{e_n\}$ of $\mathcal{B}$ with $e_{n+1}e_n = e_n$, we can choose d_k and t_ε that satisfy the above conditions and such that for every $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ for which $e_n \sum_N^\infty d_k = 0$.

For the next lemma, notice that in [22] we did set $F(B) = \mathrm{co}\{\tau \in F_\infty : B \notin I_\tau\}$ and then $T(B) = F_\infty \setminus (F(B) \cap \partial_e(\mathcal{T}(\mathcal{A})))$.

LEMMA 3.5 ([22], Lemma 5.1). *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra, with strict comparison of positive elements by traces, let $a_i, b_i \in \mathcal{A}_+$ be such that $\sum_{i=1}^\infty a_i$ and $\sum_{i=1}^\infty b_i$ are two bi-diagonal series in $\mathcal{M}(\mathcal{A})_+$, let F be a closed face of $\mathcal{T}(\mathcal{A})$, F' be its complementary face (either F or F' can be empty), and assume that $|F \cap \partial_e(\mathcal{T}(\mathcal{A}))| < \infty$. Assume also that for some $\varepsilon, \delta, \alpha > 0$ we have:*

(i) $\left(\sum_{i=1}^\infty b_i - \delta\right)_+ \notin \mathcal{A}$;

(ii) $d_\tau\left(\left(\sum_{i=m}^\infty b_i - \delta\right)_+\right) = \infty \; \forall \, \tau \in F,\ m \in \mathbb{N}$;

(iii) $d_\tau\left(\left(\sum_{i=1}^\infty a_i - \varepsilon\right)_+\right) + \alpha \leqslant d_\tau\left(\left(\sum_{i=1}^\infty b_i - \delta\right)_+\right) < \infty \; \forall \, \tau \in F'$;

(iv) $d_\tau\Big(\Big(\sum_{i=m}^{n} b_i - \delta\Big)_+\Big) \to d_\tau\Big(\Big(\sum_{i=m}^{\infty} b_i - \delta\Big)_+\Big)$ *uniformly on* F', $\forall\, m \in \mathbb{N}$;

(v) $d_\tau\Big(\Big(\sum_{i=n}^{\infty} a_i - \varepsilon\Big)_+\Big) \to 0$ *uniformly on* F'.

Then $\Big(\sum_{i=1}^{\infty} a_i - 2\varepsilon\Big)_+ \preceq \Big(\sum_{i=1}^{\infty} b_i - \delta'\Big)_+$ *for all* δ' *with* $0 < \delta' < \delta$.

LEMMA 3.6 ([22], Lemma 6.4). *Let* $\mathcal{A}$ *be a simple,* σ*-unital, non-unital,* C^**-algebra,* $P \in \mathcal{M}(\mathcal{A})$ *be a projection,* $K \subset \mathcal{T}(\mathcal{A})$ *be a closed set such that* $\widehat{P}|_K$ *is continuous, and let* $\sum_{j=1}^{\infty} A_j$ *be the strictly converging sum of elements* $A_j \in (P\mathcal{M}(\mathcal{A})P)_+$. *Assume furthermore that there exists an increasing approximate identity* $\{e_n\}_{n=1}^{\infty}$ *for* $(P\mathcal{A}P)_+$ *with* $e_{n+1}e_n = e_n$ *for all* $n \in \mathbb{N}$ *such that for all* $m \geqslant 1$, *there exists* $N \in \mathbb{N}$ *with* $e_m \sum_{j=N}^{\infty} A_j = 0$. *Then for every* $\delta \geqslant 0$,

(i) $d_\tau\Big(\Big(\sum_{j=n}^{\infty} A_j - \delta\Big)_+\Big) \to 0$ *uniformly on* K;

(ii) $d_\tau\Big(\Big(\sum_{j=1}^{n} A_j - \delta\Big)_+\Big) \to d_\tau\Big(\Big(\sum_{j=1}^{\infty} A_j - \delta\Big)_+\Big)$ *uniformly on* K.

The above two lemmas are based on the following result which we also will need in our paper.

PROPOSITION 3.7 ([22], Proposition 4.4). *Let* $\mathcal{B}$ *be a* C^**-algebra,* $A = \sum_1^{\infty} A_n$, $B = \sum_1^{\infty} B_n$ *where* $A_n, B_n \in \mathcal{M}(\mathcal{B})_+$, $A_nA_m = 0$, $B_nB_m = 0$ *for* $n \neq m$ *and the two series converge in the strict topology. If* $A_n \preceq (B_n - \delta)_+$ *for some* $\delta > 0$ *and for all* n, *then* $A \preceq (B - \delta')_+$ *for all* $0 < \delta' < \delta$.

The proof of Theorem 3.3, which is based on the above two lemmas, is inspired by the proof of Theorems 5.3 and 6.6 in [22].

Proof of Theorem 3.3. Assume that $T := T(B) \neq \emptyset$, i.e., $B \in I_\tau$ for some $\tau \in F_\infty$, leaving to the reader the similar (and simpler) case when $T = \emptyset$. Set

$$F = \mathrm{co}\{F_\infty \setminus T\}.$$

Then

$$|F \cap \partial_e(\mathcal{T}(\mathcal{A}))| \leqslant |F_\infty| < \infty. \tag{3.1}$$

Being finite dimensional, the face F is closed and hence split, i.e., $\mathcal{T}(\mathcal{A}) = F \dot{+} F'$ where F' is the complementary face of F. Then

$$F' = \mathrm{co}(T) \dot{+} F'_\infty, \tag{3.2}$$

is also closed since F'_∞ is closed by hypothesis and $\mathrm{co}(T)$ is closed because it is finite dimensional. Since $I(B) \subset \bigcap\{I_\tau : \tau \in T\}$, we need to prove that if

$A \in \mathcal{M}(\mathcal{A})_+$ and $A \in I_\tau$ for all $\tau \in T$, then $A \in I(B)$. We can assume that $\|A\| = \|B\| = 1$ and by using Theorem 3.4, we reduce to the case that $A = \sum_{k=1}^{\infty} a_k$ and $B = \sum_{k=1}^{\infty} b_k$ are bidiagonal series for an approximate identity $\{e_n\}_{n=1}^{\infty}$ and for all $m \geqslant 1$, there exists $N \in \mathbb{N}$ with $e_m \sum_{k=N}^{\infty} a_k = 0$ and $e_m \sum_{k=N}^{\infty} b_k = 0$. Since A decomposes into the sum of two diagonal series $A = \sum_{k=1}^{\infty} a_{2k-1} + \sum_{k=1}^{\infty} a_{2k}$, to simplify notations we can assume that A is diagonal. Choose $\delta > 0$ such that

$$(B-\delta)_+ \notin \mathcal{A} \quad \text{and} \quad (B-2\delta)_+ \notin I_\tau \quad \forall \tau \in F_\infty \setminus T. \tag{3.3}$$

Let $\varepsilon > 0$. Since

$$\left(A - \frac{\varepsilon}{2}\right)_+ + \left(I_{\mathcal{M}(\mathcal{A})} - \left(A - \frac{\varepsilon}{2}\right)_+\right) = I_{\mathcal{M}(\mathcal{A})},$$

$(\widehat{A - \frac{\varepsilon}{2}})_+$ is complemented under the scale $\mathcal{S}$ and hence it is continuous on F'_∞. As it is continuous also on the finite dimensional face $\mathrm{co}(T)$, it follows that

$$\left(\widehat{A - \frac{\varepsilon}{2}}\right)_+ = \sum_{k=1}^{\infty} \left(\widehat{a_k - \frac{\varepsilon}{2}}\right)_+ \in \mathrm{Aff}(F')_{++}.$$

By Dini's theorem, the series $\sum_{k=1}^{\infty} (\widehat{a_k - \frac{\varepsilon}{2}})_+$ converges uniformly on F'. Let

$$2\alpha := \min\{d_\tau((B-\delta)_+) : \tau \in \mathcal{T}(\mathcal{A})\}.$$

Choose N such that $\sum_{k=N}^{\infty} (\widehat{a_k - \frac{\varepsilon}{2}})_+(\tau) \leqslant \frac{\varepsilon}{2}\alpha$ for all $\tau \in F'$. Then

$$d_\tau\Big(\sum_{k=N}^{\infty} a_k - \varepsilon\Big)_+ = \sum_{k=N}^{\infty} d_\tau(a_k - \varepsilon)_+ \leqslant \frac{2}{\varepsilon} \sum_{k=N}^{\infty} \tau\Big(a_k - \frac{\varepsilon}{2}\Big)_+ \leqslant \alpha \quad \forall\, \tau \in F'$$

and thus

$$d_\tau\Big(\sum_{k=N}^{\infty} a_k - \varepsilon\Big)_+ + \alpha \leqslant 2\alpha \leqslant d_\tau((B-\delta)_+) \quad \forall \tau \in F'. \tag{3.4}$$

Now we are in the position to verify that all the hypotheses (i)–(v) of Lemma 3.5 are satisfied for the diagonal series $A_N = \sum_{k=N}^{\infty} a_k$, the bidiagonal series $B = \sum_{k=1}^{\infty} b_k$, the face F, and the scalars ε, δ, and α.

By (3.3), the hypothesis (i) of Lemma 3.5 holds and also $(B-2\delta)_+ \notin I_\tau$ for every $\tau \in F$. Since by (1.10)

$$d_\tau\Big(\Big(\sum_{k=1}^{\infty} b_k - 2\delta\Big)_+\Big) \leqslant d_\tau\Big(\Big(\sum_{k=1}^{m-1} b_k - \delta\Big)_+\Big) + d_\tau\Big(\Big(\sum_{k=m}^{\infty} b_k - \delta\Big)_+\Big)$$

and since

$$d_\tau\Big(\Big(\sum_{k=1}^{m-1} b_k - \delta\Big)_+\Big) \leqslant \frac{2}{\delta}\tau\Big(\Big(\sum_{k=1}^{m-1} b_k - \frac{\delta}{2}\Big)_+\Big) < \infty$$

as $\sum\limits_{k=1}^{m-1} b_k \in \mathcal{A}$, it follows that

$$d_\tau\Big(\Big(\sum_{k=m}^{\infty} b_k - \delta\Big)_+\Big) = \infty \quad \forall \tau \in F, m \in \mathbb{N} \tag{3.5}$$

which establishes hypothesis (ii) of Lemma 3.5. Hypothesis (iii) was established in (3.4). Since $\widehat{1_{\mathcal{M}(\mathcal{A})}} = \mathcal{S}$ is by hypothesis continuous on F'_∞ and since $\sum\limits_{k=N}^{\infty} a_k$ converge strictly, it follows from Lemma 3.6 that $d_\tau\Big(\Big(\sum\limits_{k=m}^{\infty} a_k - \varepsilon\Big)_+\Big) \to 0$ uniformly on F'_∞. By Lemma 3.2 of [22], $d_\tau\Big(\Big(\sum\limits_{k=m}^{\infty} a_k - \varepsilon\Big)_+\Big) \to 0$ for every trace $\tau \in T$. Since T is finite, $d_\tau\Big(\Big(\sum\limits_{k=n}^{\infty} a_k - \varepsilon\Big)_+\Big) \to 0$ uniformly also on $\mathrm{co}(T)$ and hence by (3.2) the convergence is uniform also on F'.

By the same argument, for every $m \in \mathbb{N}$, the strict convergence of

$$\Big(\sum_{k=m}^{n} b_k - \delta\Big)_+ \to \Big(\sum_{k=m}^{\infty} b_k - \delta\Big)_+$$

implies the uniform convergence over F' of

$$d_\tau\Big(\Big(\sum_{k=m}^{n} b_k - \delta\Big)_+\Big) \to d_\tau\Big(\Big(\sum_{k=m}^{\infty} b_k - \delta\Big)_+\Big).$$

Thus conditions (v) and (iv) of Lemma 3.5 are also established. Therefore,

$$\Big(\sum_{k=N}^{\infty} a_k - 2\varepsilon\Big)_+ \preceq \sum_{k=1}^{\infty} b_k$$

and hence $\Big(\sum\limits_{k=N}^{\infty} a_k - 2\varepsilon\Big)_+ \in I(B)$. Since

$$(A - 2\varepsilon)_+ = \Big(\sum_{k=1}^{\infty} a_k - 2\varepsilon\Big)_+ = \Big(\sum_{k=1}^{N-1} a_k - 2\varepsilon\Big)_+ + \Big(\sum_{k=N}^{\infty} a_k - 2\varepsilon\Big)_+$$

and $\Big(\sum\limits_{k=1}^{N-1} a_k - 2\varepsilon\Big)_+ \in \mathcal{A} \subset I(B)$, it follows that $(A - 2\varepsilon)_+ \in I(B)$. As ε is arbitrary, we conclude that $A \in I(B)$. ∎

As a consequence we obtain the following corollary.

COROLLARY 3.8. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra, with strict comparison of positive elements by traces, and with quasicontinuous* *scale, and let* $m := |F_\infty|$*. Then* $\mathcal{M}(\mathcal{A})$ *has precisely* $2^m - 1$ *proper ideals properly containing* $\mathcal{A}$*, each being an intersection of ideals* I_τ *for* $\tau \in F_\infty$*.*

Notice that if $\mathcal{A} = \mathcal{K}$ then $m = 1$ but there are no proper ideals properly containing $\mathcal{A}$, thus for the exact count of the ideals in $\mathcal{M}(\mathcal{A})$ we need indeed to assume that $\mathcal{A}$ is non-elementary.

4. PROJECTION SURJECTIVITY AND INJECTIVITY

We find it convenient to introduce the following terminology for properties that have appeared in various forms in the study of multiplier algebras of C^*-algebras.

DEFINITION 4.1. Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra, with non empty tracial simplex $\mathcal{T}(\mathcal{A})$.

(i) $\mathcal{A}$ is 1-projection surjective if for every $f \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++}$ that is complemented under $\mathcal{S} = \widehat{1_{\mathcal{M}(\mathcal{A})}}$ (i.e., there is $g \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++} \sqcup \{0\}$ such that $f + g = \mathcal{S}$) there is a projection $P \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ such that $f = \widehat{P}$.

(ii) $\mathcal{A}$ is 1-projection injective if $P \sim Q$ whenever $P, Q \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ are projections such that $\widehat{P} = \widehat{Q}$.

(iii) $\mathcal{A}$ is n-projection surjective (respectively, n-projection injective) if the algebra $M_n(\mathcal{A})$ is 1-projection surjective (respectively, 1-projection injective).

(iv) $\mathcal{A}$ is projection surjective and injective if it is 1-projection surjective and 2-projection injective.

Notice that $\mathcal{K}$ is obviously not 1-projection surjective, thus whenever we assume 1-projection surjectivity it is redundant to require that the algebra be non-elementary. We start with some simple relations between n-projection surjectivity and m-projection injectivity for various m and n.

LEMMA 4.2. *Let* $\mathcal{A}$ *be simple,* σ*-unital, non-unital* C^**-algebra, with nonempty and metrizable tracial simplex* $\mathcal{T}(\mathcal{A})$*.*

(i) *If* $\mathcal{A}$ *is* n*-projection surjective for some* $n \in \mathbb{N}$*, then it is* kn*-projection surjective for every* $k \in \mathbb{N}$ *and* $\mathcal{A} \otimes \mathcal{K}$ *is* 1*-projection surjective.*

(ii) *If* $\mathcal{A}$ *is* n*-projection injective (respectively,* $\mathcal{A} \otimes \mathcal{K}$ *is* 1*-projection injective), then it is* k*-projection injective for every* $k < n$ *(respectively, every* $k \in \mathbb{N}$*).*

(iii) *Let* $\mathcal{A}$ *be* n*-projection surjective (respectively,* $\mathcal{A} \otimes \mathcal{K}$ *is* 1*-projection surjective). If* $\mathcal{A}$ *is* 2n*-projection injective (respectively,* $\mathcal{A} \otimes \mathcal{K}$ *is* 1*-projection injective), then* $\mathcal{A}$ *is* 1*-projection surjective.*

(iv) *If* $\mathcal{A} \otimes \mathcal{K}$ *is* 1*-projection injective and surjective, then* $\mathcal{A}$ *is* n*-projection injective and surjective for every* n*.*

Proof. (i) Assume that $\mathcal{A}$ is n-projection surjective, let $k \in \mathbb{N}$, and let $f + g = kn\mathcal{S}$ for some $f \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ and $g \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++} \sqcup \{0\}$. Then $\frac{f}{k}$ is complemented under $n\mathcal{S}$ and hence there is a projection $P \in \mathcal{M}(M_n(\mathcal{A})) \setminus M_n(\mathcal{A}) = M_n(\mathcal{M}(\mathcal{A})) \setminus M_n(\mathcal{A})$ such that $\widehat{P} = \frac{f}{k}$. Then $Q := \bigoplus_{j=1}^{k} P \in M_{kn}(\mathcal{M}(\mathcal{A})) \setminus M_{kn}(\mathcal{A})$ and $\widehat{Q} = k\widehat{P} = f$. Thus $\mathcal{A}$ is kn-projection surjective. We prove now that $\mathcal{A} \otimes \mathcal{K}$ is 1-projection surjective. Let $f \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$. By the metrizability of $\mathcal{T}(\mathcal{A})$ and Proposition 2.1, $f = \sum_{j=1}^{\infty} f_j$ with $f_j \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))_{++}$. For every j, choose $n_j > \frac{\max f_j}{\min \mathcal{S}}$ and n_j divisible by n. Then $f_j < n_j\mathcal{S}$ and since f_j is continuous, f_j is complemented under $n_j\mathcal{S}$. By the first part of the proof, $\mathcal{A}$ is n_j-projection surjective, hence there is a projection $P_j \in \mathcal{M}(M_{n_j}(\mathcal{A})) \setminus M_{n_j}(\mathcal{A})$ such that $\widehat{P_j} = f_j$. Construct a strictly converging series of mutually orthogonal projections $\widetilde{P_j}$ in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ such that $\widetilde{P_j} \sim P_j$ and the series $P = \sum_{j=1}^{\infty} \widetilde{P_j}$ converges strictly. Then $P \notin \mathcal{A} \otimes \mathcal{K}$ and

$$\widehat{P} = \sum_{j=1}^{\infty} \widehat{\widetilde{P_j}} = \sum_{j=1}^{\infty} \widehat{P_j} = \sum_{j=1}^{\infty} f_j = f.$$

(ii) Assume that $\mathcal{A}$ is n-projection injective, let $k \leqslant n$ and let P, Q be projections in $\mathcal{M}(M_k(\mathcal{A})) \setminus M_k(\mathcal{A})$ with $\widehat{P} = \widehat{Q}$. Then $P \oplus 0$, $Q \oplus 0$ are projections belonging to $\mathcal{M}(M_n(\mathcal{A})) \setminus M_n(\mathcal{A})$ and $\widehat{P \oplus 0} = \widehat{Q \oplus 0}$. Then $P \oplus 0 \sim Q \oplus 0$ and hence $P \sim Q$.

(iii) Assume that $\mathcal{A}$ is n-projection surjective and $2n$-projection injective and let $f \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ and $g \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++} \sqcup \{0\}$ such that $f + g = \mathcal{S}$. Then $f + (n-1)f + ng = n\mathcal{S}$, i.e., f is complemented under $n\mathcal{S}$ and so is g. Thus there are projections $P, Q \in M_n(\mathcal{M}(\mathcal{A}))$ with $\widehat{P} = f$ and $\widehat{Q} = g$. Hence there are mutually orthogonal projection $P', Q' \in M_{2n}(\mathcal{M}(\mathcal{A}))$ with $P' \sim P$, $Q' \sim Q$. But then

$$\widehat{P' \oplus Q'} = \widehat{P} + \widehat{Q} = f + g = \mathcal{S} = \widehat{1_{\mathcal{M}(\mathcal{A})}}.$$

Since $M_{2n}(\mathcal{A})$ is 1-projection injective by hypothesis, $P' + Q' \sim 1_{\mathcal{M}(\mathcal{A})}$. Thus we can choose P', Q' with $P' + Q' = 1_{\mathcal{M}(\mathcal{A})}$ and hence $P', Q' \in \mathcal{M}(\mathcal{A})$. In particular, $\widehat{P'} = f$.

The case when $\mathcal{A} \otimes \mathcal{K}$ is 1-projection surjective and 1-projection injective is similar and is left to the reader.

(iv) Follows immediately from (ii) and (iii). ∎

In all cases where we could determine projection surjectivity and injectivity, the property holds for every n. Does 1-projection injectivity imply 2-projection injectivity and hence n-projection injectivity for every n? The answer is affirmative in the case when the algebra has real rank zero.

LEMMA 4.3. *Let $\mathcal{A}$ be a simple σ-unital, non-unital, non-elementary C^*-algebra of real rank zero and let $P \in M_2(\mathcal{M}(\mathcal{A}))$ be a projection. Then $P \sim P'' \oplus P''$ for some projection $P'' \in \mathcal{M}(\mathcal{A})$.*

Proof. Let $\{e_n\}$ be an increasing approximate identity for $\mathcal{A}$ consisting of projections and set $e_0 = 0$. Then $\{e_n \oplus e_n\}$ is an increasing approximate identity of projections for $M_2(\mathcal{A})$. By Theorem 4.1 of [62], and passing if necessary to a subsequence of $\{e_n\}$, we can find projections $p_n \leqslant (e_n - e_{n-1}) \oplus (e_n - e_{n-1})$ such that $P \sim \sum_{n=1}^{\infty} p_n$ and the series converges in the strict operator topology. By Theorem 3.3 of [62], we can further assume that for all n,

$$p_n = \begin{pmatrix} s_n & 0 \\ 0 & s_n + r_n \end{pmatrix}$$

for some projections in $\mathcal{A}$, $s_n, r_n \leqslant e_n - e_{n-1}$. By a slight adjustment of the proof, we can also assume that $r_n \neq 0$ for all n. By Theorem 1.1 of [63] we can approximately halve r_1, that is decompose it into the sum of three mutually orthogonal projections $r_1 = t_1 + t_1' + q_2'$ where $t_1 \sim t_1'$ and $q_2' \sim q_2 \lneqq r_2$. Then since

$$\begin{pmatrix} 0 & 0 \\ 0 & t_1' \end{pmatrix} \sim \begin{pmatrix} t_1 & 0 \\ 0 & 0 \end{pmatrix}$$

and both are orthogonal to $\begin{pmatrix} s_1 & 0 \\ 0 & s_1 + t_1 \end{pmatrix}$, it follows that

$$\begin{pmatrix} s_1 & 0 \\ 0 & s_1 + t_1 + t_1' \end{pmatrix} \sim \begin{pmatrix} s_1 + t_1 & 0 \\ 0 & s_1 + t_1 \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} 0 & 0 \\ 0 & q_2' \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & q_2 \end{pmatrix} \sim \begin{pmatrix} q_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and}$$
$$p_1 = \begin{pmatrix} s_1 & 0 \\ 0 & s_1 + t_1 + t_1' + q_2' \end{pmatrix} \sim \widetilde{p}_1 := \begin{pmatrix} s_1 + t_1 & 0 \\ 0 & s_1 + t_1 \end{pmatrix} + \begin{pmatrix} q_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Next, approximately halve $r_2 - q_2 = t_2 + t_2' + q_3'$ with $t_2 \sim t_2'$ and $q_3' \sim q_3 \lneqq r_3$. Then reasoning as above,

$$p_2 = \begin{pmatrix} s_2 & 0 \\ 0 & s_2 + t_2 + q_2 + t_2' + q_3' \end{pmatrix} \sim p_2' := \begin{pmatrix} s_2 + t_2 & 0 \\ 0 & s_2 + t_2 + q_2 \end{pmatrix} + \begin{pmatrix} q_3 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $p_1 p_2 = 0$ and $p_1' p_2' = 0$, it follows that $p_1 + p_2 \sim p_1' + p_2'$, namely

$$p_1 + p_2 \sim \begin{pmatrix} s_1 + t_1 & 0 \\ 0 & s_1 + t_1 \end{pmatrix} + \begin{pmatrix} s_2 + t_2 + q_2 & 0 \\ 0 & s_2 + t_2 + q_2 \end{pmatrix} + \begin{pmatrix} q_3 & 0 \\ 0 & 0 \end{pmatrix}.$$

Iterating, we find a sequence of mutually orthogonal projections

$$s_n, t_n, q_n \leqslant e_n - e_{n-1}$$

such that for every n

$$p_n \sim p'_n := \begin{pmatrix} s_n + t_n & 0 \\ 0 & s_n + t_n + q_n \end{pmatrix} + \begin{pmatrix} q_{n+1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $p'_n \leqslant e_{n+1} - e_{n-1}$, the series $P' := \sum_{n=1}^{\infty} p'_n$ converges strictly. Choose partial isometries $v_n \in M_2(\mathcal{A})$ such that $p_n = v_n^* v_n$ and $p'_n = v_n v_n^*$. Then the series $V := \sum_{n=1}^{\infty} p'_n v_n p_n$ also converges strictly to the partial isometry $V \in M_2(\mathcal{M}(\mathcal{A}))$. Then $P = V^*V$, $P' = VV^*$, and hence $P \sim P'$ within $M_2(\mathcal{M}(\mathcal{A}))$. Setting $q_1 := 0$ we have for every k that

$$\sum_{n=1}^{k} p'_n = \sum_{n=1}^{k} \begin{pmatrix} s_n + t_n + q_n & 0 \\ 0 & s_n + t_n + q_n \end{pmatrix} + \begin{pmatrix} q_{k+1} & 0 \\ 0 & 0 \end{pmatrix}$$

and hence

$$P' = \sum_{n=1}^{\infty} \begin{pmatrix} s_n + t_n + q_n & 0 \\ 0 & s_n + t_n + q_n \end{pmatrix}.$$

Let

$$P'' = \sum_{n=1}^{\infty} s_n + t_n + q_n,$$

then $P'' \in \mathcal{M}(\mathcal{A})$ is a projection and $P' = P'' \oplus P''$, which completes the proof. ∎

PROPOSITION 4.4. *Let $\mathcal{A}$ be a simple, separable, non-unital, non-elementary C^*-algebra of real rank zero . Then if $\mathcal{A}$ is* 1*-projection injective it is also n-projection injective for all $n \geqslant 1$.*

Proof. In view of Lemma 4.2, it is sufficient to prove the statement for $n = 2$. Assume that P, Q are projections in $\mathcal{M}(M_2(\mathcal{A}))$ and that $\widehat{P} = \widehat{Q}$. By Lemma 4.3,

$$P \sim P'' \oplus P'', \quad Q \sim Q'' \oplus Q''$$

for some projections P'' and Q'' in $\mathcal{M}(\mathcal{A})$. Hence $\widehat{P''} = \widehat{Q''}$ and hence $P'' \sim Q''$ whence $P \sim Q$. ∎

We proceed now to ascertain projection surjectivity and injectivity for some important classes of simple, σ-unital, non-unital, non-elementary C^*-algebras.

We start with the case of real rank zero algebras with stable rank one which was long well-known ([13], [19], [35], [36], [40], [60]). A nice exposition can be found in Theorem 3.9 of [48].

THEOREM 4.5. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra, with real rank zero, stable rank one, and such that $\mathcal{A}$ has strict comparison of positive element by traces. Then $\mathcal{A}$ is n-projection surjective and n-projection injective for every n.*

Proof. The hypotheses in Theorem 3.9 of [48] on the C^*-algebra $\mathcal{A}$ are that $\mathcal{A}$ is simple, σ-unital, non-unital, non-elementary, has real rank zero, stable rank one, and that the monoid $V(\mathcal{A})$ of equivalent classes of projections in $M_\infty(\mathcal{A})$ is strictly unperforated.

The latter hypothesis is equivalent to the condition that $\mathcal{A}$ has strict comparison of positive elements by 2-quasitraces (see Lemma 3.5, Corollary 3.10 and its proof in [47]). Obviously, strict comparison of positive elements by traces implies strict comparison by quasitraces, so the hypotheses of Theorem 3.9 in [48] are satisfied. The thesis of Theorem 3.9 in [48] is expressed in terms of a monoid isomorphism of $V(\mathcal{M}(\mathcal{A}))$ which implies n-projection surjectivity and injectivity of $\mathcal{A}$ for every n. ∎

The condition that $\mathcal{A}$ has real rank zero can be dropped in the case when $\mathcal{A}$ is separable.

THEOREM 4.6 ([42], Proposition 4.2). *Let $\mathcal{A}$ be a simple, non-unital, separable C^*-algebra, with stable rank one, and with strict comparison of positive elements by traces. Then $\mathcal{A}$ is 1-projection injective.*

Proposition 4.2 of [42] assumes the algebra to be stable, but an examination of its proof shows that stability is not necessary. Next we consider projection surjectivity.

THEOREM 4.7 ([42], Corollary 4.6). *Let $\mathcal{A}$ be a simple, separable C^*-algebra with non empty tracial simplex $\mathcal{T}(\mathcal{A})$ such that*

(•) *for every bounded function $f \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ there exists an $a \in (\mathcal{A} \otimes \mathcal{K})_+$ which is not Cuntz equivalent to a projection and such that $\widehat{[a]} = f$.*

Then $\mathcal{A} \otimes \mathcal{K}$ is 1-projection surjective.

The property (•) in the above theorem plays an important role in the study of Cuntz semigroups and is succinctly formulated in [7] as the surjectivity of the map $\iota : W(\mathcal{A})_+ \mapsto \mathrm{LAff}_b(\mathcal{T}(\mathcal{A}))_{++}$ where $W(\mathcal{A})_+$ is the sub-semigroup of equivalence classes of elements of $M_\infty(\mathcal{A})_+$ not equivalent to projections and $\iota[a](\tau) = d_\tau(a)$ for all $a \in M_\infty(\mathcal{A})_+$ and $\tau \in \mathcal{T}(\mathcal{A})$.

The property (•) was first shown to hold for C^*-algebras that are simple, unital, separable and are either exact, stably finite, and $\mathcal{Z}$-stable ([7], Theorem 5.5) or are infinite-dimensional AH-algebras of stable rank one with strict comparison of positive elements ([7], Theorem 5.3).

Condition (•) is also satisfied by some stably projectionless algebras, e.g., the monotracial Razak algebra (see for example [54]).

The $\mathcal{Z}$-stability condition in Theorem 5.5 of [7] was recently replaced by the weaker condition of having stable rank one.

THEOREM 4.8 ([57], Theorem 8.11). *Let $\mathcal{A}$ be a simple, separable, unital, non-elementary C^*-algebra with stable rank one. Then for every $f \in \mathrm{LAff}(\mathcal{QT}(\mathcal{A}))_{++}$ there exists $x \in \mathcal{A} \otimes \mathcal{K}_+$ such that $d_\tau(x) = f(\tau)$ for all $\tau \in \mathcal{Q}T(\mathcal{A})$.*

Here $\mathcal{QT}(\mathcal{A})$ denotes the Choquet simplex of 2-quasitraces of $\mathcal{A}$, which contains $\mathcal{T}(\mathcal{A})$ as a face. Notice that the hypothesis can be reformulated by asking $\mathcal{A}$ to be stable and to contain a non-zero projection.

The statement of this theorem does not state explicitly that x can be chosen to be not equivalent to a projection. It is easy to see that this can be done when $\mathcal{A}$ has real rank zero.

PROPOSITION 4.9. *Let $\mathcal{A}$ be a simple, real rank zero C^*-algebra with non-empty tracial simplex $\mathcal{T}(\mathcal{A})$. Then for every projection $q \in \mathcal{A}_+$ there exists $a \in \mathcal{A}_+$ such that $a \leqslant q$, a is not equivalent to a projection, and $\widehat{[a]} = \widehat{q}$.*

Proof. We can assume that $\mathcal{A}$ is nonelementary, as the elementary case is trivial. By Theorem 1.1 of [63], we can decompose q into a sum of projections $q = q_1 + q_1'$ with $0 \neq q_1' \preceq q_1$ and hence with $\widehat{q}_1' \leqslant \frac{1}{2}\widehat{q}$. Then decompose similarly $q_1' = q_2 + q_2'$ with $0 \neq \widehat{q}_2' \leqslant \frac{1}{2^2}\widehat{q}$. Iterating the process, we find an infinite sequence of mutually orthogonal projections $q_n \leqslant q$ such that

$$q - \sum_{n=1}^{m} q_n = q_m' \quad \text{and} \quad \widehat{q}_m' \leqslant \frac{1}{2^m}\widehat{q}.$$

Since $\widehat{q}$ is continuous, it follows that $\widehat{q} = \sum\limits_{n=1}^{\infty} \widehat{q}_n$. Then $a := \sum\limits_{n=1}^{\infty} \frac{1}{n} q_n \in \mathcal{A}_+, a \leqslant q, \widehat{[a]} = \widehat{q}$ and a is not equivalent to a projection because 0 is an accumulation point in the spectrum of a. ∎

The same result holds also for (stable, separable) algebras that do not have real rank zero due to the work [4], presented in Proposition 2.9 of [57], that states that for a countably based, simple, stably finite, non-elementary Cuntz semigroup S satisfying axioms (O5) and (O6) (and hence for the concrete Cuntz semigroup of the stable C^*-algebra $\mathcal{A}$ considered) for every $[a] \in S$ there is $[x] \in S$, $[x] \leqslant [a]$ and $[x]$ *soft* (and hence x is not equivalent to a projection) such that $\tau(x) = \tau(a)$ holds for all 2-quasitraces and hence a fortiori for all traces τ. We summarize this result for our setting.

PROPOSITION 4.10. *Let $\mathcal{A}$ be a simple, separable, non-elementary, stable C^*-algebra with non-empty tracial simplex $\mathcal{T}(\mathcal{A})$. Then for every $x \in \mathcal{A}_+$ there exists $a \in \mathcal{A}_+$ such that $a \preceq x$, a is not equivalent to a projection, and $\widehat{[a]} = \widehat{[x]}$.*

Combining Theorem 4.8, Proposition 4.10, and Theorem 4.7 we obtain the following corollary.

COROLLARY 4.11. *Let $\mathcal{A}$ be a simple, separable, unital, non-elementary C^*-algebra with stable rank one. Then $\mathcal{A} \otimes \mathcal{K}$ is* 1*-projection surjective. If furthermore $\mathcal{A} \otimes \mathcal{K}$ has strict comparison of positive elements by traces, then $\mathcal{A} \otimes \mathcal{K}$ is also* 1*-projection injective.*

Thus the class of C^*-algebras $\mathcal{A}$ with both projection injectivity and projection surjectivity for $\mathcal{A} \otimes \mathcal{K}$ includes among others:

(i) The Jiang–Su algebra $\mathcal{Z}$ and more generally, all simple, unital, separable, exact, stably finite $\mathcal{Z}$-stable C^*-algebras. (The proof of strict comparison and stable rank one is found in [56].)

We note that this is a large class and includes all simple unital finite nuclear C^*-algebras that have recently been classified in the Elliott program ($\mathcal{Z}$-stability is an axiom) (see, e.g., [17] and the references therein).

Special cases (all from the Elliott program) include important and interesting C^*-algebras like irrational rotation algebras, simple C^*-algebras coming from Cantor minimal systems, and crossed products of the form $C(X) \times_\alpha \mathcal{Z}$, where X is a compact metric space with finite topological dimension, and $\alpha : X \to X$ is a minimal homeomorphism. For all of the above algebras, the proof of simplicity can be found in [10]. A proof that the above minimal crossed products (including from minimal Cantor systems) are $\mathcal{Z}$-stable can be found in [16]. That the irrational rotation algebra has stable rank one is proven in [51]. That the irrational rotation algebras have strict comparison can be found in [3].

(ii) The reduced C^*-algebra of the free group on infinitely many generators $C^*_{\mathrm{r}}(F_\infty)$. That this C^*-algebra is simple can be found in [50]. That it has stable rank one can be found in [11]. A proof of strict comparison can be found in [53].

(iii) The monotracial Razak algebra (stably projectionless). The relevant properties are proven in [52].

Notice that all the C^*-algebras listed above also have strict comparison of positive elements (by traces). We will prove that under the additional hypothesis of separability and stability, strict comparison is indeed necessary for projection surjectivity and injectivity. We need first a simple consequence of the definition of projection surjectivity and injectivity and of the argument in the proof of Lemma 4.2(iii) that will be useful throughout the rest of the paper.

LEMMA 4.12. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary, C^*-algebra, and let P, Q be projections in $\mathcal{M}(\mathcal{A})$.*

(i) *If $P \preceq Q$ then $\widehat{P}$ is complemented under $\widehat{Q}$.*

Assume now that $\mathcal{A}$ is projection surjective and injective and that $Q \notin \mathcal{A}$.

(ii) *If $f + g = \widehat{Q}$ for some $f, g \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++}$, then there is a decomposition of $Q = P_1 + P_2$ into projections $P_1, P_2 \notin \mathcal{A}$ with $\widehat{P_1} = f$ and $\widehat{P_2} = g$.*

(iii) *If $\widehat{P}$ is complemented under $\widehat{Q}$, then $P \preceq Q$.*

Proof. (i) There is a projection $P' \in \mathcal{M}(\mathcal{A})$ with $P \sim P' \leqslant Q$ and hence $\widehat{P} = \widehat{P'}$. Let $P'' = Q - P'$, then $\widehat{Q} = \widehat{P} + \widehat{P''}$ and since $\widehat{P''}$ is either 0 (if $P'' = 0$) or strictly positive (if $P'' \neq 0$), it follows that $\widehat{P}$ is complemented under $\widehat{Q}$.

(ii) Since $f + g + \widehat{(1_{\mathcal{M}(\mathcal{A})} - Q)} = \widehat{1}_{\mathcal{M}(\mathcal{A})} = \mathcal{S}$ both f and g are complemented under $\mathcal{S}$. Thus there are projections $R_1, R_2 \notin \mathcal{A}$ such that $\widehat{R_1} = f$ and $\widehat{R_2} = g$.

Then $\widehat{R_1 \oplus R_2} = g + f = \widehat{Q}$. Since neither $R_1 \oplus R_2 \in M_2(\mathcal{A})$ nor $Q \oplus 0 \in M_2(\mathcal{A})$, by 2-projection injectivity, $R_1 \oplus R_2 \sim Q \oplus 0$ and hence $Q = P_1 + P_2$ for some mutually orthogonal projections $P_1 \sim R_1$ and $P_2 \sim R_2$. Thus $\widehat{P_1} = f$ and $\widehat{P_2} = g$.

(iii) Let $g \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++}$ be such that $\widehat{P} + g = \widehat{Q}$. Reasoning as in the proof of (ii), there is a projection $R_2 \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ such that $\widehat{P \oplus R_2} = \widehat{Q}$. Since neither $P \oplus R_2$ not Q are in $\mathcal{A}$, it follows that $P \oplus R_2 \sim Q \oplus 0$ and hence $P \prec Q$. ∎

Next, we list the following facts that are routine, but for completeness we add a short proof.

LEMMA 4.13. *Let $\mathcal{B}$ be a C^*-algebra.*

(i) *Let $T \in \mathcal{M}(\mathcal{B})_+$, $T_n \in \mathcal{M}(\mathcal{B})_+$ such that $T_n \to T$ strictly. If $a \in \mathcal{B}_+$ and $a \preceq T$, then for every $\varepsilon > 0$ there is an n such that $(a - \varepsilon)_+ \preceq T_n$.*

(i) *Let $Q \in \mathcal{M}(\mathcal{B})$ be a projection and assume that $Q\mathcal{B}Q$ has a strictly positive element b. If $a \in \mathcal{B}_+$ and $a \preceq Q$, then $a \preceq b$.*

Proof. (i) Choose an $X \in \mathcal{M}(\mathcal{B})$ such that $\|a - XTX^*\| < \frac{\varepsilon}{3}$, an $e \in \mathcal{B}_+$ with $\|e\| = 1$ such that $\|a - eae\| < \frac{\varepsilon}{3}$, and an integer n such that $\|eX(T - T_n)X^*e\| < \frac{\varepsilon}{3}$. Then $\|a - eXT_nX^*e\| < \varepsilon$ and hence

$$(a - \varepsilon)_+ \preceq eXT_nX^*e \preceq T_n.$$

(ii) Let $\varepsilon > 0$. Since $b^{1/n}$ converges strictly to Q, by (i) there is an integer n such that $(a - \varepsilon)_+ \preceq b^{1/n} \sim b$. Since ε is arbitrary, then $a \preceq b$. ∎

We need also a standard application of Kasparov's absorption theorem which has appeared in many places over the years (e.g., [28], [42]). The precise form of the argument that we require can be found in Lemma 4.3 and in the proof of Proposition 4.4 in [42].

LEMMA 4.14. *Let $\mathcal{A}$ be a simple, stable, separable C^*-algebra and let $a \in \mathcal{A}_+$. Then there is $a' \in \mathcal{A}_+$ with $a \sim a'$ and with its range projection $R_{a'} \in \mathcal{A}^{**}$ belonging to $\mathcal{M}(\mathcal{A})$. Furthermore, $R_{a'} \in \mathcal{A}$ if and only if a is equivalent to a projection.*

Proof. By Kasparov's absorption theorem and Lemma 4.3 of [42], there is a projection $P \in \mathcal{M}(\mathcal{A})$ such that the Hilbert modules $\overline{a\mathcal{A}}$ and $P\mathcal{A}$ are isomorphic, i.e., there there is a unitary $\Phi : \overline{a\mathcal{A}} \mapsto P\mathcal{A}$. If b is a strictly positive element in $\mathcal{A}$, then $a' := PbP$ is a strictly positive element in $P\mathcal{A}P$ and $R_{a'} = P$. Moreover, $P\mathcal{A} = \overline{a'\mathcal{A}}$. Then by a standard argument (see for example Proposition 4.3 of [45], see also [9], [41]), $a \sim a'$. If $P \in \mathcal{A}$, then $a' \sim P$ and hence a is equivalent to a projection. Conversely, if a equivalent to a projection P, then we can choose $a' = P$. ∎

THEOREM 4.15. *Let $\mathcal{A}$ be a simple, stable, separable, C^*-algebra with projection surjectivity and injectivity. Then $\mathcal{A}$ has strict comparison of positive elements by traces.*

Notice that by our definition, projection surjectivity or injectivity implies that $\mathcal{A}$ has non-empty tracial simplex and, clearly, projection surjectivity implies that $\mathcal{A}$ is non-elementary.

Proof. Let $a, b \in \mathcal{A}_+$ and assume that $d_\tau(a) < d_\tau(b)$ for every $\tau \in \mathcal{T}(\mathcal{A})$ such that $d_\tau(b) < \infty$. Assume without loss of generality that $\|a\| \leqslant 1$. By Lemma 4.14 we can assume that $R_a \in \mathcal{M}(\mathcal{A})$. For the first step of the proof, we construct for every $\varepsilon > 0$ a projection $P \in I_{\text{cont}} \setminus \mathcal{A}$, such that $(a-\varepsilon)_+ \preceq P$ and $\widehat{P} < \widehat{[b]}$.

Consider first the case when $R_a \notin \mathcal{A}$, namely when a is not equivalent to a projection. Since $\widehat{R}_a \in \text{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++}$, by Proposition 2.1(i) we can decompose $\widehat{R}_a$ into the pointwise converging sum $\widehat{R}_a = \sum_{n=1}^{\infty} f_n$ of functions $f_n \in \text{Aff}(\mathcal{T}(\mathcal{A}))_{++}$. By projection surjectivity, we can find projections $R_n'' \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ such that $\widehat{R}_n'' = f_n$ for every n. Since $\mathcal{A}$ is stable, we can find mutually orthogonal projections $R_n' \sim R_n''$ such that $R' = \sum_{n=1}^{\infty} R_n'$ converges strictly. As $\widehat{R}' = \sum_{n=1}^{\infty} f_n = \widehat{R}_a$, by projection injectivity we have $R_a \sim R'$. This provides a strictly converging decomposition of $R_a = \sum_{n=1}^{\infty} R_n$ into projections $R_n \in I_{\text{cont}} \setminus \mathcal{A}$. Let $\varepsilon > 0$. Then by Lemma 4.13, there is an n such that

$$(a-\varepsilon)_+ \preceq P := \sum_{k=1}^{n} R_k.$$

Thus $P \in I_{\text{cont}} \setminus \mathcal{A}$, and $\widehat{P} < \widehat{R}_a = \widehat{[a]} \leqslant \widehat{[b]}$.

Next consider the case when $R_a \in \mathcal{A}$. Then $\widehat{[a]} = \widehat{R}_a$ is continuous, hence

$$\widehat{[b]} - \widehat{R}_a \in \text{LAff}(\mathcal{T}(\mathcal{A}))_{++} \quad \text{and} \quad \mathcal{S} - \widehat{R}_a \in \text{LAff}(\mathcal{T}(\mathcal{A}))_{++}.$$

Let $0 < \alpha < \min(\widehat{[b]} - \widehat{R}_a)$. Then also $\mathcal{S} - \widehat{R}_a - \alpha \in \text{LAff}(\mathcal{T}(\mathcal{A}))_{++}$, hence the constant function α is complemented under $\mathcal{S} - \widehat{R}_a = \widehat{1_{\mathcal{M}(\mathcal{A})} - R_a}$. By Lemma 4.12, there is a projection $P_o \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ with $P_o \leqslant I_{\mathcal{M}(\mathcal{A})} - R_a$ and $\widehat{P}_o = \alpha$. Let $P = R_a + P_o$. Then $P \in I_{\text{cont}} \setminus \mathcal{A}$, $(a-\varepsilon)_+ \leqslant a \leqslant R_a \leqslant P$ and $\widehat{P} < \widehat{[b]}$.

For the second step of the proof, by Proposition 2.9 of [57], there is $c' \in \mathcal{A}_+$, $c' \preceq b$, with $\widehat{[c']} = \widehat{[b]}$ and c' not equivalent to a projection. Again by Lemma 4.14, there is a $c \in \mathcal{A}_+$ with $c \sim c'$ and such that $R_c \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$. Since $\widehat{P} < \widehat{R}_c = \widehat{[b]}$ and $\widehat{P}$ is continuous, it follows that $P \preceq R_c$ (see Corollary 5.4 below). As c is strictly positive in $R_c \mathcal{A} R_c$, it follows by Lemma 4.13 that $(a-2\varepsilon)_+ \preceq c \preceq b$. As ε is arbitrary, it follows that $a \preceq b$. ∎

If $\mathcal{A}$ is just σ-unital and/or if $\mathcal{A}$ is not stable, we cannot invoke Proposition 4.10. However, if $\mathcal{A}$ has real rank zero, we can still prove strict comparison for $\mathcal{A}$.

PROPOSITION 4.16. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra with real rank zero and with projection surjectivity and injectivity. Then $\mathcal{A}$ has strict comparison of positive elements by traces.*

Proof. It is well-known (e.g., see Corollary 3.10 of [47] and its proof) that it suffices to prove that A has strict comparison of projections by traces. Let p, q be projections in $\mathcal{A}$, and assume that $\widehat{p}(\tau) < \widehat{q}(\tau)$ for all $\tau \in \mathcal{T}(\mathcal{A})$. Since $\widehat{p}$ and $\widehat{q}$ are continuous, $\widehat{q} - \widehat{p}$ is continuous and $\mathcal{S} - \widehat{p}$ is lower semicontinuous. Choose

$$0 < \alpha < \min\{\widehat{q}(\tau) - \widehat{p}(\tau) : \tau \in \mathcal{T}(\mathcal{A})\}.$$

Then the constant function α is complemented under $\mathcal{S} - \widehat{p} = \widehat{1_{\mathcal{M}(\mathcal{A})} - p}$. By Lemma 4.12, there is a $P_o \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ orthogonal to p and such that $\widehat{P}_o = \alpha$. Thus $P := p + P_o \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$, $\widehat{P}$ is continuous, and $\widehat{P}(\tau) < \min\{q(\tau) : \tau \in \mathcal{T}(\mathcal{A})\}$.

Reasoning as in the proof of Proposition 4.9 we can find a sequence of mutually orthogonal nonzero projections $q_n \leqslant q$ such that $\widehat{q} = \sum_{n=1}^{\infty} \widehat{q}_n$. By Dini's theorem the convergence is uniform, so there is N such that $\sum_{n=1}^{N} \widehat{q}_n > \widehat{P}$. To simplify notations, assume that $N = 1$. Now choose an approximate identity $\{e_n\}$ of $\mathcal{A}$ consisting of projections and such that $e_1 = q_1$. Since $\mathcal{A}$ is simple and of real rank zero, we can find for every $n \geqslant 2$ projections $0 \neq q'_n \leqslant e_n - e_{n-1}$ and $q'_n \sim q''_n \leqslant q_n$. Set $q'_1 := q_1$ and $Q' := \sum_{n=1}^{\infty} q'_n$. Since the series converges strictly, $Q' \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ and

$$\widehat{Q}' > \widehat{q}_1 > \widehat{P}.$$

$\widehat{P}$ being continuous, it is complemented under $\widehat{Q}'$. By Lemma 4.12, $P \preceq Q'$ and hence $p \preceq Q'$. By Lemma 4.13, there is a n such that we have the following which completes the proof:

$$p \sim \left(p - \frac{1}{2}\right)_+ \preceq \sum_{k=1}^{n} q'_k \sim \sum_{k=1}^{n} q''_k \leqslant \sum_{k=1}^{n} q_k \leqslant q. \quad \blacksquare$$

5. PROJECTION SURJECTIVITY AND INJECTIVITY AND IDEALS IN $\mathcal{M}(\mathcal{A})$

As this section will illustrate, assuming that a C^*-algebra is projection surjective and injective greatly facilitates the study of the ideal structure of its multiplier algebra.

PROPOSITION 5.1. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra, which is projection surjective and injective and let $P \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ be a projection. Then for every*

$n \in \mathbb{N}$ there are mutually orthogonal projections $P_1 \sim P_2 \sim \cdots \sim P_n$ in $\mathcal{M}(\mathcal{A})$ such that $P = \sum_{j=1}^{n} P_j$.

Proof. Since $\widehat{P} = \frac{1}{n}\widehat{P} + \frac{n-1}{n}\widehat{P}$ and both functions are in $\mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++}$, by Lemma 4.12 there are two mutually orthogonal projections P_1 and P_1' not in $\mathcal{A}$ such that $P = P_1 + P_1'$, $\widehat{P_1} = \frac{1}{n}\widehat{P}$, and $\widehat{P_1'} = \frac{n-1}{n}\widehat{P}$. By the same reasoning, P_1' is the sum of two orthogonal projections $P_1' = P_2 + P_2'$ not in $\mathcal{A}$ such that $P_2 \sim P_1$ and $\widehat{P_2'} = \frac{n-2}{n}\widehat{P}$. After $n-1$ steps we get a decomposition of P into mutually orthogonal projections not in A,

$$P = P_1 + \cdots + P_{n-1} + P_n \quad \text{with } P_1 \sim P_2 \sim \cdots \sim P_{n-1} \text{ and with } \widehat{P_n} = \frac{1}{n}\widehat{P}.$$

Then $P_n \sim P_1$ by projection injectivity, which completes the proof. ∎

Compare this result with the case when $\mathcal{A}$ has real rank zero where it was shown in [63] that projections in $\mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ are divisible by 2^m. Notice that as a consequence, for every n, $\mathcal{M}(\mathcal{A}) \simeq M_n(\mathcal{M}(\mathcal{B}))$ for some hereditary subalgebra $\mathcal{B} \subset \mathcal{A}$.

COROLLARY 5.2. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra, which is n-projection surjective and injective for all n and let $[P] \in V(\mathcal{M}(\mathcal{A})) \setminus V(\mathcal{A})$. Then there is an $n \in \mathbb{N}$ such that $[P] = \sum_{j=1}^{n} [P_j]$ for some projections $P_j \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$.*

Proof. Since $P \in M_n(\mathcal{M}(\mathcal{A})) = \mathcal{M}(M_n(\mathcal{A}))$ for some $n \in \mathbb{N}$, and $P \notin V(\mathcal{A})$ and hence in particular, $P \notin M_n(\mathcal{A})$, by Proposition 5.1, $P = \sum_{j=1}^{n} P_j$ with $P_j \sim P_1 \in M_n(\mathcal{M}(\mathcal{A}))$. Then $\widehat{P}$ is complemented under $n\mathcal{S}$, i.e., there is a function $f \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++} \sqcup \{0\}$ such that $\widehat{P} + f = n\mathcal{S}$. But then $\widehat{P_1} + \frac{f}{n} = \mathcal{S}$ and hence by Lemma 4.12, $\widehat{P_1} = \widehat{P_1'}$ for some projection $P_1' \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$. By n-projection injectivity, $P_1 \sim P_1'$ and hence the conclusion follows. ∎

Another simple consequence of Lemma 4.12 is the following proposition.

PROPOSITION 5.3. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra, which is projection surjective and injective, and let P and Q be projections in $\mathcal{M}(\mathcal{A})$ with $Q \notin \mathcal{A}$. Then the following conditions are equivalent:*

(i) *$P \in I(Q)$ (the principal ideal generated by Q);*

(ii) *$\widehat{P} + f = m\widehat{Q}$ for some $m \in \mathbb{N}$ and some $f \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++} \sqcup \{0\}$.*

Proof. Assume that (i) holds, then for some $m \in \mathbb{N}$, by Lemma 1.4,

$$P \preceq \bigoplus_{k=1}^{m} Q \in M_m(\mathcal{M}(\mathcal{A})) = \mathcal{M}(M_m(\mathcal{A})).$$

Hence by Lemma 4.12(i), there is an $f \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++} \sqcup \{0\}$ such that

$$\widehat{P} + f = \widehat{\bigoplus_{k=1}^{m} Q} = m\widehat{Q}.$$

Assume that (ii) holds, then by Proposition 5.1 we can decompose P into the sum of m mutually orthogonal and equivalent projections, $P = \sum_{k=1}^{m} P_k$ and hence $\widehat{P_k} = \frac{1}{m}\widehat{P}$ for every k. Then $\widehat{P_k} + \frac{1}{m}f = \widehat{Q}$. By Lemma 4.12(ii), $P_k \preceq Q$ and hence $P_k \in I(Q)$ for every k, whence $P \in I(Q)$. ∎

In the case when $\mathcal{A}$ is simple, σ-unital, non-unital, non-elementary, and has strict comparison of positive elements by traces, we proved in Theorem 6.4 of [23] that strict comparison of positive elements holds for I_{cont}. The proof depended on the technique developed in [22] and used in the present paper in Theorem 3.3. As the following corollary illustrates, in the presence of projection surjectivity and injectivity, strict comparison of projections for I_{cont} can be obtained with a considerably simpler proof and without requiring explicitly strict comparison for the underlying algebra $\mathcal{A}$ (which however holds automatically by Theorem 4.15 if we further assume that $\mathcal{A}$ is separable and stable).

COROLLARY 5.4. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra, which is projection surjective and injective, and let $P \in I_{\mathrm{cont}}$ and $Q \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ be projections.*

(i) *If $\widehat{P}(\tau) < \widehat{Q}(\tau)$ for all τ, then $P \preceq Q$.*

(ii) *If $P \notin \mathcal{A}$ and $Q \in I_{\mathrm{cont}}$, then $I(P) = I(Q)$.*

Proof. (i) Since $\widehat{P} \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))_{++}$ by (1.14), it follows that $f := \widehat{Q} - \widehat{P} \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$, and hence $\widehat{P}$ is complemented (by f) under $\widehat{Q}$. Thus $P \preceq Q$ by Lemma 4.12(iii).

(ii) Since also $\widehat{Q} \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))_{++}$ we can choose n such that $\max \widehat{P} < n \min \widehat{Q}$. By Proposition 5.1, decompose P into the sum of n mutually orthogonal equivalent projections, $P = \sum_{k=1}^{n} P_k$, with $\widehat{P_k} = \frac{1}{n}\widehat{P}$. Every $\widehat{P_k}$ is continuous, hence $P_k \in I_{\mathrm{cont}}$. By (i), $P_k \preceq Q$ for every k, hence $P_k \in I(Q)$ and thus $P \in I(Q)$. Interchanging the role of P and Q, we conclude that $I(P) = I(Q)$. ∎

In Theorem 6.6 of [22] we proved that if $\mathcal{A}$ is simple, σ-unital, has quasi-continuous scale, and has strict comparison of positive elements, then $\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements (see Definition 1.3) (see also [21] for the real rank zero case). In the presence of projection surjectivity and injectivity, Corollary 5.5 here below will show that strict comparison of projections for $\mathcal{M}(\mathcal{A})$ can be obtained much more easily and without requiring explicitly strict comparison for the underlying algebra $\mathcal{A}$. We will use the notation introduced in Theorem 3.3 for a projection $P \in \mathcal{M}(\mathcal{A})$:

$$T(P) = \{\tau \in F_\infty : \tau(P) < \infty\}.$$

COROLLARY 5.5. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra, which is projection surjective and injective and has quasicontinuous scale, and let $P, Q \in \mathcal{M}(\mathcal{A})$ be projections with $Q \notin \mathcal{A}$.*

(i) *If $\widehat{P}(\tau) < \widehat{Q}(\tau)$ for all τ such that $\widehat{Q}(\tau) < \infty$, then $P \preceq Q$.*

(ii) *If $P \notin \mathcal{A}$ and $T(P) = T(Q)$, then $I(P) = I(Q)$.*

Proof. (i) Set $T := T(Q)$ and $F := \mathrm{co}(F_\infty \setminus T)$. We will assume that $T \neq \emptyset$ and $F \neq \emptyset$, as the case when one of the two sets is empty is similar but simpler and will be left to the reader. The face F is finite dimensional, and thus closed (and hence split) by (2.1). Its complementary face F' itself splits as $F' = \mathrm{co}(T) \dot{+} F'_\infty$. As F' is the direct sum of the finite dimensional and hence closed face $\mathrm{co}(T)$ and the face F'_∞ which is closed by hypothesis, it is also closed. Since $\widehat{Q} + \widehat{I-Q} = \mathcal{S}$, $\widehat{P} + \widehat{I-P} = \mathcal{S}$ and $\mathcal{S}$ is continuous on F'_∞, by Lemma 2.3, $\widehat{Q}$ and $\widehat{P}$ are continuous on F'_∞. Both functions are also continuous on $\mathrm{co}(T)$ since $\widehat{P}(\tau) < \widehat{Q}(\tau) < \infty$ for every $\tau \in T$ and T is finite. Thus $\widehat{Q} - \widehat{P} \in \mathrm{Aff}(F')_{++}$. Then by Lemma 2.4(iii),

$$f := \widehat{Q}|_F \dot{+} (\widehat{Q} - \widehat{P})|_{F'} \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}.$$

Since

$$\widehat{P}(\tau) + f(\tau) = \begin{cases} \widehat{P}(\tau) + \widehat{Q}(\tau) = \infty & \tau \in F, \\ \widehat{Q}(\tau) & \tau \in F', \end{cases} = \widehat{Q}(\tau),$$

$\widehat{P}$ is complemented under $\widehat{Q}$ and hence $P \preceq Q$ by Lemma 4.12.

(ii) By the first part of the proof, both $\widehat{Q}$ and $\widehat{P}$ are continuous on the closed face F'. Thus we can find n such that

$$\max_{\tau \in F'} \Big(\frac{1}{n} \widehat{P}(\tau) \Big) < \min_{\tau \in F'} \widehat{Q}(\tau).$$

Reasoning as in the proof of Corollary 5.4(ii), $P = \sum_{k=1}^{n} P_k$ and $\widehat{P_k}(\tau) < \widehat{Q}(\tau)$ for $\tau \in F'$, i.e., for all τ such that $\widehat{Q}(\tau) < \infty$. By part (i), $P_k \preceq Q$ and hence $P \in I(Q)$. Interchanging the role of P and Q, we conclude that $I(P) = I(Q)$. ∎

COROLLARY 5.6. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra with quasicontinuous scale, and metrizable tracial simplex $\mathcal{T}(\mathcal{A})$, and which is n-projection surjective and n-projection injective for every integer n. Let H be an order ideal of $V(\mathcal{M}(\mathcal{A}))$ not contained in $V(\mathcal{A})$. Let*

$$S := \{\tau \in F_\infty : \exists [P] \in H \text{ such that } \tau(P) = \infty\}.$$

Then $H = \{[Q] \in V(\mathcal{M}(\mathcal{A})) : \tau(Q) < \infty$ for all $\tau \in F_\infty \setminus S\}$. In particular, $V(\mathcal{M}(\mathcal{A}))$ has only finitely many order ideals.

Proof. By definition, $H \subset \{[Q] \in V(\mathcal{M}(\mathcal{A})) : \tau(Q) < \infty$ for all $\tau \in F_\infty \setminus S\}$. To prove the opposite inclusion, for every $\tau \in S$, choose $[P_\tau] \in H$ such that $\tau(P_\tau) = \infty$. Let $P := \bigoplus_{\tau \in S} P_\tau$. Since $S \subset F_\infty$ is finite it follows that $[P] \in H$ and that $\tau(P) = \infty$ for all $\tau \in S$. Since $P \in M_k(\mathcal{M}(\mathcal{A}))$ for some k, it is complemented under $k\mathcal{S}$ and reasoning as in the proof of Corollary 5.5, it is continuous on $\mathrm{co}(S)' = \mathrm{co}(F_\infty \setminus S) \dot{+} F'_\infty$. Let $[Q] \in V(\mathcal{M}(\mathcal{A}))$ be such $\tau(Q) < \infty$ for all $\tau \in F_\infty \setminus S$. By the same reasoning as for $\widehat{P}$, $\widehat{Q}$ is continuous on $\mathrm{co}(S)'$. By Lemma 2.5(iii), $\widehat{Q}$ is complemented under $m\widehat{P}$ for some integer m. By the assumption of n-projection surjectivity and injectivity for every n and by Lemma 4.12, it follows that $[Q] \leqslant m[P]$ and hence $[Q] \in H$. ∎

As a further consequence of projection surjectivity and injectivity we obtain the maximality for the ideals I_τ when $\tau \in F_\infty = \{\tau \in \partial_e(\mathcal{T}(\mathcal{A})) : \mathcal{S}(\tau) = \infty\}$. Maximality for I_τ was obtained for the stable case by Rørdam ([55], Theorem 4.4) for $\mathcal{A} \otimes \mathcal{K}$, $\mathcal{A}$ unital, with strict comparison of positive elements by traces and finite extremal boundary. The same result was also obtained by Perera in the proof of Theorem 6.6 of [48] for quasitraces and σ-unital, non-unital, non-elementary C^*-algebras with real rank zero, stable rank one, and weakly unperforated K_0 group. These results generalized earlier work by [12], [30].

THEOREM 5.7. *Let $\mathcal{A}$ be a simple, separable, non-unital, C^*-algebra, such that $\mathcal{A} \otimes \mathcal{K}$ is projection surjective and injective and let $\tau_o \in F_\infty$.*

(i) *The ideal I_{τ_o} of $\mathcal{M}(\mathcal{A})$ is generated by any projection $P \notin \mathcal{A}$ such that $\widehat{P}(\tau)$:* $\begin{cases} < \infty & \tau = \tau_o, \\ = \frac{\mathcal{S}}{2} & \tau \in \{\tau_o\}'. \end{cases}$ *Such projections exist.*

(ii) *I_{τ_o} is a maximal ideal.*

Proof. We first prove both these statements under the additional hypothesis that $\mathcal{A}$ is stable, in which case $\mathcal{S}(\tau) = \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$ and $F_\infty = \partial_e(\mathcal{T}(\mathcal{A}))$. Notice that by separability of $\mathcal{A}$, $\mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++} = \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ by Proposition 2.1(i) and every function in $\mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ is complemented under $\mathcal{S}$.

(i) Let $g := 1|_{\{\tau_o\}} \dot{+} \frac{\mathcal{S}}{2}|_{\{\tau_o\}'}$, or, more explicitly,

$$g(\tau) = \begin{cases} 1 & \tau = \tau_o, \\ \infty & \tau \neq \tau_o. \end{cases}$$

By Corollary 2.5 (or directly from the definition), $g \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$. Thus by Lemma 4.12, there is a projection $P \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ such that $\widehat{P} = g$. By (1.16), $P \in I_{\tau_o}$. We claim that every positive $A \in I_{\tau_o}$ belongs to $I(P)$. By expressing A as an $\mathcal{A}$ perturbation of the sum of two positive diagonal elements (Theorem 3.4) and then reasoning as in the proof of Theorem 3.3, we can assume that A itself is diagonal, i.e., $A = \sum_1^\infty a_n$, where $a_n \in \mathcal{A}_+$, $a_n a_m = 0$ for $n \neq m$, and the series converges in the strict topology. Fix $\varepsilon > 0$, then by (1.9) and (1.16),

$$\sum_{n=1}^\infty d_{\tau_o}((a_n - \varepsilon)_+) = d_{\tau_o}((A - \varepsilon)_+) \leqslant \frac{2}{\varepsilon}\tau_o\Big(\Big(A - \frac{\varepsilon}{2}\Big)_+\Big) < \infty \quad \forall\, \varepsilon > 0.$$

Choose N such that $\sum_{n=N}^{\infty} d_{\tau_o}((a_n-\varepsilon)_+) < 1$. As it is enough to prove that $A_N := \sum_{n=N}^{\infty} a_n \in I(P)$, to simplify notations assume that $N=1$. By the stability of $\mathcal{A}$, decompose $I_{\mathcal{M}(\mathcal{A})} = \sum_{k=1}^{\infty} E_k$ into a sum of mutually orthogonal projections $E_k \sim I_{\mathcal{M}(\mathcal{A})}$. Let

$$\alpha_n := d_{\tau_o}((a_n-\varepsilon)_+) + \frac{1-d_{\tau_o}((A-\varepsilon)_+)}{2^n},$$

so that $\sum_{n=1}^{\infty} \alpha_n = 1$. Then, again by Corollary 2.5,

$$g_n := \alpha_n|_{\{\tau_o\}} \dot{+} \frac{\mathcal{S}}{2}|_{\{\tau_o\}'} \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$$

and is complemented under $\mathcal{S}$. By the 1-projection surjectivity of $\mathcal{A}$ there is a projection $P_n \notin \mathcal{A}$ with $\widehat{P_n} = g_n$ and since $E_n \sim I_{\mathcal{M}(\mathcal{A})}$, we can take $P_n \leqslant E_n$. Then the series $\sum_{n=1}^{\infty} P_n$ converges strictly to a projection R and

$$\widehat{R} = \sum_{n=1}^{\infty} g_n = g = \widehat{P}.$$

Then $R \sim P$ by the 1-projection injectivity of $\mathcal{A} \otimes \mathcal{K}$, so assume without loss of generality that $R = P$. By the separability of $\mathcal{A}$, for every n we can find a strictly positive element $b_n \in P_n\mathcal{A}P_n$. Then $d_\tau(b_n) = \widehat{P_n}(\tau)$ for all τ. Since

$$d_\tau(a_n-\varepsilon)_+ < \begin{cases} \alpha_n = g_n(\tau_o) = \widehat{P_n}(\tau_o) & \tau = \tau_o, \\ \infty = g_n(\tau) = \widehat{P_n}(\tau) & \tau \neq \tau_o, \end{cases} = d_\tau(b_n),$$

by Theorem 4.15, we obtain from the strict comparison for $\mathcal{A}$ that $(a_n-\varepsilon)_+ \preceq b_n$.

Now $b_n \leqslant \|b_n\| P_n \sim P_n \sim (P_n - \frac{1}{2})_+$ and hence $(a_n-\varepsilon)_+ \preceq (P_n-\frac{1}{2})_+$ for every n. By Proposition 3.7, $(A-\varepsilon)_+ \preceq P$. Since $\varepsilon > 0$ is arbitrary, $A \preceq P$ and hence $A \in I(P)$. This proves that $I_{\tau_o} = I(P)$.

(ii) Let P be a projection for which $I_{\tau_o} = I(P)$, $\mathcal{J}$ be a closed two-sided ideal of $\mathcal{M}(\mathcal{A})$ such that $I_{\tau_o} \subsetneq \mathcal{J}$, and let $A \in \mathcal{J}_+ \setminus I_{\tau_o}$.

Invoking Theorem 3.4 and reasoning as in the first part of the proof, we can assume that $A = \sum_{n=1}^{\infty} a_n$ with $a_n \in \mathcal{A}_+$ mutually orthogonal and $A \notin I_{\tau_o}$. Choose $\varepsilon > 0$ such that $(A-\varepsilon)_+ \notin I_{\tau_o}$. As a consequence

$$d_{\tau_o}((A-\varepsilon)_+) = \sum_{n=1}^{\infty} d_{\tau_o}((a_n-\varepsilon)_+) = \infty.$$

Let e_n be an approximate identity of $\mathcal{A}$ such that $e_{n+1}e_n = e_n$ for all n and all $\tau \in \mathcal{T}(\mathcal{A})$. Recall that all e_n are in the Pedersen ideal of $\mathcal{A}$, and by Lemma 1.2, $d_\tau(e_n) < \infty$ for all n. By regrouping if necessary finite sums of a_n terms, assume that $d_{\tau_o}((a_n-\varepsilon)_+) > d_{\tau_o}(e_n - e_{n-1})$ for all n, where we set $e_0 = 0$. Reasoning as in part (i), decompose P into a sum of $P = \sum_{n=1}^{\infty} P_n$ with $\begin{cases} \widehat{P}_n(\tau)) < \infty & \tau = \tau_o, \\ \widehat{P}_n(\tau)) = \infty & \tau \neq \tau_o. \end{cases}$ Since $\mathcal{A}$ is separable, there is a strictly positive $b_n \in P_n\mathcal{A}P_n$ and we can assume that $\|b_n\| = 1$. Then for every n and every τ

$$d_\tau(e_n - e_{n-1}) < d_\tau(b_n \oplus (a_n-\varepsilon)_+).$$

Indeed, for $\tau = \tau_o$, $d_\tau(e_n - e_{n-1}) < d_\tau((a_n-\varepsilon)_+)$, while $d_\tau(b_n) = \tau(P_n) = \infty$ for every other τ. Since $M_2(\mathcal{A})$ has strict comparison, it follows that for every n

$$e_n - e_{n-1} \preceq b_n \oplus (a_n-\varepsilon)_+ \leqslant P_n \oplus (a_n-\varepsilon)_+ \sim ((P_n \oplus a_n) - \varepsilon)_+.$$

Since $1_{\mathcal{M}(\mathcal{A})} = \sum_{n=1}^{\infty} e_n - e_{n-1}$ and $\sum_{n=1}^{\infty} P_n \oplus a_n = P \oplus A$ where both series converge strictly, again by Proposition 3.7 we obtain that $1_{\mathcal{M}(\mathcal{A})} \preceq P \oplus A$. As $P \in I_{\tau_o} \subset \mathcal{J}$ and $A \in \mathcal{J}$, we have $P \oplus A \in \mathcal{J}$, thus $1_{\mathcal{M}(\mathcal{A})} \in \mathcal{J}$ and hence $\mathcal{J} = \mathcal{M}(\mathcal{A})$. We thus conclude that I_{τ_o} is maximal.

Finally, we remove the hypothesis that $\mathcal{A}$ is stable. There is a projection $R \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ such that $\mathcal{A}$ is isomorphic to $R(\mathcal{A}\otimes\mathcal{K})R$ and hence, by identifying $1_{\mathcal{M}(\mathcal{A})}$ with R, $\mathcal{M}(\mathcal{A})$ can be identified with $R\mathcal{M}(\mathcal{A}\otimes\mathcal{K})R$. As usual, we identify the tracial simplex $\mathcal{T}(\mathcal{A})$ of $\mathcal{A}$ with the tracial simplex of $\mathcal{A}\otimes\mathcal{K}$. Every ideal $\mathcal{J}$ of $R\mathcal{M}(\mathcal{A}\otimes\mathcal{K})R$ is the compression $\mathcal{J} = R\widetilde{\mathcal{J}}R$ of an ideal $\widetilde{\mathcal{J}}$ of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$. For every $\tau \in \mathcal{T}(\mathcal{A})$, denote by $I_{\tau,\mathcal{A}}$ (respectively, $I_{\tau,\mathcal{A}\otimes\mathcal{K}}$) the ideal of $R\mathcal{M}(\mathcal{A}\otimes\mathcal{K})R$ (respectively, of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$). It is then immediate to verify that $I_{\tau,\mathcal{A}} = RI_{\tau,\mathcal{A}\otimes\mathcal{K}}R$. Similarly, if $P \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ is a projection and $P \leqslant R$, then $I_{\mathcal{A}}(P) = RI_{\mathcal{A}\otimes\mathcal{K}}(P)R$ where we denote by $I_{\mathcal{A}}(P)$ (respectively, by $I_{\mathcal{A}\otimes\mathcal{K}}(P)$) the principal ideal of $R\mathcal{M}(\mathcal{A}\otimes\mathcal{K})R$ (respectively, of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$) generated by P. Since $\tau_o \in F_\infty$, the function

$$g := 1|_{\{\tau_o\}} \dot{+} \frac{\mathcal{S}}{2}|_{\{\tau_o\}'} \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$$

constructed at the beginning of the proof is complemented under $\mathcal{S}$ by Corollary 2.5 and hence there is a projection $P \in R\mathcal{M}(\mathcal{A}\otimes\mathcal{K})R$ with $\widehat{P} = g$. Since $\mathcal{A}\otimes\mathcal{K}$ satisfies the hypotheses, by the first part of the proof, $I_{\tau_o,\mathcal{A}\otimes\mathcal{K}} = I_{\mathcal{A}\otimes\mathcal{K}}(P)$ and then

$$I_{\tau_o,\mathcal{A}} = RI_{\tau_o,\mathcal{A}\otimes\mathcal{K}}R = RI_{\mathcal{A}\otimes\mathcal{K}}(P)R = I_{\mathcal{A}}(P).$$

Furthermore, since $I_{\tau_o,\mathcal{A}\otimes\mathcal{K}}$ is maximal and $I_{\tau_o,\mathcal{A}}$ is proper, it follows that $I_{\tau_o,\mathcal{A}}$ is also maximal, which proves (i) and (ii) also for the case when $\mathcal{A}$ is not stable. ∎

6. CHARACTERIZATION OF PURELY INFINITE CORONA ALGEBRAS

In this section we examine the link between pure infiniteness of the corona algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$ and other properties of the algebra $\mathcal{A}$ and its multiplier algebra

$\mathcal{M}(\mathcal{A})$. Not all the implications require the same hypotheses on the algebra $\mathcal{A}$. Denote by $\pi : \mathcal{M}(\mathcal{A}) \mapsto \mathcal{M}(\mathcal{A})/\mathcal{A}$ the quotient map.

PROPOSITION 6.1. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary, C^*-algebra, with non-empty tracial simplex $\mathcal{T}(\mathcal{A})$ and with strict comparison of positive elements by traces for $\mathcal{M}(\mathcal{A})$. Then $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is purely infinite.*

Proof. Since no non-zero quotient of $\mathcal{M}(\mathcal{A})$ can be abelian, the corona algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$ has no characters, hence by Definition 4.1 of [26], to obtain that $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is purely infinite it is (necessary and) sufficient to prove that if $A, B \in \mathcal{M}(\mathcal{A})_+$ and $\pi(A) \in I(\pi(B))$, then $\pi(A) \preceq \pi(B)$. Clearly, $A \in I(B)$. By Theorem 3.4, $A = \sum_1^\infty a_k + b_o$ where $b_o = b_o^* \in \mathcal{A}$, $0 \neq a_n \in \mathcal{A}_+$ and the series are bidiagonal ($a_n a_m = 0$ for $|n-m| > 1$ and converges strictly). Now $a_1 \sum_3^\infty a_k = 0$ and $\pi(A) = \pi\big(\sum_3^\infty a_k\big)$, so to simplify notation simply assume that there is an $0 \neq a \in \mathcal{A}_+$ such that $aA = 0$. Choose a strictly positive element $b \in \mathcal{A}$, then for all $\tau \in \mathcal{T}(\mathcal{A})$ $d_\tau(b) = d_\tau(1_{\mathcal{M}(\mathcal{A})}) = \mathcal{S}(\tau)$ and

$$d_\tau(A) \leqslant d_\tau(A+a) \leqslant d_\tau(b) = d_\tau(B+b)$$

where the first inequality is strict for all τ for which $d_\tau(b) < \infty$ and thus $d_\tau(A) < \infty$. Since $A \in I(B) = I(B+b)$, by the assumption of strict comparison on $\mathcal{M}(\mathcal{A})$, we have $A \preceq B + b$ and hence $\pi(A) \preceq B$. ∎

PROPOSITION 6.2. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary, C^*-algebra, with non-empty tracial simplex $\mathcal{T}(\mathcal{A})$. Assume there exists a projection P in I_{fin} but not in I_{cont}. Then $\pi(P) \in \mathcal{M}(\mathcal{A})/\mathcal{A}$ is not properly infinite. In particular, $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is not purely infinite.*

Proof. Assume by contradiction that $\pi(P) \oplus \pi(P) \preceq \pi(P)$. Then there is some $X \in \mathcal{M}(M_2(\mathcal{A}))$ such that

$$\|\pi(X)\pi(P)\pi(X)^* - \pi(P) \oplus \pi(P)\| < \frac{1}{2}$$

and hence there is some $a = a^* \in M_2(\mathcal{A})$ for which $\|XPX^* + a - P \oplus P\| < \frac{1}{2}$. Let $a = a_+ - a_-$ with $a_-, a_+ \geqslant 0$, then $(P \oplus P + a_- - \frac{1}{2})_+ \preceq XPX^* + a_+$. Hence

$$P \oplus P \sim \left(P \oplus P - \frac{1}{2}\right)_+ \preceq XPX^* + a_+ \preceq P \oplus a_+.$$

It is well known that then there is a $\delta > 0$ and a projection

$$Q \in \mathrm{Her}((P \oplus a_+ - \delta)_+) = \mathrm{Her}(P \oplus (a_+ - \delta)_+)$$

such that $P \oplus P \sim Q$. Notice that $(a_+ - \delta)_+$ belongs to the Pedersen ideal of $M_2(\mathcal{A})$ and has also a (positive) local unit b in the same Pedersen ideal, that is $b(a_+ - \delta)_+ = (a_+ - \delta)_+$. Then $P \oplus b$ is a local unit for $P \oplus (a_+ - \delta)_+$ and hence also for Q, that is $(P \oplus b)Q = Q$. Thus $Q \leqslant P \oplus b$. Let $g := \widehat{P \oplus b - Q}$. Then $g \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_+$ and

$$2\widehat{P} + g = \widehat{Q} + g = \widehat{P} + \widehat{b}.$$

Since $P \in I_{\mathrm{fin}}$, $\widehat{P}(\tau)$ is finite for every $\tau \in \partial_e(\mathcal{T}(\mathcal{A}))$ and hence

$$\widehat{b}(\tau) = \widehat{P}(\tau) + g(\tau) \quad \forall \tau \in \partial_e(\mathcal{T}(\mathcal{A})).$$

By Proposition 2.1, $\widehat{b} = \widehat{P} + g$. Since $\widehat{b}$ is continuous because b belongs to the Pedersen ideal and since both functions $\widehat{P}$ and g are lower semicontinuous, it follows by Lemma 2.3 that $\widehat{P}$ must be continuous. By (1.14) this contradicts the hypothesis that $P \notin I_{\mathrm{cont}}$. ∎

If $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is purely infinite, it thus follows that all the projections of I_{fin} are in I_{cont}. If $\mathcal{A}$ is 1-projection surjective this is sufficient to guarantee that $I_{\mathrm{fin}} = I_{\mathrm{cont}}$.

LEMMA 6.3. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra, and assume that $\mathcal{A}$ is* 1*-projection surjective. If $I_{\mathrm{fin}} \neq I_{\mathrm{b}}$ (respectively, $I_{\mathrm{b}} \neq I_{\mathrm{cont}}$), then there is a projection $P \in I_{\mathrm{fin}} \setminus I_{\mathrm{b}}$ (respectively, $P \in I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$).*

Proof. Let $A \in (I_{\mathrm{fin}})_+ \setminus I_{\mathrm{b}}$. Without loss of generality, assume that $\|A\| \leqslant 1$. By Lemma 1.7 there is some $\delta > 0$ and $\mu \in \mathcal{T}(\mathcal{A})$ for which $\widehat{(A-\delta)}_+(\mu) = \infty$. $\widehat{(A-\delta)}_+ \in \mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++}$ and since $(A-\delta)_+ \leqslant I$, the evaluation function $f := I - \widehat{(A-\delta)}_+$ also belongs to $\mathrm{LAff}_\sigma(\mathcal{T}(\mathcal{A}))_{++}$. As $\mathcal{S} = \widehat{1_{\mathcal{M}(\mathcal{A})}} = \widehat{(A-\delta)}_+ + f$, by Lemma 4.12 there is a projection P such that $\widehat{P} = \widehat{(A-\delta)}_+$. As $\widehat{P}(\tau) < \infty$ for all $\tau \in \partial_e(\mathcal{T}(\mathcal{A}))$ and $\widehat{P}(\mu) = \infty$, it follows that $P \in I_{\mathrm{fin}} \setminus I_{\mathrm{b}}$ by Lemma 1.7.

The case when $I_{\mathrm{b}} \neq I_{\mathrm{cont}}$ is similar: there is $A \in I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$ with $\|A\| \leqslant 1$ and $\delta > 0$ and a projection P such that $\widehat{P} = \widehat{(A-\delta)}_+$ is bounded but not continuous, and hence $P \in I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$ by (1.14). ∎

LEMMA 6.4. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra, with metrizable $\mathcal{T}(\mathcal{A})$, and with projection surjectivity and injectivity. Assume that F is a closed face, $\mathcal{S}(\tau) = \infty$ for all $\tau \in F$, and the complementary face F' is not closed. Then $I_{\mathrm{cont}} \neq I_{\mathrm{b}}$.*

Proof. Let $0 < \gamma < \min \mathcal{S}$. Then by Corollary 2.5(ii), the function $\frac{\gamma}{2}|_F \dot{+} \gamma|_{F'}$ belongs to $\mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ and is complemented under $\mathcal{S}$. Therefore there is a projection $P \notin \mathcal{A}$ such that $\widehat{P}(\tau) = \begin{cases} \frac{\gamma}{2} & \tau \in F, \\ \gamma & \tau \in F'. \end{cases}$ Since $\widehat{P}(\tau) \leqslant \gamma$ for all τ, $P \in I_{\mathrm{b}}$. Notice that $\widehat{P}(\tau) < \gamma$ for every $\tau \notin F'$. Since F' is not closed, $\widehat{P}$ is not continuous and hence $P \notin I_{\mathrm{cont}}$ by (1.14). ∎

If the scale of $\mathcal{A}$ is not quasicontinuous and $\mathcal{A}$ is projection surjective and injective, then that at least one of the inclusions $I_{\mathrm{cont}} \subset I_{\mathrm{b}} \subset I_{\mathrm{fin}}$ must be proper.

PROPOSITION 6.5. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra, with metrizable tracial simplex, and projection surjectivity and injectivity. Then*

(i) *if F_∞ is finite and F'_∞ is not closed then $I_{\text{cont}} \neq I_{\text{b}}$;*

(ii) *if F_∞ is finite, F'_∞ is closed, and $\mathcal{S}\,|_{F'_\infty}$ is not continuous, then $I_{\text{fin}} \neq I_{\text{cont}}$;*

(iii) *if F_∞ is infinite and countable, then $I_{\text{b}} \neq I_{\text{fin}}$;*

(iv) *if F_∞ is uncountable, then $I_{\text{cont}} \neq I_{\text{b}}$;*

Thus if $I_{\text{cont}} = I_{\text{fin}}$, then $\mathcal{A}$ has quasicontinuous scale.

Proof. (i) If F_∞ is finite, then $F = \text{co}(F_\infty)$ is closed and the conclusion is given by Lemma 6.4.

(ii) By Corollary 2.5 the function $1|_{F_\infty} \dot{+} \frac{\mathcal{S}}{2} \in \text{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ is complemented under $\mathcal{S}$ and therefore there is a projection P such that $\widehat{P} = 1|_{F_\infty} \dot{+} \frac{\mathcal{S}}{2}$. As

$$\widehat{P}(\tau) = \begin{cases} 1 & \tau \in F_\infty, \\ \frac{\mathcal{S}(\tau)}{2} < \infty & \tau \in F'_\infty \cap \partial_e(\mathcal{T}(\mathcal{A})), \end{cases}$$

we see that $P \in I_{\text{fin}}$. However $\widehat{P} = \frac{\mathcal{S}}{2}$ on F'_∞ is not continuous, and hence $P \notin I_{\text{cont}}$.

(iii) Let $F_\infty = \{\tau_n\}$ and apply Lemma 2.8 to the function $h = \mathcal{S}$ and the sequence $\{\tau_n\} = F_\infty$. Then $\mathcal{S} = G + F$ where G and F are in $\text{LAff}(\mathcal{T}(\mathcal{A}))_{++}$. G being complemented under $\mathcal{S}$, there is a projection $P \notin \mathcal{A}$ such that $\widehat{P} = G$. Then $P \in I_{\text{fin}}$ because $G(\tau_n) < \infty$ for all n, but $P \notin I_{\text{b}}$ because $\widehat{P}(\tau_n)$ is unbounded.

(iv) By the assumption that $\mathcal{T}(\mathcal{A})$ is metrizable, we can find an element $x \in F_\infty$ that belongs to the closure of $F_\infty \setminus \{x\}$. Then $F := \{x\}$ is closed, but $F' \supset (F_\infty \setminus \{x\})$ is not closed, hence the conclusion follows again from Lemma 6.4. ∎

Notice that the proof of (i) and (ii) did not require metrizability.

We can sharpen the result of Proposition 6.5 in the case when $\mathcal{A}$ is stable and hence $F_\infty = \partial_e(\mathcal{T}(\mathcal{A}))$. Then $\mathcal{A}$ has quasicontinuous scale if and only if $\partial_e(\mathcal{T}(\mathcal{A}))$ is finite.

PROPOSITION 6.6. *Let $\mathcal{A}$ be a simple, σ-unital, C^*-algebra, with metrizable tracial simplex, and projection surjectivity and injectivity and assume that $F_\infty = \partial_e(\mathcal{T}(\mathcal{A}))$.*

(i) *$\partial_e(\mathcal{T}(\mathcal{A}))$ is finite if and only if $I_{\text{cont}} = I_{\text{b}}$.*

(ii) *If furthermore $\mathcal{T}(\mathcal{A})$ is a Bauer simplex, then $\partial_e(\mathcal{T}(\mathcal{A}))$ is finite if and only if $I_{\text{b}} = I_{\text{fin}}$*

Proof. The necessity in both cases is given by Corollary 1.10.

(i) For the sufficiency, by Proposition 6.5(iv), it is enough to prove that if $\partial_e(\mathcal{T}(\mathcal{A}))$ is infinite and countable then $I_{\text{cont}} \neq I_{\text{b}}$. To obtain that it is sufficient (and by Lemma 6.3 also necessary) to find a projection $P \in I_{\text{b}} \setminus I_{\text{cont}}$. By the surjectivity of $\mathcal{A}$ and the fact that every function in $\text{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ is complemented under $\mathcal{S}$ because $\mathcal{S}(\tau) = \infty$ for all τ, by Corollary 1.9 and (1.14) we just need to construct a bounded function $g \in \text{LAff}(\mathcal{T}(\mathcal{A}))_{++} \setminus \text{Aff}(\mathcal{T}(\mathcal{A}))$.

Let $\{\tau_j\}_1^\infty$ be an enumeration of $\partial_e(\mathcal{T}(\mathcal{A}))$. Let $X_n := \mathrm{co}\{\tau_1, \dots, \tau_n\}$, then X_n is closed and hence it is a split face. Define

$$f_n := 1|_{X_n} \dot{+} 2|_{X_n'}.$$

By Lemma 2.4, $f_n \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$, $f_n \leqslant 2$ and clearly, f_n is monotone noincreasing. If for some n, the function f_n is not continuous, then we are done. Assume therefore that all the functions f_n are continuous and let $f := \lim_n f_n$. We claim that f is not continuous. Indeed by the compactness of $\mathcal{T}(\mathcal{A})$, there is a subsequence τ_{j_k} that converges to some $\mu \in \mathcal{T}(\mathcal{A})$. Then for every n, $f_n(\tau_{j_k}) \to f_n(\mu)$. Since $f_n(\tau_{j_k}) = 2$ for $j_k \geqslant n$, we thus have $f_n(\mu) = 2$ and hence $f(\mu) = 2$. On the other hand, for every k,

$$f(\tau_{j_k}) = \lim_n f_n(\tau_{j_k}) = 1.$$

As a consequence the function $g := 3 - f$ is bounded but also not continuous. Since $g = \lim_n (3 - f_n)$ is an increasing limit of functions in $\mathrm{Aff}(\mathcal{T}(\mathcal{A}))$, it follows that $g \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ which concludes the proof.

(ii) Reasoning as in part (i), it is enough to assume that $\partial_e(\mathcal{T}(\mathcal{A}))$ is infinite (and uncountable) and then construct a function $f \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ which is finite but unbounded on $\partial_e(\mathcal{T}(\mathcal{A}))$. We start by choosing a strictly positive lower semicontinuous function $\widetilde{f} : \partial_e(\mathcal{T}(\mathcal{A})) \mapsto (0, \infty)$ on the compact set $\partial_e(\mathcal{T}(\mathcal{A}))$ which is finite and unbounded. For instance, let d be the metric of $\mathcal{T}(\mathcal{A})$ restricted to $\partial_e(\mathcal{T}(\mathcal{A}))$, $\tau_o \in \partial_e(\mathcal{T}(\mathcal{A}))$ be an accumulation point of $\partial_e(\mathcal{T}(\mathcal{A}))$ and set

$$\widetilde{f}(\tau) := \begin{cases} \frac{1}{d(\tau,\tau_o)} & \tau \neq \tau_o, \\ 1 & \tau = \tau_o. \end{cases}$$

It is easily seen that $\widetilde{f}$ satisfies the required conditions. Decompose $\widetilde{f} = \sum_{n=1}^{\infty} \widetilde{f}_n$ as a pointwise converging sum of functions $\widetilde{f}_n \in \mathrm{Aff}(\partial_e(\mathcal{T}(\mathcal{A})))_{++}$ (Proposition 2.1). By Corollary 11.15 of [18], for each n, there is a $f_n \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))$ such that $f_n|_{\partial_e(\mathcal{T}(\mathcal{A}))} = \widetilde{f}_n$, and it is easy to see that f_n must be strictly positive. Then $f := \sum_{n=1}^{\infty} f_n \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ and $f|_{\partial_e(\mathcal{T}(\mathcal{A}))} = \widetilde{f}$. Thus f satisfies the required conditions. ∎

If $I_{\mathrm{cont}} \neq I_{\mathrm{fin}}$ we can draw several conclusions about $\mathcal{M}(\mathcal{A})$.

LEMMA 6.7. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra, with metrizable $\mathcal{T}(\mathcal{A})$, and with projection surjectivity and injectivity. Assume P is a projection in $I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$. Then there is a projection $Q \in I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$ such that:*

(i) *$\widehat{P}(\tau) < \widehat{Q}(\tau)$ for every $\tau \in \mathcal{T}(\mathcal{A})$;*

(ii) *$I(P) = I(Q)$;*

(iii) *$P \not\precsim Q$.*

In particular, strict comparison of projections by traces does not hold on $\mathcal{M}(\mathcal{A})$.

Proof. Let $P \in I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$. By (1.14) and Lemma 1.7, $\widehat{P}$ is bounded but not continuous. By invoking Proposition 5.1 and recalling that if we divide P into n equivalent projection summands, each summand generates the same ideal as P, we can assume without loss of generality that $\sup \widehat{P} < \min \mathcal{S}$. Let $\sup \widehat{P} < c < \min \mathcal{S}$. Then $\frac{1}{2}(\widehat{P} + c) \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++} \setminus \mathrm{Aff}(\mathcal{T}(\mathcal{A}))$. Moreover, both $\mathcal{S} - c \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ and $\mathcal{S} - \widehat{P} = \widehat{1_{\mathcal{M}(\mathcal{A})} - P} \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{+}$. Since

$$\mathcal{S} = \frac{1}{2}(\widehat{P} + c) + \frac{1}{2}(\mathcal{S} - \widehat{P} + \mathcal{S} - c),$$

$\frac{1}{2}(\widehat{P} + c)$ is complemented under $\mathcal{S}$ and hence by the 1-projection surjectivity of $\mathcal{A}$ there is a projection Q such that $\widehat{Q} = \frac{1}{2}(\widehat{P} + c)$. Again by (1.14) and Lemma 1.7, $Q \in I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$. Condition (i) holds as $\widehat{Q}(\tau) - \widehat{P}(\tau) = \frac{1}{2}(c - \widehat{P}(\tau)) > 0$ for all τ. As $\widehat{P} + c = 2\widehat{Q}$, it follows by Proposition 5.3 that $P \in I(Q)$. Furthermore, let $m \in \mathbb{N}$ be such that $(2m-1)\inf \widehat{P} > c$. Then $g := \frac{1}{2}((2m-1)\widehat{P} - c) \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ and $\widehat{Q} + g = m\widehat{P}$. Thus $Q \in I(P)$ by Proposition 5.3 and hence $I(P) = I(Q)$, which establishes condition (ii).

To prove (iii) assume by contradiction that $P \preceq Q$. Then there is a function $f \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ such that $\widehat{P} + f = \widehat{Q}$. But then $f = \frac{1}{2}(c - \widehat{P})$ by the boundedness of $\widehat{P}$, whence f is also upper semicontinuous and hence it is continuous. This implies that $\widehat{P}$ is continuous, a contradiction. ∎

LEMMA 6.8. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, C^*-algebra, with metrizable $\mathcal{T}(\mathcal{A})$, and with projection surjectivity and injectivity. Assume that there is a projection $P \in I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$ (respectively, $P \in I_{\mathrm{fin}} \setminus I_{\mathrm{b}}$). Then there is a projection $P_1 \in I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$ (respectively, $P_1 \in I_{\mathrm{fin}} \setminus I_{\mathrm{b}}$) such that $I(P_1) \subsetneq I(P)$. Therefore I_{b} (respectively I_{fin}), contains an infinite decreasing chain of principal ideals.*

Proof. Assume first that $P \in I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$. By (1.14) and Lemma 1.7, $\widehat{P}$ is a bounded function in $\mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ and it has at least one point of discontinuity $\mu \in \mathcal{T}(\mathcal{A})$. Then by Lemma 2.7, $\widehat{P} = G + F$ where $G, F \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ are both discontinuous at μ but for which there is a sequence $\tau_n \to \mu$ such that $G(\tau_n) \to G(\mu)$, and $\widehat{P}(\tau_n) \not\to \widehat{P}(\mu)$. By Lemma 4.12, there is a projection P_1 such that $\widehat{P_1} = G$ and $P_1 \preceq P$. Then $P_1 \in I_{\mathrm{b}} \setminus I_{\mathrm{cont}}$ and $I(P_1) \subset I(P)$. If $I(P_1) = I(P)$, we would have $P \in I(P_1)$ and hence by Proposition 5.3 there would be an $m \in \mathbb{N}$ and a function $f \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ such that $\widehat{P} + f = m\widehat{P_1} = mG$. However, since $mG(\tau_n) \to mG(\mu)$ and both $\widehat{P}$ and f are lower semicontinuous, we would conclude by Lemma 2.3 that $\widehat{P}(\tau_n) \to \widehat{P}(\mu)$, a contradiction.

Assume now that $P \in I_{\mathrm{fin}} \setminus I_{\mathrm{b}}$. By Lemma 2.8 there is sequence $\tau_n \in \partial_e(\mathcal{T}(\mathcal{A}))$ such that $\widehat{P}(\tau_n)$ is finite for every n but the sequence is unbounded. Apply Lemma 2.8 to the function $h := \widehat{P}$ and the sequence $\{\tau_n\}$ to decompose $h = G + F$ into the sum of $G, F \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$, with G unbounded but $\sup_n \frac{\widehat{P}(\tau_n)}{G(\tau_n)} = \infty$. Then there is a projection $P_1 \preceq P$ with $\widehat{P_1} = G$ and hence

$P_1 \in I_{\text{fin}} \setminus I_{\text{b}}$. Furthermore, $P \notin I(P_1)$. Indeed, otherwise there would be a $g \in \text{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ and $m \in \mathbb{N}$ such that $\widehat{P} + f = mG$. But then $\frac{\widehat{P}(\tau_n)}{G(\tau_n)} \leqslant m$ for every n, a contradiction. ∎

COROLLARY 6.9. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra, with metrizable $\mathcal{T}(\mathcal{A})$, and with projection surjectivity and injectivity. If $I_{\text{cont}} \neq I_{\text{fin}}$ then $\mathcal{M}(\mathcal{A})$ has infinitely many (principal) ideals and therefore $V(\mathcal{M}(\mathcal{A}))$ contains infinitely many (principal) order ideals.*

Proof. If $I_{\text{cont}} \neq I_{\text{fin}}$, then at least one of the inclusions $I_{\text{cont}} \subset I_{\text{b}} \subset I_{\text{fin}}$ must be proper. By Lemma 6.3, there must be a projection in $I_{\text{fin}} \setminus I_{\text{b}}$ or in $I_{\text{b}} \setminus I_{\text{cont}}$. In either case the conclusion follows from Lemma 6.8. By Lemma 1.11 we see that $V(\mathcal{M}(\mathcal{A}))$ contains infinitely many (principal) order ideals. ∎

Notice that the chains of principal ideal constructed in Lemma 6.8 are decreasing. If $\mathcal{A}$ is stable and has countably infinite extremal boundary, we can also construct increasing chains.

PROPOSITION 6.10. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, stable C^*-algebra with metrizable tracial simplex $\mathcal{T}(\mathcal{A})$, countably infinite extremal boundary $\partial_e(\mathcal{T}(\mathcal{A}))$, and projection surjectivity and injectivity. For every projection $P \in I_{\text{fin}}$ there is a continuous chain of projections $P_t \in I_{\text{fin}} \setminus I_{\text{b}}$ for $t \geqslant 1$ such that*

$$I(P) \subsetneq I(P_1) \subsetneq I(P_s) \subsetneq I(P_t) \quad \forall\, 1 < s < t.$$

Proof. Let $\{\tau_j\}_1^\infty$ be an enumeration of $\partial_e(\mathcal{T}(\mathcal{A}))$. Since $0 < \widehat{P}(\tau_n) < \infty$, we can find a sequence β_n such that:

(i) $1 < (\beta_n - 1)\widehat{P}(\tau_n)$ is monotone nondecreasing;

(ii) $\beta_n \widehat{P}(\tau_n)$ is monotone nondecreasing;

(iii) $\beta_n \to \infty$.

By Corollary 2.6 there exist a projection P_1 and a function $g \in \text{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ such that for every n

$$\widehat{P}_1(\tau_n) = \beta_n \widehat{P}(\tau_n), \quad g(\tau_n) = (\beta_n - 1)\widehat{P}(\tau_n).$$

Since $\widehat{P}(\tau_n) + g(\tau_n) = \widehat{P}_1(\tau_n)$, it follows by Proposition 2.1(iii) that $\widehat{P} + g = \widehat{P}_1$. Then $I(P) \subset I(P_1)$ by Lemma 4.12 and Proposition 5.3. On the other hand $\sup\limits_n \frac{\widehat{P}_1(\tau_n)}{\widehat{P}(\tau_n)} = \infty$ and hence by Proposition 5.3 it follows that $P_1 \notin I(P)$.

Next, for every $t > 1$, let P_t be the projection for which $\widehat{P}_t(\tau_n) := (\beta_n \widehat{P}(\tau_n))^t$. Since for $1 \leqslant s < t < \infty$ the sequence

$$(\beta_n \widehat{P}(\tau_n))^t - (\beta_n \widehat{P}(\tau_n))^s = (\beta_n \widehat{P}(\tau_n))^s((\beta_n \widehat{P}(\tau_n))^{t-s} - 1)$$

is monotone nondecreasing, again by Corollary 2.6, there exists a function $g \in \text{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ that achieves the values of that sequence at τ_n, that is

$$\widehat{P}_s(\tau_n) + g(\tau_n) = \widehat{P}_t(\tau_n).$$

But then, again by Proposition 2.1(iii) and Proposition 5.3 it follows that $\widehat{P}_s + g = \widehat{P}_t$, hence $P_s \preceq P_t$ and thus $I(P_s) \subset I(P_t)$. Since $\sup_n \frac{\widehat{P}_t(\tau_n)}{\widehat{P}_s(\tau_n)} = \infty$, again by Proposition 5.3 it follows that $P_t \notin I(P_s)$. ∎

We collect now the results obtained in this section in our main theorem.

THEOREM 6.11. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra, with metrizable tracial simplex $\mathcal{T}(\mathcal{A})$, projection surjectivity and injectivity, and strict comparison of positive elements by traces. Then the following are equivalent:*

(i) *$\mathcal{A}$ has quasicontinuous scale;*
(ii) *$\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements by traces;*
(iii) *$\mathcal{M}(\mathcal{A})/\mathcal{A}$ is purely infinite;*
(iii′) *$\frac{\mathcal{M}(\mathcal{A})}{I_{\min}}$ is purely infinite;*
(iv) *$\mathcal{M}(\mathcal{A})$ has finitely many ideals;*
(v) *$I_{\min} = I_{\text{fin}}$.*

Consider in addition

(vi) *$V(\mathcal{M}(\mathcal{A}))$ has finitely many order ideals.*

Then (vi) *implies* (i)–(v). *If $\mathcal{A}$ is n-projection surjective and n-projection injective for every n, then* (vi) *is equivalent to* (i)–(v).

We will always assume that $\mathcal{A}$ is simple, σ-unital but not unital, non-elementary, and with non-empty tracial simplex (and hence stably finite), but not all of the other three hypotheses (metrizability of $\mathcal{T}(\mathcal{A})$, projection surjectivity and injectivity of $\mathcal{A}$, and strict comparison of positive elements of $\mathcal{A}$), will be necessary for all the implications. In the proofs of the various implications, we will list which of these other hypotheses are used and/or which ones can be weakened.

Proof. (i) $\Rightarrow$ (ii) By Theorem 3.2 ([22], Theorem 6.6). For this implication we need only strict comparison of positive elements for $\mathcal{A}$.

(ii) $\Rightarrow$ (iii) By Proposition 6.1. For this implication we do not require any of the other three hypotheses.

(iii) $\Leftrightarrow$ (iii′) In view of the exact sequence

$$0 \to \frac{I_{\min}}{\mathcal{A}} \to \mathcal{M}(\mathcal{A})/\mathcal{A} \to \frac{\mathcal{M}(\mathcal{A})}{I_{\min}} \to 0$$

the conclusion follows from the "two out of three" property ([26], Theorem 4.19) provided that $\frac{I_{\min}}{\mathcal{A}}$ is purely infinite. By Theorem 4.8 of [23], a sufficient condition for $\frac{I_{\min}}{\mathcal{A}}$ to be purely infinite is that $\mathcal{A}$ is non-elementary and that $I_{\min} \neq \mathcal{A}$, which follows from the strict comparison of positive elements in $\mathcal{A}$ ([23], Corollary 3.15, Proposition 5.4, Theorem 5.6). If $\mathcal{A}$ is separable then also $I_{\min} \neq \mathcal{A}$ ([23], Corollary 3.15) so we can replace the condition of strict comparison of positive elements in $\mathcal{A}$ with the separability of $\mathcal{A}$.

(iii) $\Rightarrow$ (v) Proposition 6.2, which does not require any of three additional hypotheses, guarantees that if $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is purely infinite, then all the projections of I_{fin} belong to I_{cont}. This in turn implies that $I_{\text{cont}} = I_{\text{fin}}$ by Lemma 6.3, which makes use only 1-projection surjectivity for $\mathcal{A}$. Finally, $I_{\min} = I_{\text{cont}}$ by Theorem 5.6 of [23]) which depends only on strict comparison of positive elements.

(v)$\Rightarrow$ (i) By Proposition 6.5. Projection surjectivity and injectivity for $\mathcal{A}$ and metrizability of $\mathcal{T}(\mathcal{A})$ are used for obtaining that $I_{\text{cont}} = I_{\text{fin}}$ implies quasicontinuity of the scale. As above, strict comparison of positive elements is used for obtaining that $I_{\min} = I_{\text{cont}}$ ([23], Theorem 5.6).

(i)$\Rightarrow$ (iv) By Corollary 3.8, which makes use only of strict comparison of positive elements for $\mathcal{A}$.

(iv)$\Rightarrow$ (v) By Corollary 6.9. For this implication we use projection surjectivity and injectivity and the metrizability of $\mathcal{T}(\mathcal{A})$ to obtain that $I_{\text{cont}} = I_{\text{fin}}$, and again strict comparison of positive elements to obtain that $I_{\min} = I_{\text{cont}}$.

(vi)$\Rightarrow$ (v) Strict comparison on $\mathcal{A}$ guarantee that $I_{\min} = I_{\text{cont}}$ and metrizability of $\mathcal{T}(\mathcal{A})$ and projection surjectivity and injectivity permit to apply Lemma 6.3 and Lemma 6.8. Thus if $I_{\min} \neq I_{\text{fin}}$ then I_{fin} contains an infinite chain of *principal* ideals and hence $V(\mathcal{M}(\mathcal{A}))$ has an infinite chain of (principal) order ideals.
If $\mathcal{A}$ is n-projection surjective and n-projection injective for every n, then (i) $\Rightarrow$ (vi) by Corollary 5.6 which requires metrizability of $\mathcal{T}(\mathcal{A})$. ∎

When the algebra $\mathcal{A}$ is separable and stable, asking for strict comparison is redundant (Theorem 4.15) and we see that $I_{\min} = I_{\text{fin}}$ if and only if $I_{\min} = I_{\text{b}}$.

COROLLARY 6.12. *Let $\mathcal{A}$ be a simple, separable, stable, C^*-algebra, with projection surjectivity and injectivity. Then the following are equivalent:*

(i) *the extremal boundary $\partial_e(\mathcal{T}(\mathcal{A}))$ is finite;*
(ii) *$\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements by traces;*
(iii) *$\mathcal{M}(\mathcal{A})/\mathcal{A}$ is purely infinite;*
(iii′) *$\frac{\mathcal{M}(\mathcal{A})}{I_{\min}}$ is purely infinite;*
(iv) *$\mathcal{M}(\mathcal{A})$ has finitely many ideals;*
(v) *$I_{\min} = I_{\text{fin}}$;*
(v′) *$I_{\min} = I_{\text{b}}$;*
(vi) *$V(\mathcal{M}(\mathcal{A}))$ has finitely many order ideals.*

Proof. All the hypotheses of Theorem 6.11 are satisfied: metrizability is implied by the separability of $\mathcal{A}$, strict comparison of positive elements for $\mathcal{A}$ is implied projection surjectivity and injectivity (Theorem 4.15). Thus conditions (i), (ii), (iii), (iii′), (iv), (v), and (vi) are equivalent, where for (i) we notice that for stable C^*-algebras quasicontinuity of the scale is equivalent to finiteness of the extremal boundary.

(i) $\Leftrightarrow$ (v′) By Proposition 6.6. ∎

REMARK 6.13. For the class of algebras $\mathcal{A}$ that are simple, separable, non-unital, have real rank zero, stable rank one, and have weakly unperforated $K_0(\mathcal{A})$, the equivalence of (i), (iii), (iii′), and (v) was established in Theorem 3.4 of [27] under the additional condition that $\mathcal{A}$ has finitely many infinite extremal quasitraces. For the same class of algebras, the equivalence of the above conditions with (iv) was established in Theorem 3.6 of [27] under the additional condition that $\mathcal{A}$ is exact, is the stabilization of a unital algebra, and $\mathcal{T}(\mathcal{A})$ is a Bauer simplex. In [29] The equivalence of (i), (iii), (iv) was established under the condition that $\mathcal{A}$ is the stabilization of a simple, unital algebra, is separable, and is either exact and $\mathcal{Z}$-stable or an AH-algebra with slow dimension growth. These results, in turn, are generalizations of earlier work in [12], [30], [33], [34], [39], [55], [61], [65].

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