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THE SQUARE ROOT OF NONNEGATIVE SELFADJOINT LINEAR RELATIONS IN HILBERT SPACES

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ABSTRACT. An elementary construction of the square root of nonnegative selfadjoint linear relations in Hilbert spaces is presented.

KEYWORDS: *Hilbert space, nonnegative linear relation, selfadjoint linear relation, square root.*

MSC (2010): 47A06, 47B25, 47B65.

INTRODUCTION

Assume that A is a nonnegative bounded linear operator in a real or a complex Hilbert space $\mathfrak{H}$. Then there exists a unique nonnegative bounded linear operator S in the same Hilbert space $\mathfrak{H}$ such that $S^2 = A$. It is called the square root of A; it is denoted by $A^{1/2}$ and it commutes with every bounded operator that commutes with A, cf. [9].

S.J. Bernau presented in [2] a way to define the square root of an unbounded nonnegative selfadjoint linear operator in the setting of Hilbert spaces. Another proof of the existence of the square root of a nonnegative operator was given by Wouk in [13]. Also, an elegant factorization of the square root of unbounded nonnegative selfadjoint linear operators was recently proposed by Z. Sebestyén and Zs. Tarcsay in [12].

The concept of the square root of an operator was generalized to the case of multivalued linear operators (linear relations) and it was systematically used to solve specific problems in the operator theory and its applications, cf. [4], [5], [6], [7] and the references therein. The classical way of defining the square root of a nonnegative selfadjoint linear relation is to reduce it to the case of its operator part, cf. [4]. It is the main goal of this note to present a construction of the square root of a nonnegative selfadjoint linear relation in the setting of real or complex

Hilbert spaces. The method is rather algebraic, it is motivated by the following elementary factorization

$$\sqrt{x} = \sqrt{1-(1+x)^{-1}}\Big(\sqrt{(1+x)^{-1}}\Big)^{-1},$$

and it is quite closed to the ones used in [12], [13]. However, the techniques used for the proofs are specific to the case of multivalued linear operators. More precisely, for a nonnegative selfadjoint linear relation A in a real or a complex Hilbert space one shows that its square root exists, it is unique and it can be factorized by

$$A^{1/2} = UV^{-1},$$

where U and V stand for the square roots of the nonnegative bounded linear operators $I-(I+A)^{-1}$ and $(I+A)^{-1}$, respectively.

The note is organized as follows. Section 1 contains basic facts concerning linear relations in Hilbert spaces. In Section 2 nonnegative linear relations are studied. The main result of this note is proven in Section 3.

1. LINEAR RELATIONS IN HILBERT SPACES

Following [1] some preparatory ingredients are presented in this section. More details concerning the theory of linear relations in Hilbert spaces can be found for instance in [3], [4], [7], [8].

Throughout the paper $\mathfrak{H}$ stands for a real or a complex Hilbert space whose inner product is denoted by $\langle\cdot,\cdot\rangle$. A linear relation T in $\mathfrak{H}$ is a linear subspace of the Hilbert space Cartesian product $\mathfrak{H}\times\mathfrak{H}$. The elements of T will be denoted by $\{f,g\}$ with $f,g\in\mathfrak{H}$. For a linear relation T the following self-explanatory notions domain, range, kernel, and multi-valued part of T will be used from now on:

$$\operatorname{dom} T = \{f\in\mathfrak{H} : \{f,f'\}\in T\},\quad \operatorname{ran} T = \{f'\in\mathfrak{H} : \{f,f'\}\in T\},$$
$$\ker T = \{f\in\mathfrak{H} : \{f,0\}\in T\},\quad \operatorname{mul} T = \{f'\in\mathfrak{H} : \{0,f'\}\in T\}.$$

One says that the linear relation T is closed if it is closed as a subspace of $\mathfrak{H}\times\mathfrak{H}$. Furthermore, it is easily seen that if T is closed then the subspaces $\ker T$ and $\operatorname{mul} T$ are closed. The graph of an operator T is a linear relation; in this case $\operatorname{mul} T=\{0\}$. The notation I stands for the identity operator in $\mathfrak{H}$. In what follows a linear operator T in $\mathfrak{H}$ is identified with its graph. It is said to be closable if its closure is again the graph of an operator. The formal inverse T^{-1} of T is defined as $T^{-1}=\{\{f',f\} : \{f,f'\}\in T\}$; it is a linear relation in $\mathfrak{H}$. The following identities

$$\operatorname{dom} T^{-1} = \operatorname{ran} T,\quad \ker T^{-1} = \operatorname{mul} T \tag{1.1}$$

express the formal "duality" between T and its inverse. It is not difficult to see that

$$TT^{-1} = I\restriction_{\operatorname{ran} T}\ \widehat{+}\ (\{0\}\times\operatorname{mul} T),\quad T^{-1}T = I\restriction_{\operatorname{dom} T}\ \widehat{+}\ (\{0\}\times\ker T).$$

If λ is a real or a complex number then λT is a linear relation in $\mathfrak{H}$ given by

$$\lambda T = \{\{f, \lambda f'\} : \{f, f'\} \in T\}.$$

The adjoint of a linear relation T in $\mathfrak{H}$ is the closed linear relation T^* in $\mathfrak{H}$ defined by

$$T^* = \{\{f, f'\} \in \mathfrak{H} \times \mathfrak{H} : \langle f', h \rangle = \langle f, h' \rangle \text{ for all } \{h, h'\} \in T\}.$$

Geometrically the adjoint T^* is given by

$$T^* = JT^{\perp} = (JT)^{\perp}, \tag{1.2}$$

where the operator J from $\mathfrak{H} \times \mathfrak{H}$ onto $\mathfrak{H} \times \mathfrak{H}$ is defined by

$$J\{f, g\} = \{g, -f\}, \quad \{f, g\} \in \mathfrak{H} \times \mathfrak{H}. \tag{1.3}$$

Moreover,

$$(T^{-1})^* = (T^*)^{-1}, \tag{1.4}$$

so that

$$(\mathrm{dom}\, T)^{\perp} = \mathrm{mul}\, T^*, \quad (\mathrm{ran}\, T)^{\perp} = \ker\, T^*. \tag{1.5}$$

It is clear that the double adjoint T^{**} is the closure of the relation T.

A linear relation T in a Hilbert space $\mathfrak{H}$ is said to be symmetric if $T \subset T^*$. A symmetric linear relation T in a Hilbert space $\mathfrak{H}$ is said to be nonnegative if $\langle f', f \rangle \geqslant 0$ for all $\{f, f'\} \in T$. A linear relation T in a Hilbert space $\mathfrak{H}$ is said to be selfadjoint if $T^* = T$, so that it is automatically closed.

The sum of two linear relations T_1 and T_2 in $\mathfrak{H}$ is a linear relation defined by

$$T_1 + T_2 = \{\{f, f' + f''\} : \{f, f'\} \in T_1, \{f, f''\} \in T_2\}. \tag{1.6}$$

The adjoint of the sum of linear relations in (1.6) satisfies

$$T_1^* + T_2^* \subset (T_1 + T_2)^*. \tag{1.7}$$

Moreover, if T_2 is a bounded linear operator in $\mathfrak{H}$ then there is actually equality:

$$T_1^* + T_2^* = (T_1 + T_2)^*. \tag{1.8}$$

If T_1 and T_2 are two linear relations in a Hilbert space $\mathfrak{H}$ then their product $T_2 T_1$ is a linear relation in $\mathfrak{H}$ defined by

$$T_2 T_1 = \{\{f, f'\} : \{f, \varphi\} \in T_1, \{\varphi, f'\} \in T_2 \text{ for some } \varphi \in \mathfrak{H}\}, \tag{1.9}$$

and further one has

$$(T_2 T_1)^{-1} = (T_1)^{-1} (T_2)^{-1}. \tag{1.10}$$

The adjoint of the product of linear relations in (1.9) satisfies

$$T_1^* T_2^* \subset (T_2 T_1)^*. \tag{1.11}$$

Moreover, if T_2 is a bounded linear operator in $\mathfrak{H}$ then there is actually equality:

$$T_1^* T_2^* = (T_2 T_1)^*. \tag{1.12}$$

2. NONNEGATIVE LINEAR RELATIONS IN HILBERT SPACES

The following two preparatory results can be found with their proofs in [10].

LEMMA 2.1. *Let S be a symmetric linear relation in a Hilbert space $\mathfrak{H}$ and assume that* $\operatorname{dom} S = \mathfrak{H}$. *Then S is a selfadjoint linear relation in $\mathfrak{H}$.*

COROLLARY 2.2. *Let S be a symmetric linear relation in a Hilbert space $\mathfrak{H}$ and assume that* $\operatorname{ran} S = \mathfrak{H}$. *Then S is a selfadjoint linear relation in $\mathfrak{H}$.*

The next result offers a range characterization for the selfadjointness of nonnegative linear relations. The proof is valid for both real and complex Hilbert spaces. The operator case version of this result can be found in Theorem 2.5 of [11].

THEOREM 2.3. *Let $\mathfrak{H}$ be a real or a complex Hilbert space and let S be a nonnegative linear relation in $\mathfrak{H}$. The following statements are equivalent:*
(i) *S is a nonnegative selfadjoint linear relation in $\mathfrak{H}$;*
(ii) *$I + S$ is of full range, i.e.* $\operatorname{ran}(I + S) = \mathfrak{H}$.

Proof. In order to see that (i) implies (ii) it will be shown that $(I+S)^{-1}$ is a bounded selfadjoint linear operator in $\mathfrak{H}$. Let $\{e, f\} \in (I+S)^{-1}$ so that $\{f, e - f\} \in S$. Since S is nonnegative it follows that $\langle e - f, f\rangle \geqslant 0$, so that $\langle e, f\rangle \geqslant \|f\|^2$. This inequality further implies that $\|f\| \leqslant \|e\|$ on the basis of Cauchy's inequality. Consequently, if $e = 0$ then $f = 0$ which shows that $\operatorname{mul}(I+S)^{-1} = 0$ so that $(I+S)^{-1}$ is the graph of an operator. Furthermore, the inequality $\|f\| \leqslant \|e\|$ for all $\{e, f\} \in (I+S)^{-1}$ shows that the operator $(I+S)^{-1}$ is bounded. Using (1.4) and (1.8) one has

$$((I+S)^{-1})^* = ((I+S)^*)^{-1} = (I^* + S^*)^{-1} = (I+S)^{-1},$$

which shows that $(I+S)^{-1}$ is selfadjoint. Since it is also bounded it follows that $(I+S)^{-1}$ is an everywhere defined selfadjoint bounded operator. Consequently, $\mathfrak{H} = \operatorname{dom}(I+S)^{-1} = \operatorname{ran}(I+S)$ which implies that $I + S$ is a full range linear relation, as claimed.

For the converse implication assume that $\operatorname{ran}(I+S) = \mathfrak{H}$. It follows from Corollary 2.2 that $I + S$ is a selfadjoint linear relation. Using (1.8) one successively has

$$S^* = (I + S + (-I))^* = (I+S)^* + (-I)^* = I + S + (-I) = S,$$

which shows that S is a nonnegative selfadjoint linear relation. This argument completes the proof. ∎

A direct consequence of the proof of Theorem 2.3 follows.

COROLLARY 2.4. *Let $\mathfrak{H}$ be a real or a complex Hilbert space and let A be a nonnegative selfadjoint linear relation in $\mathfrak{H}$. Then both $(I+A)^{-1}$ and $I - (I+A)^{-1}$ are bounded nonnegative selfadjoint contractions in $\mathfrak{H}$.*

A range criteria for the self–adjointness of symmetric linear relations in real or complex Hilbert spaces will be next provided. Some preparatory results are firstly proven or presented.

The useful result of algebraic nature in the following lemma can be found in [1].

LEMMA 2.5. *Let S_1 and S_2 be linear relations in a Hilbert space $\mathfrak{H}$ and assume that $S_1 \subset S_2$. Then the following statements are equivalent:*

(i) $S_1 = S_2$*;*

(ii) $\operatorname{dom} S_1 = \operatorname{dom} S_2$ *and* $\operatorname{mul} S_1 = \operatorname{mul} S_2$.

The next result is a particular case of Theorem 3.3 in [10].

LEMMA 2.6. *Let S be a closed linear relation in a real or a complex Hilbert space $\mathfrak{H}$. Then the product S^*S is a nonnegative selfadjoint linear relation in $\mathfrak{H}$.*

The square of a symmetric linear relation in a Hilbert space is analyzed in the next result.

LEMMA 2.7. *Assume that S is a symmetric linear relation in a real or a complex Hilbert space $\mathfrak{H}$. Then S^2 is a nonnegative linear relations in $\mathfrak{H}$. Furthermore:*

$$\operatorname{mul} S^2 = \operatorname{mul} S. \tag{2.1}$$

Proof. Clearly,

$$S^2 = SS \subset S^*S^* \subset (SS)^* = (S^2)^*,$$

so that S^2 is a symmetric linear relation in $\mathfrak{H}$. Let $\{x, y\} \in S^2$, so that $\{x, z\} \in S$ and $\{z, y\} \in S$ for some $z \in \operatorname{ran} S \cap \operatorname{dom} S$. This implies that $\langle y, x\rangle = \langle z, z\rangle = \|z\|^2 \geqslant 0$ which shows that S^2 is a nonnegative linear relation in $\mathfrak{H}$, as claimed.

In order to prove the equality $\operatorname{mul} S^2 = \operatorname{mul} S$ assume that $m \in \operatorname{mul} S^2$ so that $\{0, m\} \in S^2$. This further implies that $\{0, q\} \in S$ and $\{q, m\} \in S$ so that $\langle q, q\rangle = \langle m, 0\rangle = 0$. Thus $q = 0$ which leads to $\{0, m\} \in S$. Then $\operatorname{mul} S^2 \subset \operatorname{mul} S$. Conversely, let $m \in \operatorname{mul} S$ so that $\{0, m\} \in S$. Since $\{0, 0\} \in S$ and $\{0, m\} \in S$ it follows that $\{0, m\} \in S^2$ which shows that $\operatorname{mul} S \subset \operatorname{mul} S^2$. Consequently the equality in (2.1) has been proved. ∎

The next result offers a criterion for the self–adjointness of symmetric linear relations in Hilbert spaces using their squares. The operator case can be found in Theorem 2.2 of [11]. The proof uses specific techniques from the theory of linear relations and it works for both real and complex Hilbert spaces.

THEOREM 2.8. *Let $\mathfrak{H}$ be a real or a complex Hilbert space and let S be a symmetric linear relation in $\mathfrak{H}$. Then the following are equivalent:*

(i) *S is a selfadjoint linear relation in $\mathfrak{H}$;*

(ii) *S^2 is a nonnegative selfadjoint linear relation in $\mathfrak{H}$;*

(iii) *$I + S^2$ is of full range, i.e.* $\operatorname{ran}(I + S^2) = \mathfrak{H}$.

Proof. It will be proven next that (i) implies (ii). Since S is selfadjoint it follows that S is closed. Using Lemma 2.6 one has that S^*S is a nonnegative selfadjoint linear relation in $\mathfrak{H}$. Therefore, $S^2 = SS = S^*S$ is a nonnegative selfadjoint linear relation in $\mathfrak{H}$, as claimed.

The equivalence between (ii) and (iii) is a direct consequence of Lemma 2.7 and Theorem 2.3. It only remains to show that (iii) implies (i). Assume that $\operatorname{ran}(I + S^2) = \mathfrak{H}$. Since $S \subset S^*$ it follows that

$$\operatorname{dom} S \subset \operatorname{dom} S^*. \tag{2.2}$$

It will be proven that

$$\operatorname{dom} S^* \subset \operatorname{dom} S. \tag{2.3}$$

To see this, let $w \in \operatorname{dom} S^*$ so that $\{w, t\} \in S^*$ for some $t \in \operatorname{ran} S^*$. Then for some $x, y \in \mathfrak{H}$ one has $\{x, t\} \in I + S^2$ and $\{y, -w\} \in I + S^2$. This implies that $\{x, t - x\} \in S^2$ and $\{y, -w - y\} \in S^2$ which further leads to

$$\{x, a\} \in S, \quad \{a, t - x\} \in S, \quad \text{and} \tag{2.4}$$

$$\{y, b\} \in S, \quad \{b, -w - y\} \in S, \tag{2.5}$$

for some $a, b \in \operatorname{dom} S \cap \operatorname{ran} S$. Then

$$\{a - y, t - x - b\} = \{a, t - x\} - \{y, b\} \in S \subset S^*. \tag{2.6}$$

Since $\{w, t\} \in S^*$, it follows from (2.6) that

$$\{w - a + y, x + b\} = \{w, t\} - \{a - y, t - x - b\} \subset S^*. \tag{2.7}$$

Furthermore, it follows from (2.4) and (2.5) that

$$\{x + b, a - w - y\} = \{x, a\} + \{b, -w - y\} \in S. \tag{2.8}$$

A combination of (2.7) and (2.8) leads to

$$(x + b, x + b) = (w - a + y, -(w - a + y)),$$

which is equivalent to

$$\|x + b\|^2 + \|w - a + y\|^2 = 0,$$

so that $x + b = w - a + y = 0$. This implies that $w = a - y \in \operatorname{dom} S$. Consequently, (2.3) has been proved. A combination of (2.2) and (2.3) leads to

$$\operatorname{dom} S^* = \operatorname{dom} S. \tag{2.9}$$

Moreover, it follows from the equivalence between (ii) and (iii) that S^2 is selfadjoint. Also, using (1.11) one has

$$S^2 = SS \subset S^*S^* \subset (SS)^* = (S^2)^* = S^2,$$

which shows that $S^2 = S^*S$. Similarly it can be proven that $S^2 = SS^*$, so that

$$S^2 = S^*S = SS^*. \tag{2.10}$$

It follows from (2.1) and (2.10) that

$$\operatorname{mul} S = \operatorname{mul} S^2 = \operatorname{mul} S^*S. \tag{2.11}$$

It will be next shown that

$$\operatorname{mul} S^*S = \operatorname{mul} S^*. \tag{2.12}$$

Assume that $m \in \operatorname{mul} S^*S$ so that $\{0, m\} \in S^*S$. This further implies that $\{0, q\} \in S$ and $\{q, m\} \in S^*$ so that $\langle q, q \rangle = \langle m, 0 \rangle = 0$. Thus $q = 0$ which leads to $\{0, m\} \in S^*$. Then $\operatorname{mul} S^*S \subset \operatorname{mul} S^*$. Conversely, let $m \in \operatorname{mul} S^*$ so that $\{0, m\} \in S^*$. Since $\{0, 0\} \in S$ and $\{0, m\} \in S^*$ it follows that $\{0, m\} \in S^*S$ which shows that $\operatorname{mul} S^* \subset \operatorname{mul} S^*S$. Consequently the equality in (2.12) has been proved. Using (2.11) and (2.12) one has

$$\operatorname{mul} S^* = \operatorname{mul} S. \tag{2.13}$$

Finally, it follows from Lemma 2.5, (2.9) and (2.13) that $S^* = S$, as claimed. This completes the proof. ∎

COROLLARY 2.9. *Let $\mathfrak{H}$ be a real or a complex Hilbert space, let S be a symmetric linear relation in $\mathfrak{H}$ and let A be a selfadjoint linear relation in $\mathfrak{H}$. Assume the S^2 extends A. Then $A = S^2$ and S itself is a selfadjoint linear relation in $\mathfrak{H}$.*

Proof. It follows from Lemma 2.7 that S^2 is a nonnegative linear relation in $\mathfrak{H}$, so that

$$S^2 \subset (S^2)^*. \tag{2.14}$$

Using the hypothesis $A \subset S^2$ one has

$$(S^2)^* \subset A^*. \tag{2.15}$$

A combination of (2.14) and (2.15) leads to

$$S^2 \subset (S^2)^* \subset A^* = A \subset S^2,$$

which shows that $A = S^2$. Thus S^2 is a nonnegative selfadjoint linear relation in $\mathfrak{H}$ and then it follows from Theorem 2.8 that S is a selfadjoint linear relation in $\mathfrak{H}$, as claimed. ∎

Following [4] and [7] some results concerning the decomposition of a nonnegative selfadjoint linear relation in a Hilbert space are presented in the next lemma. They will be used for the proof of the main result within the next section. Assume that A is a nonnegative selfadjoint linear relation in the Hilbert space $\mathfrak{H}$. Define $A_s = A \cap (\overline{\operatorname{dom}} A \times \overline{\operatorname{dom}} A)$. Then A_s is a nonnegative selfadjoint linear operator in the Hilbert space $\overline{\operatorname{dom}} A$ and, furthermore

$$A = A_s \oplus (\{0\} \times \operatorname{mul} A).$$

LEMMA 2.10. *Assume that S is a nonnegative selfadjoint linear relation in a real or a complex Hilbert space $\mathfrak{H}$. Then*

(i) $A = S^2$ *is also a nonnegative selfadjoint linear relation in* $\mathfrak{H}$*;*

(ii) $A_s = (S_s)^2$*;*

(iii) $A = (S_s)^2 \oplus (\{0\} \times \operatorname{mul} A)$.

3. THE MAIN RESULT

The main result of this note offers a factorization of the square root of a nonnegative selfadjoint linear relation in a real or a complex Hilbert space.

THEOREM 3.1. *Assume that A is a nonnegative selfadjoint linear relation in a real or a complex Hilbert space $\mathfrak{H}$.*

(i) *There exists an unique nonnegative selfadjoint linear relation S such that $S^2 = A$.*

(ii) *Moreover, if B is a bounded linear operator in $\mathfrak{H}$ which commutes with A then it commutes with S, namely, if $BA \subset AB$ then $BS \subset SB$.*

Proof. (i) Let U and V be the square roots of the nonnegative bounded linear operators $I - (I + A)^{-1}$ and $(I + A)^{-1}$, respectively. Clearly, $U^2V^2 = V^2U^2$, so that

$$U^{2/2^n} V^{2/2^t} = V^{2/2^t} U^{2/2^n},$$

for all nonnegative integers n and t. Define the linear relation $S = UV^{-1}$ in $\mathfrak{H}$.

It will be proven next that S is nonnegative. To see this let $\{x, y\} \in S$ so that $\{x, z\} \in V^{-1}$ and $\{z, y\} \in U$ for some $z \in \mathfrak{H}$. This implies that $\{z, x\} \in V$ so that $x = Vz$ and $y = Uz$. Therefore,

$$\begin{aligned} \langle y, x\rangle &= \langle Uz, Vz\rangle = \langle Uz, V^{1/2}V^{1/2}z\rangle \\ &= \langle V^{1/2}Uz, V^{1/2}z\rangle = \langle UV^{1/2}z, V^{1/2}z\rangle \geqslant 0. \end{aligned} \tag{3.1}$$

The relation (3.1) shows that S is a nonnegative linear relation in $\mathfrak{H}$.

To see that S is symmetric let $\{x, y\}$, $\{f, g\} \in S$. Then using similar arguments as above one has $x = Vz$, $y = Uz$, $f = Vt$ and $g = Ut$, for some z, $t \in \mathfrak{H}$. One successively has

$$\langle y, f\rangle = \langle Uz, Vt\rangle = \langle VUz, t\rangle = \langle UVz, t\rangle = \langle Vz, Ut\rangle = \langle x, g\rangle,$$

which shows that S is a symmetric linear relation in $\mathfrak{H}$.

It will be shown that $A \subset S^2$. One has

$$VV^{-1} = I \restriction_{\operatorname{ran} V} \widehat{+} (\{0\} \times \operatorname{mul} V) = I \restriction_{\operatorname{ran} V} . \tag{3.2}$$

Also,

$$\ker V = \ker V^2 = \ker (I + A)^{-1} = \operatorname{mul} (I + A) = \operatorname{mul} A,$$

and then

$$V^{-1}V = I \restriction_{\operatorname{dom} V} \widehat{+} (\{0\} \times \ker V) = I \,\widehat{+}\, (\{0\} \times \operatorname{mul} A). \tag{3.3}$$

It follows from $UV = VU$ that $UVV^{-1} = VUV^{-1}$ and then by using (3.2) one has $UI\restriction_{\operatorname{ran} V} = VUV^{-1}$. This identity further implies that $VUV^{-1} \subset U$, so that

$$V^{-1}VUV^{-1} \subset V^{-1}U. \tag{3.4}$$

Using (3.3) and (3.4) one successively has

$$UV^{-1} \subset (I \mathbin{\widehat{+}} (\{0\} \times \operatorname{mul} A))UV^{-1} = V^{-1}VUV^{-1} \subset V^{-1}U,$$

so that

$$UV^{-1} \subset V^{-1}U. \tag{3.5}$$

Let $\{x, y\} \in A$, so that $\{x, x+y\} \in I + A$. This implies that

$$\{x+y, x\} \in (I+A)^{-1} = V^2,$$

which shows that

$$\{x, x+y\} \in (V^2)^{-1}, \tag{3.6}$$

and also

$$\{x+y, y\} \in I - (I+A)^{-1} = U^2. \tag{3.7}$$

A combination of (3.6) and (3.7) leads to $\{x, y\} \in U^2(V^2)^{-1}$, which shows that

$$A \subset U^2(V^2)^{-1}. \tag{3.8}$$

Using now (3.5) and (3.8) one successively has

$$A \subset U^2(V^2)^{-1} = UUV^{-1}V^{-1} \subset UV^{-1}UV^{-1} = S^2, \tag{3.9}$$

so that the inclusion $A \subset S^2$ has been proved. Since A is selfadjoint and S^2 is nonnegative it follows from Corollary 2.9 that S is a nonnegative selfadjoint linear relation in $\mathfrak{H}$ and

$$A = S^2. \tag{3.10}$$

Furthermore, a combination of (3.9) and (3.10) shows that in fact

$$A = U^2(V^2)^{-1} = S^2. \tag{3.11}$$

Next it will be shown the uniqueness of the square root. Assume that T is also a nonnegative selfadjoint linear relation such that $T^2 = A$. Then it follows from Lemma 2.10 that

$$A = (S_s)^2 \oplus (\{0\} \times \operatorname{mul} A) = (T_s)^2 \oplus (\{0\} \times \operatorname{mul} A),$$

which implies that $A_s = (S_s)^2 = (T_s)^2$, where A_s, S_s and T_s are nonnegative selfadjoint linear operators in the Hilbert space $\overline{\operatorname{dom} A}$. Using Theorem 2.3 of [11] it follows that $S_s = T_s$, which further implies that

$$S = S_s \oplus (\{0\} \times \operatorname{mul} A) = T_s \oplus (\{0\} \times \operatorname{mul} A) = T,$$

as desired.

(ii) Assume now that B is a bounded everywhere defined linear operator in $\mathfrak{H}$ which commutes with A, that is $BA \subset AB$. Then $B(I+A) \subset (I+A)B$, so that $B(I+A)(I+A)^{-1} \subset (I+A)B(I+A)^{-1}$. This implies that

$$B(I \mathbin{\widehat{+}} (\{0\} \times \operatorname{mul}(I+A))) \subset (I+A)B(I+A)^{-1},$$

which shows that

$$B \subset (I+A)B(I+A)^{-1}. \tag{3.12}$$

Furthermore, it follows from (3.12) that

$$(I+A)^{-1}B \subset (I+A)^{-1}(I+A)B(I+A)^{-1},$$

so that

$$(I+A)^{-1}B \subset B(I+A)^{-1}, \tag{3.13}$$

because $(I+A)^{-1}(I+A)$ is the identity operator in $\mathfrak{H}$ restriced to the domain of A. Since both $(I+A)^{-1}B$ and $B(I+A)^{-1}$ are bounded everywhere defined operators in $\mathfrak{H}$, it follows from (3.13) that

$$(I+A)^{-1}B = B(I+A)^{-1}. \tag{3.14}$$

From (3.14) one has

$$U^2B = BU^2, \quad V^2B = BV^2, \tag{3.15}$$

which further implies that

$$UB = BU, \quad VB = BV. \tag{3.16}$$

It follows from (3.16) that $V^{-1}VB = V^{-1}BV$ so that $B \subset V^{-1}VB = V^{-1}BV$, which implies that $BV^{-1} \subset V^{-1}BVV^{-1} \subset V^{-1}B$. Thus

$$BV^{-1} \subset V^{-1}B. \tag{3.17}$$

Finally, if follows from (3.16) and (3.17) that

$$BS = BUV^{-1} = UBV^{-1} \subset UV^{-1}B = SB,$$

so that the commutation property $BS \subset SB$ has been proved. ∎

The nonnegative selfadjoint linear relation S determined in Theorem 3.1 is called the square root of A and it is denoted by $A^{1/2}$.

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