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FELL TOPOLOGIES FOR AF-ALGEBRAS AND THE QUANTUM PROPINQUITY

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ABSTRACT. We introduce a topology on the ideal space of any C^*-inductive limit built by an inverse limit of topologies and produce conditions for when this topology agrees with the Fell topology. With this topology, we impart criteria for when convergence of ideals of an AF-algebra can provide convergence of quotients in the quantum Gromov–Hausdorff propinquity building from previous joint work with Latrémolière. This bestows a continuous map from a class of ideals of the Boca–Mundici AF-algebra equipped with various topologies, including Jacobson and Fell topologies, to the space of quotients equipped with the propinquity topology.

KEYWORDS: *Noncommutative metric geometry, quantum metric spaces, Gromov–Hausdorff convergence, Lip-norms, AF-algebras, Jacobson topology, Fell topology.*

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1. INTRODUCTION

Latrémolière's quantum Gromov–Hausdorff propinquity [24], [28], [29] provides a powerful tool for studying and constructing new continuous families of compact quantum metric spaces [35], [36], as seen in [22], [24], [25] and [38], [40], [42]. Compact quantum metric spaces, introduced by Rieffel [35] and motivated by A. Connes [9], [10], are unital C^*-algebras equipped with certain metrics on their state spaces built from noncommutative analogues of the Lipschitz seminorm on the algebra of continuous functions on metric spaces. A key contribution of the quantum propinquity to noncommutative metric geometry is that, as a metric, it is compatible with the C^*-algebraic structure [29]. The space of two-sided, closed ideals of a given C^*-algebra can be equipped with some natural topologies, and it is natural to investigate any relationship between these topologies and the propinquity. In particular, as the quotient C^*-algebras of a unital C^*-algebra by non-trivial ideals are again unital, it is reasonable to ask about the continuity of the process of taking the quotient from the space of ideals of a quantum compact metric space, to the space of its quotients endowed with the propinquity. This paper addresses this question by exhibiting sufficient conditions for the continuity of this process. Thus, it is with continuity that we will establish a nontrivial connection between topologies on ideals and the topology formed by quantum propinquity. Therefore, this paper claims to advance both the study of topologies on ideals and noncommutative metric geometry by way of the quantum propinquity topology.

The class of C^*-algebras we shall consider are AF-algebras of Bratteli [7], and in particular, unital AF-algebras with faithful tracial states. Our work with Latrémolière in [2] already provides the quantum metrics we will use on these particular AF-algebras. This will allow us to focus on the continuity question raised in this paper. After a background section, we develop a topology on the ideal space of any C^*-inductive limit. The main application of this topology is to provide a notion of convergence for inductive sequences that determine the quotient spaces as fusing families (Definition 2.12) — a notion introduced in [1] to provide sufficient conditions for convergence in quantum propinquity of AF-algebras. This topology on ideals has close connections to the Fell topology on the ideal space. The Fell topology was introduced in [17] as a topology on closed sets of a given topology. Fell then applied this topology to the closed sets of the Jacobson topology in [16] to provide a compact Hausdorff topology on the set of all ideals of a C^*-algebra. The topology on the ideal space of C^*-inductive limits introduced in this paper is always stronger than the Fell topology, and we provide conditions for when this topology agrees with the Fell topology by way of conditions on the algebraic and analytical properties on the types of ideals themselves. In particular, our topology will agree with the Fell topology for any AF-algebra, in which case we provide an explicit metric that metrizes the topology, and we note that this result is valid for both unital and non-unital AF-algebras. We make other comparisons, including taking into consideration the restriction to primitive ideals and comparison of the Jacobson topology, as well as an analysis on unital commutative AF-algebras and unital C^*-algebras with Hausdorff Jacobson topology.

Next, Section 4 provides an answer to the question of when convergence of ideals can provide convergence of quotients. In Section 4.1, we recall the definition of the Boca–Mundici AF-algebra given in [6], [31], which arises from the Farey tessellation. Next, we prove some basic results pertaining to its Bratteli diagram structure and ideal structure, and then apply our criteria for quotients converging to a subclass of ideals of the Boca–Mundici AF-algebra, in which each quotient is $*$-isomorphic to an Effros–Shen AF-algebra. In [6], Boca proved that this subclass of ideals with its relative Jacobson topology is homeomorphic to the irrationals in $(0, 1)$ with its usual topology, which provided our initial interest in our question about convergence of quotients. The main result of this section, Theorem 4.30, produces a continuous function from a subclass of ideals of the

Boca–Mundici AF-algebra to its quotients as quantum metric spaces in the quantum propinquity topology, where the topology of the subclass ideals is homeomorphic to both the Jacobson and Fell topologies and thus with the topology introduced in this paper as well. Hence, we have an explicit example of when a metric geometry on quotients is related to a metric geometry on ideals by a continuous map. We note that the results in [6] concerning the convergence of ideals motivated the work of this paper along with our results in [2], where the convergence of the Effros–Shen algebras in propinquity were a main example. However, in this paper, the convergence of the Effros–Shen algebras will be provided by different Lip-norms than those found in [2] arising from the different inductive sequences produced by the quotients of the Boca–Mundici AF-algebra, which is a main hurdle that must be overcome in Section 4.1.

2. PRELIMINARIES: QUANTUM METRIC GEOMETRY AND AF-ALGEBRAS

The purpose of this section is to discuss our progress thus far in the realm of quantum metric spaces with regard to AF-algebras, and thus it places more focus on the AF-algebra results. We also provide a cursory overview of the material on quantum compact metric spaces. We refer the reader to the survey by Latrémolière [27] for a much more detailed and insightful introduction to the study of quantum metric spaces.

NOTATION 2.1. The norm of a normed vector space E will be denoted by $\|\cdot\|_E$ by default. The unit of a unital C^*-algebra $\mathfrak{A}$ will be denoted by $1_{\mathfrak{A}}$. The state space of $\mathfrak{A}$ will be denoted by $\mathscr{S}(\mathfrak{A})$ and the self-adjoint part of $\mathfrak{A}$ will be denoted $\mathfrak{sa}(\mathfrak{A})$.

DEFINITION 2.2 ([28], [29], [35]). A (C, D)-*quasi-Leibniz quantum compact metric space* $(\mathfrak{A}, \mathsf{L})$, for some $C \geqslant 1$ and $D \geqslant 0$, is an ordered pair where $\mathfrak{A}$ is unital C^*-algebra and L is a seminorm defined on a dense Jordan–Lie subalgebra $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$ such that:

(i) $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$;

(ii) L is a (C, D)-*quasi-Leibniz Lip-norm*, i.e. for all $a, b \in \mathrm{dom}(\mathsf{L})$:

$$\max\left\{\mathsf{L}\Big(\frac{ab+ba}{2}\Big), \mathsf{L}\Big(\frac{ab-ba}{2\mathrm{i}}\Big)\right\} \leqslant C(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \|b\|_{\mathfrak{A}}\mathsf{L}(a)) + D\mathsf{L}(a)\mathsf{L}(b);$$

(iii) the *Monge–Kantorovich metric* defined, for all two states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$, by

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$$

metrizes the weak* topology of $\mathscr{S}(\mathfrak{A})$;

(iv) the seminorm L is lower semi-continuous with respect to $\|\cdot\|_{\mathfrak{A}}$.

A primary interest in developing a theory of quantum metric spaces is the introduction of various hypertopologies on classes of such spaces, thus allowing us to study the geometry of classes of C^*-algebras and perform analysis on these classes as for instance with this current paper. A classical model for these hypertopologies is given by the Gromov–Hausdorff distance [19], [20]. While several noncommutative analogues of the Gromov–Hausdorff distance have been proposed — most importantly Rieffel's original construction of a quantum version of the Gromov–Hausdorff distance [37] — we shall work with a particular metric introduced by Latrémolière, [29], as we did in [2]. This metric, known as the quantum propinquity, is designed to be best suited to quasi-Leibniz quantum compact metric spaces, and in particular, is zero between two such spaces if and only if they are quantum isometric, a notion defined in the next theorem, which implies in particular that the C^*-algebras of these spaces are $*$-isomorphic.

THEOREM-DEFINITION 2.3 ([28], [29]). *Fix $C \in \mathbb{R}, D \in \mathbb{R}$ such that $C \geqslant 1$ and $D \geqslant 0$. Let $\mathcal{QQCMS}_{C,D}$ be the class of all (C, D)-quasi-Leibniz quantum compact metric spaces. There exists a class function $\Lambda_{C,D}$ from $\mathcal{QQCMS}_{C,D} \times \mathcal{QQCMS}_{C,D}$ to $[0, \infty) \subseteq \mathbb{R}$ such that:*

(i) *for any $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$ we have:*

$$\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \max\{\mathrm{diam}(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}), \mathrm{diam}(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})\};$$

(ii) *for any $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$ we have:*

$$0 \leqslant \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = \Lambda_{C,D}((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}));$$

(iii) *for any $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{C}, \mathsf{L}_{\mathfrak{C}}) \in \mathcal{QQCMS}_{C,D}$ we have:*

$$\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{C}, \mathsf{L}_{\mathfrak{C}})) \leqslant \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) + \Lambda_{C,D}((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{C}, \mathsf{L}_{\mathfrak{C}}));$$

(iv) *for any $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$ and for any bridge γ from $\mathfrak{A}$ to $\mathfrak{B}$ defined in Definition* 3.6 *in* [29], *we have:*

$$\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \lambda(\gamma|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}),$$

where $\lambda(\gamma|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}})$ is defined in Definition 3.17 *in* [29];

(v) *for any $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$, we have:*

$$\Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$$

if and only if $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are quantum isometric, i.e. if and only if there exists a $*$-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ with $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$;

(vi) *if Ξ is a class function from $\mathcal{QQCMS}_{C,D} \times \mathcal{QQCMS}_{C,D}$ to $[0, \infty)$ which satisfies properties* (ii), (iii) *and* (iv) *above, then, for all $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ in $\mathcal{QQCMS}_{C,D}$,*

$$\Xi((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \Lambda_{C,D}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})).$$

The quantum propinquity is, in fact, a special form of the dual Gromov–Hausdorff propinquity [23], [26], [28] also introduced by Latrémolière, which is a complete metric, up to quantum isometry, on the class of Leibniz quantum compact metric spaces, and which extends the topology of the Gromov–Hausdorff distance as well. Thus, as the dual propinquity is dominated by the quantum propinquity [26], we conclude that *all the convergence results in this paper are valid for dual Gromov–Hausdorff propinquity as well.*

In this paper, all our quantum metrics will be $(2,0)$-quasi-Leibniz quantum compact metric spaces. Thus, we will simplify our notation as follows in the next convention.

CONVENTION 2.4. In this paper, Λ will be meant for $\Lambda_{2,0}$.

Now, we recall some results from [1], [2]. The Lip-norms from our work in [2] turn out to be $(2,0)$-quasi-Leibniz Lip-norms. To express our Theorem 2.6, which is Theorem 3.5 in [2], we will need the following notations.

In this paper, we will denote $\mathbb{N} = \{0, 1, 2, \ldots\}$.

NOTATION 2.5. Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ be an inductive sequence, in which $\mathfrak{A}_n$ is a C^*-algebra and $\alpha_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ is a $*$-homomorphism for all $n \in \mathbb{N}$, with limit $\mathfrak{A} = \varinjlim \mathcal{I}$. We denote the canonical $*$-homomorphisms $\mathfrak{A}_n \to \mathfrak{A}$ by $\underrightarrow{\alpha}^n$ for all $n \in \mathbb{N}$ (see Chapter 6.1 in [32]).

We display our Lip-norms built from faithful tracial states which will be utilized throughout the paper. These Lip-norms were motivated by the work of E. Christensen and C. Ivan in [8].

THEOREM 2.6 (Theorem 3.5 in [2]). *Let $\mathfrak{A}$ be a unital AF-algebra endowed with a faithful tracial state μ. Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ be an inductive sequence of finite dimensional C^*-algebras with C^*-inductive limit $\mathfrak{A}$, with $\mathfrak{A}_0 \cong \mathbb{C}$ and where α_n is unital and injective for all $n \in \mathbb{N}$. Let π be the GNS representation of $\mathfrak{A}$ constructed from μ on the space $L^2(\mathfrak{A}, \mu)$. For all $n \in \mathbb{N}$, let:*

$$\mathbb{E}(\cdot \mid \underrightarrow{\alpha}^n(\mathfrak{A}_n)) : \mathfrak{A} \to \mathfrak{A}$$

be the unique conditional expectation of $\mathfrak{A}$ onto the canonical image $\underrightarrow{\alpha}^n(\mathfrak{A}_n)$ of $\mathfrak{A}_n$ in $\mathfrak{A}$, and such that $\mu \circ \mathbb{E}(\cdot \mid \underrightarrow{\alpha}^n(\mathfrak{A}_n)) = \mu$.

Let $\beta : \mathbb{N} \to (0, \infty)$ have limit 0 at infinity. If, for all $a \in \mathfrak{sa}(\bigcup_{n\in\mathbb{N}} \underrightarrow{\alpha}^n(\mathfrak{A}_n))$, we set

$$\mathsf{L}^{\beta}_{\mathcal{I},\mu}(a) = \sup\left\{ \frac{\|a - \mathbb{E}(a \mid \underrightarrow{\alpha}^n(\mathfrak{A}_n))\|_{\mathfrak{A}}}{\beta(n)} : n \in \mathbb{N} \right\},$$

then $(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu})$ is a $(2,0)$-quasi-Leibniz quantum compact metric space. Moreover,

$$\Lambda((\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{I},\mu} \circ \underrightarrow{\alpha}^n), (\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu})) \leqslant \beta(n)$$

for all $n \in \mathbb{N}$, and thus:

$$\lim_{n\to\infty} \Lambda((\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{I},\mu} \circ \underrightarrow{\alpha}^n), (\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{I},\mu})) = 0.$$

CONVENTION 2.7. If we have a unital AF-algebra $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, where $\mathfrak{A}_n$ is a unital finite dimensional C^*-subalgebra of $\mathfrak{A}$ and $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$ for all $n \in \mathbb{N}$

with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$, equipped with a faithful tracial state μ, then we denote the sequence of subalgebras by $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$. And in Theorem 2.6, we use the inclusion mapping $\iota_n : \mathfrak{A}_n \to \mathfrak{A}$ in place of $\underrightarrow{\alpha^n}$ for each $n \in \mathbb{N}$, the same results of Theorem 2.6 will follow in this setting given the same hypotheses, where we denote the Lip-norm by $\mathsf{L}^{\beta}_{\mathcal{U},\mu}$.

In [2], the fact that the defining finite-dimensional subalgebras provide explicit approximations of the inductive limit with respect to the quantum Gromov–Hausdorff propinquity allowed us to prove that both the UHF algebras and the Effros–Shen AF-algebras are continuous images of the Baire space with respect to the quantum propinquity. Our pursuit was motivated by the fact that the Effros–Shen algebras were used by Pimsner and Voiculescu to classify the irrational rotation algebras [34], while Latrémolière showed continuity of the irrational rotation algebras in propinquity with respect to their irrational parameters in [24]. We list the Effros–Shen algebra result here, since we will utilize both the definition of the Effros–Shen algebras and the continuity result in Section 4.1 extensively.

We begin by recalling the construction of the Effros–Shen algebras, denoted by $\mathfrak{AF}_\theta$, constructed in [15] for any irrational θ in $(0,1)$. For such θ, let $(a_j)_{j\in\mathbb{N}}$ be the unique sequence in $\mathbb{N}$ such that

$$\theta = \lim_{n\to\infty}[a_0, a_1, \ldots, a_n], \tag{2.1}$$

where $[a_0, a_1, \ldots, a_n]$ denotes the standard continued fraction. The sequence of natural numbers $(a_j)_{j\in\mathbb{N}}$ is called the *continued fraction expansion* of θ, and we will simply denote it by writing $\theta = [a_0, a_1, a_2, \ldots] = [a_j]_{j\in\mathbb{N}}$. We note that $a_0 = 0$ (since $\theta \in (0,1)$) and $a_n \in \mathbb{N}\setminus\{0\}$ for $n \geqslant 1$.

We fix $\theta \in (0,1)\setminus\mathbb{Q}$, and let $\theta = [a_j]_{j\in\mathbb{N}}$ be its continued fraction decomposition. We then obtain the sequence $(p_n^\theta/q_n^\theta)_{n\in\mathbb{N}}$ of *convergents* of θ with $p_n^\theta \in \mathbb{N}$, $q_n^\theta \in \mathbb{N}\setminus\{0\}$ by setting:

$$\begin{cases} \begin{pmatrix} p_1^\theta & q_1^\theta \\ p_0^\theta & q_0^\theta \end{pmatrix} = \begin{pmatrix} a_0a_1+1 & a_1 \\ a_0 & 1 \end{pmatrix}, \\ \begin{pmatrix} p_{n+1}^\theta & q_{n+1}^\theta \\ p_n^\theta & q_n^\theta \end{pmatrix} = \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n^\theta & q_n^\theta \\ p_{n-1}^\theta & q_{n-1}^\theta \end{pmatrix} \text{ for all } n \in \mathbb{N}\setminus\{0\}. \end{cases} \tag{2.2}$$

We note that $p_n^\theta/q_n^\theta = [a_0, a_1, \ldots, a_n]$ for all $n \in \mathbb{N}$, and $(p_n^\theta/q_n^\theta)_{n\in\mathbb{N}}$ converges to θ (see [21]).

Expression (2.2) contains the crux for the construction of the Effros–Shen AF-algebras.

NOTATION 2.8. Throughout this paper, we shall employ the notation $x \oplus y \in X \oplus Y$ to mean that $x \in X$ and $y \in Y$ for any two vector spaces X and Y whenever no confusion may arise, as a slight yet convenient abuse of notation.

NOTATION 2.9. Let $\theta \in (0,1) \setminus \mathbb{Q}$ and $\theta = [a_j]_{j\in\mathbb{N}}$ be the continued fraction expansion of θ. Let $(p_n^\theta)_{n\in\mathbb{N}}$ and $(q_n^\theta)_{n\in\mathbb{N}}$ as in (2.2). We set $\mathfrak{AF}_{\theta,0} = \mathbb{C}$ and

$$\mathfrak{AF}_{\theta,n} = \mathfrak{M}(q_n^\theta) \oplus \mathfrak{M}(q_{n-1}^\theta), \quad \alpha_{\theta,n} : a \oplus b \in \mathfrak{AF}_{\theta,n} \longmapsto \begin{pmatrix} a & & & \\ & \ddots & & \\ & & a & \\ & & & b \end{pmatrix} \oplus a \in \mathfrak{AF}_{\theta,n+1},$$

where a appears a_{n+1} times on the diagonal of the right hand side matrix above. We also set α_0 to be the unique unital $*$-morphism from $\mathbb{C}$ to $\mathfrak{AF}_{\theta,1}$. We thus define the Effros–Shen C^*-algebra $\mathfrak{AF}_\theta$, after [15],

$$\mathfrak{AF}_\theta = \varinjlim (\mathfrak{AF}_{\theta,n}, \alpha_{\theta,n})_{n\in\mathbb{N}} = \varinjlim \mathcal{I}_\theta.$$

We now present our continuity result for Effros–Shen AF-algebras from [2]. We note that the Baire space is homeomorphic to the space of irrational numbers in $(0,1)$. A proof of this can be found in Proposition 5.10 in [2].

THEOREM 2.10 (Theorem 5.14 in [2]). *Using Notation 2.9, the function*

$$\theta \in ((0,1) \setminus \mathbb{Q}, |\cdot|) \longmapsto (\mathfrak{AF}_\theta, \mathsf{L}^{\beta_\theta}_{\mathcal{I}_\theta,\sigma_\theta}) \in (\mathcal{QQCMS}_{2,0}, \Lambda)$$

is continuous from $(0,1) \setminus \mathbb{Q}$, with its topology as a subset of $\mathbb{R}$, to the class of $(2,0)$-quasi-Leibniz quantum compact metric spaces metrized by the quantum propinquity Λ, where σ_θ is the unique faithful tracial state on $\mathfrak{AF}_\theta$, and $\beta_\theta(n) = \dim(\mathfrak{AF}_{\theta,n})^{-1}$ for all $n \in \mathbb{N}$.

In [1], we generalized the convergence results in [2] utilizing the notion of a fusing family of inductive sequences. We will utilize this notion and this general convergence theorem in this paper for our quotient convergence results. We list the appropriate definitions and results here.

We now define a notion of fusing inductive sequences in Definition 2.12. Fusing inductive sequences are an equivalent description for the convergence of ideals in an AF-algebra, as will be seen in Lemma 3.23.

NOTATION 2.11. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ denote the Alexandroff compactification of $\mathbb{N}$ with respect to the discrete topology of $\mathbb{N}$. For $N \in \mathbb{N}$, let $\mathbb{N}_{\geqslant N} = \{k \in \mathbb{N} : k \geqslant N\}$, and similarly, for $\overline{\mathbb{N}}_{\geqslant N}$.

DEFINITION 2.12 (Definition 3.5 in [1]). For each $k \in \overline{\mathbb{N}}$, let $(\mathfrak{A}_{k,n}, \alpha_{k,n})_{n\in\mathbb{N}}$ be an inductive sequence with inductive limit, $\mathfrak{A}^k = \varinjlim \mathcal{I}(k)$, where $\mathcal{I}(k) = (\mathfrak{A}_{k,n}, \alpha_{k,n})_{n\in\mathbb{N}}$. We say that the family of C^*-algebras $\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is a *fusing family* of C^*-algebras if:

(i) there exists $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ non-decreasing such that $\lim_{n\to\infty} c_n = \infty$, and

(ii) for all $N \in \mathbb{N}$, if $k \in \overline{\mathbb{N}}_{\geqslant c_N}$, then $(\mathfrak{A}_{k,n}, \alpha_{k,n}) = (\mathfrak{A}_{\infty,n}, \alpha_{\infty,n})$ for all $n \in \{0, 1, \ldots, N\}$.

If this occurs, then we call the sequence $(c_n)_{n\in\mathbb{N}}$ the *fusing sequence*.

CONVENTION 2.13. If for each $k \in \overline{\mathbb{N}}$ we are given a C^*-algebra $\mathfrak{A}^k = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_{k,n}}^{\|\cdot\|_{\mathfrak{A}^k}}$ such that $\mathfrak{A}_{k,n}$ is a C^*-subalgebra of $\mathfrak{A}^k$ and $\mathfrak{A}_{k,n} \subseteq \mathfrak{A}_{k,n+1}$ for all $n \in \mathbb{N}$, and there exists a C^*-algebra $\mathfrak{B}$ such that $\mathfrak{A}_{k,n}$ is a C^*-subalgebra of $\mathfrak{B}$ for all $k \in \overline{\mathbb{N}}, n \in \mathbb{N}$, then we will call $\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ a fusing family if both (i) and (ii) of Definition 2.12 are satisfied with $(\mathfrak{A}_{k,n}, \alpha_{k,n}) = (\mathfrak{A}_{\infty,n}, \alpha_{\infty,n})$ replaced with $\mathfrak{A}_{k,n} = \mathfrak{A}_{\infty,n}$ given the same conditions on k, n. This is simply done by replacing $\alpha_{k,n}, \alpha_{k,\infty}$ with the appropriate inclusion mappings.

Next, we provide our general criteria for convergence of AF-algebras in propinquity using the notion of fusing family along with suitable notions of convergence of the remaining tools used to build our faithful tracial state Lip-norms.

THEOREM 2.14 (Theorem 3.10 in [1]). *For each $k \in \overline{\mathbb{N}}$, let $(\mathfrak{A}_{k,n}, \alpha_{k,n})_{n\in\mathbb{N}}$ be an inductive sequence with inductive limit, $\mathfrak{A}^k = \varinjlim \mathcal{I}(k)$, where $\mathcal{I}(k) = (\mathfrak{A}_{k,n}, \alpha_{k,n})_{n\in\mathbb{N}}$, such that $\mathfrak{A}_{k,0} = \mathfrak{A}_{k',0} \cong \mathbb{C}$ and $\alpha_{k,n}$ is unital and injective for all $k, k' \in \overline{\mathbb{N}}, n \in \mathbb{N}$. If:*

(i) *$\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is a fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$;*

(ii) *$\{\tau^k : \mathfrak{A}^k \to \mathbb{C}\}_{k\in\overline{\mathbb{N}}}$ is a family of faithful tracial states such that for each $N \in \mathbb{N}$, we have that $(\tau^k \circ \underrightarrow{\alpha_k^N})_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $\tau^\infty \circ \underrightarrow{\alpha_\infty^N}$ in the weak* topology on $\mathscr{S}(\mathfrak{A}_{\infty,N})$; and*

(iii) *$\{\beta^k : \overline{\mathbb{N}} \to (0,\infty)\}_{k\in\overline{\mathbb{N}}}$ is a family of convergent sequences such that for all $N \in \mathbb{N}$ if $k \in \mathbb{N}_{\geqslant c_N}$, then $\beta^k(n) = \beta^\infty(n)$ for all $n \in \{0, 1, \ldots, N\}$ and there exists $B : \overline{\mathbb{N}} \to (0,\infty)$ with $B(\infty) = 0$ and $\beta^m(l) \leqslant B(l)$ for all $m, l \in \overline{\mathbb{N}}$;*

then:

$$\lim_{k\to\infty} \Lambda((\mathfrak{A}^k, \mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}), (\mathfrak{A}^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\infty),\tau^\infty})) = 0,$$

where $\mathsf{L}^{\beta^k}_{\mathcal{I}(k),\tau^k}$ is given by Theorem 2.6.

This theorem generalized the UHF and Effros–Shen algebra convergence results of [2], in which we showed this in the Effros–Shen algebra case in the proof of Theorem 3.14 in [1].

3. A TOPOLOGY ON THE IDEAL SPACE OF C^*-INDUCTIVE LIMITS

For a fixed C^*-algebra, the ideal space may be endowed with various natural topologies. We may identify each ideal with a quotient, which is a C^*-algebra itself. Now, this defines a function from the ideal space, which has natural topologies, to the class of C^*-algebras. If each quotient has a quasi-Leibniz Lip-norm, then this function becomes much more intriguing as we are able to discuss its continuity by endowing its codomain with the topology of the propinquity. Towards this, we develop a topology on ideals of any C^*-inductive limit that is compatible with this goal. The purpose of this is to allow fusing families of ideals to provide fusing families of quotients in Proposition 3.26 — a first step in providing convergence of quotients in quantum propinquity. While our new topology is strongly motivated by the Fell topology, it may be stronger, though it does coincide with the Fell topology for AF-algebras. As a side-product, we will also provide a metric for the Fell topology in the case of AF-algebras. In order to construct our topology, we will use the given inductive sequence of a C^*-inductive limit to construct an inverse limit topology from the Fell topologies of the limit of the inductive sequence. Also, our new topology is designed to fit our problem more naturally than the Fell topology — even when it actually agrees with the latter, in which case our new topology becomes a more adequate description of the Fell topology for our purpose. But, we first define the Fell topology on ideals and prove some basic results, which requires the Jacobson topology on the class of primitive ideals. In the next definition, we recall a possible definition of the Jacobson topology for our purpose and refer to [32] for a study of this topology.

DEFINITION 3.1. Let $\mathfrak{A}$ be a C^*-algebra. Denote the *set of norm closed two-sided ideals* of $\mathfrak{A}$ by $\mathrm{Ideal}(\mathfrak{A})$, including the trivial ideals $\{0\}$ and $\mathfrak{A}$. Denote by $\mathrm{Prim}(\mathfrak{A})$ the set

$$\{J \in \mathrm{Ideal}(\mathfrak{A}) : J = \ker \pi, \pi \neq 0 \text{ irreducible } *\text{-representation of } \mathfrak{A}\}.$$

Note that $\mathfrak{A} \notin \mathrm{Prim}(\mathfrak{A})$.

The *Jacobson topology* is the topology on $\mathrm{Prim}(\mathfrak{A})$, denoted by $Jacobson$, such that for each closed set F there exists a unique ideal $I_F \in \mathrm{Ideal}(\mathfrak{A})$ such that $F = \{J \in \mathrm{Prim}(\mathfrak{A}) : J \supseteq I_F\}$ (see Theorem 5.4.2, Theorem 5.4.6, and Theorem 5.4.7 in [32]).

CONVENTION 3.2. Given a C^*-algebra $\mathfrak{A}$ and $I \in \mathrm{Ideal}(\mathfrak{A})$, an element of the quotient $\mathfrak{A}/I$ will be denoted by $a + I$ for some $a \in \mathfrak{A}$. Furthermore, the quotient norm will be denoted by $\|a + I\|_{\mathfrak{A}/I} = \inf\{\|a + b\|_{\mathfrak{A}} : b \in I\}$.

Now, we may define the Fell topology, which is a topology on all ideals of a C^*-algebra. We begin by presenting the definition of the Fell topology on closed sets of any topological space along with some facts, which will help with some later proofs.

DEFINITION 3.3 ([17]). Let (X, τ) be a topological space (no further assumptions made). Let $\mathcal{C}l(X)$ denote the set of closed subsets of X. Let K be a compact set of X, and let F be a finite family of non-empty open subsets of X. Define:

$$U(K, F) = \{Y \in \mathcal{C}l(X) : Y \cap K = \emptyset \text{ and } Y \cap A \neq \emptyset \text{ for all } A \in F\}.$$

The Fell topology $\tau_{\mathcal{C}l(X)}$ on $\mathcal{C}l(X)$ is generated by the topological basis

$$\{U(K, F) \subseteq \mathcal{C}l(X) : K \subseteq X \text{ is compact and } F \subseteq \tau \setminus \{\emptyset\}, F \text{ finite}\}.$$

We list some facts about this topology and the striking conclusion that the Fell topology on $\mathcal{C}l(X)$ is always compact and Hausdorff when X is locally compact.

LEMMA 3.4 (Lemma 1 and Theorem 1 in [17]). *If (X, τ) is a topological space, then the topological space $(\mathcal{C}l(X), \tau_{\mathcal{C}l(X)})$ is compact.*

If (X,τ) is a locally compact space, then the topological space $(\mathcal{C}l(X), \tau_{\mathcal{C}l(X)})$ is compact Hausdorff.

Next, we are in a position to apply this to build a topology on the ideal space of a C^*-algebra.

DEFINITION 3.5 ([16]). Let $\mathfrak{A}$ be a C^*-algebra. Let $\mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))$ be the set of closed subsets of $(\mathrm{Prim}(\mathfrak{A}), Jacobson)$ with compact Hausdorff topology, denoted by $\tau_{\mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))}$, given by Theorem 2.2 in [16] and more generally [17] and Lemma 3.4. Define

$$fell : I \in \mathrm{Ideal}(\mathfrak{A}) \mapsto \{J \in \mathrm{Prim}(\mathfrak{A}) : J \supseteq I\},$$

where $fell(\mathrm{Ideal}(\mathfrak{A})) = \mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))$ by Theorem 5.4.7 in [32].

The *Fell topology* on $\mathrm{Ideal}(\mathfrak{A})$, denoted $Fell$, is given by

$$Fell = \{U \subseteq \mathrm{Ideal}(\mathfrak{A}) : U = fell^{-1}(V), V \in \tau_{\mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))}\}$$

and $fell$ is continuous by definition. Also, $(\mathrm{Ideal}(\mathfrak{A}), Fell)$ is compact Hausdorff since $fell$ is a bijection.

The following lemma is stated in Section 2 in [3], where the Fell topology $Fell$ is denoted τ_s. We provide a proof.

LEMMA 3.6. *Let $\mathfrak{A}$ be a C^*-algebra. Let $(I_\mu)_{\mu\in\Delta} \subseteq \mathrm{Ideal}(\mathfrak{A})$ be a net and $I \in \mathrm{Ideal}(\mathfrak{A})$. The net $(I_\mu)_{\mu\in\Delta}$ converges to I with respect to the Fell topology if and only if the net $(\|a + I_\mu\|_{\mathfrak{A}/I_\mu})_{\mu\in\Delta} \subseteq \mathbb{R}$ converges to $\|a + I\|_{\mathfrak{A}/I} \in \mathbb{R}$ for all $a \in \mathfrak{A}$.*

Proof. Let $Y \in \mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))$. By Theorem 2.2 in [16], define:

$$M_Y : a \in \mathfrak{A} \longmapsto \sup\{\|a + I\|_{\mathfrak{A}/I} : I \in Y\} \in \mathbb{R}.$$

By the first proof line of Theorem 2.2 in [16], we note that $\bigcap_{I\in Y} I \in \mathrm{Ideal}(\mathfrak{A})$ and

$$M_Y(a) = \left\| a + \bigcap_{I\in Y} I \right\|_{\mathfrak{A}/(\bigcap_{I\in Y} I)} \quad \text{for all } a \in \mathfrak{A} : . \tag{3.1}$$

Let $P \in \mathrm{Ideal}(\mathfrak{A})$, then $fell(P) = \{J \in \mathrm{Prim}(\mathfrak{A}) : J \supseteq P\} \in \mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))$ by Definition 3.5. Note that $\bigcap_{H\in fell(P)} H = P$ by Theorem 5.4.3 in [32]. Thus, by (3.1)

$$M_{fell(P)}(a) = \|a + P\|_{\mathfrak{A}/P}. \tag{3.2}$$

Now, assume that $(I_\mu)_{\mu\in\Delta} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I \in \mathrm{Ideal}(\mathfrak{A})$ with respect to the Fell topology. Since $fell$ is continuous, the net $(fell(I_\mu))_{\mu\in\Delta} \subseteq \mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))$ converges to $fell(I) \in \mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))$ with respect to the topology on $\mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))$. By Theorem 2.2 in [16], the net of functions $(M_{fell(I_\mu)})_{\mu\in\Delta}$ converges pointwise to $M_{fell(I)}$, which completes the forward implication by (3.2).

For the reverse implication, assume that $(\|a + I_\mu\|_{\mathfrak{A}/I_\mu})_{\mu\in\Delta} \subseteq \mathbb{R}$ converges to $\|a + I\|_{\mathfrak{A}/I} \in \mathbb{R}$ with respect to the usual topology on $\mathbb{R}$ for all $a \in \mathfrak{A}$ and for some net $(I_\mu)_{\mu\in\Delta} \subseteq \mathrm{Ideal}(\mathfrak{A})$ and $I \in \mathrm{Ideal}(\mathfrak{A})$. But then, by (3.2) and the assumption, the net $(M_{fell(I_\mu)})_{\mu\in\Delta}$ converges pointwise to $M_{fell(I)}$. By Theorem 2.2 in [16], the net $(fell(I_\mu))_{\mu\in\Delta} \subseteq \mathcal{Cl}(\mathrm{Prim}(\mathfrak{A}))$ converges to $fell(I) \in \mathcal{Cl}(\mathrm{Prim}(\mathfrak{A}))$ with respect to the topology on $\mathcal{Cl}(\mathrm{Prim}(\mathfrak{A}))$. However, as $fell$ is a continuous bijection between the compact Hausdorff spaces $(\mathrm{Ideal}(\mathfrak{A}), Fell)$ and $(\mathcal{Cl}(\mathrm{Prim}(\mathfrak{A})), \tau_{\mathcal{Cl}(\mathrm{Prim}(\mathfrak{A}))})$, the map $fell$ is a homeomorphism. Thus, we conclude that $(I_\mu)_{\mu\in\Delta}$ converges to I with respect to the Fell topology. ∎

Now, the Fell topology induces a topology on $\mathrm{Prim}(\mathfrak{A})$ via its relative topology. But, the set $\mathrm{Prim}(\mathfrak{A})$ can also be equipped with the Jacobson topology (see Definition 3.1). Thus, a comparison of both topologies is in order in Proposition 3.7, which can be proven using Lemma 3.6.

PROPOSITION 3.7. *The relative topology induced by the Fell topology of Definition* 3.5 *on* $\mathrm{Prim}(\mathfrak{A})$ *contains the Jacobson topology of Definition* 3.1 *on* $\mathrm{Prim}(\mathfrak{A})$.

Proof. Let $F \subseteq \mathrm{Prim}(\mathfrak{A})$ be closed in the Jacobson topology. Then, there exists a unique $I_F \in \mathrm{Ideal}(\mathfrak{A})$ with $F = \{J \in \mathrm{Prim}(\mathfrak{A}) : J \supseteq I_F\}$ by Definition 3.5.

Let $J \in \mathrm{Prim}(\mathfrak{A})$ such that there exists a net $(J^\mu)_{\mu\in\Delta} \subseteq F$ that converges to $J \in \mathrm{Prim}(\mathfrak{A})$ in the Fell topology. Let $x \in I_F$, then $x \in J^\mu$ for all $\mu \in \Delta$. Thus $(\|x + J^\mu\|_{\mathfrak{A}/J^\mu})_{\mu\in\Delta} = (0)_{\mu\in\Delta}$, and this net also converges to $\|x + J\|_{\mathfrak{A}/J}$ by Lemma 3.6. Thus $\|x + J\|_{\mathfrak{A}/J} = 0$, which implies that $x \in J$. Hence $J \supseteq I_F$, and since $J \in \mathrm{Prim}(\mathfrak{A})$ we have $J \in F$.

Thus, F is closed in the relative topology on $\mathrm{Prim}(\mathfrak{A})$ induced by the Fell topology, which verifies the containment of the topologies. ∎

Now, we want to build a topology on the ideal space of a C^*-inductive limit using the inverse limit of the Fell topologies on the ideal spaces of each C^*-algebra of the inductive sequence. In order to build an inverse limit, we need continuous maps, gifted to us by the next lemma.

LEMMA 3.8. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* C^**-algebras such that there exists a* $*$*-monomorphism* $\pi : \mathfrak{A} \longrightarrow \mathfrak{B}$, *then the following map is continuous with respect to the associated Fell topologies:*

$$\pi_i : J \in \mathrm{Ideal}(\mathfrak{B}) \mapsto \pi^{-1}(J) \in \mathrm{Ideal}(\mathfrak{A}).$$

Proof. The map π_i is well-defined since π is a $*$-homomorphism. For continuity, we first prove the following helpful claim.

CLAIM 3.9. *Let* $\mathfrak{A}$ *be a* C^**-algebra and let* $\mathfrak{A}_k$ *be a* C^**-subalgebra. If* $J \in \mathrm{Ideal}(\mathfrak{A})$, *then the following map is a* $*$*-monomorphism and thus an isometry:*

$$\phi_J^k : a + J \cap \mathfrak{A}_k \in \mathfrak{A}_k/(J \cap \mathfrak{A}_k) \longmapsto a + J \in \mathfrak{A}/J. \tag{3.3}$$

Proof of claim. Assume that $a, b \in \mathfrak{A}_k$ such that $a + J \cap \mathfrak{A}_k = b + J \cap \mathfrak{A}_k$, which implies that $a - b \in J \cap \mathfrak{A}_k \subseteq J \Rightarrow a + J = b + J$, and thus ϕ_J^k is well-defined. Next, assume that $a, b \in \mathfrak{A}_k$ such that $a + J = b + J$, which implies that $a - b \in J$. But, we have $a - b \in \mathfrak{A}_k \Rightarrow a - b \in J \cap \mathfrak{A}_k$ and $a + J \cap \mathfrak{A}_k = b + J \cap \mathfrak{A}_k$, which provides injectivity. Thus, for each $k \in \mathbb{N}$, we have ϕ_J^k is a well-defined injective $*$-homomorphism since J is an ideal. Hence, the map ϕ_J^k is an isometry for any $J \in \mathrm{Ideal}(\mathfrak{A})$, which proves the claim. ∎

To continue with continuity, let $(J_\mu)_{\mu\in\Pi} \subseteq \mathrm{Ideal}(\mathfrak{B})$ be a net that converges to $J \in \mathrm{Ideal}(\mathfrak{B})$ with respect to the Fell topology. Fix $a \in \mathfrak{A}$ and $\mu \in \Pi$, then since π is injective, isometric, and surjects onto $J_\mu \cap \pi(\mathfrak{A})$:

$$\begin{aligned}\|a+\pi_i(J_\mu)\|_{\mathfrak{A}/\pi_i(J_\mu)} &= \|a+\pi^{-1}(J_\mu\cap\pi(\mathfrak{A}))\|_{\mathfrak{A}/\pi_i(J_\mu)} = \inf_{a'\in\pi^{-1}(J_\mu\cap\pi(\mathfrak{A}))}\|a-a'\|_{\mathfrak{A}}\\ &= \inf_{\pi(a')\in J_\mu\cap\pi(\mathfrak{A})}\|\pi(a)-\pi(a')\|_{\mathfrak{B}} = \inf_{b\in J_\mu\cap\pi(\mathfrak{A})}\|\pi(a)-b\|_{\mathfrak{B}}\\ &= \|\pi(a)+(J_\mu\cap\pi(\mathfrak{A}))\|_{\pi(\mathfrak{A})/(J_\mu\cap\pi(\mathfrak{A}))} = \|\pi(a)+J_\mu\|_{\mathfrak{B}/J_\mu},\end{aligned}$$

where we used Claim 3.9 in the last equality. Thus, by Lemma 3.6, we have that $(\pi_i(J_\mu))_{\mu\in\Pi} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $\pi_i(J) \in \mathrm{Ideal}(\mathfrak{A})$ in the Fell topology, which concludes the proof. ∎

We now present a construction which will be applied later on to inductive limits to obtain our new topology. Following [43], we define the inverse limit sequence of topological spaces and its limit.

DEFINITION 3.10. A family $(X_n, \tau_n, f_{n+1})_{n\in\mathbb{N}}$ is an *inverse limit sequence* of topological spaces if $(X_n, \tau_n)_{n\in\mathbb{N}}$ is a family of topological spaces and $(f_{n+1})_{n\in\mathbb{N}}$ is a family of continuous functions such that $f_{n+1} : X_{n+1} \to X_n$ for all $n \in \mathbb{N}$. The *inverse limit space* of $(X_n, \tau_n, f_{n+1})_{n\in\mathbb{N}}$, denoted by (X_∞, τ_∞), is the subset X_∞ of $\prod_{n\in\mathbb{N}} X_n$ defined by:

$$X_\infty = \left\{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : f_{n+1}(x_{n+1}) = x_n \text{ for all } n \in \mathbb{N}\right\},$$

where τ_∞ is the topology on X_∞ given by the relative topology induced by the product topology on $\prod_{n\in\mathbb{N}} X_n$ with respect to the given topologies τ_n on X_n for all $n \in \mathbb{N}$.

Our topology on the ideal space will be induced by an initial topology by the following map once our inverse limit is established.

PROPOSITION 3.11. *Let $\mathfrak{A}$ be a C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. The following map is a well-defined injection:*

$$i(\cdot,\mathcal{U}) : I \in \mathrm{Ideal}(\mathfrak{A}) \longmapsto (I\cap\mathfrak{A}_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathrm{Ideal}(\mathfrak{A}_n).$$

Proof. Since $I \in \mathrm{Ideal}(\mathfrak{A})$ and $\mathfrak{A}_n$ is a C^*-subalgebra for all $n \in \mathbb{N}$, we have that $I \cap \mathfrak{A}_n \in \mathrm{Ideal}(\mathfrak{A}_n)$ for all $n \in \mathbb{N}$. Thus, the map $i(\cdot,\mathcal{U})$ is well-defined.

Next, for injectivity, assume that $I, J \in \text{Ideal}(\mathfrak{A})$ such that $i(I,\mathcal{U}) = i(J,\mathcal{U})$. Hence, $I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n$ for all $n \in \mathbb{N}$, which implies $\bigcup_{n\in\mathbb{N}} (I \cap \mathfrak{A}_n) = \bigcup_{n\in\mathbb{N}} (J \cap \mathfrak{A}_n)$. Therefore, $\overline{\bigcup_{n\in\mathbb{N}} (I \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\bigcup_{n\in\mathbb{N}} (J \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}}$. But, by Lemma III.4.1 in [12], we conclude that $I = J$. ∎

The following produces the remaining ingredients for our topology.

LEMMA 3.12. *Let $\mathfrak{A}$ be a C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. For each $n \in \mathbb{N}$, we denote $\iota_{n+1} : J \in \text{Ideal}(\mathfrak{A}_{n+1}) \to J \cap \mathfrak{A}_n \in \text{Ideal}(\mathfrak{A}_n)$. Then the family $(\text{Ideal}(\mathfrak{A}_n), Fell, \iota_{n+1})_{n\in\mathbb{N}}$ is an inverse limit sequence with non-empty compact Hausdorff inverse limit space*

$$\text{Ideal}(\mathfrak{A})_\infty = \left\{ (J_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \text{Ideal}(\mathfrak{A}_n) : J_{n+1} \cap \mathfrak{A}_n = J_n \textit{ for all } n \in \mathbb{N} \right\}$$

equipped with the topology $Fell_\infty$ of Definition 3.10*, and thus, using notation from Proposition* 3.11

$$i(\text{Ideal}(\mathfrak{A}), \mathcal{U}) \subseteq \text{Ideal}(\mathfrak{A})_\infty.$$

Proof. The conclusions follow immediately from Lemma 3.8, Definition 3.10, and Proposition 3.11. The non-empty compact Hausdorff conclusion follows from Theorem 29.11 in [43] and the fact that $\text{Ideal}(\mathfrak{A}_n)$ equipped with the Fell topology, is a non-empty compact Hausdorff space for each $n \in \mathbb{N}$. ∎

DEFINITION 3.13. Let $\mathfrak{A}$ be a C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. By Lemma 3.12, the initial topology induced by $i(\cdot,\mathcal{U})$ and the topological space $(\text{Ideal}(\mathfrak{A})_\infty, Fell_\infty)$ on $\text{Ideal}(\mathfrak{A})$ exists and is Hausdorff (by injectivity of $i(\cdot,\mathcal{U})$), which we denote it by $Fell_{i(\mathcal{U})}$.

We will now provide some sufficient conditions for when $Fell_{i(\mathcal{U})}$ agrees with $Fell$. We first show that it is always the case that $Fell \subseteq Fell_{i(\mathcal{U})}$.

PROPOSITION 3.14. *If $\mathfrak{A}$ is a C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, then the Fell topology on $\text{Ideal}(\mathfrak{A})$ is contained in the topology $Fell_{i(\mathcal{U})}$ of Definition* 3.13.

Proof. Let $(J_\mu)_{\mu\in\Pi} \subseteq \text{Ideal}(\mathfrak{A})$ be a net that converges to $J \in \text{Ideal}(\mathfrak{A})$ with respect to $Fell_{i(\mathcal{U})}$. Hence, the net $(i(J_\mu,\mathcal{U}))_{\mu\in\Pi} \subseteq \text{Ideal}(\mathfrak{A})_\infty$ converges to $i(J,\mathcal{U}) \in \text{Ideal}(\mathfrak{A})_\infty$ with respect to $Fell_\infty$ by definition. Again, by definition, the net $(i(J_\mu,\mathcal{U}))_{\mu\in\Pi} \subseteq \text{Ideal}(\mathfrak{A})_\infty$ converges to $i(J,\mathcal{U}) \in \text{Ideal}(\mathfrak{A})_\infty$ with respect to the product topology on $\prod_{n\in\mathbb{N}} \text{Ideal}(\mathfrak{A}_n)$, where each $\text{Ideal}(\mathfrak{A}_n)$ is equipped with topology $Fell$.

First, fix $a \in \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$. So, there exists $N \in \mathbb{N}$ such that $a \in \mathfrak{A}_N$. Let $\mu \in \Pi$. Thus, by Claim 3.9,

$$\|a + J_\mu\|_{\mathfrak{A}/J_\mu} = \|a + J_\mu \cap \mathfrak{A}_N\|_{\mathfrak{A}_N/(J_\mu\cap\mathfrak{A}_N)}.$$

Since the projection maps are continuous for the product topology, we conclude that the net $(\|a + J_\mu \cap \mathfrak{A}_N\|_{\mathfrak{A}_N/(J_\mu\cap\mathfrak{A}_N)})_{\mu\in\Pi}$ converges to $\|a + J \cap \mathfrak{A}_N\|_{\mathfrak{A}_N/(J\cap\mathfrak{A}_N)}$ in the usual topology on $\mathbb{R}$.

Hence, the net $(\|a+J_\mu\|_{\mathfrak{A}/J_\mu})_{\mu\in\Pi}$ converges to $\|a+J\|_{\mathfrak{A}/J}$ in the usual topology on $\mathbb{R}$ for all $a \in \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$. Now, let $a \in \mathfrak{A}$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ and $a_N \in \mathfrak{A}_N \subseteq \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ such that $\|a - a_N\|_{\mathfrak{A}} < \varepsilon/3$. Thus, there exists $\mu_0 \in \Pi$ such that for all $\mu \geqslant \mu_0$, we have:

$$|\|a_N + J_\mu\|_{\mathfrak{A}/J_\mu} - \|a_N + J\|_{\mathfrak{A}/J}| < \frac{\varepsilon}{3}.$$

Hence, if $\mu \geqslant \mu_0$, then:

$$\begin{aligned}
&|\|a + J_\mu\|_{\mathfrak{A}/J_\mu} - \|a + J\|_{\mathfrak{A}/J}| \\
&\qquad \leqslant |\|a + J_\mu\|_{\mathfrak{A}/J_\mu} - \|a_N + J_\mu\|_{\mathfrak{A}/J_\mu}| \\
&\qquad\qquad +|\|a_N+J_\mu\|_{\mathfrak{A}/J_\mu}-\|a_N+J\|_{\mathfrak{A}/J}|+|\|a_N+J\|_{\mathfrak{A}/J}-\|a+J\|_{\mathfrak{A}/J}| \\
&\qquad < \|a - a_N+J_\mu\|_{\mathfrak{A}/J_\mu}+\frac{\varepsilon}{3}+\|a_N-a+J\|_{\mathfrak{A}/J}\leqslant 2\|a-a_N\|_{\mathfrak{A}}+\frac{\varepsilon}{3}<\varepsilon,
\end{aligned}$$

which completes the proof by Lemma 3.6. ∎

Thus, by this proposition and Lemma 3.12, if it is also the case that the topology $Fell_{i(\mathcal{U})}$ is compact, it must agree with the topology *Fell* by maximal compactness of Hausdorff spaces. An obvious way that this would be true is if the map $i(\cdot,\mathcal{U})$ surjected onto $\mathrm{Ideal}(\mathfrak{A})_\infty$. It turns out that this is the case for all AF-algebras, and we provide a characterization of this scenario by a condition on the algebraic ideals of a C^*-algebra motivated by Bratteli's work in [7]. This is the next lemma that follows after the following notation.

NOTATION 3.15. Let $\mathfrak{A}$ be a C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Let $\mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)$ denote the set of two-sided ideals of $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ that are not necessarily closed in $\mathfrak{A}$. Denote:

$$\mathrm{algIdeal}\Big(\bigcup_{n\in\mathbb{N}}\mathfrak{A}_n\Big)_{\mathrm{prod}}=\Big\{J\in \mathrm{algIdeal}\Big(\bigcup_{n\in\mathbb{N}}\mathfrak{A}_n\Big) : J\cap\mathfrak{A}_n \in \mathrm{Ideal}(\mathfrak{A}_n) \text{ for all } n\in\mathbb{N}\Big\}.$$

LEMMA 3.16. *If $\mathfrak{A}$ is a C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, then using notation from Proposition* 3.11 *and Lemma* 3.12 *and Notation* 3.15*, the map*

$$J \in \mathrm{algIdeal}\Big(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\Big)_{\mathrm{prod}} \mapsto \overline{J}^{\|\cdot\|_{\mathfrak{A}}} \in \mathrm{Ideal}(\mathfrak{A})$$

is a well-defined surjection onto $\mathrm{Ideal}(\mathfrak{A})$.

Furthermore, the following two statements are equivalent:

(i) *the function* $J \in \mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)_{\mathrm{prod}} \mapsto \overline{J}^{\|\cdot\|_{\mathfrak{A}}} \in \mathrm{Ideal}(\mathfrak{A})$ *is injective, and thus a bijection onto* $\mathrm{Ideal}(\mathfrak{A})$*;*

(ii) *the function* $i(\cdot,\mathcal{U})$ *is surjective onto* $\mathrm{Ideal}(\mathfrak{A})_\infty$*, and thus a bijection onto* $\mathrm{Ideal}(\mathfrak{A})_\infty$.

In particular, if $\mathfrak{A}$ is AF *and $\mathcal{U}$ is chosen to be a family of finite-dimensional C^*-algebras, then the map $i(\cdot,\mathcal{U})$ surjects onto* $\mathrm{Ideal}(\mathfrak{A})_\infty$.

Proof. We first show that the map $J \in \mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)_{\mathrm{prod}} \mapsto \overline{J}^{\|\cdot\|_{\mathfrak{A}}} \in \mathrm{Ideal}(\mathfrak{A})$ is a well-defined surjection onto $\mathrm{Ideal}(\mathfrak{A})$. This map is clearly well-defined. For surjectivity, let $J \in \mathrm{Ideal}(\mathfrak{A})$. Then, we have that $J \cap \mathfrak{A}_n \in \mathrm{Ideal}(\mathfrak{A}_n)$ and thus, if we define $I = \bigcup_{n\in\mathbb{N}} (J \cap \mathfrak{A}_n)$, then $I \in \mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)_{\mathrm{prod}}$ and $\overline{I}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\bigcup_{n\in\mathbb{N}} (J \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}} = J$ by Lemma III.4.1 in [12].

Assume (i). The map $i(\cdot,\mathcal{U})$ is already a well-defined injection by Proposition 3.11 and Proposition 3.12. For surjectivity, let $(J_n)_{n\in\mathbb{N}} \in \mathrm{Ideal}(\mathfrak{A})_\infty$. Thus $J_n \in \mathrm{Ideal}(\mathfrak{A}_n)$ and $J_n \subseteq J_{n+1}$ for all $n \in \mathbb{N}$, and so if we let $J = \bigcup_{n\in\mathbb{N}} J_n$, then $J \in \mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)_{\mathrm{prod}}$. Thus $\overline{J}^{\|\cdot\|_{\mathfrak{A}}} \in \mathrm{Ideal}(\mathfrak{A})$. We claim that $i(\overline{J}^{\|\cdot\|_{\mathfrak{A}}},\mathcal{U}) = (J_n)_{n\in\mathbb{N}}$. Indeed, define $I_n = \overline{J}^{\|\cdot\|_{\mathfrak{A}}} \cap \mathfrak{A}_n$ for each $n \in \mathbb{N}$. We have $I_n \in \mathrm{Ideal}(\mathfrak{A}_n)$ and $I_n \subseteq I_{n+1}$ for all $n \in \mathbb{N}$. Again, if we let $I = \bigcup_{n\in\mathbb{N}} I_n$, then we have $I \in \mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)_{\mathrm{prod}}$ and $\overline{I}^{\|\cdot\|_{\mathfrak{A}}} \in \mathrm{Ideal}(\mathfrak{A})$. However:

$$\overline{I}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\bigcup_{n\in\mathbb{N}} I_n}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\bigcup_{n\in\mathbb{N}} (\overline{J}^{\|\cdot\|_{\mathfrak{A}}} \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}} = \overline{J}^{\|\cdot\|_{\mathfrak{A}}}$$

by Lemma III.4.1 in [12]. Hence $\bigcup_{n\in\mathbb{N}} I_n = I = J = \bigcup_{n\in\mathbb{N}} J_n$ by the assumption that the map of (1) is injective, which implies that $\overline{J}^{\|\cdot\|_{\mathfrak{A}}} \cap \mathfrak{A}_n = I_n = J_n$ for all $n \in \mathbb{N}$. Thus $i(\overline{J}^{\|\cdot\|_{\mathfrak{A}}},\mathcal{U}) = (I_n)_{n\in\mathbb{N}} = (J_n)_{n\in\mathbb{N}}$, which completes this direction.

Next, assume (ii). Let $I, J \in \mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)_{\mathrm{prod}}$ such that $I \neq J$. Thus there exists $N \in \mathbb{N}$ such that $I \cap \mathfrak{A}_N \neq J \cap \mathfrak{A}_N$. Hence $(I \cap \mathfrak{A}_n)_{n\in\mathbb{N}} \neq (J \cap \mathfrak{A}_n)_{n\in\mathbb{N}}$ where $(I \cap \mathfrak{A}_n)_{n\in\mathbb{N}}, (J \cap \mathfrak{A}_n)_{n\in\mathbb{N}} \in \mathrm{Ideal}(\mathfrak{A})_\infty$. By the assumption that the map of (ii) is a surjection, there exist $K_I, K_J \in \mathrm{Ideal}(\mathfrak{A})$ with $K_I \neq K_J$ and $i(K_I,\mathcal{U}) = (I \cap \mathfrak{A}_n)_{n\in\mathbb{N}}$ and $i(K_J,\mathcal{U}) = (J \cap \mathfrak{A}_n)_{n\in\mathbb{N}}$ since $i(\cdot,\mathcal{U})$ is well-defined. However, this implies that $K_I \cap \mathfrak{A}_n = I \cap \mathfrak{A}_n$ and $K_J \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n$ for all $n \in \mathbb{N}$. But, then:

$$K_I = \overline{\bigcup_{n\in\mathbb{N}} (K_I \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\bigcup_{n\in\mathbb{N}} (I \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}} = \overline{I}^{\|\cdot\|_{\mathfrak{A}}}$$

by Lemma III.4.1 in [12]. Similarly, we have $K_J = \overline{J}^{\|\cdot\|_{\mathfrak{A}}}$, and therefore $\overline{I}^{\|\cdot\|_{\mathfrak{A}}} \neq \overline{J}^{\|\cdot\|_{\mathfrak{A}}}$, which completes the proof of the equivalence between (i) and (ii).

Finally, assume $\mathfrak{A}$ is AF and $\mathcal{U}$ is a family of finite-dimensional C^*-algebras. If $J \in \mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)$, then $J \cap \mathfrak{A}_n$ is finite-dimensional and thus closed for all $n \in \mathbb{N}$. Hence $J \cap \mathfrak{A}_n \in \mathrm{Ideal}(\mathfrak{A}_n)$ for all $n \in \mathbb{N}$. Therefore $\mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n) = \mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)_{\mathrm{prod}}$. However, the map $J \in \mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n) \mapsto \overline{J}^{\|\cdot\|_{\mathfrak{A}}} \in \mathrm{Ideal}(\mathfrak{A})$ is a bijection onto $\mathrm{Ideal}(\mathfrak{A})$ by Theorem 3.3 in [7], which completes the proof by the established equivalence of (i) and (ii). ∎

We will now see that Lemma 3.16 produces natural sufficient conditions for our topology to agree with the Fell topology.

THEOREM 3.17. *Let $\mathfrak{A}$ be a C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Using Notation* 3.15*, if the function $J \in \mathrm{algIdeal}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)_{\mathrm{prod}} \mapsto \overline{J}^{\|\cdot\|_{\mathfrak{A}}} \in \mathrm{Ideal}(\mathfrak{A})$ is injective, and thus a bijection onto* $\mathrm{Ideal}(\mathfrak{A})$*, then the topology $Fell_{i(\mathcal{U})}$ of Definition* 3.13 *agrees with the topology Fell on* $\mathrm{Ideal}(\mathfrak{A})$.

In particular, if $\mathfrak{A}$ is AF *and $\mathcal{U}$ is chosen to be a family of finite-dimensional C^*-algebras, then the topology $Fell_{i(\mathcal{U})}$ agrees with the topology Fell on* $\mathrm{Ideal}(\mathfrak{A})$.

Proof. By Lemma 3.16, the map $i(\cdot, \mathcal{U})$ is a bijection onto $\mathrm{Ideal}(\mathfrak{A})_\infty$. Hence, since the topological space $(\mathrm{Ideal}(\mathfrak{A})_\infty, Fell_\infty)$ is compact Hausdorff by Lemma 3.12, $Fell_{i(\mathcal{U})}$ is compact Hausdorff on $\mathrm{Ideal}(\mathfrak{A})$ since it is the initial topology induced by a bijection onto a compact Hausdorff space. However, by Proposition 3.14 and maximal compactness of Hausdorff spaces, the proof is complete. ∎

We are going to define a metric on $\mathrm{Ideal}(A)$ using our maps to and from $\prod_{n\in\mathbb{N}} \mathrm{Ideal}(\mathfrak{A}_n)$. As per our work so far, this metric will metrize the Fell topology on the ideal space of AF-algebras. While it is known that the Fell topology is metrizable when the C^*- algebra is separable (we state this fact in Lemma 3.18), our particular choice of metric will be adapted to our current problem.

LEMMA 3.18. *If $\mathfrak{A}$ is a separable C^*-algebra, then the Fell topology on* $\mathrm{Ideal}(\mathfrak{A})$ *is compact metrizable.*

Proof. The Fell topology on $\mathrm{Ideal}(\mathfrak{A})$ is already compact Hausdorff (Definition 3.5). Since $\mathfrak{A}$ is separable, the Jacobson topology on $\mathrm{Prim}(\mathfrak{A})$ is second countable by Corollary 4.3.4 in [33]. However, by (III) pg. 474 in [17], the Fell topology has a countable basis. Thus, the Fell topology is second countable compact Hausdorff, which completes the proof by Urysohn's metrization theorem (Theorem 23.1 in [43]). ∎

PROPOSITION 3.19. *If $\mathfrak{A}$ is a separable C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, then for each $n \in \mathbb{N}$, the Fell topology on* $\mathrm{Ideal}(\mathfrak{A}_n)$ *is metrized by a metric d_n with diameter at most* 1, *and the $[0,\infty)$-valued map on $\prod_{n\in\mathbb{N}} \mathrm{Ideal}(\mathfrak{A}_n) \times \prod_{n\in\mathbb{N}} \mathrm{Ideal}(\mathfrak{A}_n)$ defined by:*

$$d((I_n)_{n\in\mathbb{N}}, (J_n)_{n\in\mathbb{N}}) = \sum_{n=0}^{\infty} \frac{d_n(I_n, J_n)}{2^n}$$

is a compact metric on the product topology of $\prod_{n\in\mathbb{N}} \mathrm{Ideal}(\mathfrak{A}_n)$ with respect to the Fell topology on each $\mathrm{Ideal}(\mathfrak{A}_n)$ *and induces a totally bounded metric on* $\mathrm{Ideal}(\mathfrak{A})$ *defined by:*

$$\mathsf{m}_{\prod(Fell),\mathcal{U}}(I, J) = \sum_{n=0}^{\infty} \frac{d_n(I \cap \mathfrak{A}_n, J \cap \mathfrak{A}_n)}{2^n},$$

which metrizes the topology $Fell_{i(\mathcal{U})}$ on $\mathrm{Ideal}(\mathfrak{A})$ *of Definition* 3.13.

Proof. Since $\mathfrak{A}$ is separable, the subspace $\mathfrak{A}_n$ is separable for all $n \in \mathbb{N}$. Thus, by Lemma 3.18, the Fell topology of each $\mathrm{Ideal}(\mathfrak{A}_n)$ is metrized by some metric d_n. If d_n has diameter more than 1, then simply use the metric $d_n/(1 + d_n)$ instead, which metrizes the same topology and has diameter at most 1, and thus the metric d defined in the statement of the proposition metrizes the product topology. The fact that $\mathsf{m}_{\prod(Fell),\mathcal{U}}$ is a totally bounded metric follows from the fact that $\mathsf{m}_{\prod(Fell),\mathcal{U}} = d \circ (i(\cdot,\mathcal{U}) \times i(\cdot,\mathcal{U}))$ and $i(\cdot,\mathcal{U})$ is an injection by Proposition 3.11. The fact that $\mathsf{m}_{\prod(Fell),\mathcal{U}}$ metrizes $Fell_{i(\mathcal{U})}$ follows by construction. ∎

In Theorem 3.21, we will show that the metric $\mathsf{m}_{\prod(Fell),\mathcal{U}}$ above metrizes the Fell topology when $\mathfrak{A}$ is AF and the inductive sequence $\mathcal{U}$ contains only finite-dimensional C^*-algebras. The next corollary shows that we can simplify the metric d and thus $\mathsf{m}_{\prod(Fell),\mathcal{U}}$, when $\mathfrak{A}$ is AF.

COROLLARY 3.20. *If $\mathfrak{A}$ is a separable C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ and* $\mathrm{Ideal}(\mathfrak{A}_n)$ *is finite for each $n \in \mathbb{N}$, then the compact metric product topology of $\prod_{n\in\mathbb{N}} \mathrm{Ideal}(\mathfrak{A}_n)$ with respect to the Fell topology on each* $\mathrm{Ideal}(\mathfrak{A}_n)$ *is metrized by the metric:*

$$d_{i(\mathcal{U})}((I_n)_{n\in\mathbb{N}}, (J_n)_{n\in\mathbb{N}}) = \begin{cases} 0 & \text{if } I_n = J_n \text{ for all } n \in \mathbb{N}, \\ 2^{-\min\{m\in\mathbb{N}: I_m \neq J_m\}} & \text{otherwise,} \end{cases}$$

and induces a totally bounded metric on Ideal$(\mathfrak{A})$ *defined by:*

$$\mathsf{m}_{i(\mathcal{U})}(I,J) = \begin{cases} 0 & \text{if } I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n \text{ for all } n \in \mathbb{N}, \\ 2^{-\min\{m \in \mathbb{N} : I \cap \mathfrak{A}_m \neq J \cap \mathfrak{A}_m\}} & \text{otherwise,} \end{cases}$$

that metrizes the same topology of $\mathsf{m}_{\prod(Fell),\mathcal{U}}$ *of Propsition* 3.19 *on* Ideal$(\mathfrak{A})$ *and the topology* $Fell_{i(\mathcal{U})}$ *of Definition* 3.13.

Proof. Since the Fell topology is always compact Hausdorff, the topology on Ideal$(\mathfrak{A}_n)$ is discrete as the set is finite, and thus we may take our metrics d_n from the previous proposition to be the discrete metric (that assigns 1 to distinct points) for all $n \in \mathbb{N}$. Finally, the topology given by $d_{i(\mathcal{U})}$ and d of Theorem 3.19 on $\prod_{n \in \mathbb{N}}$ Ideal$(\mathfrak{A}_n)$ agree in this setting as these metrics are equivalent, which completes the proof by construction of $Fell_{i(\mathcal{U})}$. ∎

Now, we may prove the main result of this section in which the metric of the previous corrollary does in fact metrize the Fell topology for AF-algebras.

THEOREM 3.21. *If* $\mathfrak{A}$ *is an* AF*-algebra then, for any non-decreasing sequence of finite-dimensional* C^**-subalgebras* $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ *such that* $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, *the metric* $\mathsf{m}_{i(\mathcal{U})}$ *of Corollary* 3.20 *metrizes the Fell topology on* Ideal$(\mathfrak{A})$.

For the proof observe that finite-dimensional C^*-algebras have finitely many ideals and apply Theorem 3.17 to Corollary 3.20.

An immediate consequence of Theorem 3.21 is that, although the metric is built using a fixed inductive sequence, the metric topology with respect to an inductive sequence is homeomorphic to the metric topology on the same AF-algebra with respect to any other inductive sequence. In particular, concerning continuity or convergence results, Corollary 3.22 provides that one may choose any inductive sequence to suit the needs of the problem at hand.

COROLLARY 3.22. *Let* $\mathfrak{A}, \mathfrak{B}$ *be* AF*-algebras and fix any non-decreasing sequences of finite dimensional* C^**-subalgebras* $\mathcal{U}_{\mathfrak{A}} = (\mathfrak{A}_n)_{n \in \mathbb{N}}, \mathcal{U}_{\mathfrak{B}} = (\mathfrak{B}_n)_{n \in \mathbb{N}}$, *respectively, such that* $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ *and* $\mathfrak{B} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{B}_n}^{\|\cdot\|_{\mathfrak{B}}}$. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* $*$*-isomorphic, then the metric spaces* $(\text{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U}_{\mathfrak{A}})})$ *and* $(\text{Ideal}(\mathfrak{B}), \mathsf{m}_{i(\mathcal{U}_{\mathfrak{B}})})$ *are homeomorphic.*

In particular, if $\mathfrak{A}$ *is* AF *and* $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_{1,n}}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_{2,n}}^{\|\cdot\|_{\mathfrak{A}}}$, *where* $\mathcal{U}_1 = (\mathfrak{A}_{1,n})_{n \in \mathbb{N}}, \mathcal{U}_2 = (\mathfrak{A}_{2,n})_{n \in \mathbb{N}}$ *are any non-decreasing sequences of finite dimensional* C^**-subalgebras of* $\mathfrak{A}$, *then the metric spaces* $(\text{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U}_1)})$ *and* $(\text{Ideal}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U}_2)})$ *are homeomorphic.*

Proof. By construction of the Fell and the Jacobson topologies Definition 3.5, if $\mathfrak{A}$ and $\mathfrak{B}$ are $*$-isomorphic then the Fell topologies are homeomorphic. Thus, the conclusion follows by Theorem 3.21. ∎

In the context of this paper, a main motivation for the metric of Corollary 3.20 is to provide a fusing family of quotients via convergence of ideals. First, for a fixed ideal of an inductive limit of the form $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, we provide an inductive limit in the sense of Notation 2.5 that is $*$-isomorphic to the quotient. The reason for this is that given $I \in \mathrm{Ideal}(\mathfrak{A})$, then $\mathfrak{A}/I = \overline{\bigcup_{n\in\mathbb{N}} ((\mathfrak{A}_n + I)/I)}^{\|\cdot\|_{\mathfrak{A}/I}}$ (see Proposition 3.26), but if two ideals satisfy $I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n$ for some $n \in \mathbb{N}$, then even though this provides that $(\mathfrak{A}_n + I)/I$ is $*$-isomorphic to $(\mathfrak{A}_n + J)/J$ as they are both $*$-isomorphic to $(\mathfrak{A}_n/(I \cap \mathfrak{A}_n))$ (see Proposition 3.26), the two algebras $(\mathfrak{A}_n + J)/J$ and $(\mathfrak{A}_n + I)/I$ are not equal in any way if $I \neq J$, yet, equality is a requirement for fusing families (see Definition 2.12). Thus, Notation 3.25 will allow us to present, up to $*$-isomorphism, quotients as fusing families from convergence of ideals in the metric of Corollary 3.20 as we will see in Proposition 3.26.

Before moving to fusing families of quotients, we show that a fusing family of ideals is equivalent to convergence in the metric on ideals of Corollary 3.20.

LEMMA 3.23. *Let $\mathfrak{A}$ be* AF*-algebra and fix any non-decreasing sequence of finite dimensional C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$.*

If $(I^k)_{k\in\overline{\mathbb{N}}} \subseteq$ Ideal$(\mathfrak{A})$, then the following are equivalent:

(i) $\left\{ I^k = \overline{\bigcup_{n\in\mathbb{N}} I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}} \right\}$ *is a fusing family of Definition* 2.12*;*

(ii) $(I^k)_{k\in\mathbb{N}}$ *converges to I^∞ with respect to the metric $\mathsf{m}_{i(\mathcal{U})}$ of Corollary* 3.20*;*

(iii) $(I^k)_{k\in\mathbb{N}}$ *converges to I^∞ in the Fell topology.*

Proof. The equivalence between (ii) and (iii) is given by Theorem 3.21. We begin with showing that (ii) $\Rightarrow$ (i). Assume $(I^k)_{k\in\mathbb{N}} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \mathrm{Ideal}(\mathfrak{A})$ with respect to $\mathsf{m}_{i(\mathcal{U})}$, which is equivalent to convergence in *Fell* by Theorem 3.21. Thus, we have $\lim_{k\to\infty} \mathsf{m}_{i(\mathcal{U})}(I^k, I^\infty) = 0$. From this, we construct an increasing sequence $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{N} \setminus \{0\}$ such that:

$$\mathsf{m}_{i(\mathcal{U})}(I^k, I^\infty) \leqslant 2^{-(n+1)}$$

for all $k \geqslant c_n$. In particular, fix $N \in \mathbb{N}$, if $k \in \mathbb{N}_{\geqslant c_N}$, then $I^k \cap \mathfrak{A}_n = I^\infty \cap \mathfrak{A}_n$ for all $n \in \{0, \ldots, N\}$, which implies that $\left\{ I^k = \overline{\bigcup_{n\in\mathbb{N}} I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}} \right\}$ is a fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$ by Definition 2.12.

For (i) $\Rightarrow$ (ii), assume that $\left\{ I^k = \overline{\bigcup_{n\in\mathbb{N}} I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}} \right\}$ is a fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$. Therefore, for all $N \in \mathbb{N}$, if $k \in \mathbb{N}_{\geqslant c_N}$, then $I^k \cap \mathfrak{A}_n = I^\infty \cap \mathfrak{A}_n$ for all $n \in \{0, \ldots, N\}$. Hence, let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. If $k \geqslant c_N \in \mathbb{N}$, then we have the following which completes the proof:

$$\mathsf{m}_{i(\mathcal{U})}(I^k, I^\infty) \leqslant 2^{-(N+1)} < 2^{-N} < \varepsilon. \quad \blacksquare$$

REMARK 3.24. Clearly, the metric $\mathsf{m}_{i(\mathcal{U})}$ of Corollary 3.20 can be defined on any C^*-inductive limit even without the assumption of AF or separability. And, in general, this metric would produce an even finer topology than $Fell_{i(\mathcal{U})}$ as $\mathsf{m}_{i(\mathcal{U})}$ is given by a metric on the product topology induced by the discrete topology on the ideal space of each $\mathfrak{A}_n$. Furthermore, we note the equivalence between (i) and (ii) in Lemma 3.23 would still hold for this metric in this more general setting. This connection with fusing families was another strong motivation for the pursuit of this metric.

NOTATION 3.25. Let $\mathfrak{A}$ be a C^*-algebra with a non-decreasing sequence of C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Let $I \in \mathrm{Ideal}(\mathfrak{A})$. For $n \in \mathbb{N}$

$$\gamma_{I,n} : a + I \cap \mathfrak{A}_n \in \mathfrak{A}_n/(I\cap\mathfrak{A}_n) \longmapsto a + (I \cap \mathfrak{A}_{n+1}) \in \mathfrak{A}_{n+1}/(I\cap\mathfrak{A}_{n+1}),$$

is a$*$-monomorphism by the same argument of Claim 3.9.

Let $\mathcal{I}(\mathfrak{A}/I) = (\mathfrak{A}_n/(I\cap\mathfrak{A}_n), \gamma_{I,n})_{n\in\mathbb{N}}$, and denote its C^*-inductive limit by $\varinjlim \mathcal{I}(\mathfrak{A}/I)$.

If $\mathfrak{B} \subseteq \mathfrak{A}$ is a C^*-subalgebra and $I \in \mathrm{Ideal}(\mathfrak{A})$, then denote:

$$\mathfrak{B} + I = \overline{\{b + c : b \in \mathfrak{B}, c \in I\}}^{\|\cdot\|_{\mathfrak{A}}}.$$

PROPOSITION 3.26. *Let $\mathfrak{A}$ be* AF *and fix any non-decreasing sequence of finite-dimensional C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ of $\mathfrak{A}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Using Notation* 3.25, *if $I \in$* Ideal$(\mathfrak{A})$, *then there exists a $*$-isomorphism*

$$\phi_I : \varinjlim \mathcal{I}(\mathfrak{A}/I) \to \mathfrak{A}/I$$

such that for all $n \in \mathbb{N}$ the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A}_n/(I\cap\mathfrak{A}_n) & \xrightarrow{\ \overrightarrow{\gamma_I^n}\ } & \varinjlim \mathcal{I}(\mathfrak{A}/I)\ , \\ & \searrow^{\phi_I^n} & \downarrow \phi_I \\ & & \mathfrak{A}/I \end{array}$$

where all maps $\phi_I^n : a + (I\cap\mathfrak{A}_n) \in \mathfrak{A}_n/(I\cap\mathfrak{A}_n) \longmapsto a + I \in (\mathfrak{A}_n + I)/I \subseteq \mathfrak{A}/I$ are $*$-monomorphisms onto $(\mathfrak{A}_n + I)/I$, in which $\mathfrak{A}_n + I = \{a + b \in \mathfrak{A} : a \in \mathfrak{A}_n, b \in I\}$ is a C^*-subalgebra of $\mathfrak{A}$ containing I as an ideal and $\bigcup_{n\in\mathbb{N}} ((\mathfrak{A}_n + I)/I)$ is a dense $*$-subalgebra of $\mathfrak{A}/I$ with $((\mathfrak{A}_n + I)/I)_{n\in\mathbb{N}}$ non-decreasing.

Furthermore, if $(I^k)_{k\in\mathbb{N}} \subseteq$ Ideal$(\mathfrak{A})$ *converges to $I^\infty \in$* Ideal$(\mathfrak{A})$ *with respect to $\mathsf{m}_{i(\mathcal{U})}$ of Corollary* 3.20 *or the Fell topology, then using Definition* 2.12, *we have that $\left\{ I^k = \overline{\bigcup_{n\in\mathbb{N}} I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}} \right\}$ is a fusing family with respect to some fusing sequence*

$(c_n)_{n\in\mathbb{N}}$ *and that* $\{\varinjlim \mathcal{I}(\mathfrak{A}/I^k) : k \in \overline{\mathbb{N}}\}$ *is a fusing family with fusing sequence* $(c_n)_{n\in\mathbb{N}}$.

Proof. Let $I \in \mathrm{Ideal}(\mathfrak{A})$. Fix $n \in \mathbb{N}$. Note that $\mathfrak{A}_n + I$ is a C*-subalgebra of $\mathfrak{A}$ since $I \in \mathrm{Ideal}(\mathfrak{A})$, and furthermore $I \in \mathrm{Ideal}(\mathfrak{A}_n + I)$. Now, we have $\mathfrak{A}_n + I = \{a + b \in \mathfrak{A} : a \in \mathfrak{A}_n, b \in I\}$ since $\mathfrak{A}_n$ and I are both closed in $\mathfrak{A}$ and $\mathfrak{A}_n$ is finite dimensional. Next, ϕ_I^n is an injective $*$-homomorphism by Claim 3.9. If $a \in \mathfrak{A}_n$, then $\phi_I^n(a + \mathfrak{A}_n/(I \cap \mathfrak{A}_n)) = a + I$ and the composition $\phi_I^{n+1}(\gamma_{I,n}(a + (I \cap \mathfrak{A}_n))) = \phi_I^{n+1}(a + (I \cap \mathfrak{A}_{n+1})) = a + I$. Hence, for all $n \in \mathbb{N}$, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A}_n/(I\cap\mathfrak{A}_n) & \xrightarrow{\gamma_{I,n}} & \mathfrak{A}_{n+1}/(I\cap\mathfrak{A}_{n+1}) \\ & \searrow^{\phi_I^n} & \downarrow{\scriptstyle \phi_I^{n+1}} \\ & & \mathfrak{A}/I \end{array} .$$

Thus, by Theorem 6.1.2 in [32], the definition of inductive limit in Chapter 6.1 in [32], and the fact that each map in the above diagram is an isometry, there exists a unique $*$-monomorphism $\phi_I : \varinjlim \mathcal{I}(\mathfrak{A}/I) \to \mathfrak{A}/I$ such that for all $n \in \mathbb{N}$ the diagram in the statement of this theorem commutes.

Next, fix $n \in \mathbb{N}$. Let $x \in (\mathfrak{A}_n + I)/I$, and so $x = a + b + I$, where $a \in \mathfrak{A}_n, b \in I$. Thus, we have $a + b - a = b \in I \Rightarrow x - (a + I) = 0 + I \Rightarrow x = a + I$. But then $\phi_I^n(a + (I \cap \mathfrak{A}_n)) = x$. Hence, the map ϕ_I^n is onto $(\mathfrak{A}_n + I)/I$ and thus:

$$\phi_I\Big(\bigcup_{n\in\mathbb{N}} \underrightarrow{\gamma_I^n}(\mathfrak{A}_n/(I \cap \mathfrak{A}_n))\Big) = \bigcup_{n\in\mathbb{N}} ((\mathfrak{A}_n + I)/I),$$

where the right-hand side is a dense $*$-subalgebra of $\mathfrak{A}/I$ by continuity of the quotient map and the assumption that $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ is dense in $\mathfrak{A}$. Hence, since the normed space $\varinjlim \mathcal{I}(\mathfrak{A}/I)$ is complete and ϕ_I is a linear isometry on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$, ϕ_I surjects onto $\mathfrak{A}/I$. Thus $\phi_I : \varinjlim \mathcal{I}(\mathfrak{A}/I) \to \mathfrak{A}/I$ is a $*$-isomorphism.

Next, assume that $(I^k)_{n\in\mathbb{N}} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \mathrm{Ideal}(\mathfrak{A})$ with respect to $\mathsf{m}_{i(\mathcal{U})}$, which is equivalent to convergence in *Fell* by Theorem 3.21. By Lemma 3.23, the family $\left\{I^k = \overline{\bigcup_{n\in\mathbb{N}} I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}}\right\}$ is a fusing family with fusing sequence $(b_n)_{n\in\mathbb{N}}$ by Definition 2.12.

Let $c_n = b_{n+1}$ for all $n \in \mathbb{N}$. Then, the sequence $(c_n)_{n\in\mathbb{N}}$ is a fusing sequence for $\left\{I^k = \overline{\bigcup_{n\in\mathbb{N}} I^k \cap \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}} : k \in \overline{\mathbb{N}}\right\}$. Fix $N \in \mathbb{N}, n \in \{0, \ldots, N\}$, and $k \in \mathbb{N}_{\geqslant c_N}$. Then, the equality $I^k \cap \mathfrak{A}_n = I^\infty \cap \mathfrak{A}_n$ implies that $\mathfrak{A}_n/(I^k \cap \mathfrak{A}_n) = \mathfrak{A}_n/(I^\infty \cap \mathfrak{A}_n)$. But, also, we gather $\gamma_{I^k,n} = \gamma_{I^\infty,n}$ since $\mathfrak{A}_{n+1}/(I^k \cap \mathfrak{A}_{n+1}) = \mathfrak{A}_{n+1}/(I^\infty \cap \mathfrak{A}_{n+1})$ as $c_n = b_{n+1}$. Hence, the familiy of inductive limits $\{\varinjlim \mathcal{I}(\mathfrak{A}/I^k) : k \in \overline{\mathbb{N}}\}$ is a fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$. ∎

Now that we have this identification with our metric and the Fell topology, we finish our discussion on the metric topology by considering it in the unital commutative case of AF-algebras in Corollary 3.31. It will be the case that the relative metric topology of Corollary 3.20, the relative Fell topology, and the Jacobson topology all agree on the primitive ideals. However, we begin with a more general scenario, where we only assume that the Jacobson topology is Hausdorff on a unital C^*-algebra since in this case the relative Fell topology and the Jacobson topology all agree on the primitive ideals. First, a remark on restricting to the unital case.

REMARK 3.27. In the following results of this section, we restrict our attention to unital C^*-algebras since in this case $\mathrm{Prim}(\mathfrak{A})$ is a compact subset of the Fell topology, as seen in Lemma 3.29. However, although the Jacobson topology is still locally compact in the non-unital case (see Corollary 3.3.8 in [13]) and one can form the Alexandroff compactification in the Hausdorff case of the Jacobson topology, the fact that $\mathfrak{A} \in \mathrm{Ideal}(\mathfrak{A})$ (note that $\mathfrak{A}$ plays the role of the point at infinity of the Alexandroff compactification by Definition 3.5 and Corollary (1) on page 475 in [17]) may not be isolated in the Fell topology in general diminishes any reasonable expectation that the relative Fell topology on $\mathrm{Prim}(\mathfrak{A})$ would agree with the Jacobson topology in this generality. An example of when $\mathfrak{A}$ is not isolated in the Fell topology is when $\mathfrak{A} = C_0(Y)$, where $Y = \{1/n \in \mathbb{R} : n \in \mathbb{N} \setminus \{0\}\} \subset (0,1]$. Indeed, if we define for all $m \in \mathbb{N}$ the ideal $I_m = \{g \in \mathfrak{A} : g(\{1/(n+2) \in \mathbb{R} : n \geqslant m\}) = 0\} \subsetneq \mathfrak{A}$, then the sequence $(I_m)_{m\in\mathbb{N}} \subset \mathrm{Ideal}(\mathfrak{A}) \setminus \{\mathfrak{A}\}$ converges to $\mathfrak{A}$ in the Fell topology by Lemma 3.6 and definition of $C_0(Y)$.

On the other hand, the element $\mathfrak{A} \in \mathrm{Ideal}(\mathfrak{A})$ is always isolated in the Fell topology when $\mathfrak{A}$ is unital regardless of any separation condition on the Jacobson topology. Indeed, if $J \in \mathrm{Ideal}(\mathfrak{A}) \setminus \{\mathfrak{A}\}$, then $\|1_{\mathfrak{A}} + J\|_{\mathfrak{A}/J} \geqslant 1$ since the set $\{a \in \mathfrak{A} : \|a + 1_{\mathfrak{A}}\|_{\mathfrak{A}} < 1\}$ contains only invertible elements by Corollary VII.2.3 in [11]. Hence, no net of ideals in $\mathrm{Ideal}(\mathfrak{A}) \setminus \{\mathfrak{A}\}$ may converge to $\mathfrak{A}$ by Lemma 3.6 since $\|1_{\mathfrak{A}} + \mathfrak{A}\|_{\mathfrak{A}/\mathfrak{A}} = 0$.

Before we move to the C^*-algebra setting, we present a fact about the Fell topology in the context of topological spaces. The following is mentioned in [17], but we provide a detailed proof here.

LEMMA 3.28. *If (X, τ) is a compact Hausdorff space, then the map*

$$s : x \in X \longmapsto \{x\} \in \mathcal{C}l(X)$$

is a well-defined homeomorphism onto its image with respect to the relative Fell topology on $\mathcal{C}l(X)$ of Defintion 3.3, and moreover the set

$$s(X) = \{\{x\} \in \mathcal{C}l(X) : x \in X\}$$

is a compact, and thus a closed subset of $\mathcal{C}l(X)$ with respect to the Fell topology.

Proof. Since (X, τ) is compact Hausdorff and the space $\mathcal{C}l(X)$ equipped with the Fell topology is compact Hausdorff by Lemma 3.4, we only have to check that s is continuous and note that s is well-defined since (X, τ) is Hausdorff.

Let $(x_\lambda)_{\lambda \in \Lambda} \subseteq X$ be a net that converges to some $x \in X$ with respect to the topology τ. We claim that $(\{x_\lambda\})_{\lambda \in \Lambda} \subseteq \mathcal{C}l(X)$ converges to $\{x\} \in \mathcal{C}l(X)$ with respect to the Fell topology.

Let $K \subseteq X$ be a compact set with respect to τ and let $n \in \mathbb{N}$ and $A_0, \ldots, A_n \in \tau \setminus \{\emptyset\}$ and let $F = \bigcup_{j=0}^{n} \{A_j\} \subseteq \tau$. Assume that $\{x\} \in U(K, F) = \{Y \in \mathcal{C}l(X) : Y \cap K = \emptyset$ and $Y \cap A_j \neq \emptyset$ for all $j \in \{0, \ldots, n\}\}$. Thus

$$x \in (X \setminus K) \cap \Big(\bigcap_{j=0}^{n} A_j \Big) \in \tau$$

since K is closed as (X, τ) is Hausdorff. Therefore there exists $\alpha \in \Lambda$ such that for all $\lambda \geqslant \alpha$, we have that

$$x_\lambda \in (X \setminus K) \cap \Big(\bigcap_{j=0}^{n} A_j \Big) \in \tau,$$

giving that $\{x_\lambda\} \in U(K, F)$ for all $\lambda \geqslant \alpha$, which completes that proof. ∎

LEMMA 3.29. *If $\mathfrak{A}$ is a unital C^*-algebra such that* $\mathrm{Prim}(\mathfrak{A})$ *equipped with its Jacobson topology is Hausdorff, then on* $\mathrm{Prim}(\mathfrak{A})$ *the relative Fell topology agrees with the Jacobson topology and* $\mathrm{Prim}(\mathfrak{A})$ *is a compact and thus closed subset in the Fell topology.*

Proof. By the fact that the Jacobson topology on $\mathrm{Prim}(\mathfrak{A})$ is compact in the unital case by Proposition 3.1.8 in [13] and Lemma 3.28, the map

$$s : P \in (\mathrm{Prim}(\mathfrak{A}), Jacobson) \mapsto \{P\} \in (\mathcal{C}l(\mathrm{Prim}(\mathfrak{A})), \tau_{\mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))})$$

is a well-defined homeomorphism onto its image with respect to the relative topology such that $s(\mathrm{Prim}(\mathfrak{A})) \subset \mathcal{C}l(\mathrm{Prim}(\mathfrak{A})$ is compact and thus closed in the topology $\tau_{\mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))}$, and note that $(\mathcal{C}l(\mathrm{Prim}(\mathfrak{A})), \tau_{\mathcal{C}l(\mathrm{Prim}(\mathfrak{A}))})$ is also compact.

Next, let $P \in \mathrm{Prim}(\mathfrak{A})$. Since the Jacobson topology is Hausdorff, we have that $\{P\}$ is closed in the Jacobson topology. Hence, by Definition 3.5, there exists a unique ideal $I \in \mathrm{Ideal}(\mathfrak{A})$ such that $fell(I) = \{P\}$. However, Theorem 5.4.3 in [32] implies that $I = \bigcap_{J \in fell(I)} J = P$, and thus $fell(P) = \{P\}$ for all $P \in \mathrm{Prim}(\mathfrak{A})$. Hence since $fell$ is a bijection, we gather that:

$$fell^{-1}(\{\{J\} \in \mathcal{C}l(\mathrm{Prim}(\mathfrak{A})) : J \in \mathrm{Prim}(\mathfrak{A})\}) = \mathrm{Prim}(\mathfrak{A}).$$

Hence, the map

$$fell^{-1} \circ s : P \in (\mathrm{Prim}(\mathfrak{A}), Jacobson) \mapsto P \in (\mathrm{Prim}(\mathfrak{A}), Fell)$$

is a homeomorphism onto $\text{Prim}(\mathfrak{A})$ since the map $fell$ is a homeomorphism by the end of the proof of Lemma 3.6, where $(\text{Prim}(\mathfrak{A}), Fell)$ denotes the relative Fell topology on $\text{Prim}(\mathfrak{A})$, which completes the proof. ∎

Before providing the final result of this section, we present a classical result with proof, in which the Jacobson topology on the primitive ideals of a unital commutative $\mathfrak{A}$ is homeomorphic to the maximal ideal space with its weak* topology. This is true, of course, with non-unital as well and the following proof is exactly the same in this case, but we only consider the unital case. Of course, $\text{Prim}(\mathfrak{A})$ is compact on any unital C^*-algebra, commutative or not (see Proposition 3.1.8 in [13]), so the main purpose of the following theorem is to provide Hausdorff separation in the commutative case.

THEOREM 3.30. *If $\mathfrak{A}$ is a unital commutative C^*-algebra and $M_{\mathfrak{A}}$ denotes its space of non-zero multiplicative linear functionals with its weak* topology, then the map*

$$\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \text{Prim}(\mathfrak{A})$$

is a homeomorphism onto $\text{Prim}(\mathfrak{A})$ *with its Jacobson topology, and therefore* $\text{Prim}(\mathfrak{A})$ *with its Jacobson topology is a compact Hausdorff space.*

Proof. By Theorem 5.4.4 in [32], the set $\text{Prim}(\mathfrak{A})$ is the set of maximal ideals. However, for all $\varphi \in M_{\mathfrak{A}}$, the ideal $\ker \varphi$ is maximal. Hence, the map $\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \text{Prim}(\mathfrak{A})$ is a bijection by Theorem I.2.5 in [12]. Furthermore, by Theorem 5.1.6 in [32], the set of pure states on $\mathfrak{A}$ is equal to $M_{\mathfrak{A}}$. Therefore, by Proposition 4.3.3 in [33], the map $\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \text{Prim}(\mathfrak{A})$ is a homeomorphism onto $\text{Prim}(\mathfrak{A})$ since it is a continuous and open bijection. Since $M_{\mathfrak{A}}$ is locally compact Hausdorff by Corollary I.2.6 in [12], the set $\text{Prim}(\mathfrak{A})$ with its Jacobson topology is a compact Hausdorff space. ∎

COROLLARY 3.31. *Let $\mathfrak{A}$ be a unital* AF*-algebra and fix any non-decreasing sequence of finite-dimensional C^*-subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Let $(\text{Prim}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})})$ denote $\text{Prim}(\mathfrak{A})$ equipped with the relative topology induced by the metric topology of $\mathsf{m}_{i(\mathcal{U})}$ of Corollary* 3.20.

(i) *If the Jacobson topology on $\text{Prim}(\mathfrak{A})$ is Hausdorff, then $(\text{Prim}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})})$ has the same topology as the Jacobson topology or the relative Fell topology on $\text{Prim}(\mathfrak{A})$.*

(ii) *If $\mathfrak{A}$ is a unital commutative* AF*-algebra, then $(\text{Prim}(\mathfrak{A}), \mathsf{m}_{i(\mathcal{U})})$ is homeomorphic to the space of non-zero multiplicative linear functionals on $\mathfrak{A}$ denoted $M_{\mathfrak{A}}$ with its weak* topology, where the homeomorphism is given by:*

$$\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \text{Prim}(\mathfrak{A}).$$

Proof. For (i), combine Theorem 3.21 with Lemma 3.29. For (ii), combine Theorem 3.21 with Lemma 3.29 and Theorem 3.30. ∎

REMARK 3.32. The metric in Corollary 3.20 can be seen as an explicit presentation of a metric on a metrizable topology on ideals [4]. However, the metrizable topology in [4] is presented only in the case of AF-algebras. This topology still metrizes the Fell topology in the AF case, which we also proved for the metric of Corollary 3.20 via a different approach in Theorem 3.21 by our inverse limit topology. An advantage of our inverse limit topology approach is that it also allowed us provide a suitable topology for the ideal space of any C^*-algebra formed by an inductive limit and many other possibilities for future study on its own. Also, we note that the metric of Corollary 3.20 allows us to explicitly calculate distances between ideals in Remark 4.22. Therefore, one can make interesting comparisons with certain classical metrics on irrationals, and this metric also serves the purpose of providing fusing families of quotients in Proposition 3.26.

4. CONVERGENCE OF QUOTIENTS OF AF-ALGEBRAS IN QUANTUM PROPINQUITY

In the case of unital AF-algebras, we provide criteria for when convergence of ideals in the Fell topology provides convergence of quotients in the quantum propinquity topology, where the quotients are equipped with faithful tracial states. But, first, as we saw in Proposition 3.26, it seems that an inductive limit is suitable for describing fusing families with regard to convergence of ideals. Thus, in order to avoid the notational trouble of too many inductive limits, we will phrase many results in this section in terms of closure of unions. Now, when a quotient has a faithful tracial state, it turns out that the $*$-isomorphism provided in Proposition 3.26 is a quantum isometry (cf. Theorem-Definition 2.3) between the induced quantum compact metric spaces of Theorem 2.6, which preserves the finite-dimensional structure as well in Theorem 4.1. The purpose of this is to apply Theorem 2.14 directly to the quotient spaces. This utilizes our criteria for quantum isometries between AF-algebras in [1].

THEOREM 4.1. *Let $\mathfrak{A}$ be a unital* AF-*algebra with unit $1_{\mathfrak{A}}$ such that $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ is an increasing sequence of unital finite dimensional C^*-subalgebras with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$ and $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Let $I \in \mathrm{Ideal}(\mathfrak{A}) \setminus \{\mathfrak{A}\}$. By Proposition 3.26 we have $\mathfrak{A}/I = \overline{\bigcup_{n\in\mathbb{N}} ((\mathfrak{A}_n+I)/I)}^{\|\cdot\|_{\mathfrak{A}/I}}$. Denote $\mathcal{U}/I = ((\mathfrak{A}_n+I)/I)_{n\in\mathbb{N}}$, and note that $(\mathfrak{A}_0 + I)/I = \mathbb{C}1_{\mathfrak{A}/I}$.*

If $\mathfrak{A}/I$ is equipped with a faithful tracial state μ and using notation from Proposition 3.26, the map $\mu \circ \phi_I$ is a faithful tracial state on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$.

Furthermore, let $\beta : \mathbb{N} \to (0,\infty)$ have limit 0 at infinity. Using Theorem 2.6, if $\mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I}$ is the $(2,0)$-quasi-Leibniz Lip norm on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$ and $\mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}$ is the $(2,0)$-quasi-Leibniz Lip norm on $\mathfrak{A}/I$, then:

$$\phi_I^{-1} : (\mathfrak{A}/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}) \to (\varinjlim \mathcal{I}(\mathfrak{A}/I), \mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I})$$

is a quantum isometry of Theorem-Definition 2.3 *and*

$$\Lambda((\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I), \mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I}), (\mathfrak{A}/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu})) = 0.$$

Moreover, for all $n \in \mathbb{N}$, we have:

$$\Lambda((\mathfrak{A}_n/(I\cap\mathfrak{A}_n), \mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I} \circ \underrightarrow{\gamma^n_I}), ((\mathfrak{A}_n + I)/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu})) = 0.$$

Proof. Since $I \neq \mathfrak{A}$, the AF-algebra $\mathfrak{A}/I$ is unital and $(\mathfrak{A}_0 + I)/I = \mathbb{C}1_{\mathfrak{A}/I}$ as $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Since μ is faithful on $\mathfrak{A}/I$, $\mu\circ\phi_I$ is faithful on $\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)$ since ϕ_I is a $*$-isomorphism by Proposition 3.26.

By Notation 3.25, define $\mathcal{U}(\mathfrak{A}/I) = (\underrightarrow{\gamma^m_I}(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)))_{m\in\mathbb{N}}$. By Chapter 6.1 in [32], the sequence $\mathcal{U}(\mathfrak{A}/I) = (\underrightarrow{\gamma^m_I}(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)))_{m\in\mathbb{N}}$ is an increasing sequence of unital finite dimensional C^*-subalgebras of $\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)$ such that

$$\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I) = \overline{\bigcup_{m\in\mathbb{N}} \underrightarrow{\gamma^m_I}(\mathfrak{A}_m/(I\cap\mathfrak{A}_m))}^{\|\cdot\|_{\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)}}$$

and $\underrightarrow{\gamma^0_I}(\mathfrak{A}_0/(I\cap\mathfrak{A}_0)) = \mathbb{C}1_{\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)}$. Thus, let $\mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I}$ on $\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)$ and $\mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}$ on $\mathfrak{A}/I$ be given by Theorem 2.6,.

Now, fix $m \in \mathbb{N}$. Since $\phi_I \circ \underrightarrow{\gamma^m_I} = \phi^m_I$ by Proposition 3.26 we thus have:

$$\underrightarrow{\gamma^m_I}(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)) = \phi_I^{-1}\circ\phi^m_I(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)) = \phi_I^{-1}((\mathfrak{A}_m + I)/I).$$

Since the chosen faithful tracial state on $\underrightarrow{\lim}\mathcal{I}(\mathfrak{A}/I)$ is $\mu\circ\phi_I$, by Theorem 5.3 in [1] we have that $(\underrightarrow{\gamma^m_I}(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)), \mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I})$ is quantum isometric to $((\mathfrak{A}_m + I)/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu})$ by the map ϕ_I^{-1} restricted to $(\mathfrak{A}_m + I)/I$ for all $m \in \mathbb{N}$. However, the quantum metric spaces $(\underrightarrow{\gamma^m_I}(\mathfrak{A}_m/(I\cap\mathfrak{A}_m)), \mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I})$ and $((\mathfrak{A}_m/(I\cap\mathfrak{A}_m)), \mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I} \circ \underrightarrow{\gamma^m_I})$ are quantum isometric via the map $\underrightarrow{\gamma^m_I}$. Since quantum isometry is an equivalence relation, we conclude by Theorem-Definition 2.3

$$\Lambda((\mathfrak{A}_m/(I\cap\mathfrak{A}_m), \mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I} \circ \underrightarrow{\gamma^m_I}), ((\mathfrak{A}_m + I)/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu})) = 0.$$

Moreover, Theorem 5.3 in [1] also implies that

$$\phi_I^{-1} : (\mathfrak{A}/I, \mathsf{L}^{\beta}_{\mathcal{U}/I,\mu}) \to (\underrightarrow{\lim}\mathcal{U}(\mathfrak{A}/I), \mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I})$$

is a quantum isometry. Next, define $\mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I}$ from Theorem 2.6. By Proposition 5.2 in [1], we may replace $\mathsf{L}^{\beta}_{\mathcal{U}(\mathfrak{A}/I),\mu\circ\phi_I}$ with $\mathsf{L}^{\beta}_{\mathcal{I}(\mathfrak{A}/I),\mu\circ\phi_I}$, which completes the proof. ∎

Thus, the quantum isometry ϕ_I in Theorem 4.1 is in some sense the best one could hope for, since it preserves the finite-dimensional approximations in the quantum propinquity. Next, we give criteria for when a family of quotients converge in the quantum propinquity with respect to ideal convergence.

THEOREM 4.2. *Let $\mathfrak{A}$ be a unital AF-algebra with unit $1_{\mathfrak{A}}$ such that $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ is an increasing sequence of unital finite dimensional C^*-subalgebras with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$ such that $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Let $(I^n)_{n\in\overline{\mathbb{N}}} \subseteq \mathrm{Ideal}(\mathfrak{A}) \setminus \{\mathfrak{A}\}$ such that $\{\mu_k : \mathfrak{A}/I^k \to \mathbb{C} : k \in \overline{\mathbb{N}}\}$ is a family of faithful tracial states. Let $Q^k : \mathfrak{A} \to \mathfrak{A}/I^k$ denote the quotient map for all $k \in \overline{\mathbb{N}}$. If:*

(i) *$(I^n)_{n\in\mathbb{N}} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \mathrm{Ideal}(\mathfrak{A})$ with respect to $\mathsf{m}_{i(\mathcal{U})}$ of Corollary 3.20 or the Fell topology (Definition 3.5) with fusing sequence $(c_n)_{n\in\mathbb{N}}$ for the fusing family $\{I^n = \overline{\bigcup_{k\in\mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}}\}$,*

(ii) *for each $N \in \mathbb{N}$, we have that $(\mu_k \circ Q^k)_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $\mu_\infty \circ Q^\infty$ in the weak* topology on $\mathscr{S}(\mathfrak{A}_N)$, and*

(iii) *$\{\beta^k : \overline{\mathbb{N}} \to (0,\infty)\}_{k\in\overline{\mathbb{N}}}$ is a family of convergent sequences such that for all $N \in \mathbb{N}$ if $k \in \mathbb{N}_{\geqslant c_N}$, then $\beta^k(n) = \beta^\infty(n)$ for all $n \in \{0,1,\ldots,N\}$ and there exists $B : \overline{\mathbb{N}} \to (0,\infty)$ with $B(\infty) = 0$ and $\beta^m(l) \leqslant B(l)$ for all $m,l \in \overline{\mathbb{N}}$,*
then using notation from Theorem 4.1:

$$\lim_{n\to\infty} \Lambda\left((\mathfrak{A}/I^n, \mathsf{L}^{\beta^n}_{\mathcal{U}/I^n,\mu_n}), (\mathfrak{A}/I^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{U}/I^\infty,\mu_\infty})\right) = 0.$$

Proof. By Lemma 3.23, the assumption that $(I^n)_{n\in\mathbb{N}} \subseteq \mathrm{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \mathrm{Ideal}(\mathfrak{A})$ with respect to $\mathsf{m}_{i(\mathcal{U})}$ or the Fell topology implies that

$$\left\{ I^n = \overline{\bigcup_{k\in\mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}} \right\}$$

is a fusing family with some fusing sequence $(c_n)_{n\in\mathbb{N}}$ where also the family $\{\varinjlim \mathcal{I}(\mathfrak{A}/I^n) : n \in \overline{\mathbb{N}}\}$ is also a fusing family with fusing sequence $(c_n)_{n\in\mathbb{N}}$.

Fix $N \in \mathbb{N}$ and $k \in \mathbb{N}_{\geqslant c_N}$. Let $x \in \mathfrak{A}_N$, and let $Q^k_N : \mathfrak{A}_N \to \mathfrak{A}_N/(I^k \cap \mathfrak{A}_N)$ and $Q^\infty_N : \mathfrak{A}_N \to \mathfrak{A}_N/(I^\infty \cap \mathfrak{A}_N)$ denote the quotient maps, and let $\phi_{I^k} : \varinjlim \mathcal{I}(\mathfrak{A}/I^k) \to \mathfrak{A}/I^k$ denote the $*$-isomorphism given in Proposition 3.26 and recall that $\mathcal{I}(\mathfrak{A}/I^k) = (\mathfrak{A}_n/(I^k \cap \mathfrak{A}_n), \gamma_{I^k,n})_{n\in\mathbb{N}}$ from Notation 3.25. Now, by Proposition 3.26 and its commuting diagram, we gather:

$$\begin{aligned}\mu_k \circ \phi_{I^k} \circ \underrightarrow{\gamma^N_{I^k}} \circ Q^k_N(x) &= \mu_k \circ \phi^N_{I^k} \circ Q^k_N(x)\\ &= \mu_k \circ \phi^N_{I^k}(x + I^k \cap \mathfrak{A}_N) = \mu_k(x + I^k) = \mu_k \circ Q^k(x).\end{aligned}$$

Therefore, by hypothesis (ii), $(\mu_k \circ \phi_{I^k} \circ \underrightarrow{\gamma^N_{I^k}} \circ Q^k_N)_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $\mu_\infty \circ \phi_{I^\infty} \circ \underrightarrow{\gamma^N_{I^\infty}} \circ Q^\infty_N$ in the weak* topology on $\mathfrak{A}_N$.

Hence, the sequence $(\mu_k \circ \phi_{I^k} \circ \underrightarrow{\gamma^N_{I^k}})_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $\mu_\infty \circ \phi_{I^\infty} \circ \underrightarrow{\gamma^N_{I^\infty}}$ in the weak* topology on $\mathscr{S}(\mathfrak{A}_N/(I^\infty \cap \mathfrak{A}_N))$ by Theorem V.2.2 in [11]. Thus, by hypothesis (iii) and by Theorem 2.14, we have that:

$$\lim_{n\to\infty} \Lambda((\varinjlim \mathcal{I}(\mathfrak{A}/I^n), \mathsf{L}^{\beta^n}_{\mathcal{I}(\mathfrak{A}/I^n),\mu_n\circ\phi_{I^n}}), (\varinjlim \mathcal{I}(\mathfrak{A}/I^\infty), \mathsf{L}^{\beta^\infty}_{\mathcal{I}(\mathfrak{A}/I^\infty),\mu_\infty\circ\phi_{I^\infty}})) = 0.$$

But, as $\phi^{-1}_{I^n}$ is an isometric isomorphism for all $n \in \overline{\mathbb{N}}$ by Theorem 4.1, we conclude:

$$\lim_{n\to\infty} \Lambda((\mathfrak{A}/I^n, \mathsf{L}^{\beta^n}_{\mathcal{U}/I^n,\mu_n}), (\mathfrak{A}/I^\infty, \mathsf{L}^{\beta^\infty}_{\mathcal{U}/I^\infty,\mu_\infty})) = 0,$$

which completes the proof. ∎

4.1. THE BOCA–MUNDICI AF-ALGEBRA. The Boca–Mundici AF-algebra $\mathfrak{F}$ (for Farey) is constructed from the Farey tessellation and was discovered independently in [6] and [31]. In both [6], [31], it was shown that all Effros–Shen AF-algebras (Notation 2.9) arise as quotients of certain primitive ideals of $\mathfrak{F}$. This is the main motivation for our convergence result in [2]. In both [6], [31], it was also shown that the center of the AF-algebra $\mathfrak{F}$ is $*$-isomorphic to C([0,1]), which provided the framework for Eckhardt to introduce a noncommutative analogue to the Gauss map in [14].

We present the construction of this algebra as presented in the paper by Boca [6]. We refer mostly to Boca's work as his unique results pertaining to the Jacobson topology (for example Corollary 12 in [6], which is the result that led us to begin this paper) are more applicable to our work (see Proposition 4.19). As in [6], we define the AF-algebra $\mathfrak{F}$ recursively by the following relations (4.1). We note that the relations presented here are the same as in Section 1 of [6], but instead of starting at $n = 0$, these relations begin at $n = 1$, so that this formulation of the AF-algebra $\mathfrak{F}$ as an inductive limit begins with $\mathbb{C}$. For every $n \in \mathbb{N} \setminus \{0\}$ and $k \in \mathbb{N}$ define:

$$\text{(4.1)} \quad \begin{cases} q(n,0)=q(n,2^{n-1})=1, \;\; p(n,0)=0, \;\; p(n,2^{n-1})=1; & \\ q(n+1,2k) = q(n,k), \;\; p(n+1,2k) = p(n,k), & 0 \leqslant k \leqslant 2^{n-1}; \\ q(n+1,2k+1)=q(n,k)+q(n,k+1), & 0\leqslant k\in\leqslant 2^{n-1}-1; \\ p(n+1,2k+1)=p(n,k)+p(n,k+1), & 0\leqslant k\in\leqslant 2^{n-1}-1; \\ r(n,k) = \frac{p(n,k)}{q(n,k)}, & 0 \leqslant k \leqslant 2^{n-1} - 1. \end{cases}$$

DEFINITION 4.3. For $n \in \mathbb{N} \setminus \{0\}$, define the *finite dimensional C^*-algebras*,

$$\mathfrak{F}_n = \bigoplus_{k=0}^{2^{n-1}} \mathfrak{M}(q(n,k)) \quad \text{and} \quad \mathfrak{F}_0 = \mathbb{C},$$

where $\mathfrak{M}(N)$ denotes the C^*-algebra of complex $N \times N$-matrices.

Next, we define $*$-homomorphisms to complete the inductive limit recipe. We utilize partial multiplicity matrices.

DEFINITION 4.4. For $n \in \mathbb{N} \setminus \{0\}$, let F_n be the $(2^n+1) \times (2^{n-1}+1)$ matrix with entries in $\{0,1\}$ determined entry-wise by:

$$(F_n)_{h,j} = \begin{cases} 1 & \text{if } (h = 2k+1, k \in \{0,\dots,2^{n-1}\}, \text{ and } j = k+1) \\ & \quad \text{or } (h = 2k, k \in \{1,\dots,2^{n-1}\} \text{ and } (j = k \vee j = k+1)); \\ 0 & \text{otherwise.} \end{cases}$$

For example,

$$F_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We would like these matrices to determine unital $*$-monomorphisms, so that our inductive limit is a unital C^*-algebra, which motivates the following lemma.

LEMMA 4.5. *Using Definition* 4.4 *and relations* (4.1), *if* $n \in \mathbb{N} \setminus \{0\}$, *then:*

$$F_n \begin{pmatrix} q(n,0) \\ q(n,1) \\ \vdots \\ q(n,2^{n-1}) \end{pmatrix} = \begin{pmatrix} q(n+1,0) \\ q(n+1,1) \\ \vdots \\ q(n+1,2^n) \end{pmatrix}.$$

Proof. Let $n \in \mathbb{N} \setminus \{0\}$. Let $k \in \{1,\dots,2^{n-1}\}$ and consider $q(n+1,2k-1)$. Now, by Definition 4.4, row $2k-1+1 = 2k$ of F_n has 1 in entry k and $k+1$, and 0 elsewhere. Thus:

$$\begin{aligned} ((F_n)_{2k,1},\dots,(F_n)_{2k,2^{n-1}+1}) \cdot \begin{pmatrix} q(n,0) \\ q(n,1) \\ \vdots \\ q(n,2^{n-1}) \end{pmatrix} &= q(n,k-1) + q(n,k-1+1) \\ &= q(n+1,2k-1) \end{aligned}$$

by (4.1). Next, let $k \in \{0,\dots,2^{n-1}\}$ and consider $q(n+1,2k)$. By Definition 4.4, row $2k+1$ of F_n has 1 in entry $k+1$ and 0 elsewhere. Thus, by (4.1):

$$((F_n)_{2k+1,1},\dots,(F_n)_{2k+1,2^{n-1}+1}) \cdot \begin{pmatrix} q(n,0) \\ q(n,1) \\ \vdots \\ q(n,2^{n-1}) \end{pmatrix} = q(n,2k) = q(n+1,2k).$$

Hence, by matrix multiplication, the proof is complete. ∎

DEFINITION 4.6 ([6], [31]). Define $\varphi_0 : \mathfrak{F}_0 \to \mathfrak{F}_1$ by $\varphi_0(a) = a \oplus a$. For $n \in \mathbb{N} \setminus \{0\}$, by Lemma III.2.1 in [12] and Lemma 4.5, let $\varphi_n : \mathfrak{F}_n \to \mathfrak{F}_{n+1}$ be the unital $*$-monomorphism determined by F_n in Definition 4.4. Using Definition 4.3, we let the unital C^*-inductive limit (Notation 2.5)

$$\mathfrak{F} = \varinjlim (\mathfrak{F}_n, \varphi_n)_{n \in \mathbb{N}}$$

denote the *Boca–Mundici AF-algebra*.

Let $\mathfrak{F}^n = \underrightarrow{\varphi^n}(\mathfrak{F}_n)$ for all $n \in \mathbb{N}$ and $\mathcal{U}_{\mathfrak{F}} = (\mathfrak{F}^n)_{n \in \mathbb{N}}$, which is a non-decreasing sequence of C^*-subalgebras of $\mathfrak{F}$ with $\mathfrak{F}^0 = \mathbb{C}1_{\mathfrak{F}}$ and $\mathfrak{F} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{F}^n}^{\|\cdot\|_{\mathfrak{F}}}$ (see Chapter 6.1 in [32]).

We note that in [6], the AF-algebra $\mathfrak{F}$ is constructed by a Bratteli diagram displayed as Figure 2 in [6], so in order to utilize the results of [6], we verify that we have the same Bratteli diagram up to adding one vertex of label 1 at level $n = 0$ satisfying the conditions at the beginning of Section 1 in [6]. But, first, we fix some notation for Bratteli diagrams and state some well-known results that will prove useful.

DEFINITION 4.7 ([7]). A *Bratteli diagram* is given by a directed graph $\mathcal{D} = (V^{\mathcal{D}}, E^{\mathcal{D}})$ with labelled vertices and multiple edges between two vertices is allowed. The set $V^{\mathcal{D}} \subset \mathbb{N}^2$ is the set of labeled vertices and $E^{\mathcal{D}} \subset \mathbb{N}^2 \times \mathbb{N}^2$ is the set of edges, which consist of ordered pairs from $V^{\mathcal{D}}$. For each $n \in \mathbb{N}$, let $v_n^{\mathcal{D}} \in \mathbb{N}$.

Define $V^{\mathcal{D}} = \bigcup_{n \in \mathbb{N}} V_n^{\mathcal{D}}$, where for $n \in \mathbb{N}$, we let

$$V_n^{\mathcal{D}} = \{(n,k) \in \mathbb{N} \times \mathbb{N} : k \in \{0, \ldots, v_n^{\mathcal{D}}\}\},$$

and denote the label of the vertices $(n,k) \in V^{\mathcal{D}}$ by $[n,k]_{\mathcal{D}} \in \mathbb{N} \setminus \{0\}$.

Next, let $E^{\mathcal{D}} \subset V^{\mathcal{D}} \times V^{\mathcal{D}}$. Now, we list some axioms for $V^{\mathcal{D}}$ and $E^{\mathcal{D}}$.

(i) For all $n \in \mathbb{N}$, if $m \in \mathbb{N} \setminus \{n+1\}$, then $((n,k),(m,q)) \notin E^{\mathcal{D}}$ for all $k \in \{0, \ldots, v_n^{\mathcal{D}}\}$ and $q \in \{0, \ldots, v_m^{\mathcal{D}}\}$.

(ii) If $(n,k) \in V^{\mathcal{D}}$, then there exists $q \in \{0, \ldots, v_{n+1}^{\mathcal{D}}\}$ such that $((n,k),(n+1,q)) \in E^{\mathcal{D}}$.

(iii) If $n \in \mathbb{N} \setminus \{0\}$ and $(n,k) \in V^{\mathcal{D}}$, then there exists $q \in \{0, \ldots, v_{n-1}^{\mathcal{D}}\}$ such that $((n-1,q),(n,k)) \in E^{\mathcal{D}}$.

If $\mathcal{D}$ satisfies all properties above, then we call $\mathcal{D}$ a *Bratteli diagram*. The set of all Bratteli diagrams is denoted by $\mathscr{BD}$.

We also introduce the following notation. For each $n \in \mathbb{N}$, let:

$$E_n^{\mathcal{D}} = (V_n^{\mathcal{D}} \times V_{n+1}^{\mathcal{D}}) \cap E^{\mathcal{D}}.$$

By axiom (i) we have $E^{\mathcal{D}} = \bigcup_{n \in \mathbb{N}} E_n^{\mathcal{D}}$. Also, for $((n,k),(n+1,q)) \in E_n^{\mathcal{D}}$, we denote by $[(n,k),(n+1,q)]_{\mathcal{D}} \in \mathbb{N} \setminus \{0\}$ the number of edges from (n,k) to $(n+1,q)$. Let $(n,k) \in V^{\mathcal{D}}$, define

$$R_{(n,k)}^{\mathcal{D}} = \{(n+1,q) \in V_{n+1}^{\mathcal{D}} : ((n,k),(n+1,q)) \in E^{\mathcal{D}}\},$$

which is non-empty by axiom (ii). We refer to $V_n^{\mathcal{D}}, E_n^{\mathcal{D}}$, and $(V_n^{\mathcal{D}}, E_n^{\mathcal{D}})$ as the vertices at level n, edges at level n, and diagram at level n, respectively.

REMARK 4.8. It is easy to see that this definition coincides with Bratteli's of Section 1.8 in [7], in that we simply trade his arrow notation with that of edges and number of edges. That is, given a Bratteli diagram $\mathcal{D}$, the correspondence is: $(n,k) \searrow^p (n+1,q)$ if and only if $((n,k),(n+1,q)) \in E^{\mathcal{D}}$ and $[(n,k),(n+1,q)]_{\mathcal{D}} = p$.

DEFINITION 4.9 ([7]). Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n\in\mathbb{N}}$ be an inductive sequence of finite dimensional C^*-algebras with C^*-inductive limit $\mathfrak{A}$, where α_n is injective for all $n \in \mathbb{N}$. Thus, $\mathfrak{A}$ is an AF-algebra by Chapter 6.1 in [32]. Let $\mathcal{D}_b(\mathfrak{A})$ be a diagram associated to $\mathfrak{A}$ constructed as follows.

Fix $n \in \mathbb{N}$. Since $\mathfrak{A}_n$ is finite dimensional, $\mathfrak{A}_n \cong \bigoplus_{k=0}^{a_n} \mathfrak{M}(n(k))$ such that $a_n \in \mathbb{N}$ and $n(k) \in \mathbb{N} \setminus \{0\}$ for $k \in \{0, \ldots, a_n\}$. Define

$$v_n^{\mathcal{D}_b(\mathfrak{A})} = a_n, \quad V_n^{\mathcal{D}_b(\mathfrak{A})} = \{(n,k) \in \mathbb{N}^2 : k \in \{0, \ldots, v_n^{\mathcal{D}_b(\mathfrak{A})}\}\},$$

and label $[n,k]_{\mathcal{D}_b(\mathfrak{A})} = \sqrt{\dim(\mathfrak{M}(n(k)))}$ for $k \in \{0, \ldots, v_n^{\mathcal{D}_b(\mathfrak{A})}\}$.

Let A_n be the $(a_{n+1}+1) \times (a_n+1)$-partial multiplicity matrix associated to $\alpha_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ with entries $(A_n)_{i,j} \in \mathbb{N}, i \in \{1, \ldots, a_{n+1}+1\}, j \in \{1, \ldots, a_n+1\}$ given by Lemma III.2.2 in [12]. Define

$$E_n^{\mathcal{D}_b(\mathfrak{A})} = \{((n,k),(n+1,q)) \in \mathbb{N}^2 \times \mathbb{N}^2 : (A_n)_{q+1,k+1} \neq 0\},$$

and if $((n,k),(n+1,q)) \in E_n^{\mathcal{D}_b(\mathfrak{A})}$, then let the number of edges be $[(n,k),(n+1,q)]_{\mathcal{D}_b(\mathfrak{A})} = (A_n)_{q+1,k+1}$.

Let $V^{\mathcal{D}_b(\mathfrak{A})} = \bigcup_{n\in\mathbb{N}} V_n^{\mathcal{D}_b(\mathfrak{A})}$, $E^{\mathcal{D}_b(\mathfrak{A})} = \bigcup_{n\in\mathbb{N}} E_n^{\mathcal{D}_b(\mathfrak{A})}$, and denote $\mathcal{D}_b(\mathfrak{A}) = (V^{\mathcal{D}_b(\mathfrak{A})}, E^{\mathcal{D}_b(\mathfrak{A})})$. By Section 1.8 in [7], we conclude $\mathcal{D}_b(\mathfrak{A}) \in \mathscr{BD}$ is a Bratteli diagram as in Definition 4.7.

If $\mathfrak{A}$ is an AF-algebra of the form $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ where $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence of finite dimensional C^*-subalgebras of $\mathfrak{A}$, then the diagram $\mathcal{D}_b(\mathfrak{A})$ has the same vertices as the one above, and the edges are formed by the partial multiplicity matrix built from the partial multiplicities of the inclusion mappings $\iota_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ for all $n \in \mathbb{N}$.

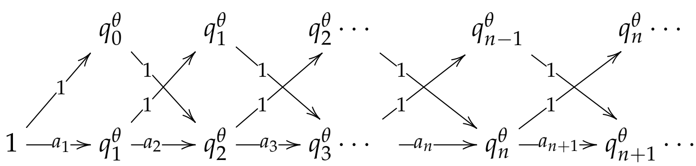
REMARK 4.10. We note that the converse of Definition 4.9 is true in the sense that given a Bratteli diagram, one may construct an AF-algebra associated to it. The process is described in Section 1.8 in [7], and in particular, one may construct partial multiplicity matrices from the edge set, which then provide injective $*$-homomorphisms to build an inductive limit.

As an example, which will be used in Proposition 4.23, we display the Bratteli diagram for the Effros–Shen AF-algebras of Notation 2.9.

EXAMPLE 4.11. Fix $\theta \in (0,1) \setminus \mathbb{Q}$ with continued fraction expansion $\theta = [a_j]_{j\in\mathbb{N}}$ using (2.1) with convergents $(p_n^\theta / q_n^\theta)_{n\in\mathbb{N}}$ given by (2.2). Let $\mathfrak{AF}_\theta$ be the Effros–Shen AF-algebra from Notation 2.9. Thus, $v_0^{\mathcal{D}_b(\mathfrak{AF}_\theta)} = 0$ and $V_0^{\mathcal{D}_b(\mathfrak{AF}_\theta)} = \{(0,0)\}$ with $[0,0]_{\mathcal{D}_b(\mathfrak{AF}_\theta)} = 1$. For $n \in \mathbb{N} \setminus \{0\}$, we have $v_n^{\mathcal{D}_b(\mathfrak{AF}_\theta)} = 1$ and $V_n^{\mathcal{D}_b(\mathfrak{AF}_\theta)} = \{(n,0),(n,1)\}$ with $[n,0]_{\mathcal{D}_b(\mathfrak{AF}_\theta)} = q_n^\theta, [n,1]_{\mathcal{D}_b(\mathfrak{AF}_\theta)} = q_{n-1}^\theta$. The partial multiplicity matrices are

$$A_0 = \begin{pmatrix} a_1 \\ 1 \end{pmatrix} \quad \text{and} \quad A_n = \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } n \in \mathbb{N} \setminus \{0\}$$

by Notation 2.9 and Lemma III.2.1 in [12], which determines the edges. We now provide the diagram as a graph. The label in the edges denotes the number of edges and the top row contains the vertices $(n,1)$ with their labels with n increasing from left to right, and the bottom row having vertices $(n,0)$ with their labels with n increasing from left to right:



Returning to the diagram setting, we define what an ideal of a diagram is.

DEFINITION 4.12. Let $\mathcal{D} = (V^\mathcal{D}, E^\mathcal{D})$ be a Bratteli diagram. We call $\mathcal{D}(I) = (V^I, E^I)$ an *ideal diagram* of $\mathcal{D}$ if $V^I \subseteq V^\mathcal{D}, E^I \subseteq E^\mathcal{D}$ and:

(i) (directed) if $(n,k) \in V^I$ and $((n,k),(n+1,q)) \in E^\mathcal{D}$, then $(n+1,q) \in V^I$;

(ii) (hereditary) if $(n,k) \in V^\mathcal{D}$ and $R^\mathcal{D}_{(n,k)} \subseteq V^I$, then $(n,k) \in V^I$;

(iii) (edges) if $(n,k),(n+1,q) \in V^I$ such that $((n,k),(n+1,q)) \in E^\mathcal{D}$, then $((n,k),(n+1,q)) \in E^I$.

Furthermore, if $(n,k) \in V^\mathcal{D} \cap V^I$, then $[n,k]_\mathcal{D} = [n,k]_{\mathcal{D}(I)}$. And, if $((n,k),(n+1,q)) \in E^\mathcal{D} \cap E^I$, then $[(n,k),(n+1,q)]_\mathcal{D} = [(n,k),(n+1,q)]_{\mathcal{D}(I)}$.

Also, for $n \in \mathbb{N}$, denote $V_n^I = V_n^\mathcal{D} \cap V^I$ and $E_n^I = E_n^\mathcal{D} \cap E^I$ with $I_n = (V_n^I, E_n^I)$ to also include all associated labels and number of edges, and refer to V_n^I as the vertices at level n of the diagram. Let $\mathrm{Ideal}(\mathcal{D})$ denote the set of ideals of $\mathcal{D}$.

NOTATION 4.13. Let $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_\mathfrak{A}}$ be an AF-algebra where $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence of finite dimensional C^*-subalgebras of $\mathfrak{A}$. Using Definition 4.9, let $\mathcal{D}_b(\mathfrak{A})$ be the associated diagram.

Let $I \in \mathrm{Ideal}(\mathfrak{A})$ be a norm closed two-sided ideal of $\mathfrak{A}$. Then by Lemma 3.2 in [7], the subset Λ of $\mathcal{D}_b(\mathfrak{A})$ formed by I is an ideal in the sense of Definition 4.12. Denote this by $\mathcal{D}_b(\mathfrak{A})(I) \in \mathrm{Ideal}(\mathcal{D}_b(\mathfrak{A}))$, where $\mathrm{Ideal}(\mathcal{D}_b(\mathfrak{A}))$ is the set of ideals of $\mathcal{D}_b(\mathfrak{A})$ from Definition 4.12.

PROPOSITION 4.14 (Lemma 3.2 in [7]). *Let* $\mathfrak{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ *be an* AF-*algebra, where* $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$ *is a non-decreasing sequence of finite dimensional* C^**-subalgebras of* $\mathfrak{A}$ *and Bratteli diagram* $\mathcal{D}_b(\mathfrak{A})$ *from Definition* 4.9*. Using Notation* 4.13 *and Definition* 4.12*, the map*

$$i(\cdot,\mathcal{D}_b(\mathfrak{A})) : I \in \mathrm{Ideal}(\mathfrak{A}) \longmapsto \mathcal{D}_b(\mathfrak{A})(I) \in \mathrm{Ideal}(\mathcal{D}_b(\mathfrak{A}))$$

given by Lemma 3.2 *in* [7] *is a well-defined bijection, where the vertices of* $V_n^{\mathcal{D}_b(\mathfrak{A})(I)}$ *are uniquely determined by* $I \cap \mathfrak{A}_n$ *for each* $n \in \mathbb{N}$*.*

PROPOSITION 4.15. *The Bratteli diagram* $\mathcal{D}_b(\mathfrak{F}) = (V^{\mathcal{D}_b(\mathfrak{F})}, E^{\mathcal{D}_b(\mathfrak{F})})$ *of* $\mathfrak{F}$ *of Definition* 4.9*, satisfies for all* $n \in \mathbb{N} \setminus \{0\}$*:*

(i) $V_n^{\mathcal{D}_b(\mathfrak{F})} = \{(n,k) : k \in \{0,\ldots,2^{n-1}\}\}$;

(ii) $((n,k),(n+1,l)) \in E_n^{\mathcal{D}_b(\mathfrak{F})}$ *if and only if* $|2k-l| \leqslant 1$*.*

Moreover, two such vertices are either connected at exactly one edge or are not connected at all.

Proof. Property (i) is clear by Definition 4.3. By the definition of *Bratteli diagram* in Section III.2 in [12], an edge exists from (n,s) to $(n+1,t)$ if and only if its associated entry in the partial multiplicity matrix $(F_n)_{t+1,s+1}$ is non-zero.

Now, assume that $|2s-t| \leqslant 1$. First, assume $t = 2k+1$ for some $k \in \{0,\ldots,2^{n-1}-1\}$. We thus have $|2s-t| \leqslant 1 \Leftrightarrow k \leqslant s \leqslant k+1 \Leftrightarrow s \in \{k,k+1\}$, since $s \in \mathbb{N}$.

Next, assume that $t = 2k$ for some $k \in \{0,\ldots,2^{n-1}\}$. We thus have

$$|2s-t| \leqslant 1 \Leftrightarrow \frac{-1}{2} + k \leqslant s \leqslant \frac{1}{2} + k \Leftrightarrow |s-k| \leqslant \frac{1}{2} \Leftrightarrow s = k$$

since $s \in \mathbb{N}$. But, considering both t odd and even, these equivalences are also equivalent to the conditions for $(F_n)_{t+1,s+1}$ to be non-zero by Definition 4.4, which determine the edges of $\mathcal{D}_b(\mathfrak{F})$. Furthermore, since the non-zero entries of F_n are all 1, only one edge exists between vertices for which there is an edge. ∎

Next, we describe the ideals of $\mathfrak{F}$, whose quotients are $*$-isomorphic to the Effros–Shen AF-algebras.

DEFINITION 4.16 ([6]). Let $\theta \in (0,1) \setminus \mathbb{Q}$. We define the ideal $I_\theta \in \mathrm{Ideal}(\mathfrak{F})$ diagrammatically. By the proof of Proposition 4.i in [6], for each $n \in \mathbb{N} \setminus \{0\}$, there exists a unique $j_n(\theta) \in \{0,\ldots,2^{n-1}-1\}$ such that $r(n,j_n(\theta)) < \theta < r(n,j_n(\theta)+1)$ of relations (4.1). The set of vertices is defined by:

$$V^{\mathcal{D}_b(\mathfrak{F})} \setminus (\{(n,j_n(\theta)),(n,j_n(\theta)+1) : n \in \mathbb{N} \setminus \{0\} \cup \{(0,0)\}\})$$

and will be denoted by $V^{\mathcal{D}(I_\theta)}$. Let $E^{\mathcal{D}(I_\theta)}$ be the set of edges of $\mathcal{D}_b(\mathfrak{F})$, which are between the vertices in $V^{\mathcal{D}(I_\theta)}$ and let $\mathcal{D}(I_\theta) = (V^{\mathcal{D}(I_\theta)}, E^{\mathcal{D}(I_\theta)})$. By Proposition 4.i in [6], the diagram $\mathcal{D}(I_\theta) \in \mathrm{Ideal}(\mathcal{D}_b(\mathfrak{F}))$ is an ideal diagram as in Definition 4.12.

Using Proposition 4.14, define:

$$I_\theta = i(\cdot, \mathcal{D}_b(\mathfrak{A}))^{-1}(\mathcal{D}(I_\theta)) \in \mathrm{Ideal}(\mathfrak{A}).$$

By Proposition 4.i in [6], if $n \in \mathbb{N} \setminus \{0,1\}$ and $1 \leqslant j_n(\theta) \leqslant 2^{n-1} - 2$, then:

$$I_\theta \cap \mathfrak{F}^n = \underset{\rightarrow}{\varphi^n}\Big(\Big(\bigoplus_{k=0}^{j_n(\theta)-1} \mathfrak{M}(q(n,k))\Big) \oplus \{0\} \oplus \{0\} \oplus \Big(\bigoplus_{k=j_n(\theta)+2}^{2^{n-1}} \mathfrak{M}(q(n,k))\Big)\Big).$$

If $j_n(\theta) = 0$, then:

$$I_\theta \cap \mathfrak{F}^n = \underset{\rightarrow}{\varphi^n}\Big(\{0\} \oplus \{0\} \oplus \Big(\bigoplus_{k=j_n(\theta)+2}^{2^{n-1}} \mathfrak{M}(q(n,k))\Big)\Big).$$

If $j_n(\theta) = 2^{n-1} - 1$, then

$$I_\theta \cap \mathfrak{F}^n = \underset{\rightarrow}{\varphi^n}\Big(\Big(\bigoplus_{k=0}^{j_n(\theta)-1} \mathfrak{M}(q(n,k))\Big) \oplus \{0\} \oplus \{0\}\Big),$$

and if $n \in \{0,1\}$, then $I_\theta \cap \mathfrak{F}^n = \{0\}$. We note that $I_\theta \in \mathrm{Prim}(\mathfrak{F})$ by Proposition 4.i in [6].

Before moving to describing the quantum metric structure of quotients of the ideals of Definition 4.16, the following lemma captures more properties of the structure of the ideals introduced in Definition 4.16; its proof is omitted as it is elementary and follows basic ingredients from [6].

LEMMA 4.17. *Using notation from Definition* 4.16, *if* $n \in \mathbb{N} \setminus \{0\}, \theta \in (0,1) \setminus \mathbb{Q}$, *then* $j_{n+1}(\theta) \in \{2j_n(\theta), 2j_n(\theta)+1\}$.

DEFINITION 4.18 ([30]). The *Baire space* $\mathcal{N}$ is the set $(\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ endowed with the metric d defined, for any two $(x(n))_{n\in\mathbb{N}}, (y(n))_{n\in\mathbb{N}}$ in $\mathcal{N}$, by:

$$\mathsf{d}((x(n))_{n\in\mathbb{N}}, (y(n))_{n\in\mathbb{N}}) = \begin{cases} 0 & x(n) = y(n) \text{ for all } n \in \mathbb{N}, \\ 2^{-\min\{n\in\mathbb{N}: x(n)\neq y(n)\}} & \text{otherwise.} \end{cases}$$

In the next two results on the subset of ideals of Definition 4.16, we provide a useful topological result about the metric on ideals of Corollary 3.20, in which the equivalence of (i) and (iii) is a consequence of Corollary 12 in [6], which is unique to Boca's work on the AF-algebra $\mathfrak{F}$. Furthermore, it was shown in Remark 8(ii) in [6] that $\mathrm{Prim}(\mathfrak{F})$ with the Jacobson topology is not T_1 and therefore not Hausdorff. Thus the following proposition does not immediately follow from Corollary 3.31. However, in the next proposition, a direct advantage of the metric of Corollary 3.20 is to recover the Jacobson topology from the Fell topology on the subset of ideals of Definition 4.16.

PROPOSITION 4.19. *If* $(\theta_n)_{n\in\overline{\mathbb{N}}} \subseteq (0,1) \setminus \mathbb{Q}$, *then using notation from Definition* 4.6 *and Definition* 4.16, *the following are equivalent:*

(i) $(\theta_n)_{n\in\mathbb{N}}$ *converges to* θ_∞ *with respect to the usual topology on* $\mathbb{R}$*;*

(ii) $(\mathsf{cf}(\theta_n))_{n\in\mathbb{N}}$ *converges to* $\mathsf{cf}(\theta_\infty)$ *with respect to the Baire space,* $\mathscr{N}$ *and its metric from Definition* 4.18*, where* cf *denotes the unique continued fraction expansion of an irrational;*

(iii) $(I_{\theta_n})_{n\in\mathbb{N}}$ *converges to* I_{θ_∞} *with respect to the Jacobson topology (Definition* 3.1*) on* $\mathrm{Prim}(\mathfrak{F})$*;*

(iv) $(I_{\theta_n})_{n\in\mathbb{N}}$ *converges to* I_{θ_∞} *with respect to the metric topology of* $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ *of Corollary* 3.21 *or the Fell topology of Definition* 3.5*.*

Proof. The equivalence between (i) and (ii) is a classic result, for which a proof can be found in Proposition 5.10 in [2]. The equivalence between (i) and (iii) is immediate from Corollary 12 in [6]. And, therefore, (ii) is equivalent to (iii). Thus, it remains to prove that (iii) is equivalent to (iv).

(iv) implies (iii) is an immediate consequence of Proposition 3.7 and Theorem 3.21 as the Fell topology is stronger. Hence, assume (iii). Then, since we have already established (iii) implies (ii), we may assume (ii). For each $n \in \overline{\mathbb{N}}$, let $\mathsf{cf}(\theta_n) = [a_j^n]_{j\in\mathbb{N}}$. By assumption, the coordinates $a_0^n = 0$ for all $n \in \overline{\mathbb{N}}$. Now, assume that there exists $N \in \mathbb{N}\setminus\{0\}$ such that $a_j^n = a_j^\infty$ for all $n \in \mathbb{N}$ and $j \in \{0,\ldots,N\}$. Assume without loss of generality that N is odd. Thus, using Figure 5 in [6], we have that

$$L_{a_1^n-1}\circ R_{a_2^n}\circ\cdots\circ L_{a_N^n} = L_{a_1^\infty-1}\circ R_{a_2^\infty}\circ\cdots\circ L_{a_N^\infty} \tag{4.2}$$

for all $n \in \mathbb{N}$. But, equation (4.2) determines the vertices for the diagram of the quotient $\mathfrak{F}/I_{\theta_n}$ for all $n \in \overline{\mathbb{N}}$ by the proof of Proposition 4.i in [6] and the vertices of the diagram of the quotient $\mathfrak{F}/I_{\theta_n}$ are simply the complement of the vertices of the diagram of I_{θ_n} by Theorem III.4.4 in [12]. Now, primitive ideals must have the same vertices at level 0 of the diagram since they cannot equal $\mathfrak{A}$ by Definition 3.1 and are thus non-unital. But, for any $\eta \in (0,1)\setminus\mathbb{Q}$, the ideals I_η must always have the same vertices at level 1 of the diagram as well since the only two vertices are $(1,0),(1,1)$ and $r(1,0) = 0 < \theta < 1 = r(1,1)$ by relations (4.1) for all $\theta \in (0,1)\setminus\mathbb{Q}$. Thus, by (4.2) we gather that $I_{\theta_n}\cap\mathfrak{F}^j = I_{\theta_\infty}\cap\mathfrak{F}^j$ for all $n \in \mathbb{N}$ and

$$j \in \left\{0,\ldots,\max\left\{1, a_1^\infty - 1 + \left(\sum_{k=2}^N a_k^N\right)\right\}\right\},$$

where $\max\left\{1, a_1^\infty - 1 + \left(\sum_{k=2}^N a_k^N\right)\right\} \geqslant N$ as the terms of the continued fraction expansion are all positive integers for coordinates greater than 0. Thus, by the definition of the metric on the Baire space and the metric $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$, we conclude that convergence in the Baire space metric of $(\mathsf{cf}(\theta_n))_{n\in\mathbb{N}}$ to $\mathsf{cf}(\theta_\infty)$ implies convergence of $(I_{\theta_n})_{n\in\mathbb{N}}$ to I_{θ_∞} with respect to the metric $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ or the Fell topology by Theorem 3.21. ∎

Although the next result follows from Proposition 4.19 and the proofs of Proposition 4.i and Lemma 11 in [6], we provide a proof here.

PROPOSITION 4.20. *The map:*

$$\theta \in (0,1) \setminus \mathbb{Q} \longmapsto I_\theta \in \operatorname{Prim}(\mathfrak{A})$$

is a homeomorphism onto its image when $(0,1) \setminus \mathbb{Q}$ *is equipped with the topology induced by the usual topology on* $\mathbb{R}$ *and* $\operatorname{Prim}(\mathfrak{A})$ *is equipped with either the Jacobson topology, Fell topology, or the metric topology of* $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ *of Corollary* 3.21.

Proof. By Proposition 4.19, the fact that the Jacobson topology of a separable C^*-algebra is second countable (see Corollary 4.3.4 in [33]), and the Fell topology of an AF-algebra being metrizable (see Theorem 3.21 or more generally Lemma 3.18), sequential continuity suffices. Thus we only need to verify that the map defined in this proposition is a well-defined bijection onto its image. However, it is well-defined by Definition 4.16. Thus, injectivity remains, which will follow from the next claim.

CLAIM 4.21. *If* $\theta \in (0,1) \setminus \mathbb{Q}$, *then*

$$\lim_{n\to\infty} r(n, j_n(\theta)) = \theta,$$

where the quantity $r(n, j_n(\theta))$ *is defined in* (4.1) *and in Definition* 4.16.

Proof of claim. Fix $\theta \in (0,1) \setminus \mathbb{Q}$. Let $(p_n^\theta / q_n^\theta)_{n\in\mathbb{N}}$ denote convergents of θ. Now, the proofs of Proposition 4.i and Lemma 11 in [6] show the existence of an increasing sequence $(k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N} \setminus \{0\}$ such that for all $n \in \mathbb{N} \setminus \{0\}$:

$$(r(k_n, j_{k_n}(\theta)), r(k_n, j_{k_n}(\theta)+1)) \in \left\{ \left(\frac{p_n^\theta}{q_n^\theta}, \frac{p_{n-1}^\theta}{q_{n-1}^\theta} \right), \left(\frac{p_{n-1}^\theta}{q_{n-1}^\theta}, \frac{p_n^\theta}{q_n^\theta} \right) \right\}. \tag{4.3}$$

Next, fix $n \in \mathbb{N} \setminus \{0\}$. Consider $r(n, j_n(\theta))$. By Lemma 4.17, first assume that $j_{n+1}(\theta) = 2j_n(\theta)$. Then, by (4.1) we have:

$$r(n+1, j_{n+1}(\theta)) = \frac{p(n+1, 2j_n(\theta))}{q(n+1, 2j_n(\theta))} = r(n, j_n(\theta)).$$

Also, we have

$$\begin{aligned} r(n+1, j_{n+1}(\theta)+1) &= \frac{p(n+1, 2j_n(\theta)+1)}{q(n+1, 2j_n(\theta)+1)} \\ &= \frac{p(n, j_n(\theta)) + p(n, j_n(\theta)+1)}{p(n, j_n(\theta)) + p(n, j_n(\theta)+1)} \leqslant r(n, j_n(\theta)+1) \end{aligned}$$

by (4.1) and the fact that

$$p(n, j_n(\theta)+1) q(n, j_n(\theta)) - p(n, j_n(\theta)) q(n, j_n(\theta)+1) = 1 > 0$$

from Section 1 in [6]. For the case $j_{n+1}(\theta) = 2j_n(\theta)+1$, a similar argument shows that $r(n+1, j_{n+1}(\theta)) \geqslant r(n, j_n(\theta))$ and $r(n+1, j_{n+1}(\theta)+1) = r(n, j_n(\theta)+1)$. Hence, for all $n \in \mathbb{N} \setminus \{0\}$, we gather that:

$$r(n+1, j_{n+1}(\theta)+1) - r(n+1, j_{n+1}(\theta)) \leqslant r(n, j_n(\theta)+1) - r(n, j_n(\theta)). \tag{4.4}$$

For all $n \in \mathbb{N} \setminus \{0\}$ such that $n \geqslant k_1$, define $N_n = \max\{k_m : k_m \leqslant n\}$. Note that since $(k_n)_{n\in\mathbb{N}}$ is increasing, we have that $(N_n)_{n\geqslant k_1}$ is non-decreasing and $\lim_{n\to\infty} N_n = \infty$. Now, fix $n \in \mathbb{N} \setminus \{0\}$ such that $n \geqslant k_1$, combining expressions (4.3) and (4.4) we have by Definition 4.16,

$$0 < \theta - r(n, j_n(\theta)) < r(n, j_n(\theta) + 1) - r(n, j_n(\theta)) \leqslant r(N_n, j_{N_n}(\theta) + 1) - r(N_n, j_{N_n}(\theta)) = \left| \frac{p^\theta_{N_n}}{q^\theta_{N_n}} - \frac{p^\theta_{N_n-1}}{q^\theta_{N_n-1}} \right|,$$

and therefore $\lim_{n\to\infty} r(n, j_n(\theta)) = \theta$ since $\lim_{n\to\infty} (p^\theta_n / q^\theta_n) = \theta$ and $(N_n)_{n\geqslant k_1}$ is non-decreasing with $\lim_{n\to\infty} N_n = \infty$. ∎

Next, let $\theta, \eta \in (0,1) \setminus \mathbb{Q}$. Assume that $I_\theta = I_\eta$ and thus their diagrams agree (Theorem 3.3 in [7]). Hence, we have that $j_n(\theta) = j_n(\eta)$ for all $n \in \mathbb{N}$, and thus $r(n, j_n(\theta)) = r(n, j_n(\eta))$ for all $n \in \mathbb{N} \setminus \{0\}$. Therefore, by the claim we have

$$\theta = \lim_{n\to\infty} r(n, j_n(\theta)) = \lim_{n\to\infty} r(n, j_n(\eta)) = \eta,$$

which completes the proof. ∎

REMARK 4.22. An immediate consequence of Proposition 4.20 is that if: $(0,1) \setminus \mathbb{Q}$ is equipped with its relative topology from the usual topology on $\mathbb{R}$, the set $\{I_\theta \in \mathrm{Prim}(\mathfrak{A}) : \theta \in (0,1) \setminus \mathbb{Q}\}$ is equipped with its relative topology induced by the Jacobson topology, and the set $\{I_\theta \in \mathrm{Prim}(\mathfrak{A}) : \theta \in (0,1) \setminus \mathbb{Q}\}$ is equipped with its relative topology induced by the metric topology of $\mathsf{m}_{i(\mathcal{U}_\mathfrak{F})}$ of Corollary 3.20 or the Fell topology of Definition 3.5, then all these spaces are homeomorphic to the Baire space $\mathscr{N}$ with its metric topology from Definition 4.18. In particular, the totally bounded metric $\mathsf{m}_{i(\mathcal{U}_\mathfrak{F})}$ topology on the set of ideals $\{I_\theta \in \mathrm{Prim}(\mathfrak{A}) : \theta \in (0,1) \setminus \mathbb{Q}\}$ is homeomorphic to $(0,1) \setminus \mathbb{Q}$ with its totally bounded metric topology inherited from the usual topology on $\mathbb{R}$. Hence, in some sense, the metric $\mathsf{m}_{i(\mathcal{U}_\mathfrak{F})}$ topology shares more metric information with $(0,1) \setminus \mathbb{Q}$ and its metric than the Baire space metric topology as the Baire space is complete and not totally bounded (see Theorem 6.5 in [2]) (since the Baire space is complete, if it were totally bounded, then it would be compact, which would therefore contradict the fact that it is homeomorphic to the irrationals). This can also be displayed in metric calculations.

Indeed, consider $\theta, \mu \in (0,1) \setminus \mathbb{Q}$ with continued fraction expansions $\theta = [a_j]_{j\in\mathbb{N}}$ and $\mu = [b_j]_{j\in\mathbb{N}}$, in which $a_0 = 0, a_1 = 1000, a_j = 1, \forall j \geqslant 2$ and $b_0, b_1 = 1, b_j = 1\ \forall j \geqslant 2$, and thus $\theta \approx 0.001, \mu \approx 0.618, |\theta - \mu| \approx 0.617$. In the Baire metric $\mathsf{d}(\mathsf{cf}(\theta), \mathsf{cf}(\mu)) = 0.5$, and in the ideal metric $\mathsf{m}_{i(\mathcal{U}_\mathfrak{F})}(I_\theta, I_\mu) = 0.25$ since at level $n = 1$ the diagram for $\mathfrak{F}/I_\theta$ begins with L_{999} and for $\mathfrak{F}/I_\mu$ begins with R_{b_2} by Proposition 4.i in [6], so the ideal diagrams differ first at $n = 2$. Now, assume that for μ we have instead $b_1 = 999, b_j = 1\ \forall j \geqslant 2$, and thus $|\theta - \mu| \approx 0.000000998$, but in the Baire metric we still have that $\mathsf{d}(\mathsf{cf}(\theta), \mathsf{cf}(\mu)) = 0.5$, while $\mathsf{m}_{i(\mathcal{U}_\mathfrak{F})}(I_\theta, I_\mu) =$

2^{-1000} since at level $n = 1$ the diagram for $\mathfrak{F}/I_\theta$ begins with L_{999} and for $\mathfrak{F}/I_\mu$ begins with L_{998} and then transitions to R_{b_2} by Proposition 4.i in [6], so the ideal diagrams differ first at $n = 1000$. In conclusion, in this example, the absolute value metric $|\cdot|$ behaves much more like the metric $\mathsf{m}_{i(\mathcal{U}_\mathfrak{F})}$ than the Baire metric.

Fix $\theta \in (0,1) \setminus \mathbb{Q}$. We present a $*$-isomorphism from $\mathfrak{F}/I_\theta$ to $\mathfrak{AF}_\theta$ as a proposition to highlight a useful property for our purposes. Of course, Proposition 4.i in [6] already established that $\mathfrak{F}/I_\theta$ and $\mathfrak{AF}_\theta$ are $*$-isomorphic, but here we simply provide an explicit example of such a $*$-isomorphism, which will serve us in the results pertaining to tracial states in Lemma 4.29.

PROPOSITION 4.23. *If $\theta \in (0,1) \setminus \mathbb{Q}$ with continued fraction expansion $\theta = [a_j]_{j\in\mathbb{N}}$ as in* (2.1)*, then using Notation* 2.9 *and Definition* 4.16*, there exists a $*$-isomorphism $\mathfrak{af}_\theta : \mathfrak{F}/I_\theta \to \mathfrak{AF}_\theta$ such that if $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$, then:*

$$\mathfrak{af}_\theta(\underrightarrow{\varphi}^{a_1}(x) + I_\theta) = \underrightarrow{\alpha_\theta}^1(x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)}) \in \underrightarrow{\alpha_\theta}^1(\mathfrak{AF}_{\theta,1}).$$

Proof. By the proof of Proposition 4.i in [6], the Bratteli diagram of $\mathfrak{F}/I_\theta$ begins with the diagram L_{a_1-1} of Figure 5 in [6] at level $n = 1$. Now, the diagram $C_a \circ C_b$ of Figure 6 in [6] is a section of the diagram of Example 4.11, in which the left column of $C_{a_1-1} \circ C_{a_2}$ is the bottom row of the first two levels from left to right after level $n = 0$ of Example 4.11. Therefore, by the placement of $\circledast$ at level a_1 in Figure 6 in [6], define a map $f : (\mathfrak{F}^{a_1} + I_\theta)/I_\theta \to \underrightarrow{\alpha_\theta}^1(\mathfrak{AF}_{\theta,1})$ by:

$$f : (\underrightarrow{\varphi}^{a_1}(x) + I_\theta) \mapsto \underrightarrow{\alpha_\theta}^1(x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)}),$$

where $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$. We show that f is a $*$-isomorphism from $(\mathfrak{F}^{a_1} + I_\theta)/I_\theta$ onto $\underrightarrow{\alpha_\theta}^1(\mathfrak{AF}_{\theta,1})$.

We first show that f is well-defined. Let $c, e \in (\mathfrak{F}^{a_1} + I_\theta)/I_\theta$ such that $c = e$. Now, we have $c = \underrightarrow{\varphi}^{a_1}(c') + I_\theta, e = \underrightarrow{\varphi}^{a_1}(e') + I_\theta$ where $c' = c'_0 \oplus \cdots \oplus c'_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ and $e' = e'_0 \oplus \cdots \oplus e'_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$. But, the assumption $c = e$ implies that $\underrightarrow{\varphi}^{a_1}(c' - e') \in I_\theta \cap \mathfrak{F}^{a_1}$. Thus, by Definition 4.16 of I_θ, we have that $c'_{j_{a_1}(\theta)+1} \oplus c'_{j_{a_1}(\theta)} = e'_{j_{a_1}(\theta)+1} \oplus e'_{j_{a_1}(\theta)}$, and since $j_{a_1}(\theta) = q_0^\theta$ and $j_{a_1}(\theta) + 1 = q_1^\theta$ by Proposition 4.i in [6] and the discussion at the start of the proof, we gather that f is a well-defined $*$-homomorphism since the canonical maps $\underrightarrow{\alpha_\theta}^1$ and $\underrightarrow{\varphi}^{a_1}$ are $*$-homomorphisms.

For surjectivity of f, let $x = \underrightarrow{\alpha_\theta}^1(x_{q_1^\theta} \oplus x_{q_0^\theta})$, where $x_{q_1^\theta} \oplus x_{q_0^\theta} \in \mathfrak{AF}_{\theta,1}$. Define $y = y_0 \oplus \cdots y_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ such that $y_{j_{a_1}(\theta)} = x_{q_0^\theta}$ and $y_{j_{a_1}(\theta)+1} = x_{q_1^\theta}$ with $y_k = 0$ for all $k \in \{0, \ldots, 2^{a_1-1}\} \setminus \{j_{a_1}(\theta), j_{a_1}(\theta) + 1\}$. Hence $f(\underrightarrow{\varphi}^{a_1}(y) + I_\theta) = x$.

For injectivity of f, let $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ and $y = y_0 \oplus \cdots \oplus y_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ such that $f(\underrightarrow{\varphi}^{a_1}(x) + I_\theta) = f(\underrightarrow{\varphi}^{a_1}(y) + I_\theta)$. Thus, since $\underrightarrow{\alpha_\theta}^1$ is injective, we have that $x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)} = y_{j_{a_1}(\theta)+1} \oplus y_{j_{a_1}(\theta)}$. But, this then implies that $\underrightarrow{\varphi}^{a_1}(x - y) \in$

$I_\theta \cap \mathfrak{F}^{a_1} \subseteq I_\theta$ by Definition 4.16, and therefore, the terms $\underset{\longrightarrow}{\varphi^{a_1}}(x) + I_\theta = \underset{\longrightarrow}{\varphi^{a_1}}(y) + I_\theta$, which completes the argument that f is a $*$-isomorphism from $(\mathfrak{F}^{a_1} + I_\theta)/I_\theta$ onto $\underset{\rightarrow}{\alpha_\theta^1}(\mathfrak{AF}_{\theta,1})$.

Lastly, using Definition 4.9, consider the Bratteli diagram of $\mathfrak{F}/I_\theta$ given by the sequence of unital C^*-subalgebras $((\mathfrak{F}^{x_{j+1}} + I_\theta)/I_\theta)_{j\in\mathbb{N}}$, where $x_{j+1} = \sum_{k=1}^{j+1} a_k$ for all $j \in \mathbb{N}$. Hence, the proof of Proposition 4.i in [6] and Figure 6 in [6] provide that this diagram of $\mathfrak{F}/I_\theta$ is equivalent to the Bratteli diagram of $\mathfrak{AF}_\theta$ beginning at $\mathfrak{AF}_{\theta,1}$ given by Example 4.11, where this equivalence of Bratteli diagrams is given by Section 23.3 and Theorem 23.3.7 in [5]. Therefore, combining the equivalence relation of Section 23.3 and Theorem 23.3.7 in [5] and the construction of the $*$-isomorphism in Proposition III.2.7 in [12], we conclude that there exists a $*$-isomorphism $\mathfrak{af}_\theta : \mathfrak{F}/I_\theta \to \mathfrak{AF}_\theta$ such that $\mathfrak{af}_\theta(z) = f(z)$ for all $z \in (\mathfrak{F}^{a_1} + I_\theta)/I_\theta$, which completes the proof. ∎

From the $*$-isomorphism of Proposition 4.23, we may provide a faithful tracial state for the quotient $\mathfrak{F}/I_\theta$ from the unique faithful tracial state of $\mathfrak{AF}_\theta$.

NOTATION 4.24. Fix $\theta \in (0,1) \setminus \mathbb{Q}$. There is a unique faithful tracial state on $\mathfrak{AF}_\theta$ denoted σ_θ (see Lemma 5.3 and Lemma 5.5 in [2]). Thus,

$$\tau_\theta = \sigma_\theta \circ \mathfrak{af}_\theta$$

is the unique faithful tracial state on $\mathfrak{F}/I_\theta$ with $\mathfrak{af}_\theta$ from Proposition 4.23.

Let $Q_\theta : \mathfrak{F} \to \mathfrak{F}/I_\theta$ denote the quotient map. Thus, by Theorem V.2.2 in [11], there exists a unique linear functional on $\mathfrak{F}$ denoted ρ_θ such that $\ker \rho_\theta \supseteq I_\theta$ and $\tau_\theta \circ Q_\theta(x) = \rho_\theta(x)$ for all $x \in \mathfrak{F}$. Since τ_θ is a tracial state and

$$\tau_\theta \circ Q_\theta(x) = \rho_\theta(x)$$

for all $x \in \mathfrak{F}$, we conclude that ρ_θ is also a tracial state that vanishes on I_θ. Furthermore, ρ_θ is faithful on $\mathfrak{F} \setminus I_\theta$ since τ_θ is faithful on $\mathfrak{F}/I_\theta$.

One more ingredient remains before we define the quantum metric structure for the quotient spaces $\mathfrak{F}/I_\theta$.

LEMMA 4.25. *Let $\theta \in (0,1) \setminus \mathbb{Q}$. Using notation from Definition* 4.6 *and Definition* 4.16, *if we define*

$$\beta^\theta : n \in \mathbb{N} \longmapsto \frac{1}{\dim((\mathfrak{F}^n + I_\theta)/I_\theta)} \in (0, \infty),$$

then $\beta^\theta(n) = 1/(q(n, j_n(\theta))^2 + q(n, j_n(\theta)+1)^2) \leqslant 1/n^2$ *for all* $n \in \mathbb{N} \setminus \{0\}$ *and* $\beta^\theta(0) = 1$.

Proof. First, $(\mathfrak{F}^0 + I_\theta)/I_\theta = \mathbb{C}1_{\mathfrak{F}/I_\theta}$, hence, $\beta^\theta(0) = 1$. Fix $n \in \mathbb{N} \setminus \{0\}$. Since $(\mathfrak{F}^n + I_\theta)/I_\theta$ is $*$-isomorphic to $\mathfrak{F}^n/(I_\theta \cap \mathfrak{F}^n)$ (see Proposition 3.26), we have that

$$\dim((\mathfrak{F}^n + I_\theta)/I_\theta) = \dim(\mathfrak{F}^n/(I_\theta \cap \mathfrak{F}^n)) = q(n, j_n(\theta))^2 + q(n, j_n(\theta)+1)^2$$

by Definition 4.16 and the dimension of the quotient is the difference in dimensions of $\mathfrak{F}^n$ and $I_\theta \cap \mathfrak{F}^n$. Therefore, $\beta^\theta(n) = 1/(q(n, j_n(\theta))^2 + q(n, j_n(\theta)+1)^2)$.

To prove the inequality of the lemma, we claim that for all $n \in \mathbb{N} \setminus \{0\}$, we have $q(n, j_n(\theta)) \geqslant n$ or $q(n, j_n(\theta)+1) \geqslant n$. We proceed by induction. If $n = 1$, then $q(1, j_1(\theta)) = 1$ and $q(1, j_1(\theta)+1) = 1$ by relations (4.1). Next assume the statement of the claim is true for $n = m$. Thus, we have that $q(m, j_m(\theta)) \geqslant m$ or $q(m, j_m(\theta)+1) \geqslant m$. First, assume that $q(m, j_m(\theta)) \geqslant m$. By Lemma 4.17, assume that $j_{m+1}(\theta) = 2j_m(\theta)$. Thus, we gather $q(m+1, j_{m+1}(\theta)+1) = q(m+1, 2j_m(\theta)+1) = q(m, j_m(\theta)) + q(m, j_m(\theta)+1) \geqslant m+1$ by relations (4.1) and since $q(m, j_m(\theta)+1) \in \mathbb{N} \setminus \{0\}$. The case $j_{m+1}(\theta) = 2j_m(\theta)+1$ follows similarly as well as the case $q(m, j_m(\theta)+1) \geqslant m$, which completes the induction argument .

In particular, we have $q(n, j_n(\theta)) \geqslant n$ or $q(n, j_n(\theta)+1) \geqslant n$, which implies that $q(n, j_n(\theta))^2 \geqslant n^2$ or $q(n, j_n(\theta)+1)^2 \geqslant n^2$. And thus, for all $n \in \mathbb{N} \setminus \{0\}$,

$$\frac{1}{q(n, j_n(\theta))^2 + q(n, j_n(\theta)+1)^2} \leqslant \frac{1}{n^2},$$

which completes the proof. ∎

Hence, we have all the ingredients to define the quotient quantum metric spaces of the ideals of Definition 4.16.

NOTATION 4.26. Fix $\theta \in (0,1) \setminus \mathbb{Q}$. Using Definition 4.6, Definition 4.16, Notation 4.24, and Lemma 4.25, let

$$(\mathfrak{F}/I_\theta, \mathsf{L}^{\beta^\theta}_{\mathcal{U}_{\mathfrak{F}}/I_\theta, \tau_\theta})$$

denote the $(2,0)$-quasi-Leibniz quantum compact metric space given by Theorem 4.1 associated to the ideal I_θ, faithful tracial state τ_θ, and $\beta^\theta : \mathbb{N} \to (0, \infty)$ having limit 0 at infinity by Lemma 4.25.

REMARK 4.27. Fix $\theta \in (0,1) \setminus \mathbb{Q}$. Although $\mathfrak{F}/I_\theta$ and $\mathfrak{AF}_\theta$ are $*$-isomorphic, it is unlikely that $(\mathfrak{F}/I_\theta, \mathsf{L}^{\beta^\theta}_{\mathcal{U}_{\mathfrak{F}}/I_\theta, \tau_\theta})$ is quantum isometric to $(\mathfrak{AF}_\theta, \mathsf{L}^{\beta_\theta}_{\mathcal{I}_\theta, \sigma_\theta})$ of Theorem 2.10 based on the Lip-norm constructions. Thus, one could not simply apply Proposition 4.19 to Theorem 2.10 to achieve Theorem 4.30.

To provide our continuity results, we describe the faithful tracial states on the quotients in sufficient detail through Lemma 4.28 and Lemma 4.29.

LEMMA 4.28. *Fix $\theta \in (0,1) \setminus \mathbb{Q}$. Let tr_d be the unique tracial state of $\mathfrak{M}(d)$. Using notation from Definition 4.6 and Definition 4.16 and from Notation 4.24, if $n \in \mathbb{N} \setminus \{0\}$, then there exists $c(n,\theta) \in (0,1)$ such that*

$$\rho_\theta \circ \underset{\to}{\varphi}^n(a) = c(n,\theta)\mathrm{tr}_{q(n,j_n(\theta))}(a_{j_n(\theta)}) + (1 - c(n,\theta))\mathrm{tr}_{q(n,j_n(\theta)+1)}(a_{j_n(\theta)+1}),$$

for all $a = a_0 \oplus \cdots \oplus a_{2^n-1} \in \mathfrak{F}_n$, and $\rho_\theta \circ \underset{\to}{\varphi}^0(a) = a$ for all $a \in \mathfrak{F}_0$.

Furthermore, for all $n \in \mathbb{N} \setminus \{0\}$, we have:

$$c(n+1,\theta) = \begin{cases} \frac{(q(n,j_n(\theta))+q(n,j_n(\theta)+1))c(n,\theta)-q(n,j_n(\theta))}{q(n,j_n(\theta)+1)} & \text{if } j_{n+1}(\theta) = 2j_n(\theta), \\ (1+\frac{q(n,j_n(\theta)+1)}{q(n,j_n(\theta))})c(n,\theta) & \text{if } j_{n+1}(\theta) = 2j_n(\theta)+1. \end{cases}$$

Proof. Fix $\theta \in (0,1) \setminus \mathbb{Q}$. Let $n \in \mathbb{N}$. If $n = 0$, then $\rho_\theta \circ \underset{\rightarrow}{\varphi}^0(a) = a$ for all $a \in \mathfrak{F}_0$ since $\mathfrak{F}_0 = \mathbb{C}$. Next, let $n \in \mathbb{N} \setminus \{0\}$ and $a = a_0 \oplus \cdots \oplus a_{2^{n-1}} \in \mathfrak{F}_n$. By Example IV.5.4 in [12], we have

$$\rho_\theta \circ \underset{\rightarrow}{\varphi}^n(a) = \sum_{k=0}^{2^{n-1}} c(n,k)\mathrm{tr}_{q(n,k)}(a_k),$$

for some $c(n,k) \in [0,1]$ for $k \in \{0,\ldots,2^{n-1}\}$ such that $\sum_{k=0}^{2^{n-1}} c(n,k) = 1$. Now, if $k = j_n(\theta)$, then we denote $c(n,\theta) = c(n, j_n(\theta))$.

Now, denoting by τ_θ the unique tracial state on $\mathfrak{F}/I_\theta$ (see Notation 4.24) and by $e_{(n,k)}$ the mimimal projection in $\{0\} \oplus \cdots \oplus \mathfrak{M}(q(n,k)) \oplus \{0\} \oplus \cdots \oplus \{0\}$ we have $c(n,k) = q(n,k)\tau_\theta(e_{(n,k)})$ for $k \in \{0,\ldots,2^{n-1}\}$ and clearly $c(n,k) = 0$ unless $k \in \{j_n(\theta), j_n(\theta)+1\}$. Clearly:

$$c(n,j_n(\theta)) + c(n,j_n(\theta)+1) = 1. \tag{4.5}$$

There are two cases to consider: (i) $j_{n+1}(\theta) = 2j_n(\theta)$ and (ii) $j_{n+1}(\theta) = 2j_n(\theta)+1$.

In case (i), $c(n,\theta) = c(n,j_n(\theta))$ and $c(n+1,\theta) = c(n+1,2j_n(\theta))$. From well-known considerations about traces on AF-algebras (see, e.g., Proposition 2.5.1 in [18]) we have:

$$\begin{aligned} \frac{c(n+1,j_{n+1}(\theta))}{2q(n,j_n(\theta))+1} &= \frac{c(n,j_n(\theta)+1)}{q(n,j_n(\theta)+1)} \\ \frac{c(n,j_n(\theta))}{q(n,j_n(\theta))} &= \frac{c(n+1,2j_n(\theta))}{q(n,j_n(\theta))} + \frac{c(n+1,2j_n(\theta)+1)}{q(n,j_n(\theta))+q(n,j_n(\theta)+1)}. \end{aligned} \tag{4.6}$$

The second equality in (4.6) and (4.5) lead to:

$$\begin{aligned} c(n+1,\theta) &= c(n+1,2j_n(\theta)) \\ &= \frac{(q(n,j_n(\theta))+q(n,j_n(\theta)+1))c(n,j_n(\theta)) - q(n,j_n(\theta))}{q(n,j_n(\theta)+1)}. \end{aligned}$$

In case (ii), $c(n,\theta) = c(n,j_n(\theta))$ and $c(n+1,\theta) = c(n+1,2j_n(\theta)+1)$. In this case we have:

$$\begin{aligned} \frac{c(n+1,2j_n(\theta)+1)}{q(n,j_n(\theta))+q(n,j_n(\theta)+1)} &= \frac{c(n,j_n(\theta))}{q(n,j_n(\theta))} \\ \frac{c(n,j_n(\theta)+1)}{q(n,j_n(\theta)+1)} &= \frac{c(n+1,2j_n(\theta)+2)}{q(n,j_n(\theta)+1)} + \frac{c(n+1,2j_n(\theta)+1)}{q(n+1,2j_n(\theta)+1)}. \end{aligned} \tag{4.7}$$

The first equality in (4.7) gives:

$$c(n+1,\theta)=c(n+1,2j_n(\theta)+1)=\frac{q(n,j_n(\theta))+q(n,j_n(\theta)+1)}{q(n,j_n(\theta))}c(n,j_n(\theta)). \quad \blacksquare$$

LEMMA 4.29. *Using notation from Lemma* 4.28, *if* $\theta\in(0,1)\setminus\mathbb{Q}$, *then:*

$$c(1,\theta)=1-\theta.$$

Moreover, using notation from Definition 4.16, *if* $\theta,\mu\in(0,1)\setminus\mathbb{Q}$ *such that there exists* $N\in\mathbb{N}\setminus\{0\}$ *with* $I_\theta\cap\mathfrak{F}^N=I_\mu\cap\mathfrak{F}^N$, *then there exists* $a,b\in\mathbb{R}, a\neq 0$ *such that:*

$$c(N,\theta)=a\theta+b, \quad c(N,\mu)=a\mu+b.$$

Proof. Let $\theta\in(0,1)\setminus\mathbb{Q}$, and denote its continued fraction expansion by $\theta=[a_j]_{j\in\mathbb{N}}$. Let $(p_n^\theta/q_n^\theta)_{n\in\mathbb{N}}$ denote convergents of θ. Now, the unique trace on the Effros–Shen AF-algebra $\mathfrak{AF}_\theta=\varinjlim(\mathfrak{M}(q_n^\theta)\oplus\mathfrak{M}(q_{n-1}^\theta),\alpha_{\theta,n})_{n\in\mathbb{N}}$ assigns the values $(-1)^n q_n^\theta(p_{n-1}^\theta-q_{n-1}^\theta\theta)\in(0,1)$ and $1-(-1)^n q_n^\theta(p_{n-1}^\theta-q_{n-1}^\theta\theta)=(-1)^{n+1}q_{n-1}^\theta(p_n^\theta-q_n^\theta\theta)\in(0,1)$ to $1_{\mathfrak{M}(q_n^\theta)}\oplus 0$ and $0\oplus 1_{\mathfrak{M}(q_{n-1}^\theta)}$, respectively (see Lemma 5.5 in [2]). Using notation from Lemma 4.28, we have

$$c(a_1+\cdots+a_n,\theta)=(-1)^{n+1}q_{n-1}^\theta(p_n^\theta-q_n^\theta\theta), \quad \forall n\geqslant 1.$$

and in particular,

$$c(a_1,\theta)=1-a_1\theta.$$

By reverse induction, we find $c(m,\theta)=1-m\theta$ for $m=a_1,a_1-1,\ldots,1$, and in particular $c(1,\theta)=1-\theta$.

Let $N\in\mathbb{N}\setminus\{0,1\}$. Assume that $I_\mu\cap\mathfrak{F}^{N+1}=I_\theta\cap\mathfrak{F}^{N+1}$. Now, since $\mathfrak{F}^N\subseteq\mathfrak{F}^{N+1}$, we thus have $I_\mu\cap\mathfrak{F}^N=I_\theta\cap\mathfrak{F}^N$. Hence, by the induction hypothesis, there exist $a,b\in\mathbb{R}, a\neq 0$ such that $c(N,\mu)=a\mu+b$ and $c(N,\theta)=a\theta+b$. But, as $I_\mu\cap\mathfrak{F}^{N+1}=I_\theta\cap\mathfrak{F}^{N+1}$, the vertices at level $N+1$ agree in the ideal diagrams by Proposition 4.14. In particular, by Definition 4.16, we have $j_{N+1}(\theta)=j_{N+1}(\mu)$, and similarly, the term $j_N(\theta)=j_N(\mu)$ by $I_\mu\cap\mathfrak{F}^N=I_\theta\cap\mathfrak{F}^N$. The conclusion follows by Lemma 4.28. $\blacksquare$

We can now prove the main result of this section.

THEOREM 4.30. *Using Definition* 4.16 *and Notation* 4.26, *the map*

$$I_\theta\in(\mathrm{Prim}(\mathfrak{F}),\tau)\longmapsto(\mathfrak{F}/I_\theta,\mathsf{L}^{\beta^\theta}_{\mathcal{U}_{\mathfrak{F}/I_\theta},\tau_\theta})\in(\mathcal{QQCMS}_{2,0},\Lambda)$$

is continuous to the class of $(2,0)$*-quasi-Leibniz quantum compact metric spaces metrized by the quantum propinquity* Λ, *where* τ *is either the Jacobson topology, the relative metric topology of* $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ *of Corollary* 3.20, *or the relative Fell topology of Definition* 3.5.

Proof. By Proposition 4.19 and Proposition 4.20, we only need to show continuity with respect to the metric $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$ with sequential continuity. Thus, let

$(I_{\theta_n})_{n\in\overline{\mathbb{N}}} \subset \mathrm{Prim}(\mathfrak{F})$ be a sequence such that $(I_{\theta_n})_{n\in\mathbb{N}}$ converges to I_{θ_∞} with respect to $\mathsf{m}_{i(\mathcal{U}_{\mathfrak{F}})}$. Therefore, by Lemma 3.23, this implies that

$$\left\{ I_{\theta_n} = \overline{\bigcup_{k\in\mathbb{N}} I_{\theta_n} \cap \mathfrak{F}^k}^{\|\cdot\|_{\mathfrak{F}}} : n \in \overline{\mathbb{N}} \right\}$$

is a fusing family with some fusing sequence $(c_n)_{n\in\mathbb{N}}$. Thus, condition (i) of Theorem 4.2 is satisfied.

For condition (ii) of Theorem 4.2, let $N \in \mathbb{N}$, then by definition of fusing sequences, if $k \in \mathbb{N}_{\geqslant c_N}$, then $I_{\theta_k} \cap \mathfrak{F}^N = I_{\theta_\infty} \cap \mathfrak{F}^N$. Now, let $k \in \overline{\mathbb{N}}_{\geqslant c_N}$. Consider ρ_{θ_k} on $\mathfrak{F}^N$. By Lemma 4.29, there exist $a, b \in \mathbb{R}, a \neq 0$, such that $c(N, \theta_k) = a\theta_k + b$ for all $k \in \overline{\mathbb{N}}_{\geqslant c_N}$. But, by Proposition 4.19, $(\theta_n)_{n\in\mathbb{N}}$ converges to θ_∞ with respect to the usual topology on $\mathbb{R}$. Hence, the sequence $(c(N, \theta_k))_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to $c(N, \theta_\infty)$ with respect to the usual topology on $\mathbb{R}$ and the same applies to $(1 - c(N, \theta_k))_{k\in\mathbb{N}_{\geqslant c_N}}$. However, by Lemma 4.28, the coefficient $c(N, \theta_k)$ determines ρ_{θ_k} for all $k \in \overline{\mathbb{N}}_{\geqslant c_N}$. Hence, Lemma 3.3 in [1] provides that $(\rho_{\theta_k})_{k\in\mathbb{N}_{\geqslant c_N}}$ converges to ρ_{θ_∞} in the weak* topology on $\mathscr{S}(\mathfrak{F}^N)$.

Condition (iii) of Theorem 4.2 follows a similar argument as in the proof of condition (ii) since the sequences β^θ of Lemma 4.25 are determined by the terms $j_n(\theta)$. Also, all β^θ are uniformly bounded by the sequence $(1/n^2)_{n\in\mathbb{N}}$ which converges to 0. Therefore, the proof is complete. ∎

As an aside to Remark 4.27, we obtain the following analogue to Theorem 2.10 in terms of quotients.

COROLLARY 4.31. *Using Notation* 4.26, *the map*

$$\theta \in ((0,1) \setminus \mathbb{Q}, |\cdot|) \longmapsto (\mathfrak{F}/I_\theta, \mathsf{L}^{\beta^\theta}_{\mathcal{U}_{\mathfrak{F}}/I_\theta, \tau_\theta}) \in (\mathcal{QQCMS}_{2,0}, \Lambda)$$

is continuous from $(0,1) \setminus \mathbb{Q}$*, with its topology as a subset of* $\mathbb{R}$ *to the class of* $(2,0)$*-quasi-Leibniz quantum compact metric spaces metrized by the quantum propinquity* Λ*.*

For the proof apply Proposition 4.20 to Theorem 4.30.

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