J. OPERATOR THEORY
85:1(2021), 5–20
doi: 10.7900/jot.2020oct18.2295

DAN-VIRGIL VOICULESCU AT SEVENTY

HARI BERCOVICI, KEN DYKEMA, ALEXANDRU NICA

For Dan Voiculescu, teacher, mentor, and friend

ABSTRACT. We present a biographical sketch of Dan-Virgil Voiculescu as well as a description of some of his contributions to two directions of research that he has pursued over a long period of time. A current list of his publications is included.

KEYWORDS: *Dan-Virgil Voiculescu, biography, survey, perturbation of operators, free probability.*

MSC (2010): Primary 01A70; Secondary 47-02, 46-02.

BIOGRAPHICAL SKETCH

Dan-Virgil Voiculescu was born on June 14, 1949 in Bucharest, Romania. His talent and passion for mathematics manifested themselves early, as evidenced by the three medals (one silver and two gold) obtained in successive years at the International Mathematical Olympiad. His first mathematical publication also dates from his high school years [1].

During 1967–1972 Voiculescu was a student in the Faculty of Mathematics and Mechanics at the University of Bucharest and, upon graduation, he was immediately hired in a permanent teaching position in the same department. From 1973 to 1986, he worked as a researcher, first at the Institute of Mathematics of the Romanian Academy, then, after that institute was disbanded, in the Mathematics Section of a research Institute of Scientific and Technical Creation. During this period, Voiculescu completed his doctoral dissertation under the direction of Ciprian Foiaş and eventually defended his degree in 1977. He was among the first editors of *Journal of Operator Theory* (started in 1979). In Romania he was also involved in initiating a series of international operator theory conferences that continues to this day.

In 1986, he emigrated to the United States, using the opportunity provided by the International Congress of Mathematicians. After one year as a visiting professor, he became a permanent member of the Mathematics Department at the University of California, Berkeley, where he continues to teach today. He also held a number of visiting positions in Paris, Montréal, and Zürich.

FIGURE 1. Dan-Virgil Voiculescu in 1981.

The work of Dan-Virgil Voiculescu was recognized in many ways over the years. He was awarded the Gheorghe Ţiţeica prize of the Romanian Academy in 1976, received the medal of the Collège de France in 1989, and was awarded the 2004 Award in Mathematics by the National Academy of Sciences of the United States. He also became a member of the National Academy of Sciences in 2006 and a fellow of the American Mathematical Society in 2013 (the first year that fellows were elected). In 2014 he received an honorary doctorate from the University of Waterloo, Canada. Dan was invited twice to speak at the International Congress of Mathematicians (Warsaw, 1983 and Zürich, 1994; the second address was plenary). He was also invited to speak at the Congress of the International Association for Mathematical Physics (Leipzig, 1991) and the Bernoulli Congress (Guanajuato, 2000). He was invited to give numerous distinguished lecture series all over the world.

As a doctoral adviser, Dan-Virgil Voiculescu had numerous graduates, many of whom went on to distinguished careers in mathematics. Here is a list of his Ph.D. students, in the order of graduation:

Sorin Popa, Mihai Pimsner, Dan Timotin, David Bernier, Victor Nistor, Kenneth Dykema, Emmanuel Germain, Alexandru Nica, Dimitri Shlyakhtenko, Trent Walker, Eric Hsu, Michael Anshelevich, Kenley Jung, Stephen Curran, Adam Merberg, Weihua Liu.

The Math Genealogy site lists a total of 78 mathematical descendants of Dan Voiculescu.

ON THE CONTRIBUTIONS OF DAN-VIRGIL VOICULESCU TO MATHEMATICS

As of today, Dan Voiculescu's publications span 54 years and impact an astonishing number of mathematical fields. We focus here on two important areas to which he returned repeatedly throughout these years. Even within these areas, our account is selective.

PERTURBATION THEORY FOR OPERATORS. One of the earliest invariant subspace theorems, attributed to von Neumann but first published by Aronszajn and Smith in 1955, states that arbitrary compact operators on a separable Hilbert space have nontrivial invariant subspaces. An essential ingredient in the proof is the fact that a compact operator T has arbitrarily large, finite dimensional, approximately invariant projections. More precisely, there exists an increasing sequence $(P_n)_{n\in\mathbb{N}}$ of finite rank projections, strongly convergent to the identity operator I, such that $\lim_{n\to\infty} \|(I-P_n)TP_n\| = 0$. Halmos called this property of T *quasitriangularity* and asked (in his paper *Ten problems in Hilbert space*) whether this property alone suffices to prove the existence of invariant subspaces. It was shown by Douglas and Pearcy that an operator T, such that $\lambda I - T$ has negative Fredholm index for some scalar λ, is not quasitriangular. Using new approximation techniques, Apostol, Foiaş, and Voiculescu [12], [13], [14], [15], [16] proved the astonishing converse: if T is *not* quasitriangular, then there exists λ such that $\lambda I - T$ has negative Fredholm index, and thus T has nontrivial *hyper*invariant subspaces, for instance the range of $\lambda I - T$. This reduces the general invariant subspace problem to the class of *biquasitriangular,* that is, operators T such that both T and T^* are quasitriangular. In particular, Halmos's question is now equivalent to the invariant subspace problem.

The techniques developed in the study of quasitriangularity proved to be useful in an array of approximation problems, both for single operators and for operator algebras. For instance, Apostol, Foiaş and Voiculescu [20] proved that every quasinilpotent operator can be approximated arbitrarily well by nilpotent ones. In fact, they provided a spectral description of the norm closure of the set of nilpotent operators on a Hilbert space. In a related work, Voiculescu proved [18] that the set of biquasitriangular operators equals the norm closure of the set of algebraic operators (that is, operators that satisfy a polynomial equation). Eventually, these results led to a program, whose main contributors were Voiculescu, Apostol, Fialkow, and Herrero, aiming at a description and classification of those sets of operators that are closed under norm limits and similarity, for instance the closures of the similarity orbits of single operators. Many of the results of this program were collected in the monograph [54].

FIGURE 2. Dan-Virgil Voiculescu in 1995.

A classical result of Weyl states that arbitrary selfadjoint operators on a Hilbert space can be turned into diagonal operators by a compact (additive) perturbation of arbitrarily small norm. Von Neumann improved this result by showing that the perturbation can be made to be Hilbert–Schmidt with arbitrarily small Hilbert–Schmidt norm. Kuroda further improved the result by replacing the Hilbert–Schmidt class by an arbitrary (symetrically) normed ideal of compact operators, other than the trace class. That one cannot include the trace class in this result follows from the work of Kato and Rosenblum in scattering theory.

The original result of Weyl was later subsumed by the work of Brown, Douglas and Fillmore on extensions of the compact operators by the algebra of continuous functions on a compact metric space X. These authors introduced an object, now denoted $\mathrm{Ext}(C(X))$, consisting of the equivalence classes of such extensions, and showed that $\mathrm{Ext}(C(X))$ has a natural semigroup structure (induced essentially by direct sums), and is in fact a group. The existence of a unit in this group implies the fact that an arbitrary commuting sequence of selfadjoint operators on a Hilbert space can be turned into diagonal operators (relative to the same basis) by compact additive perturbations. The study of this group also leads, among other results, to a classification of essentially normal operators, that is, operators T such that $T^*T - TT^*$ is compact.

The object $\mathrm{Ext}(A)$ can also be defined if A is a C^*-algebra (which we assume separable and unital), and this is naturally endowed with a semigroup structure. A breakthrough result of Voiculescu [24], [26] has the deceptively simple statement that this semigroup has an additive unit. This result proved to be seminal and we illustrate below a few of the ways in which it changed the landscape of operator theory. One immediate consequence is that every operator on a Hilbert space can be approximated arbitrary well by operators that have reducing subspaces of infinite dimension and codimension, thus answering another one of Halmos's *Ten questions*.

Returning to the extensions of the von Neumann improvement of Weyl's theorem, we saw above that every commuting pair of selfadjoint operators can be made to be a pair of diagonal operators (in the same basis) by a compact perturbation of arbitrarily small norm. The results of Kato and Rosenblum seemed to hint that the perturbation cannot generally be made to be Hilbert–Schmidt. A refinement of the noncommutative Weyl–von Neumann theorem of [24], [26] allowed Voiculescu to show [38], [45] that the perturbation can in fact be supposed to have arbitrarily small Hilbert–Schmidt norm, but there is a smaller ideal of compact operators, which he denoted $\mathcal{C}_2^-$ (and which can be alternatively described as the Lorentz type Schatten ideal $\mathfrak{S}_{2,1}$) that provides the natural obstruction to perturbation. The work of [38], [45] applies in fact to commuting n-tuples of selfadjoint operators, and it is a subtle blend of abstract operator theory and harmonic analysis; normal operators correspond with $n = 2$. (See also [132] for recent developments in the study of perturbations of commuting n-tuples.)

An important concept that emerged from this work is the existence of approximate units (say, for a finite number of operators) relative to a normed ideal of operators. This is tested by evaluating a certain numerical function, depending on the operator (or operators) under consideration and on the normed ideal. It is this numerical function that makes it possible to bring harmonic analysis into the study of perturbations of commuting selfadjoint operators. Voiculescu was also able to apply this idea to noncommuting operators.

A further development is related with an extension of the program of Brown, Douglas, and Fillmore by considering operators T such that $T^*T - TT^*$ belongs to a normed ideal of compact operators. In the case of the Hilbert–Schmidt ideal, this leads to the study of new operator algebras whose study is explored in recent work [123], [127].

The work started in [26] led to developments in the study of separable C^*-algebras, a field in which this work is considered fundamental. Among these developments is, for instance, the study of continuous fields of extensions of the ideal of compact operators [37], [44]. This work can be viewed as a precursor of Kasparov's *KK*-theory.

FREE PROBABILITY. The study of von Neumann algebras of type II_1 was initiated by Murray and von Neumann almost 90 years ago. Among the early results are the first two examples of nonisomorphic factors of type II_1: the hyperfinite factor $\mathcal{R}$ and the von Neumann algebra $\mathcal{L}\mathbb{F}_2$ of the free (noncommutative) group with two generators. Factors that, like $\mathcal{R}$, have the so-called property Γ were studied rather extensively and many examples of distinct factors of this kind were constructed. Practically the only thing that was known about $\mathcal{L}\mathbb{F}_2$ was that it *does not* have property Γ. Voiculescu started rectifying this situation in the early 1980s. Algebras of type II_1 were generally understood to be noncommutative versions of the algebra of random variables on a probability space in which a normalized trace plays the role of the expected value. The concept of independence, when naively extended to the noncommutative context, is not as useful as classical independence, but it still allows for the formulation of noncommutative analogs of some of the classical limit theorems. Voiculescu realized that an effective study of $\mathcal{L}\mathbb{F}_n$ requires a new concept of independence which he dubbed *freeness*. Eventually, this developed into a wide ranging theory that is now known as *free probability theory* and it has its own classification number 46L54 assigned by Math. Reviews and Zentralblatt. Voiculescu is rightly recognized as the founder and most important contributor to this theory.

One of the earliest results of free probability is a free version of the central limit theorem [56]. Thus, suppose that $\mathcal{A}$ is a von Neumann algebra, τ is a normalized trace defined on $\mathcal{A}$, and $(a_n)_{n\in\mathbb{N}}$ is a sequence of freely independent (relative to τ) elements of $\mathcal{A}$ such that $\tau(a_n) = 0$ and $\tau(a_n^2) = 1$ for every $n \in \mathbb{N}$. Then, under mild additional conditions, the variables

$$b_n = \frac{a_1 + \cdots + a_n}{\sqrt{n}}$$

converge *in moments*, not to the standard Gaussian distribution, but to a standard semicircular distribution, that is,

$$\lim_{n\to\infty} \tau(b_n^k) = \frac{1}{2\pi}\int_{-2}^{2} t^k \sqrt{4-t^2}\,\mathrm{d}t, \quad k \in \mathbb{N}.$$

An important role in classical probability theory is played by characteristic functions which are the Fourier transforms of probability distributions. The characteristic function corresponding to the sum of two independent random variables is simply the product of the characteristic functions corresponding to the two summands. In a fundamental paper [57], Voiculescu shows that a similar result is true for sums of free random variables. An analogous result for products of free random variables appears in [61]. The analogs of the Fourier transforms in these two cases, called the R- and S-transforms (and also referred to as the Voiculescu transforms) involve resolvents and inverses of analytic functions rather than merely integration. These seminal results allowed for the development of a version of harmonic analysis in which there are new ways to combine probability measures, namely, the additive and multiplicative free convolutions that correspond with the addition and multiplication of free random variables.

FIGURE 3. Dan-Virgil Voiculescu in 2015.

Many results of classical probabilistic harmonic analysis have found more-or-less complete free analogs (for instance, see [57], [61], [75] concerning infinite divisibility and [118] concerning max limit laws).

The semicircular law, that appears as the free analog of the normal distribution, also appears in a classical probability context. Wigner showed that this is the limit as $N \to \infty$ of the eigenvalue distributions of $N \times N$ selfadjoint matrices that are distributed normally. (More precisely, one looks at random matrices $X_N = \Re[x_{i,j}^{(N)}]_{i,j=1}^N$, where $x_{i,j}^{(N)}$ are complex, identically distributed, centered normal distributions with variance $2/N$. The variance condition is needed for the existence of a limit law.) The appearance of the semicircular law in both of these contexts is not coincidental, as shown in the breakthrough work [70]. There, Voiculescu establishes a firm connection between free probability and the theory of random matrices by showing, for instance, that independent Hermitian random matrices A_N and B_N, which have invariant distributions relative to conjugation by unitary operators, and these distributions have a limit as $N \to \infty$, are *asymptotically free*. As a consequence, the asymptotic distributional behavior of, say, $A_N + B_N$ can be determined using the tools of free probability theory. For instance, the result of Wigner can be seen to follow naturally from the fact that, given A_N and B_N as above that are identically distributed and have centered normal entries, the matrix $(A_N + B_N)/\sqrt{2}$ has the same distribution as A_N and B_N. The connection between free probability and random matrices discovered in

[70] had profound consequences both for the study of the von Neumann algebras $\mathcal{L}\mathbb{F}_n$ and for the theory of random matrices. For instance, work of Voiculescu, Dykema, and Rădulescu soon determined that the so-called *fundamental group* of the algebra $\mathcal{L}\mathbb{F}_\infty$ is $\mathbb{R}_+$.

The work in [70] was followed by an astounding series of papers [83], [86], [93], [95], [99], [103] in which Voiculescu uses analogies between random matrices and statistical mechanics to introduce and exploit free analogues of the classical concepts of entropy and Fisher information. The first of these papers introduces these new concepts for a single random variable. Even this simplest situation leads [83], [100] to important insights such as the relevance of the Hilbert transform (which is later extended to several variables under the name of *conjugate variables*) and the existence of anaytic subordination in the context of free additive convolution. This latter discovery, further improved by Biane (see also [104] for further improvements) proved to be one of the best tools in free harmonic analysis, leading to regularity results for free convolution [83], [100]. The work in [93] constitutes a breakthrough in the study of factors of type II_1. It is shown there that the algebra $\mathcal{L}\mathbb{F}_n$ does not have a Cartan subalgebra for $n \geqslant 2$. These were the first II_1 factors shown to have this property. (A Cartan subalgebra of a von Neumann algebra $\mathcal{A}$ is a commutative subalgebra $\mathcal{B} \subset \mathcal{A}$ such that the unitaries $u \in \mathcal{A}$ satisfying $u\mathcal{B}u^* \subset \mathcal{B}$ generate $\mathcal{A}$ as a von Neumann algebra.)

The construction of the conjugate variables in [99] amounts to a construction of the first elements of a noncommutative version of calculus, in which partial derivatives are replaced by certain derivations from an algebra of "differentiable" functions with values in the tensor square of that algebra. A major step in the development of this calculus occurs in [104] where the coalgebra structure induced by the partial derivatives is shown to have an intimate connection with freeness. More precisely, it is shown in [104] that, under appropriate freeness conditions, conditional expectations are coalgebra homomorphisms. This result provides a conceptually simple explanation of the existence of subordination functions, and it also provides a vast generalization of subordination to free "operator valued" random variables. In turn, this extension made it clear that resolvents of (operator valued) noncommutative random variables should really be regarded as being defined on sets of matrices of arbitrary sizes. This revived ideas first considered by J.L. Taylor and led to renewed interest in the study of "noncommutative functions".

Further developments of the ideas outlined above appear in [110], [114], [120]. These ideas had far reaching consequences, not only for the study of free group algebras and random matrices, but also for the study of von Neumann algebras generated by variables that are close to being free.

In yet another development of free probability, Voiculescu introduced the notion of bifreeness [122], [125], [126], [130]. This is a theory that applies to sets of variables that are divided into "left" and "right" variables and it is modeled by two different ways in which one can define creation operators on a full Fock space. There is again a new kind of harmonic analysis that is appropriate for this new concept, though the various convolutions associated with it are more difficult to calculate. It is perhaps too early to understand what the ramifications of this theory will eventually be.

Acknowledgements. All pictures have been taken by George M. Bergman, downloaded from the Archives of the Mathematisches Forschungsinstitut Oberwolfach and published by permission.

LIST OF PUBLICATIONS OF DAN-VIRGIL VOICULESCU

[1] D. VOICULESCU, An equation concerning convex bodies and applications to associated bodies of a convex body [Romanian], *Stud. Cerc. Mat.* **18**(1966), 741–745.

[2] P. FARKAŞ, D. VOICULESCU, Some observations on convergent series [Romanian], *Gaz. Mat. Ser. A* **72**(1967), 241–245.

[3] D. VOICULESCU, Spaces with convexity. I [Romanian], *Stud. Cerc. Mat.* **19**(1967), 295–301.

[4] D. VOICULESCU, Spaces with convexity. II [Romanian], *Stud. Cerc. Mat.* **19**(1967), 303–311.

[5] D. VOICULESCU, Convergent sequences in semimetric spaces *Publ. Math. Debrecen* **16**(1969), 141–144.

[6] D. VOICULESCU, Sur un théorème de Misonou, *Rev. Roumaine Math. Pures Appl.* **17**(1972), 309–310.

[7] D.V. VOICULESCU, Sur les sous-espaces parafermés invariants d'une algèbre de von Neumann, *Bull. Sci. Math (2)* **96**(1972), 161–168.

[8] D. VOICULESCU, Hyperquasitriangularité et théorèmes du type Aronszajn–Smith, *C. R. Acad. Sci. Paris Sér. A-B* **276**(1973), A1605–A1606.

[9] D. VOICULESCU, Some extensions of quasitriangularity, *Rev. Roumaine Math. Pures Appl.* **18**(1973), 1303–1320.

[10] D. VOICULESCU, Some extensions of quasitriangularity. II, *Rev. Roumaine Math. Pures Appl.* **18**(1973), 1439–1456.

[11] C. APOSTOL, C. FOIAŞ, D. VOICULESCU, Structure spectrale des opérateurs non quasi-triangulaires, *C. R. Acad. Sci. Paris Sér. A-B* **276**(1973), A1669–A1671.

[12] C. APOSTOL, C. FOIAŞ, D. VOICULESCU, Some results on non-quasitriangular operators. II, *Rev. Roumaine Math. Pures Appl.* **18**(1973), 159–181.

[13] C. APOSTOL, C. FOIAŞ, D. VOICULESCU, Some results on non-quasitriangular operators. III, *Rev. Roumaine Math. Pures Appl.* **18**(1973), 309–324.

[14] C. APOSTOL, C. FOIAŞ, D. VOICULESCU, Some results on non-quasitriangular operators. IV, *Rev. Roumaine Math. Pures Appl.* **18**(1973), 487–514.

[15] C. APOSTOL, C. FOIAŞ, D. VOICULESCU, Some results on non-quasitriangular operators. V, *Rev. Roumaine Math. Pures Appl.* **18**(1973), 1133–1149.

[16] C. APOSTOL, C. FOIAŞ, D. VOICULESCU, Some results on non-quasitriangular operators. VI, *Rev. Roumaine Math. Pures Appl.* **18**(1973), 1473–1494.

[17] C. APOSTOL, D. VOICULESCU, On a problem of Halmos, *Rev. Roumaine Math. Pures Appl.* **19**(1974), 283–284.

[18] D. VOICULESCU, Norm-limits of algebraic operators, *Rev. Roumaine Math. Pures Appl.* **19**(1974), 371–378.

[19] D. VOICULESCU, Sur les représentations factorielles finies de $\mathrm{U}(\infty)$ et autres groupes semblables, *C. R. Acad. Sci. Paris Sér. A* **279**(1974), 945–946.

[20] C. APOSTOL, C. FOIAŞ, D. VOICULESCU, On the norm-closure of nilpotents. II. *Rev. Roumaine Math. Pures Appl.* **19**(1974), 549–557.

[21] C. APOSTOL, D. VOICULESCU, Quasitriangularity in Banach space. II, *Rev. Roumaine Math. Pures Appl.* **20**(1975), 171–179.

[22] Ş. STRĂTILĂ, D. VOICULESCU, Sur les représentations factorielles infinies de $\mathrm{U}(\infty)$, *C. R. Acad. Sci. Paris Sér. A-B* **280**(1975), A555–A558.

[23] C. FOIAŞ, C. PEARCY, D. VOICULESCU, The staircase representation of biquasitriangular operators, *Michigan Math. J.* **22**(1975), 343–352.

[24] D. VOICULESCU, Un théorème du type Weyl–von Neumann non commutatif, *C. R. Acad. Sci. Paris Sér. A* **281**(1975), 735–736.

[25] Ş. STRĂTILĂ, D. VOICULESCU, *Representations of* AF-*Algebras and of the Group* $\mathrm{U}(\infty)$, Lecture Notes in Math., vol. 486, Springer-Verlag, Berlin-New York 1975.

[26] D. VOICULESCU, A non-commutative Weyl–von Neumann theorem, *Rev. Roumaine Math. Pures Appl.* **21**(1976), 97–113.

[27] Ş. STRĂTILĂ, D. VOICULESCU, L. ZSIDÓ, Sur les produits croisés, *C. R. Acad. Sci. Paris Sér. A-B* **282**(1976), A779–A782.

[28] C. APOSTOL, C. FOIAŞ, D. VOICULESCU, On strongly reductive algebras, *Rev. Roumaine Math. Pures Appl.* **21**(1976), 633–641.

[29] C. APOSTOL, C. FOIAŞ, D. VOICULESCU, Strongly reductive operators are normal, *Acta Sci. Math. (Szeged)* **38**(1976), 261–263.

[30] D. VOICULESCU, Représentations factorielles de type II_1 de $\mathrm{U}(\infty)$, *J. Math. Pures Appl. (9)* **55**(1976), 1–20.

[31] Ş. STRĂTILĂ, D. VOICULESCU, L ZSIDÓ, On crossed products. I, *Rev. Roumaine Math. Pures Appl.* **21**(1976), 1411–1449.

[32] Ş. STRĂTILĂ, D. VOICULESCU, L. ZSIDÓ, On crossed products. II, *Rev. Roumaine Math. Pures Appl.* **22**(1977), 83–117.

[33] D. VOICULESCU, E. DECIU, C. VLAD, A. PETRESCU, The use of the theory of distributions in the study of shocks. [Romanian], *Bul. Inst. Politehn. Bucureşti Ser. Mec.* **39**(1977), 15–27.

[34] H. BERCOVICI, D. VOICULESCU, Tensor operations on characteristic functions of C_0 contractions, *Acta Sci. Math. (Szeged)* **39**(1977), 205–231.

[35] Ş. STRĂTILĂ, D. VOICULESCU, On a class of KMS states for the unitary group $\mathrm{U}(\infty)$, *Math. Ann.* **235**(1978), 87–110.

[36] C. PEARCY, C. FOIAŞ, D. VOICULESCU, Biquasitriangular operators and quasisimilarity, in *Linear Spaces and Approximation (Proc. Conf., Math. Res. Inst., Oberwolfach, 1977)*, Lecture Notes in Biomath., vol. 21, Springer-Verlag, Berlin-New York 1978, pp. 47–52.

[37] M. PIMSNER, S. POPA, D. VOICULESCU, Homogeneous C^*-extensions of $C(X) \otimes K(H)$. I, *J. Operator Theory* **1**(1979), 55–108.

[38] D. VOICULESCU, Some results on norm-ideal perturbations of Hilbert space operators, *J. Operator Theory* **2**(1979), 3–37.

[39] C. FOIAŞ, C. PASNICU, D. VOICULESCU, Weak limits of almost invariant projections, *J. Operator Theory* **2**(1979), 79–93.

[40] D. VOICULESCU, Amenability and Katz algebras, in *Algèbres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille, 1977)*, Colloq. Internat. CNRS, vol. 274, CNRS, Paris 1979, pp. 451–457.

[41] D. VOICULESCU, A note on quasitriangularity and trace-class self-commutators, *Acta Sci. Math. (Szeged)* **42**(1980), 195–199.

[42] M. PIMSNER, D. VOICULESCU, Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras, *J. Operator Theory* **4**(1980), 93–118.

[43] M. PIMSNER, D. VOICULESCU, Imbedding the irrational rotation C^*-algebra into an AF-algebra, *J. Operator Theory* **4**(1980), 201–210.

[44] M. PIMSNER, S. POPA, D. VOICULESCU, Homogeneous C^*-extensions of $C(X) \otimes K(H)$. II, *J. Operator Theory* **4**(1980), 211–249.

[45] D. VOICULESCU, Some results on norm-ideal perturbations of Hilbert space operators. II, *J. Operator Theory* **5**(1981), 77–100.

[46] D. VOICULESCU, Remarks on the singular extension in the C^*-algebra of the Heisenberg group, *J. Operator Theory* **5**(1981), 147–170.

[47] R.G. DOUGLAS, D. VOICULESCU, On the smoothness of sphere extensions, *J. Operator Theory* **6**(1981), 103–111.

[48] M. PIMSNER, S. POPA, D. VOICULESCU, Remarks on ideals of the Calkin-algebra for certain singular extensions, in *Topics in Modern Operator Theory (Timişoara/Herculane, 1980)*, Oper. Theory Adv. Appl., vol. 2, Birkhäuser-Verlag, Basel-Boston, Mass. 1981, pp. 269–277.

[49] D. VOICULESCU, Remarks on Hilbert–Schmidt perturbations of almost-normal operators, in *Topics in Modern Operator Theory (Timişoara/Herculane, 1980)*, Oper. Theory Adv. Appl., vol. 2, Birkhäuser-Verlag, Basel-Boston, Mass. 1981, pp. 311–318.

[50] C. APOSTOL, D.A. HERRERO, D. VOICULESCU, The closure of the similarity orbit of a Hilbert space operator, *Bull. Amer. Math. Soc. (N.S.)* **6**(1982), 421–426.

[51] M. PIMSNER, D. VOICULESCU, K-groups of reduced crossed products by free groups, *J. Operator Theory* **8**(1982), 131–156.

[52] Ş. STRĂTILĂ, D. VOICULESCU, A survey on representations of the unitary group $\mathrm{U}(\infty)$, in *Spectral Theory (Warsaw, 1977)*, Banach Center Publ., vol. 8, PWN, Warsaw 1982, pp. 415–434.

[53] D. VOICULESCU, Asymptotically commuting finite rank unitary operators without commuting approximants, *Acta Sci. Math. (Szeged)* **45**(1983), 429–431.

[54] C. APOSTOL, L.A. FIALKOW, D.A. HERRERO, D. VOICULESCU, *Approximation of Hilbert Space Operators*, Vol. II, Res. Notes Math., vol. 102, Pitman (Adv. Publ. Program), Boston, MA 1984.

[55] D. VOICULESCU, Hilbert space operators modulo normed ideals, in *Proceedings of the International Congress of Mathematicians (Warsaw, 1983)*, Vol. 1, 2, PWN, Warsaw 1984, pp. 1041–1047.

[56] D. VOICULESCU, Symmetries of some reduced free product C^*-algebras, in *Operator Algebras and their Connections with Topology and Ergodic Theory (Buşteni, 1983)*, Lecture Notes in Math., vol. 1132, Springer-Verlag, Berlin 1985, pp. 556–588.

[57] D. VOICULESCU, Addition of certain noncommuting random variables, *J. Funct. Anal.* **66**(1986), 323–346.

[58] D. VOICULESCU, Almost inductive limit automorphisms and embeddings into AF-algebras, *Ergodic Theory Dynam. Systems* **6**(1986), 475–484.

[59] D. VOICULESCU, Dual algebraic structures on operator algebras related to free products, *J. Operator Theory* **17**(1987), 85–98.

[60] D. VOICULESCU, On a trace formula of M.G. Kreĭn, in *Operators in Indefinite Metric Spaces, Scattering Theory and other Topics (Bucharest, 1985)*, Oper. Theory Adv. Appl., vol. 24, Birkhäuser-Verlag, Basel 1987, pp. 329–332.

[61] D. VOICULESCU, Multiplication of certain noncommuting random variables, *J. Operator Theory* **18**(1987), 223–235.

[62] D. VOICULESCU, A note on quasidiagonal operators, in *Topics in Operator Theory*, Oper. Theory Adv. Appl., vol. 32, Birkhäuser-Verlag, Basel 1988, pp. 265–274.

[63] H. BERCOVICI, D. VOICULESCU, The analogue of Kuroda's theorem for n-tuples, in *The Gohberg Anniversary Collection (Calgary, AB, 1988)*, Vol. II, Oper. Theory Adv. Appl., vol. 41, Birkhäuser-Verlag, Basel 1989, pp. 57–60.

[64] D. VOICULESCU, Property T and approximation of operators, *Bull. London Math. Soc.* **22**(1990), 25–30.

[65] D. VOICULESCU, On the existence of quasicentral approximate units relative to normed ideals. I, *J. Funct. Anal.* **91**(1990), 1–36.

[66] D. VOICULESCU, Noncommutative random variables and spectral problems in free product C^*-algebras, in *Proceedings of the Seventh Great Plains Operator Theory Seminar (Lawrence, KS, 1987)*, *Rocky Mountain J. Math.* **20**(1990), 263–283.

[67] G. DAVID, D. VOICULESCU, s-Numbers of singular integrals for the invariance of absolutely continuous spectra in fractional dimensions, *J. Funct. Anal.* **94**(1990), 14–26.

[68] D. VOICULESCU, Entropy of Bogoliubov automorphisms of the canonical anticommutation relations, *Comm. Math. Phys.* **133**(1990), 521–542.

[69] D. VOICULESCU, Circular and semicircular systems and free product factors, in *Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory (Paris, 1989)*, Progr. Math., vol. 92, Birkhäuser Boston, Boston, MA 1990, pp. 45–60.

[70] D. VOICULESCU, Limit laws for random matrices and free products, *Invent. Math.* **104**(1991), 201–220.

[71] D. VOICULESCU, A note on quasi-diagonal C^*-algebras and homotopy, *Duke Math. J.* **62**(1991), 267–271.

[72] D. VOICULESCU, Entropy of dynamical systems and perturbations of operators, *Ergodic Theory Dynam. Systems* **11**(1991), 779–786.

[73] D. VOICULESCU, Entropy of dynamical systems and perturbations of operators. II, *Houston J. Math.* **17**(1991), 651–661.

[74] D. VOICULESCU, Free noncommutative random variables, random matrices and the II_1 factors of free groups, in *Quantum Probability and Related Topics*: QP-PQ, vol. VI, World Sci. Publ., River Edge, NJ 1991, pp. 473–487.

[75] H. BERCOVICI, D. VOICULESCU, Lévy–Hinčin type theorems for multiplicative and additive free convolution, *Pacific J. Math.* **153**(1992),217–248.

[76] D,V. VOICULESCU, K.J. DYKEMA, A. NICA, *Free Random Variables. A Noncommutative Probability Approach to Free Products with Applications to Random Matrices, Operator Algebras and Harmonic Analysis on Free Groups*, CRM Monogr. Ser., vol. 1, Amer. Math. Soc., Providence, RI 1992.

[77] D. VOICULESCU, Perturbations of operators, connections with singular integrals, hyperbolicity and entropy, in *Harmonic Analysis and Discrete Potential Theory (Frascati, 1991)*, 181–191, Plenum, New York 1992.

[78] D. VOICULESCU, Entropy-invariants of dynamical systems and perturbations of operators, in *Proceedings of the 10th Congress on Mathematical Physics, Leipzig, Germany, July 30-August 9, 1991*, Math. Phy., vol. 10, Springer-Verlag, Berlin 1992, pp. 303–307.

[79] D. VOICULESCU, Entropy of random walks on groups and the Macaev norm, *Proc. Amer. Math. Soc.* **119**(1993), 971–977.

[80] D. VOICULESCU, II_1-factors of free groups and random matrices, in *Representation Theory of Groups and Algebras*, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI 1993, pp. 101–107.

[81] D. VOICULESCU, The K-groups of the C^*-algebra of a semicircular family, *K-Theory* **7**(1993), 5–7.

[82] D. VOICULESCU, Around quasidiagonal operators, *Integral Equations Operator Theory* **17**(1993), 137–149.

[83] D. VOICULESCU, The analogues of entropy and of Fisher's information measure in free probability theory. I, *Comm. Math. Phys.* **155**(1993), 71–92.

[84] D. VOICULESCU, A note on the augmentation ideal of certain rings associated with free groups, *Acta Sci. Math. (Szeged)* **57**(1993), 277–287.

[85] H. BERCOVICI, D. VOICULESCU, Free convolution of measures with unbounded support, *Indiana Univ. Math. J.* **42**(1993), 733–773.

[86] D. VOICULESCU, The analogues of entropy and of Fisher's information measure in free probability theory. II, *Invent. Math.* **118**(1994), 411–440.

[87] D. VOICULESCU, Alternative Entropies in Operator Algebras, in *First European Congress of Mathematics Paris, France, July 6-10, 1992*, Vol. II, Progr. Math., vol. 120, Birkhäuser-Verlag, Basel 1994, pp. 475–483.

[88] D. VOICULESCU, Dynamical approximation entropies and topological entropy in operator algebras, *Comm. Math. Phys.* **170**(1995), 249–281.

[89] H. BERCOVICI, D. VOICULESCU, Superconvergence to the central limit and failure of the Cramér theorem for free random variables, *Probab. Theory Related Fields* **103**(1995), 215–222.

[90] P. DE LA HARPE, D. VOICULESCU, A problem on the II_1-factors of Fuchsian groups, in *Recent Advances in Operator Algebras (Orléans, 1992)*, Astérisque, vol. 232, 1995, pp. 155–158.

[91] D. VOICULESCU, Operations on certain non-commutative operator-valued random variables, in *Recent Advances in Operator Algebras (Orléans, 1992)*, Astérisque, vol. 232, 1995, pp. 243–275.

[92] D. VOICULESCU, Free probability theory: random matrices and von Neumann algebras, in *Proceedings of the International Congress of Mathematicians (Zürich, 1994)*, Vol. 1, 2, Birkhäuser-Verlag, Basel 1995, pp. 227–241.

[93] D. VOICULESCU, The analogues of entropy and of Fisher's information measure in free probability theory. III. The absence of Cartan subalgebras, *Geom. Funct. Anal.* **6**(1996), 172–199.

[94] S.J. SZAREK, D. VOICULESCU, Volumes of restricted Minkowski sums and the free analogue of the entropy power inequality, *Comm. Math. Phys*. **178**(1996), 563–570.

[95] D. VOICULESCU, The analogues of entropy and of Fisher's information measure in free probability theory. IV. Maximum entropy and freeness, in *Free Probability Theory (Waterloo, ON, 1995)*, Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI 1997, pp. 293–302.

[96] D.-V. VOICULESCU, The derivative of order $1/2$ of a free convolution by a semicircle distribution, *Indiana Univ. Math. J.* **46**(1997), 697–703.

[97] D. VOICULESCU, Topics in free entropy, in *Operator Algebras and Quantum Field Theory (Rome, 1996)*, Int. Press, Cambridge, MA 1997, pp. 458–462.

[98] D. VOICULESCU, A strengthened asymptotic freeness result for random matrices with applications to free entropy, *Int. Math. Res. Not.* **1**(1998), 41–63.

[99] D. VOICULESCU, The analogues of entropy and of Fisher's information measure in free probability theory. V. Noncommutative Hilbert transforms, *Invent. Math.* **132**(1998), 189–227.

[100] H. BERCOVICI, D. VOICULESCU, Regularity questions for free convolution, in *Nonselfadjoint Operator Algebras, Operator Theory, and Related Topics*, Oper. Theory Adv. Appl., vol. 104, Birkhäuser-Verlag, Basel 1998, pp. 37–47.

[101] D. VOICULESCU, The noncommutative Hilbert transform approach to free entropy, in *Quantum Probability (Gdańsk, 1997)*, Banach Center Publ., vol. 43, Polish Acad. Sci. Inst. Math., Warsaw 1998, pp. 421–427.

[102] D. VOICULESCU, Free entropy dimension $\leqslant 1$ for some generators of property T factors of type II_1, *J. Reine Angew. Math.* **514**(1999), 113–118.

[103] D. VOICULESCU, The analogues of entropy and of Fisher's information measure in free probability theory. VI. Liberation and mutual free information, *Adv. Math.* **146**(1999), 101–166.

[104] D. VOICULESCU, The coalgebra of the free difference quotient and free probability, *Int. Math. Res. Not.* **2**(2000), 79–106.

[105] M. EMERY, A. NEMIROVSKI, D. VOICULESCU, P. BERNARD, *Lectures on Probability Theory and Statistics. Lectures from the 28th Summer School on Probability Theory held in Saint-Flour, August 17–September 3, 1998*, Lecture Notes in Math., vol. 1738. Springer-Verlag, Berlin 2000.

[106] D. VOICULESCU, Lectures on free probability theory, in *Lectures on Probability Theory and Statistics (Saint-Flour, 1998)*, Lecture Notes in Math., vol. 1738, Springer-Verlag, Berlin 2000, pp. 279–349.

[107] S.J. SZAREK, D. VOICULESCU, Shannon's entropy power inequality via restricted Minkowski sums, in *Geometric Aspects of Functional Analysis*, Lecture Notes in Math., vol. 1745, Springer-Verlag, Berlin 2000,pp. 257–262.

[108] D. VOICULESCU, A note on cyclic gradients, *Indiana Univ. Math. J.* **49**(2000), 837–841.

[109] P. BIANE, D. VOICULESCU, A free probability analogue of the Wasserstein metric on the trace-state space, *Geom. Funct. Anal.* **11**(2001), 1125–1138.

[110] D. VOICULESCU, Cyclomorphy, *Int. Math. Res. Not.* (2002), 299–332.

[111] D. VOICULESCU, Free entropy, *Bull. London Math. Soc.* **34**(2002), 257–278.

[112] D. VOICULESCU, Analytic subordination consequences of free Markovianity, *Indiana Univ. Math. J.* **51**(2002), 1161–1166.

[113] D. VOICULESCU, The topological version of free entropy, *Lett. Math. Phys.* **62**(2002), 71–82.

[114] D. VOICULESCU, Free analysis questions. I. Duality transform for the coalgebra of $\partial X : B$, *Int. Math. Res. Not.* (2004), 793–822.

[115] D. VOICULESCU, Free probability and the von Neumann algebras of free groups, *Rep. Math. Phys.* **55**(2005), 127–133.

[116] D. VOICULESCU, Aspects of free probability, in *XIVth International Congress on Mathematical Physics*, World Sci. Publ., Hackensack, NJ 2005, pp. 145–157.

[117] D. VOICULESCU, Symmetries arising from free probability theory, in *Frontiers in Number Theory, Physics, and Geometry*. I, Springer-Verlag, Berlin 2006, pp. 231–243.

[118] G.B. AROUS, D.V. VOICULESCU, Free extreme values, *Ann. Probab.* **34**(2006), 2037–2059.

[119] D.-V. VOICULESCU, Aspects of free analysis, *Jpn. J. Math.* **3**(2008), 163–183.

[120] D.-V. VOICULESCU, Free analysis questions. II: the Grassmannian completion and the series expansions at the origin, *J. Reine Angew. Math.* **645**(2010), 155–236.

[121] P. BIANE, A. GUIONNET, D.-V. VOICULESCU, *Noncommutative Probability and Random Matrices at Saint-Flour. Probability at Saint-Flour*, Springer-Verlag, Heidelberg 2012.

[122] D.-V. VOICULESCU, Free probability for pairs of faces. I, *Comm. Math. Phys.* **332**(2014), 955–980.

[123] D.-V. VOICULESCU, Countable degree-1 saturation of certain C^*-algebras which are coronas of Banach algebras, *Groups Geom. Dyn.* **8**(2014), 985–1006.

[124] D.-V. VOICULESCU, Almost normal operators mod Hilbert–Schmidt and the K-theory of the Banach algebras $E\Lambda(\Omega)$, *J. Noncommut. Geom.* **8**(2014), 1123–1145.

[125] D.-V. VOICULESCU, Free probability for pairs of faces. II: 2-variables bi-free partial R-transform and systems with rank $\leqslant 1$ commutation, *Ann. Inst. Henri Poincaré Probab. Stat.* **52**(2016), 1–15.

[126] D.-V. VOICULESCU, Free probability for pairs of faces. III: 2-variables bi-free partial S- and T-transforms, *J. Funct. Anal.* **270**(2016), 3623–3638.

[127] D.-V. VOICULESCU, Some C^*-algebras which are coronas of non-C^*-Banach algebras, *J. Geom. Phys.* **105**(2016), 123–129.

[128] J. BOURGAIN, D.-V. VOICULESCU, The essential centre of the mod a diagonalization ideal commutant of an n-tuple of commuting Hermitian operators, in *Noncommutative Analysis, Operator Theory and Applications*, Oper. Theory Adv. Appl., vol. 252, Linear Oper. Linear Syst., Birkhäuser/Springer, Cham 2016, pp. 77–80.

[129] D.-V. VOICULESCU, The bi-free extension of free probability, in *Mathematical Analysis, Probability and Applications — Plenary Lectures*, Springer Proc. Math. Stat., vol. 177, Springer, Cham 2016, pp. 217–233.

[130] D.-V. VOICULESCU, Free probability for pairs of faces. IV: Bi-free extremes in the plane, *J. Theoret. Probab.* **30**(2017), 384–394.

[131] D.-V. VOICULESCU, Lebesgue decomposition of functionals and unique preduals for commutants modulo normed ideals, *Houston J. Math.* **43**(2017), 1251–1262.

[132] D.-V. VOICULESCU, Hybrid normed ideal perturbations of n-tuples of operators. I, *J. Geom. Phys.* **128**(2018), 169–184.

[133] D.-V. VOICULESCU, K-theory and perturbations of absolutely continuous spectra, *Comm. Math. Phys.* **365**(2019), 363–373.

[134] D.-V. VOICULESCU, A remark about supramenability and the Macaev norm, *Groups Geom. Dyn.* **13**(2019), 379–388.

[135] D.-V. VOICULESCU, A hydrodynamic exercise in free probability: setting up free Euler equations, *Expo. Math.* **38**(2020), 271–283.

HARI BERCOVICI, DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405-7106, U.S.A.
E-mail address: bercovic@indiana.edu

KEN DYKEMA, DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843-3368, U.S.A.
E-mail address: ken.dykema@math.tamu.edu

ALEXANDRU NICA, DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, 200 UNIVERSITY AVENUE WEST, WATERLOO, ONTARIO N2L 3G1, CANADA
E-mail address: anica@uwaterloo.ca

Received October 18, 2020.