J. OPERATOR THEORY
90:2(2023), 605–623
doi: 10.7900/jot.2022jan21.2393

UNBOUNDED WEYL TRANSFORM ON THE EUCLIDEAN MOTION GROUP AND HEISENBERG MOTION GROUP

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Communicated by Alexandru Aleman

ABSTRACT. In this article, we define the Weyl transform on a second countable type I locally compact group G, and as an operator on $L^2(G)$, we prove that the Weyl transform is compact when the symbol lies in $L^p(G \times \widehat{G})$ with $1 \leqslant p \leqslant 2$. Further, for the Euclidean motion group and Heisenberg motion group, we prove that the Weyl transform cannot be extended as a bounded operator for the symbol belongs to $L^p(G \times \widehat{G})$ with $2 < p < \infty$. To carry out this, we construct positive, square integrable and compactly supported function, on the respective groups, such that the $L^{p'}$ norm of its Fourier transform is infinite, where p' is the conjugate index of p.

KEYWORDS: *Euclidean motion group, Fourier transform, Heisenberg motion group, Weyl transform.*

MSC (2020): Primary 47G30; Secondary 43A30, 33C05, 33C10.

INTRODUCTION

In [18], Hermann Weyl studied the quantization problem in quantum mechanics and introduced a type of pseudo-differential operators. These operators are useful in physics and mathematics, especially in PDE, harmonic analysis, time frequency analysis. In [19], Wong called these operators as Weyl transforms. Further in [19], for the symbol in $L^p(\mathbb{R}^{2n})$ with $1 \leqslant p \leqslant 2$, compactness of the Weyl transform, as an operator on $L^2(\mathbb{R}^n)$, is studied. Moreover in [15], Simon proved that the operator is not even bounded when the symbol is in $L^p(\mathbb{R}^{2n})$ with $2 < p < \infty$.

In general, for a locally compact group, Fourier transform is an operator valued function. For the Heisenberg group, in [11], Weyl transform is defined for operator valued symbols and its boundedness is proved (even compactness) when the symbol is in the corresponding L^p-spaces with $1 \leqslant p \leqslant 2$, while unboundedness occurs for $2 < p < \infty$. Further, in [12] Weyl transform on the upper half plane, and in [3] Weyl transform on the quaternion Heisenberg group, are studied.

In this article, we consider a second countable type I locally compact group G and define Weyl transform in view of its inversion formula. We prove that the Weyl transform, as an operator on $L^2(G)$, is compact, when the symbol is in $L^p(G \times \widehat{G})$ with $1 \leqslant p \leqslant 2$, where $\widehat{G}$ is the dual of G. Further, to prove that the Weyl transform cannot be extended as a bounded operator for a symbol in $L^p(G \times \widehat{G})$ with $2 < p < \infty$, it is enough to construct a positive, square integrable and compactly supported function on G such that $L^{p'}$ norm of its Fourier transform is infinite, where $\frac{1}{p} + \frac{1}{p'} = 1$. In this perspective, we construct such type of functions for the Euclidean motion group and Heisenberg motion group.

In addition, we want to mention that these examples will not follow directly from the Euclidean space and Heisenberg group examples, respectively, for the following reasons. The Fourier transform on the Euclidean motion group is operator valued, whereas it is just a function for the Euclidean space. We have constructed appropriate examples and proved the result by correlating the Fourier transform and Bessel function. Secondly, the case becomes more difficult for the Heisenberg motion group due to the presence of metaplectic representation whose implicitness may not be visible at the first instance.

1. WEYL TRANSFORM ON LOCALLY COMPACT GROUPS

In this section, we recall some harmonic analysis results, namely Fourier inversion, Plancherel formula, and Hausdorff–Young inequality on certain locally compact groups, see [4, 10] for details. Then in terms of Wigner transform, we define Weyl transform. After that, we prove the compactness of the Weyl transform for the symbol in corresponding L^p-spaces with $1 \leqslant p \leqslant 2$. This section concludes with a sufficient condition for the unboundedness of the Weyl transform for $2 < p < \infty$.

Let G be a second countable locally compact group with type I left regular representation, and $A(G)$, $\widehat{G}$ denote the Fourier algebra and dual of G, respectively. Then there is a standard measure μ on $\widehat{G}$, called the Plancherel measure, a μ-measurable field $(\pi, \mathcal{H}_\pi)$ of representations and a measurable field $\mathcal{K} = (K_\pi)$ of nonzero positive self-adjoint operators such that K_π is semi-invariant with weight $\triangle^{-1}$, for almost all $\pi \in \widehat{G}$, where $\triangle$ is the modular function of G and $A(G)$. If G is unimodular, then $K_\pi = I_{\mathcal{H}_\pi}$, the identity operator on $\mathcal{H}_\pi$. For $f \in L^1(G) \cap A(G)$, the group Fourier transform

$$\pi(f) = \int_G f(x)\pi(x) \mathrm{d}\nu(x)$$

is a bounded operator on $\mathcal{H}_\pi$, where $\mathrm{d}\nu$ is a left Haar measure on G. Further, f can be recovered by the inversion formula.

THEOREM 1.1 ([10], Inversion theorem). *Let $f \in L^1(G) \cap A(G)$. Then*

$$f(x) = \int\limits_{\widehat{G}} \mathrm{tr}(\pi(x)^{-1}\pi(f)K_\pi)\mathrm{d}\mu(\pi).$$

Next, we discuss the Plancherel formula for the Fourier transform, and for this, we start with Schatten class operators. Let S_p be the space of all Schatten-p class operators, which is a Banach space with the norm $\|T\|_{S_p}^p = \mathrm{tr}(T^*T)^{p/2}$ for $1 \leqslant p < \infty$ and for $p = \infty$, the norm is the usual operator norm.

THEOREM 1.2 ([4], Plancherel formula). *Let $f \in L^1 \cap L^2(G)$. Then*

$$\int\limits_G |f(x)|^2 \mathrm{d}\nu(x) = \int\limits_{\widehat{G}} \|\pi(f)K_\pi^{1/2}\|_{S_2}^2 \,\mathrm{d}\mu(\pi).$$

This map $L^1 \cap L^2(G) \to L^2(\widehat{G})$ can be extended uniquely to a unitary map from $L^2(G)$ onto $L^2(\widehat{G})$.

Let $f \in L^1 \cap L^p(G)$, where $1 \leqslant p \leqslant \infty$. Define the L^p Fourier transform of f by $\mathcal{F}_p(f)\pi = \pi(f)K_\pi^{1/p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then the following Hausdorff–Young inequality holds.

THEOREM 1.3 ([10]). *If $f \in L^1 \cap L^p(G)$, where $1 < p < 2$, then $\mathcal{F}_p(f) \in L^{p'}(\widehat{G})$ and this map $f \mapsto \mathcal{F}_p(f)$ extends uniquely to a bounded linear map from $L^p(G)$ into $L^{p'}(\widehat{G})$ with norm less than or equal to* 1.

To study the Weyl transform, we need to define Wigner transform on G and investigate its boundedness properties. For this, we first describe the following product spaces and the left translation operator.

Consider the measure $\mathrm{d}\nu \otimes \mathrm{d}\mu$ on $G \times \widehat{G}$ and for $1 \leqslant p \leqslant \infty$, let $L^p(G \times \widehat{G}, S_p, \mathrm{d}\nu \otimes \mathrm{d}\mu)$ be the space of S_p valued functions satisfying:

$$\|f\|_{p,\nu\otimes\mu}^p = \int\limits_{G\times\widehat{G}} \|f(x,\pi)K_\pi^{1/p}\|_{S_p}^p \mathrm{d}\nu(x)\mathrm{d}\mu(\pi) < \infty, \quad 1 \leqslant p < \infty,$$

$$\|f\|_{\infty,\nu\otimes\mu} = \operatorname*{ess\,sup}_{(x,\pi)\in G\times\widehat{G}} \|f(x,\pi)\|_{S_\infty} < \infty.$$

The left translation operator is defined by $\tau_{x'}f(x) = f(x'^{-1}x)$ for $f \in L^p(G)$. Further, $C_c(G)$ denotes the space of all compactly supported continuous functions on G. Throughout this section, p, p' are conjugate indices.

DEFINITION 1.4. Let $f, g \in C_c(G)$ and $(x, \pi) \in G \times \widehat{G}$. Then the Wigner transform associated with f, g is defined by

$$V(f,g)(x,\pi) = \int_G f(x')\tau_{x'}g(x)\pi(x')\mathrm{d}\nu(x').$$

That is, $V(f,g)(x,\pi) = \pi(f \cdot \tau g(x))$.

The following result proves that the Wigner transforms are in certain L^p-spaces.

PROPOSITION 1.5. *Let $f, g \in C_c(G)$. Then $V(f,g) \in L^{p'}(G \times \widehat{G}, S_{p'}, \mathrm{d}\nu \otimes \mathrm{d}\mu)$ and*

$$\|V(f,g)\|_{p',\nu\otimes\mu} \leqslant \|f\|_2 \|g\|_2 \tag{1.1}$$

for $p' \in [2, \infty]$. Thus $V : C_c(G) \times C_c(G) \to L^{p'}(G \times \widehat{G}, S_{p'}, \mathrm{d}\nu \otimes \mathrm{d}\mu)$ can be extended uniquely to a bilinear operator $V : L^2(G) \times L^2(G) \to L^{p'}(G \times \widehat{G}, S_{p'}, \mathrm{d}\nu \otimes \mathrm{d}\mu)$ with

$$\|V(f,g)\|_{p',\nu\otimes\mu} \leqslant \|f\|_2 \|g\|_2.$$

Proof. Let $p' = \infty$. Then

$$\begin{aligned}\|V(f,g)\|_{\infty,\nu\otimes\mu} &= \operatorname*{ess\,sup}_{(x,\pi)\in G\times\widehat{G}} \|V(f,g)(x,\pi)\|_{S_\infty} \\ &= \operatorname*{ess\,sup}_{(x,\pi)\in G\times\widehat{G}} \Big\|\int_G f(x')\tau_{x'}g(x)\pi(x')\mathrm{d}\nu(x')\Big\|_{S_\infty} \\ &\leqslant \operatorname*{ess\,sup}_{x\in G} \int_G |f(x')\tau_{x'}g(x)|\mathrm{d}\nu(x') \leqslant \|f\|_2\|g\|_2.\end{aligned}$$

For $p' = 2$, Plancherel formula, Theorem 1.2, gives

$$\begin{aligned}\|V(f,g)\|^2_{2,\nu\otimes\mu} &= \int_G \int_{\widehat{G}} \|\pi(f \cdot \tau g(x))K_\pi^{1/2}\|^2_{S_2} \mathrm{d}\mu(\pi)\mathrm{d}\nu(x) \\ &= \int_G \int_G |f(x')\tau_{x'}g(x)|^2 \mathrm{d}\nu(x')\mathrm{d}\nu(x) = \|f\|_2^2\|g\|_2^2.\end{aligned}$$

Then Riesz–Thorin interpolation theorem completes the proof. ∎

REMARK 1.6. Let G be the Euclidean space $\mathbb{R}^n$. For $r > 2$, an upper bound for the L^r norm of the Wigner transform, associated with two square-integrable functions f and g, is obtained in [8]. Note that as a consequence of the Plancherel formula, equality holds for the $r = 2$ case. For $2 < r < \infty$, the above mentioned bounds are optimal, and the optimality will be achieved if and only if f and g are a pair of particular Gaussians. This is due to Lieb, see [8] for details and a few other sharp bounds. For a non-commutative group G, e.g., Heisenberg group, it is exciting to find out the best possible bound for (1.1).

The following proposition is a way of writing the group Fourier transform in terms of the Wigner transform.

PROPOSITION 1.7. *Let* $f, g \in L^1 \cap L^2(G)$ *and* $\mathrm{C} = \int_G g(x)\mathrm{d}\nu(x) \neq 0$. *Then* $\pi(f) = \mathrm{C}^{-1} \int_G V(f,g)(x,\pi)\mathrm{d}\nu(x)$ *for* $\pi \in \widehat{G}$.

Proof. We have:

$$\begin{aligned}\int_G V(f,g)(x,\pi)\mathrm{d}\nu(x) &= \int_G \int_G f(x')g(x'^{-1}x)\pi(x')\mathrm{d}\nu(x)\mathrm{d}\nu(x') \\ &= \Big(\int_G g(x)\mathrm{d}\nu(x)\Big)\Big(\int_G f(x')\pi(x')\mathrm{d}\nu(x')\Big) = \mathrm{C}\,\pi(f). \quad \blacksquare\end{aligned}$$

In view of Proposition 1.7, inversion formula, Theorem 1.1, can be reformulated.

COROLLARY 1.8. *Let* $f \in L^1(G) \cap A(G)$ *and* $g \in L^1 \cap L^2(G)$ *with* $\mathrm{C} = \int_G g(x)\mathrm{d}\nu(x) \neq 0$. *Then*

$$f(x) = \mathrm{C}^{-1} \int_{\widehat{G}} \mathrm{tr}\Big(\pi(x)^{-1}\Big(\int_G V(f,g)(x',\pi)\mathrm{d}\nu(x')\Big)K_\pi\Big)\mathrm{d}\mu(\pi).$$

DEFINITION 1.9. Let $\varsigma \in L^p(G \times \widehat{G}, S_p, \mathrm{d}\nu \otimes \mathrm{d}\mu)$, where $1 \leqslant p \leqslant 2$. Then corresponding to ς, Weyl transform $W_\varsigma : L^2(G) \to L^2(G)$ is defined by

$$\langle W_\varsigma f, \overline{g}\rangle = \langle V(f,g), \varsigma\rangle_{\nu\otimes\mu} = \int_G \int_{\widehat{G}} \mathrm{tr}(\varsigma^*(x,\pi)V(f,g)(x,\pi)K_\pi)\mathrm{d}\mu(\pi)\mathrm{d}\nu(x),$$

where $f, g \in L^2(G)$. Here $\varsigma^*(x,\pi) = \varsigma(x,\pi)^*$ and by the abuse of notation $\langle\cdot,\cdot\rangle_{\nu\otimes\mu}$ is used.

After a bit of calculation, we can conclude that

$$W_\varsigma f(x) = \int_{\widehat{G}} \int_G \mathrm{tr}(\varsigma^*(x'x,\pi)\pi(x')K_\pi)f(x')\mathrm{d}\nu(x')\mathrm{d}\mu(\pi). \tag{1.2}$$

Thus $W_\varsigma : L^2(G) \to L^2(G)$ is an integral operator with kernel

$$K(x,x') = \int_{\widehat{G}} \mathrm{tr}(\varsigma^*(x'x,\pi)\pi(x')K_\pi)\mathrm{d}\mu(\pi).$$

In the following proposition, we investigate the boundedness of the Weyl transform.

PROPOSITION 1.10. *Let* $\varsigma \in L^p(G \times \widehat{G}, S_p, \mathrm{d}\nu \otimes \mathrm{d}\mu)$, *where* $1 \leqslant p \leqslant 2$. *Then* $\|W_\varsigma\| \leqslant \|\varsigma\|_{p,\nu\otimes\mu}$.

Proof. Since $|\mathrm{tr}(A^*B)| \leqslant \|A\|_{S_p}\|B\|_{S_{p'}}$, for $f, g \in L^2(G)$ we have

$$|\langle W_\varsigma f, \overline{g}\rangle| \leqslant \int\limits_G \int\limits_{\widehat{G}} |\mathrm{tr}(\varsigma^*(x,\pi)V(f,g)(x,\pi)K_\pi)|\mathrm{d}\mu(\pi)\mathrm{d}\nu(x)$$

$$\leqslant \int\limits_G \int\limits_{\widehat{G}} \|\varsigma(x,\pi)(K_\pi^{1/p})^*\|_{S_p}\|V(f,g)(x,\pi)K_\pi^{1/p'}\|_{S_{p'}}\mathrm{d}\mu(\pi)\mathrm{d}\nu(x). \tag{1.3}$$

For $p = 1$, the above integral is less than or equal to

$$\|\varsigma\|_{1,\nu\otimes\mu}\|V(f,g)\|_{\infty,\nu\otimes\mu} \leqslant \|\varsigma\|_{1,\nu\otimes\mu}\|f\|_2\|g\|_2.$$

If $1 < p \leqslant 2$ by the Hölder's inequality, the integral in (1.3) is less than or equal to

$$\Big(\int\limits_G\int\limits_{\widehat{G}}\|\varsigma(x,\pi)K_\pi^{1/p}\|_{S_p}^p\mathrm{d}\mu(\pi)\mathrm{d}\nu(x)\Big)^{1/p}\Big(\int\limits_G\int\limits_{\widehat{G}}\|V(f,g)(x,\pi)K_\pi^{1/p'}\|_{S_{p'}}^{p'}\mathrm{d}\mu(\pi)\mathrm{d}\nu(x)\Big)^{1/p'}$$

$$= \|\varsigma\|_{p,\nu\otimes\mu}\|V(f,g)\|_{p',\nu\otimes\mu} \leqslant \|\varsigma\|_{p,\nu\otimes\mu}\|f\|_2\|g\|_2. \quad ∎$$

Further for $p = 2$, W_ς is a Hilbert–Schmidt operator.

PROPOSITION 1.11. *If $\varsigma \in L^2(G\times\widehat{G}, S_2, \mathrm{d}\nu\otimes\mathrm{d}\mu)$, then W_ς is a Hilbert–Schmidt operator. Furthermore, $\|W_\varsigma\|_{S_2} = \|\varsigma\|_{2,\nu\otimes\mu}$.*

Proof. Since W_ς is an integral operator, we have

$$\begin{aligned}\|W_\varsigma\|_{S_2}^2 &= \int\limits_G\int\limits_G |K(x,x')|^2\mathrm{d}\nu(x)\mathrm{d}\nu(x')\\ &= \int\limits_G\int\limits_G \Big|\int\limits_{\widehat{G}} \mathrm{tr}(\varsigma^*(x'x,\pi)\pi(x')K_\pi)\mathrm{d}\mu(\pi)\Big|^2\mathrm{d}\nu(x)\mathrm{d}\nu(x')\\ &= \int\limits_G\int\limits_G \Big|\int\limits_{\widehat{G}} \mathrm{tr}(\varsigma^*(x,\pi)\pi(x')K_\pi)\mathrm{d}\mu(\pi)\Big|^2\mathrm{d}\nu(x)\mathrm{d}\nu(x'),\end{aligned}$$

where last equality is ensured by applying the change of variable $x = x'x$, as $\mathrm{d}\nu$ is a left invariant Haar measure. Since $\mathrm{tr}(T^*) = \overline{\mathrm{tr}(T)}$ and $K_\pi^* = K_\pi$, the above integral becomes

$$\int\limits_G\int\limits_G \Big|\int\limits_{\widehat{G}} \mathrm{tr}(K_\pi\pi(x')^{-1}\varsigma(x,\pi))\mathrm{d}\mu(\pi)\Big|^2\mathrm{d}\nu(x)\mathrm{d}\nu(x').$$

Applying the Fourier inversion and Plancherel formula, we get

$$\begin{aligned}\|W_\varsigma\|_{S_2}^2 &= \int\limits_G\int\limits_G |\mathcal{F}^{-1}(\varsigma)(x,\cdot)(x')|^2\mathrm{d}\nu(x')\mathrm{d}\nu(x)\\ &= \int\limits_G\int\limits_{\widehat{G}} \|\varsigma(x,\pi)K_\pi^{1/2}\|_{S_2}^2\mathrm{d}\mu(\pi)\mathrm{d}\nu(x) = \|\varsigma\|_{2,\nu\otimes\mu}^2. \quad ∎\end{aligned}$$

THEOREM 1.12. *If* $\varsigma \in L^p(G \times \widehat{G}, S_p, d\nu \otimes d\mu)$, *where* $1 \leqslant p \leqslant 2$, *then* W_ς *is a Schatten-*p' *operator.*

Proof. Let $\varsigma \in L^p(G \times \widehat{G}, S_p, d\nu \otimes d\mu)$, where $1 \leqslant p \leqslant 2$. Then by applying complex interpolation, as a corollary of Proposition 1.10 and Proposition 1.11, we have W_ς is a Schatten-p' operator. ∎

Assuming a type of change of variable true on G, for $p = 1$, W_ς is a trace class operator. Note that we will have this assumption only on Proposition 1.13.

PROPOSITION 1.13. *Let* G *be a second countable type* I *locally compact group satisfying the following assertion. There is a constant* $c > 0$ *such that for all* $f \in L^1(G)$,

$$\int_G |f(xx)| d\nu(x) \leqslant c \int_G |f(x)| d\nu(x). \tag{1.4}$$

If $\varsigma \in L^1(G \times \widehat{G}, S_1, d\nu \otimes d\mu)$, *then* W_ς *is a trace class operator.*

Proof. Since W_ς is an integral operator

$$\|W_\varsigma\|_{S_1} = \int_G |K(x,x)| d\nu(x) = \int_G \Big| \int_{\widehat{G}} \operatorname{tr}(\varsigma^*(xx,\pi)\pi(x)K_\pi) d\mu(\pi) \Big| d\nu(x).$$

Since $\operatorname{tr}(T^*) = \overline{\operatorname{tr}(T)}$ and $\operatorname{tr}(TU) = \operatorname{tr}(UT)$ we have

$$\begin{aligned}\|W_\varsigma\|_{S_1} &\leqslant \int_G \int_{\widehat{G}} |\operatorname{tr}(\pi(x)^*\varsigma(xx,\pi)K_\pi)| d\mu(\pi) d\nu(x) \\ &\leqslant \int_G \int_{\widehat{G}} \operatorname{tr}(|\varsigma(xx,\pi)K_\pi|) d\mu(\pi) d\nu(x) \leqslant c\|\varsigma\|_{1,\nu\otimes\mu},\end{aligned}$$

where the second last inequality is true because $|\operatorname{tr}(T)| \leqslant \operatorname{tr}(T^*T)^{1/2} = \operatorname{tr}(|T|)$. ∎

REMARK 1.14. (i) A large class of groups, including the Euclidean spaces, step two nilpotent Lie groups, affine group on the upper half plane, have the property (1.4). The main advantage of Proposition 1.13 is that, in view of Proposition 1.11 and complex interpolation, we have the following better conclusion than Theorem 1.12:

If G *satisfies* (1.4) *and* $\varsigma \in L^p(G \times \widehat{G}, S_p, d\nu \otimes d\mu)$, *where* $1 \leqslant p \leqslant 2$, *then* W_ς *is a Schatten-*p *operator.*

(ii) In [9], the authors considered second countable locally compact type I unimodular groups. On the other hand, we do not have the unimodular assumption, which causes slight technical adjustments. With many interesting results, in [9], authors studied Wigner transformation, τ-pseudo-differential operators, which become Weyl transform for particular τ, and proved some compactness properties. For instance, in terms of the Weyl transform (true for τ-pseudo-differential operators), it is proved in [9] that if $\varsigma \in L^p(G \times \widehat{G}, S_p, d\nu \otimes d\mu)$, where $1 \leqslant p \leqslant 2$, then W_ς is a Schatten-p' operator. We also have the same conclusion in Theorem 1.12. In addition, for certain groups, we have a stronger conclusion in Remark 1.14(i) that W_ς is a Schatten-p operator.

However, the primary intention of the article is to study the case $p > 2$, and the rest of the article will focus on $p > 2$.

The next result gives a necessary and sufficient condition for boundedness of the Weyl transform in terms of the Wigner transform.

PROPOSITION 1.15. *Let $2 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then for all $\varsigma \in L^p(G \times \widehat{G}, S_p, \mathrm{d}\nu \otimes \mathrm{d}\mu)$, the Weyl transform W_ς is a bounded operator on $L^2(G)$ if and only if there exists a constant C such that $\|V(f,g)\|_{p',\nu\otimes\mu} \leqslant C\|f\|_2\|g\|_2$ for all $f, g \in L^2(G)$.*

Proof. A similar argument as in Proposition 1.10 proves that W_ς is bounded for all $\varsigma \in L^p(G \times \widehat{G}, S_p, \mathrm{d}\nu \otimes \mathrm{d}\mu)$, whenever there exists a constant C such that $\|V(f,g)\|_{p',\nu\otimes\mu} \leqslant C\|f\|_2\|g\|_2$ for all $f, g \in L^2(G)$.

Conversely, assume that W_ς is bounded for each $\varsigma \in L^p(G \times \widehat{G}, S_p, \mathrm{d}\nu \otimes \mathrm{d}\mu)$. Then there exists a constant C_ς such that $\|W_\varsigma f\|_2 \leqslant C_\varsigma \|f\|_2$ for all $f \in L^2(G)$. For $f, g \in C_c(G)$ with $\|f\|_2 = \|g\|_2 = 1$, define the bounded linear functional on $L^p(G \times \widehat{G}, S_p, \mathrm{d}\nu \otimes \mathrm{d}\mu)$ by $\mathcal{Q}_{f,g}(\varsigma) = \langle W_\varsigma f, g\rangle$. Then $\sup |\mathcal{Q}_{f,g}(\varsigma)| \leqslant C_\varsigma$, where the supremum is over all $f, g \in C_c(G)$ with $\|f\|_2 = \|g\|_2 = 1$. Therefore, by the uniform boundedness principle, there exists a constant C such that $\|\mathcal{Q}_{f,g}\| \leqslant C$ for all $f, g \in C_c(G)$ with $\|f\|_2 = \|g\|_2 = 1$. Hence $|\langle W_\varsigma f, g\rangle| \leqslant C\|\varsigma\|_{p,\nu\otimes\mu}\|f\|_2\|g\|_2$, that is, $\|W_\varsigma\| \leqslant C\|\varsigma\|_{p,\nu\otimes\mu}$. Thus for $f, g \in C_c(G)$,

$$\|V(f,g)\|_{p',\nu\otimes\mu} = \sup_{\|\varsigma\|_{p,\nu\otimes\mu}=1} |\langle V(f,g), \varsigma\rangle_{\nu\otimes\mu}| = \sup_{\|\varsigma\|_{p,\nu\otimes\mu}=1} |\langle W_\varsigma f, \overline{g}\rangle| \leqslant C\|f\|_2\|g\|_2.$$

The density argument completes the proof. ∎

A sufficient condition for unboundedness of the Weyl transform is obtained in the following result.

PROPOSITION 1.16. *Let $2 < p < \infty$ and f be a square integrable, compactly supported function on G with $\int_G f(x)\mathrm{d}\nu(x) \neq 0$. If W_ς is a bounded operator on $L^2(G)$ for all $\varsigma \in L^p(G \times \widehat{G}, S_p, \mathrm{d}\nu \otimes \mathrm{d}\mu)$, then $\int_{\widehat{G}} \|\mathcal{F}_p(f)\pi\|_{S_{p'}}^{p'} \mathrm{d}\mu(\pi) < \infty$.*

Proof. Let f be supported on a compact set K. Then $\widetilde{K} = KK = \{xy : x, y \in K\}$ is compact and $V(f,f)$ is supported on $\widetilde{K} \times \widehat{G}$. In view of Proposition 1.7, instead of $\int_{\widehat{G}} \|\mathcal{F}_p(f)\pi\|_{S_{p'}}^{p'} \mathrm{d}\mu(\pi)$, it is enough to consider the following integral

$$\int_{\widehat{G}} \Big\| \int_G V(f,f)(x,\pi) K_\pi^{1/p'} \mathrm{d}\nu(x) \Big\|_{S_{p'}}^{p'} \mathrm{d}\mu(\pi).$$

Applying Minkowski's integral inequality and Hölder's inequality, the above integral is less than or equal to

$$\Big(\int_{\widetilde{K}}\Big(\int_{\widehat{G}}\|V(f,f)(x,\pi)K_\pi^{1/p'}\|_{S_{p'}}^{p'}\mathrm{d}\mu(\pi)\Big)^{1/p'}\mathrm{d}\nu(x)\Big)^{p'}$$

$$\leqslant\Big(\int_{\widetilde{K}}\mathrm{d}\nu(x)\Big)^{p'/p}\int_{\widetilde{K}}\int_{\widehat{G}}\|V(f,f)(x,\pi)K_\pi^{1/p'}\|_{S_{p'}}^{p'}\mathrm{d}\mu(\pi)\mathrm{d}\nu(x).$$

Hence by Proposition 1.15, it completes the proof. ∎

To prove that the Weyl transform W_ς cannot be extended as a bounded operator for $\varsigma \in L^p(G\times\widehat{G}, S_p, \mathrm{d}\nu\otimes\mathrm{d}\mu)$, where $2<p<\infty$, it is enough to consider the following problem.

PROBLEM 1.17. For $p\in(2,\infty)$, does there exist a square integrable, compactly supported function f on G with $\int_G f(x)\mathrm{d}\nu(x)\neq 0$, such that

$$\int_{\widehat{G}}\|\mathcal{F}_p(f)\pi\|_{S_{p'}}^{p'}\,\mathrm{d}\mu(\pi)$$

is infinite?

In [15], Simon gave an example of such functions for $\mathbb{R}^n$, and later on, it is considered for the Heisenberg group [11], quaternion Heisenberg group [3] and upper half plane [12]. In this article, we define such functions for the Euclidean motion group and Heisenberg motion group. However, for general second countable type I locally compact group, this is still open.

PROBLEM 1.18. Proposition 1.11 can be thought of as the Plancherel formula for the Weyl transform. It is interesting to ask whether we can have an inversion formula, that is, can we retrieve ς from W_ς. This may lead to many exciting properties of the Weyl transform.

2. EUCLIDEAN MOTION GROUP

In this section, we briefly discuss the Fourier analysis on the Euclidean motion group. Thereafter, we shall precisely write down the formula for the Weyl transform in this setup and prove that it cannot be extended as a bounded operator for the symbol in the corresponding L^p-spaces with $2<p<\infty$.

Let $n\in\mathbb{N}\setminus\{1\}$ and $SO(n)$ be the special orthogonal group of order n. Then Euclidean motion group $M(n)$ is the semidirect product of $\mathbb{R}^n$ with $K=SO(n)$. The group law on $M(n)$ can be expressed as

$$(x_1,k_1)(x_2,k_2)=(x_1+k_1\cdot x_2,k_1k_2),$$

where $x_1, x_2 \in \mathbb{R}^n$ and $k_1, k_2 \in K$. The Haar measure on $M(n)$ can be written as $\mathrm{d}\nu(x,k) = \mathrm{d}x\mathrm{d}k$, where $\mathrm{d}x$ and $\mathrm{d}k$ are the Haar measures on $\mathbb{R}^n$ and K, respectively. The Fourier analysis on $M(n)$ can be discussed in the following two cases. For details, see [7, 13, 16].

Case 1. $n = 2$. An arbitrary element of $M(2)$ can be written as $(z, \mathrm{e}^{\mathrm{i}\varphi})$, where $z \in \mathbb{C}, \varphi \in \mathbb{R}$. Up to unitary equivalence, all the infinite dimensional irreducible unitary representations of $M(2)$ are parametrized by $a > 0$. Moreover, for each $a > 0$, the representation π_a, realized on $L^2(S^1)$, is given by

$$\pi_a(z, \mathrm{e}^{\mathrm{i}\varphi})g(\theta) = \mathrm{e}^{\mathrm{i}\mathrm{Re}(a\mathrm{e}^{\mathrm{i}\theta}z)}g(\theta - \varphi),$$

where $g \in L^2(S^1)$. The Plancherel measure on $(0, \infty)$ is $\mathrm{d}\mu(a) = a\mathrm{d}a$, where $\mathrm{d}a$ is the Lebesgue measure on $(0, \infty)$. Further, for $f \in L^1(M(2))$ the group Fourier transform, defined by

$$\pi_a(f) = \widehat{f}(a) = \int\limits_{M(2)} f(z, \mathrm{e}^{\mathrm{i}\varphi})\pi_a(z, \mathrm{e}^{\mathrm{i}\varphi})\mathrm{d}z\mathrm{d}\varphi,$$

is a bounded operator on $L^2(S^1)$. For $\varsigma \in C_{\mathrm{c}}(M(2) \times (0, \infty), S_p, \mathrm{d}\nu \otimes \mathrm{d}\mu)$ and $f \in L^2(M(2))$, the Weyl transform W_ς takes the form

$$W_\varsigma f(z, \mathrm{e}^{\mathrm{i}\varphi}) = \int\limits_0^\infty \int\limits_{M(2)} \mathrm{tr}(\varsigma^*((w, \mathrm{e}^{\mathrm{i}\psi})(z, \mathrm{e}^{\mathrm{i}\varphi}))\pi_a(w, \mathrm{e}^{\mathrm{i}\psi}))f(w, \mathrm{e}^{\mathrm{i}\psi})a\mathrm{d}w\mathrm{d}\psi\mathrm{d}a.$$

Case 2. $n \geqslant 3$. Let $M = SO(n-1)$ be the subgroup of K that fixes the point $e_1 = (1, 0, \ldots, 0)$. Let $\widehat{M}$ be the unitary dual group of M. Given a unitary irreducible representation $\sigma \in \widehat{M}$, realized on the Hilbert space $\mathcal{H}_\sigma$ of dimension d_σ, consider the space $L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$ consisting of $d_\sigma \times d_\sigma$ complex matrices valued functions φ on K such that $\varphi(uk) = \sigma(u)\varphi(k)$, where $u \in M$, $k \in K$, and satisfying

$$\int\limits_K \mathrm{tr}(\varphi(k)^*\varphi(k))\mathrm{d}k < \infty.$$

For $(a, \sigma) \in (0, \infty) \times \widehat{M}$, a unitary representation $\pi_{a,\sigma}$ of $M(n)$ is defined by

$$\pi_{a,\sigma}(x, k)(\varphi)(s) = \mathrm{e}^{\mathrm{i}a\langle s^{-1}\cdot e_1, x\rangle}\varphi(sk),$$

where $\varphi \in L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$. Up to unitary equivalence, these are all the infinite dimensional, unitary, irreducible representations that appear in the Plancherel formula. The group Fourier transform of a function $f \in L^1(M(n))$ is defined by

$$\widehat{f}(a, \sigma) = \int\limits_{M(n)} f(x, k)\pi_{a,\sigma}(x, k)\mathrm{d}x\mathrm{d}k.$$

Further, for $f \in L^1 \cap L^2(M(n))$, Plancherel formula holds

$$\int\limits_{M(n)} |f(x,k)|^2 \mathrm{d}x\mathrm{d}k = \mathrm{c}_n \int\limits_0^\infty \Big(\sum_{\sigma\in\widehat{M}} d_\sigma \|\widehat{f}(a,\sigma)\|_{HS}^2 \Big) a^{n-1}\mathrm{d}a,$$

for some constant c_n.

Next, we construct the required square integrable and compactly supported functions for $M(n)$. To do so, we need to discuss some properties of Bessel functions.

For $v \in \mathbb{R}$, first kind Bessel function J_v of order v is defined by

$$J_v(t) = \sum_{l=0}^\infty \frac{(-1)^l (t/2)^{v+2l}}{l!\,\Gamma(l+v+1)}, \tag{2.1}$$

where t is a non-negative real number. Then for $n \geqslant 2$, $a \in (0,\infty)$ and $x \in \mathbb{R}^n$, the following relation hold:

$$\int\limits_{S^{n-1}} \mathrm{e}^{\mathrm{i}ax\cdot\omega}\mathrm{d}\omega = c_n\,(a|x|)^{1-(n/2)} J_{(n/2)-1}(a|x|), \tag{2.2}$$

where $c_n > 0$ is a constant depending only on n and $\mathrm{d}\omega$ is the surface measure on S^{n-1}.

If ξ, ζ and η are complex parameters with $\eta \neq 0, -1, \dots$, then the complex power series

$$\sum_{l=0}^\infty \frac{(\xi)_l(\zeta)_l}{(\eta)_l}\,\frac{z^l}{l\,!}, \tag{2.3}$$

converges for $|z| < 1$, where $(\xi)_0 = 1$ and $(\xi)_l = \xi(\xi+1)\cdots(\xi+l-1)$ for $l \geqslant 1$. The sum of the above series is denoted by ${}_2F_1(\xi,\zeta;\eta\,;z)$, and is a hypergeometric function. Moreover, if $\mathrm{Re}(\eta-\xi-\zeta) > 0$, then the series (2.3) also converges for $|z| = 1$ and

$${}_2F_1(\xi,\zeta;\eta\,;1) = \frac{\Gamma(\eta)\Gamma(\eta-\xi-\zeta)}{\Gamma(\eta-\xi)\Gamma(\eta-\zeta)}. \tag{2.4}$$

For more details, see [1], pp. 62–66.

The following property of Bessel functions hold, see [17], p. 385.

LEMMA 2.1 ([17]). *Let $u, \alpha, v \in \mathbb{R}$ be such that $u, \alpha > 0$ and $v \geqslant 0$. Then*

$$\int\limits_0^\infty \mathrm{e}^{-ut} t^{\alpha-1} J_v(t)\mathrm{d}t = \frac{(1/2)^v \Gamma(\alpha+v)}{(1+u^2)^{(\alpha+v)/2}\Gamma(v+1)}\, {}_2F_1\Big(\frac{\alpha+v}{2}, \frac{1-\alpha+v}{2}; 1+v\,; \frac{1}{1+u^2}\Big),$$

where ${}_2F_1$ is a hypergeometric function as defined in (2.3)*.*

Next lemma is needful to prove the main result in this section.

LEMMA 2.2. *Let $n \in \mathbb{N} \setminus \{1\}$ and $\beta \in (\frac{n}{3}, \frac{n}{2})$. Then there exist $C > 0$ and $a_0 > 0$, both independent of β, such that*

$$\int_0^a t^{(n/2)-\beta} J_{(n/2)-1}(t) \mathrm{d}t \geqslant C$$

for all $a \geqslant a_0$.

Proof. Let $\frac{n}{3} < \beta < \frac{n}{2}$ and $\alpha = \frac{n}{2} - \beta + 1$, then $1 < \alpha < \frac{n}{6} + 1$. Again for $0 < u < 1$, we have $\frac{1}{2} < \frac{1}{1+u^2} < 1$. Now, we shall find an independent of u positive lower bound of the integral defined in Lemma 2.1, when $\alpha = \frac{n}{2} - \beta + 1$ and $v = \frac{n}{2} - 1$. For that, consider the following two cases.

Case 1. $n = 2$. Since $1 < \alpha < \frac{4}{3}$, except for the first term, all other terms in the series expansion of ${}_2F_1\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}; 1; \frac{1}{1+u^2}\right)$ are negative. Further, as $1 - \frac{\alpha}{2} - \frac{1-\alpha}{2} > 0$, we get

$$\begin{aligned} {}_2F_1\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}; 1; \frac{1}{1+u^2}\right) &= 1 + \sum_{l=1}^{\infty} \frac{(\alpha/2)_l ((1-\alpha)/2)_l}{(1)_l \, l!} \frac{1}{(1+u^2)^l} \\ &\geqslant 1 + \sum_{l=1}^{\infty} \frac{(\alpha/2)_l ((1-\alpha)/2)_l}{(1)_l \, l!}. \end{aligned} \tag{2.5}$$

From Lemma 2.1, using (2.4)–(2.5), we have

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-ut} t^{1-\beta} J_0(t) \mathrm{d}t = \int_0^\infty \mathrm{e}^{-ut} t^{\alpha-1} J_0(t) \mathrm{d}t &= \frac{\Gamma(\alpha)}{(1+u^2)^{\alpha/2}} \, {}_2F_1\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}; 1; \frac{1}{1+u^2}\right) \\ &\geqslant \frac{\Gamma(\alpha)}{2} \frac{\Gamma(1/2)}{\Gamma(1-(\alpha/2))\Gamma((\alpha+1)/2)} > 0, \end{aligned} \tag{2.6}$$

where the last inequality is true as $1 \leqslant (1+u^2)^{\alpha/2} \leqslant 2^{\alpha/2} \leqslant 2$ for $1 < \alpha < \frac{4}{3}$.

Cases 2. $n \geqslant 3$. Since $\frac{n}{3} < \beta < \frac{n}{2}$, all the terms in the series expansion of the function ${}_2F_1\left(\frac{n-\beta}{2}, \frac{\beta-1}{2}; \frac{n}{2}; \frac{1}{1+u^2}\right)$ are positive. Hence ${}_2F_1\left(\frac{n-\beta}{2}, \frac{\beta-1}{2}; \frac{n}{2}; \frac{1}{1+u^2}\right) \geqslant 1$. Thus from Lemma 2.1, we have

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-ut} t^{(n/2)-\beta} J_{(n/2)-1}(t) \mathrm{d}t &= \frac{(1/2)^{(n/2)-1} \Gamma(n-\beta)}{(1+u^2)^{(n-\beta)/2} \Gamma(n/2)} \, {}_2F_1\left(\frac{n-\beta}{2}, \frac{\beta-1}{2}; \frac{n}{2}; \frac{1}{1+u^2}\right) \\ &\geqslant \frac{(1/2)^{(n/2)-1} \Gamma(n-\beta)}{2^{(n-\beta)/2} \Gamma(n/2)} > 0. \end{aligned} \tag{2.7}$$

Consider arbitrary $n \geqslant 2$. Since for $x > 0$, $\Gamma(x)$ is continuous, the lower bounds in (2.6) and (2.7) are independent of α, β. Hence for $0 < u < 1$ and $\frac{n}{3} <$

$\beta < \frac{n}{2}$, there exist $a_0, C > 0$, independent of u, β, such that for all $a \geqslant a_0$,

$$\int_0^a \mathrm{e}^{-ut} t^{(n/2)-\beta} J_{(n/2)-1}(t)\mathrm{d}t \geqslant C.$$

Therefore, $\lim_{u \to 0^+} \int_0^a \mathrm{e}^{-ut} t^{(n/2)-\beta} J_{(n/2)-1}(t)\mathrm{d}t \geqslant C$ for all $a \geqslant a_0$. Thus

$$\int_0^a t^{(n/2)-\beta} J_{(n/2)-1}(t)\mathrm{d}t \geqslant C$$

for all $a \geqslant a_0$ and $\frac{n}{3} < \beta < \frac{n}{2}$. ∎

Now, we prove the main results of this section.

THEOREM 2.3. *Consider the square integrable and compactly supported function* $f_\beta(z, \mathrm{e}^{\mathrm{i}\varphi}) = \chi_{B_1(0)}(z)\frac{1}{|z|^\beta}$, *where* $\beta < 1$, *on* $M(2)$. *Then for* $1 < q < 2$, *there exist* $\beta \in (\frac{2}{3}, 1)$ *such that* $\int_0^\infty \|\widehat{f}_\beta(a)\|_{S_q}^q a\mathrm{d}a = \infty$.

Proof. Let $e_0(\theta) = 1$, then $e_0 \in L^2(S^1)$. For $a > 0$,

$$\begin{aligned}\langle \widehat{f}_\beta(a)e_0, e_0\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}_\beta(a)e_0(\theta)\overline{e_0(\theta)}\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{\mathbb{C}} \chi_{B_1(0)}(z)\frac{1}{|z|^\beta}\mathrm{e}^{\mathrm{i}\mathrm{Re}(a\mathrm{e}^{\mathrm{i}\theta}z)}e_0(\theta-\varphi)\mathrm{d}z\mathrm{d}\varphi\mathrm{d}\theta \\ &= \int_{-\pi}^{\pi}\int_0^1 \frac{1}{r^\beta}\Big(\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}ar\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\psi})}\mathrm{d}\theta\Big) r\mathrm{d}r\mathrm{d}\psi.\end{aligned}$$

In view of (2.2), we can write

$$\langle \widehat{f}_\beta(a)e_0, e_0\rangle = 2\pi c_2 \int_0^1 \frac{1}{r^{\beta-1}} J_0(ar)\mathrm{d}r = 2\pi c_2\, a^{\beta-2}\int_0^a \frac{1}{r^{\beta-1}} J_0(r)\mathrm{d}r. \tag{2.8}$$

Hence by Lemma 2.2, if $\beta \in (\frac{2}{3}, 1)$, then there exists $C > 0$ and $a_0 > 0$ such that $\langle \widehat{f}_\beta(a)e_0, e_0\rangle \geqslant 2\pi c_2\, C a^{\beta-2}$ for all $a \geqslant a_0$. Thus for $1 < q < 2$,

$$\begin{aligned}\int_0^\infty \|\widehat{f}_\beta(a)\|_{S_q}^q a\mathrm{d}a &\geqslant \int_0^\infty \|\widehat{f}_\beta(a)\|_{S_\infty}^q a\mathrm{d}a \geqslant \int_0^\infty |\langle \widehat{f}_\beta(a)e_0, e_0\rangle|^q a\mathrm{d}a \\ &\geqslant \int_{a_0}^\infty |\langle \widehat{f}_\beta(a)e_0, e_0\rangle|^q a\mathrm{d}a \geqslant \widetilde{C}\int_{a_0}^\infty a^{(\beta-2)q} a\mathrm{d}a.\end{aligned}$$

The integral $\int_{a_0}^{\infty} a^{(\beta-2)q+1}\,\mathrm{d}a$ is finite if and only if $\beta < 2-\frac{2}{q}$. Since $q<2$, we have $2-\frac{2}{q}<1$. If we choose $\beta \in \left(\max\{\frac{2}{3}, 2-\frac{2}{q}\}, 1\right)$, the corresponding function f_β will be the required function. ∎

THEOREM 2.4. *Let $n \geqslant 3$. Consider the square integrable and compactly supported function $f_\beta(x,k) = \chi_{B_1(0)}(x)\frac{1}{|x|^\beta}$, where $\beta < \frac{n}{2}$, on $M(n)$. Then for $1<q<2$, there exists $\beta \in (\frac{n}{3}, \frac{n}{2})$ such that $\int_0^\infty \sum_{\sigma\in\widehat{M}} d_\sigma \|\widehat{f}_\beta(a,\sigma)\|_{S_q}^q a^{n-1}\mathrm{d}a = \infty$.*

Proof. Let $\sigma_o \in \widehat{M}$, be the trivial representation. Consider the function φ_0 on K defined by $\varphi_0(k) = 1$ for all $k \in K$. Since $d_{\sigma_o} = 1$, we have $\varphi_0 \in L^2(K, \mathbb{C}^{d_{\sigma_o}\times d_{\sigma_o}})$. Now for $a>0$,

$$\begin{aligned}
\langle \widehat{f}_\beta(a,\sigma_o)\varphi_0, \varphi_0\rangle &= \int_K\int_K\int_{\mathbb{R}^n} f_\beta(x,k)\mathrm{e}^{\mathrm{i}a\langle s^{-1}\cdot e_1, x\rangle}\varphi_0(sk)\overline{\varphi_0(s)}\,\mathrm{d}x\mathrm{d}k\mathrm{d}s \\
&= \int_K\int_{\mathbb{R}^n} \chi_{B_1(0)}(x)\frac{1}{|x|^\beta}\mathrm{e}^{\mathrm{i}a\langle s^{-1}\cdot e_1, x\rangle}\,\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Again $S^{n-1} = \{s^{-1}\cdot e_1 : s \in K\}$. Therefore, using (2.2) we get

$$\begin{aligned}
\langle \widehat{f}_\beta(a,\sigma_o)\varphi_0, \varphi_0\rangle &= \int_{S^{n-1}}\int_{\mathbb{R}^n} \chi_{B_1(0)}(x)\frac{1}{|x|^\beta}\mathrm{e}^{\mathrm{i}a\langle \omega, x\rangle}\,\mathrm{d}x\mathrm{d}\omega \\
&= c_n\int_{\mathbb{R}^n} \chi_{B_1(0)}(x)\frac{1}{|x|^\beta}\,(a|x|)^{1-(n/2)}J_{(n/2)-1}(a|x|)\mathrm{d}x \\
&= c_n\int_0^1\int_{S^{n-1}} \frac{1}{r^\beta}\,(ar)^{1-(n/2)}J_{(n/2)-1}(ar)r^{n-1}\mathrm{d}r\mathrm{d}\omega \\
&= c_n\,a^{\beta-n}\int_0^a r^{(n/2)-\beta}J_{(n/2)-1}(r)\mathrm{d}r.
\end{aligned}$$

Hence by Lemma 2.2, if $\beta \in (\frac{n}{3}, \frac{n}{2})$, then there exists $C>0$ and $a_0>0$ such that $\langle \widehat{f}_\beta(a,\sigma_o)\varphi_0, \varphi_0\rangle \geqslant c_n\,Ca^{\beta-n}$ for all $a \geqslant a_0$. Thus for $1<q<2$,

$$\begin{aligned}
\int_0^\infty \sum_{\sigma\in\widehat{M}} d_\sigma\|\widehat{f}(a,\sigma)\|_{S_q}^q a^{n-1}\mathrm{d}a &\geqslant \int_0^\infty \|\widehat{f}(a,\sigma_o)\|_{S_q}^q a^{n-1}\mathrm{d}a \geqslant \int_0^\infty |\langle \widehat{f}_\beta(a,\sigma_o)\varphi_0,\varphi_0\rangle|^q a^{n-1}\mathrm{d}a \\
&\geqslant \int_{a_0}^\infty |\langle \widehat{f}_\beta(a,\sigma_o)\varphi_0,\varphi_0\rangle|^q a^{n-1}\mathrm{d}a \geqslant \widetilde{C}\int_{a_0}^\infty a^{(\beta-n)q+n-1}\mathrm{d}a.
\end{aligned}$$

The integral $\int_{a_0}^{\infty} a^{(\beta-n)q+n-1}\mathrm{d}a$ is infinite if $\beta \geqslant n - \frac{n}{q}$. Since $q < 2$, we have $n - \frac{n}{q} < \frac{n}{2}$. If we choose $\beta \in (\max\{\frac{n}{3}, n - \frac{n}{q}\}, \frac{n}{2})$, the corresponding function f_β will be the required function. ∎

3. HEISENBERG MOTION GROUP

The Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is a step two nilpotent Lie group having center $\mathbb{R}$ that is equipped with the group law

$$(z,t)\cdot(w,s) = \Big(z + w, t + s + \frac{1}{2}\mathrm{Im}(z \cdot \overline{w})\Big).$$

By the Stone–von Neumann theorem, the infinite dimensional irreducible unitary representations of $\mathbb{H}^n$ can be parameterized by $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. That is, each $\lambda \in \mathbb{R}^*$ defines a Schrödinger representation π_λ of $\mathbb{H}^n$ via

$$\pi_\lambda(z,t)\varphi(\xi) = \mathrm{e}^{\mathrm{i}\lambda t}\mathrm{e}^{\mathrm{i}\lambda(x\cdot\xi + (1/2)x\cdot y)}\varphi(\xi + y),$$

where $z = x + \mathrm{i}y$ and $\varphi \in L^2(\mathbb{R}^n)$.

Having chosen the sublaplacian $\mathcal{L}$ of the Heisenberg group $\mathbb{H}^n$ and its geometry, there is a larger group of isometries that commute with $\mathcal{L}$, known as Heisenberg motion group. The Heisenberg motion group G is the semidirect product of $\mathbb{H}^n$ with the unitary group $K = U(n)$. Since K defines a group of automorphisms on $\mathbb{H}^n$, via $k \cdot (z,t) = (kz, t)$, the group law on G can be expressed as

$$(z,t,k_1)\cdot(w,s,k_2) = \Big(z + k_1 w, t + s - \frac{1}{2}\mathrm{Im}(k_1 w \cdot \overline{z}), k_1 k_2\Big).$$

Since a right K-invariant function on G can be thought as a function on $\mathbb{H}^n$, the Haar measure on G is given by $\mathrm{d}g = \mathrm{d}z\mathrm{d}t\mathrm{d}k$, where $\mathrm{d}z\mathrm{d}t$ and $\mathrm{d}k$ are the normalized Haar measure on $\mathbb{H}^n$ and K, respectively.

For $k \in K$ define another set of representations of the Heisenberg group $\mathbb{H}^n$ by $\pi_{\lambda,k}(z,t) = \pi_\lambda(kz,t)$. Since $\pi_{\lambda,k}$ agrees with π_λ on the center of $\mathbb{H}^n$, it follows by the Stone–von Neumann theorem for the Schrödinger representation that $\pi_{\lambda,k}$ is equivalent to π_λ. Hence there exists an intertwining operator $\mu_\lambda(k)$ satisfying

$$\pi_\lambda(kz,t) = \mu_\lambda(k)\pi_\lambda(z,t)\mu_\lambda(k)^*.$$

By an appropriate selection of μ_λ, it becomes a unitary representation of K on $L^2(\mathbb{R}^n)$, called metaplectic representation. For details, we refer to [5], Chapter 4. Let $(\sigma, \mathcal{H}_\sigma)$ be an irreducible unitary representation of K and $\mathcal{H}_\sigma = \mathrm{span}\{e_j^\sigma : 1 \leqslant j \leqslant d_\sigma\}$. For $k \in K$, the matrix coefficients of the representation $\sigma \in \widehat{K}$ are given by

$$\varphi_{ij}^\sigma(k) = \langle \sigma(k)e_i^\sigma, e_j^\sigma \rangle,$$

where $i,j = 1,\ldots,d_\sigma$.

Let $\phi_\alpha^\lambda(x) = |\lambda|^{n/4}\phi_\alpha(\sqrt{|\lambda|}x);\ \alpha \in \mathbb{Z}_+^n$, where ϕ_α's are the Hermite functions on $\mathbb{R}^n$. Then for each $\lambda \in \mathbb{R}^*$, the set $\{\phi_\alpha^\lambda : \alpha \in \mathbb{Z}_+^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$. By letting $P_m^\lambda = \text{span}\{\phi_\alpha^\lambda : |\alpha| = m\}$, μ_λ becomes an irreducible unitary representation of K on P_m^λ. Hence, the action of μ_λ can be realized on P_m^λ by

$$\mu_\lambda(k)\phi_\gamma^\lambda = \sum_{|\alpha|=|\gamma|} \eta_{\gamma\alpha}^\lambda(k)\phi_\alpha^\lambda, \tag{3.1}$$

where $\eta_{\gamma\alpha}^\lambda$'s are the matrix coefficients of $\mu_\lambda(k)$. Define a bilinear form $\phi_\alpha^\lambda \otimes e_j^\sigma$ on $L^2(\mathbb{R}^n) \times \mathcal{H}_\sigma$ by $\phi_\alpha^\lambda \otimes e_j^\sigma = \phi_\alpha^\lambda e_j^\sigma$. Then $\{\phi_\alpha^\lambda \otimes e_j^\sigma : \alpha \in \mathbb{Z}_+^n, 1 \leqslant j \leqslant d_\sigma\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$. Denote $\mathcal{H}_\sigma^2 = L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma$.

Define a representation ρ_σ^λ of G on the space $\mathcal{H}_\sigma^2$ by

$$\rho_\sigma^\lambda(z,t,k) = \pi_\lambda(z,t)\mu_\lambda(k) \otimes \sigma(k).$$

In the article [14], it is shown that ρ_σ^λ are all possible irreducible unitary representations of G that participate in the Plancherel formula. Thus, in view of the above discussion, we shall denote the partial dual of the group G by $G' \cong \mathbb{R}^* \times \widehat{K}$. The Fourier transform of $f \in L^1(G)$ defined by

$$\widehat{f}(\lambda,\sigma) = \int_K \int_{\mathbb{R}} \int_{\mathbb{C}^n} f(z,t,k)\rho_\sigma^\lambda(z,t,k)\mathrm{d}z\mathrm{d}t\mathrm{d}k,$$

is a bounded linear operator on $\mathcal{H}_\sigma^2$. As the Plancherel formula

$$\int_K \int_{\mathbb{H}^n} |f(z,t,k)|^2 \mathrm{d}z\mathrm{d}t\mathrm{d}k = (2\pi)^{-n} \int_{\mathbb{R}\setminus\{0\}} \sum_{\sigma\in\widehat{K}} d_\sigma \|\widehat{f}(\lambda,\sigma)\|_{S_2}^2 |\lambda|^n \mathrm{d}\lambda$$

holds for $f \in L^2(G)$, it follows that $\widehat{f}(\lambda,\sigma)$ is a Hilbert–Schmidt operator on $\mathcal{H}_\sigma^2$. For detailed Fourier analysis on the Heisenberg motion group, see [2, 6, 14].

First, we recall the example of the previously mentioned required function for the Heisenberg group, which we extend to the Heisenberg motion group.

Let $A = \{(x,y,t) : |x_l| \leqslant 1, |y_l| \leqslant 1, |t| \leqslant 1; l = 1,\ldots,n\}$ be a compact subset of $\mathbb{R}^{2n} \times \mathbb{R}$ and

$$g_\xi(z,t) = g_\xi(x,y,t) = |t|^\xi \prod_{j=1}^n |x_j|^\xi \prod_{j=1}^n |y_j|^\xi \chi_A(z,t), \tag{3.2}$$

where $\xi > -\frac{1}{2}$. Then the following result holds.

THEOREM 3.1 ([11]). *For $1 < q < 2$, there exists $\xi > -\frac{1}{2}$, such that*

$$\int_{\mathbb{R}\setminus\{0\}} |\langle \widehat{g_\xi}(\lambda)\phi_0^\lambda, \phi_0^\lambda\rangle|^q |\lambda|^n \mathrm{d}\lambda$$

is infinite.

The following proposition gives the required function for the Heisenberg motion group.

THEOREM 3.2. *Consider the function* $f_\xi(z,t,k) = g_\xi(z,t)$ *on* G, *where* g_ξ *is defined in* (3.2). *Then* f_ξ *is square integrable and compactly supported. Further for* $1 < q < 2$, *there exists* $\xi > -\frac{1}{2}$ *such that* $\int\limits_{\mathbb{R}\setminus\{0\}} \sum\limits_{\sigma\in\widehat{K}} d_\sigma \|\widehat{f_\xi}(\lambda,\sigma)\|_{S_q}^q |\lambda|^n \mathrm{d}\lambda$ *is infinite.*

Proof. Let $1 < q < 2$ and $\lambda \in \mathbb{R}\setminus\{0\}$. Then

$$\begin{aligned}\sum_{\sigma\in\widehat{K}} d_\sigma \|\widehat{f_\xi}(\lambda,\sigma)\|_{S_q}^q &\geqslant \sum_{\sigma\in\widehat{K}} (d_\sigma^{1/q}\|\widehat{f_\xi}(\lambda,\sigma)\|_{S_2})^q \geqslant \Big(\sum_{\sigma\in\widehat{K}} (d_\sigma^{1/q}\|\widehat{f_\xi}(\lambda,\sigma)\|_{S_2})^2\Big)^{q/2} \\ &\geqslant \Big(\sum_{\sigma\in\widehat{K}} d_\sigma^{2/q}\|\widehat{f_\xi}(\lambda,\sigma)\|_{S_2}^2\Big)^{q/2} \geqslant \Big(\sum_{\sigma\in\widehat{K}} d_\sigma\|\widehat{f_\xi}(\lambda,\sigma)\|_{S_2}^2\Big)^{q/2}. \end{aligned} \tag{3.3}$$

It is already discussed that $\{\phi_\alpha^\lambda \otimes e_j^\sigma : \alpha \in \mathbb{Z}_+^n, 1 \leqslant j \leqslant d_\sigma\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)\otimes\mathcal{H}_\sigma$. Thus by (3.1),

$$\langle \widehat{f_\xi}(\lambda,\sigma)(\phi_\alpha^\lambda\otimes e_j^\sigma),(\phi_\beta^\lambda\otimes e_l^\sigma)\rangle = \int\limits_{\mathbb{H}^n}\int\limits_K f_\xi(z,t,k)\sum_{|\gamma|=|\alpha|}\eta_{\alpha\gamma}(k)\Phi_{\gamma\beta}^\lambda(z,t)\phi_{jl}^\sigma(k)\mathrm{d}z\mathrm{d}t\mathrm{d}k,$$

where $\Phi_{\gamma\beta}^\lambda(z,t) = \langle \pi_\lambda(z,t)\phi_\gamma^\lambda,\phi_\beta^\lambda\rangle$. Therefore, $\|\widehat{f_\xi}(\lambda,\sigma)(\phi_\alpha^\lambda\otimes e_j^\sigma)\|_2^2$ is equal to

$$\begin{aligned}&\sum_{\beta\in\mathbb{Z}_+^n}\sum_{1\leqslant l\leqslant d_\sigma}\Big|\int\limits_{\mathbb{H}^n}\int\limits_K f_\xi(z,t,k)\sum_{|\gamma|=|\alpha|}\eta_{\alpha\gamma}(k)\Phi_{\gamma\beta}^\lambda(z,t)\phi_{jl}^\sigma(k)\mathrm{d}z\mathrm{d}t\mathrm{d}k\Big|^2 \\ &\quad = \sum_{\beta\in\mathbb{Z}_+^n}\sum_{1\leqslant l\leqslant d_\sigma}\Big|\sum_{|\gamma|=|\alpha|}\langle\widehat{g_\xi}(\lambda)\phi_\gamma^\lambda,\phi_\beta^\lambda\rangle\int\limits_K\eta_{\alpha\gamma}(k)\phi_{jl}^\sigma(k)\mathrm{d}k\Big|^2.\end{aligned}$$

That is,

$$\|\widehat{f_\xi}(\lambda,\sigma)\|_{S_2}^2 = \sum_{\alpha,\beta\in\mathbb{Z}_+^n}\sum_{1\leqslant j,l\leqslant d_\sigma}\Big|\sum_{|\gamma|=|\alpha|}\langle\widehat{g_\xi}(\lambda)\phi_\gamma^\lambda,\phi_\beta^\lambda\rangle\int\limits_K\eta_{\alpha\gamma}(k)\phi_{jl}^\sigma(k)\mathrm{d}k\Big|^2.$$

Hence by the Peter–Weyl theorem (Plancherel) for compact groups, we get

$$\begin{aligned}\sum_{\sigma\in\widehat{K}} d_\sigma\|\widehat{f_\xi}(\lambda,\sigma)\|_{S_2}^2 &= \sum_{\alpha,\beta\in\mathbb{Z}_+^n}\int\limits_K\Big|\sum_{|\gamma|=|\alpha|}\langle\widehat{g_\xi}(\lambda)\phi_\gamma^\lambda,\phi_\beta^\lambda\rangle\eta_{\alpha\gamma}(k)\Big|^2\mathrm{d}k \\ &\geqslant \int\limits_K|\langle\widehat{g_\xi}(\lambda)\phi_0^\lambda,\phi_0^\lambda\rangle\eta_{00}(k)|^2\mathrm{d}k = |\langle\widehat{g_\xi}(\lambda)\phi_0^\lambda,\phi_0^\lambda\rangle|^2.\end{aligned}$$

Since $\mu_\lambda|_{P_0^\lambda}$ is irreducible, the last equality follows from Schur's orthogonality relation. Therefore, from (3.3), we can conclude that

$$\int\limits_{\mathbb{R}\setminus\{0\}} \sum_{\sigma\in\widehat{K}} d_\sigma \|\widehat{f_\xi}(\lambda,\sigma)\|_{S_q}^q |\lambda|^n \mathrm{d}\lambda \geqslant \int\limits_{\mathbb{R}\setminus\{0\}} \Big(\sum_{\sigma\in\widehat{K}} d_\sigma \|\widehat{f_\xi}(\lambda,\sigma)\|_{S_2}^2\Big)^{q/2} |\lambda|^n \mathrm{d}\lambda$$
$$\geqslant \int\limits_{\mathbb{R}\setminus\{0\}} |\langle \widehat{g_\xi}(\lambda)\phi_0^\lambda, \phi_0^\lambda\rangle|^q |\lambda|^n \mathrm{d}\lambda.$$

Thus Theorem 3.1 proves the result. ∎

REMARK 3.3. As compared to motion groups, the result in product space is straightforward. Let G be a second countable type I locally compact unimodular group and $(\pi, \mathcal{H}_\pi)$ be its representation. Consider $G_P = \mathbb{R}^n \times G$. For $f \in L^1(G_P)$, Fourier transform defined by $\widehat{f}(x,\pi) = \int\limits_{\mathbb{R}^n}\int\limits_{G} f(y,u)\mathrm{e}^{-2\pi\mathrm{i}x\cdot y}\pi(u)\mathrm{d}\nu(u)\mathrm{d}y$ is a bounded operator on $\mathcal{H}_\pi$. If we take $f(x,u) = f_1(x)f_2(u)$, then

$$\int\limits_{\mathbb{R}^n}\int\limits_{\widehat{G}} \|\widehat{f}(x,\pi)\|_{S_q}^q \mathrm{d}\mu(\pi)\mathrm{d}y = \int\limits_{\mathbb{R}^n} |\widehat{f_1}(y)|^q \mathrm{d}y \int\limits_{\widehat{G}} \|\widehat{f_2}(\pi)\|_{S_q}^q \mathrm{d}\mu(\pi). \tag{3.4}$$

It is known that, for $q \in (1,2)$, there exists a positive, square integrable and compactly supported function f_1 on $\mathbb{R}^n$ such that $\int\limits_{\mathbb{R}^n} |\widehat{f_1}(y)|^q \mathrm{d}y$ is infinite (see [15]). If we choose a square integrable and compactly supported function f_2 on G such that $\int\limits_{\widehat{G}} \|\widehat{f_2}(\pi)\|_{S_q}^q \mathrm{d}\mu(\pi) \neq 0$, then by (3.4), $\int\limits_{\mathbb{R}^n}\int\limits_{\widehat{G}} \|\widehat{f}(x,\pi)\|_{S_q}^q \mathrm{d}\mu(\pi)\mathrm{d}y$ is infinite. Hence in view of Proposition 1.16, the Weyl transform W_ζ on G_P is not bounded for $\zeta \in L^p(G_P \times \widehat{G}_P)$ with $2 < p < \infty$.

Acknowledgements. The authors would like to thank the referees for valuable suggestions. The first author gratefully acknowledges the support provided by IIT Guwahati, the Government of India.

REFERENCES

[1] G.E. ANDREWS, R. ASKEY, R. ROY, *Special Functions*, Encyclopedia Math. Appl., vol. 71, Cambridge Univ. Press, Cambridge 1999.

[2] C. BENSON, J. JENKINS, G. RATCLIFF, Bounded K-spherical functions on Heisenberg groups, *J. Funct. Anal.* **105**(1992), 409–443.

[3] L. CHEN, J. ZHAO, Weyl transform and generalized spectrogram associated with quaternion Heisenberg group, *Bull. Sci. Math.* **136**(2012), 127–143.

[4] M. DUFLO, C.C. MOORE, On the regular representation of a nonunimodular locally compact group, *J. Funct. Anal.* **21**(1976), 209–243.

[5] G.B. FOLLAND, *Harmonic Analysis in Phase Space,* Ann. Math. Stud., vol. 122, Princeton Univ. Press, Princeton, NJ 1989.

[6] S. GHOSH, R.K. SRIVASTAVA, Benedicks–Amrein–Berthier theorem for the Heisenberg motion group, *Bull. London Math. Soc.* **54**(2022), 526–539.

[7] K. KUMAHARA, K. OKAMOTO, An analogue of the Paley–Wiener theorem for the Euclidean motion group, *Osaka Math. J.* **10**(1973), 77–91.

[8] E.H. LIEB, Integral bounds for radar ambiguity functions and Wigner distributions, *J. Math. Phys.* **31**(1990), 594–599.

[9] M. MĂNTOIU, M. RUZHANSKY, Pseudo-differential operators, Wigner transform and Weyl systems on type I locally compact groups, *Doc. Math.* **22**(2017), 1539–1592.

[10] W. NASSERDDINE, L^p-Fourier inversion formula on certain locally compact groups, *C. R. Math. Acad. Sci. Paris* **357**(2019), 583–588.

[11] L. PENG, J. ZHAO, Weyl transforms associated with the Heisenberg group, *Bull. Sci. Math.* **132**(2008), 78–86.

[12] L. PENG, J. ZHAO, Weyl transforms on the upper half plane, *Rev. Mat. Comput.* **23**(2010), 77–95.

[13] R.P. SARKAR, S. THANGAVELU, On theorems of Beurling and Hardy for the Euclidean motion group, *Tohoku Math. J. (2)* **57**(2005), 335–351.

[14] S. SEN, Segal–Bargmann transform and Paley–Wiener theorems on Heisenberg motion groups, *Adv. Pure Appl. Math.* **7**(2016), 13–28.

[15] B. SIMON, The Weyl transform and L^p functions on phase space, *Proc. Amer. Math. Soc.* **116**(1992), 1045–1047.

[16] M. SUGIURA, *Unitary Representations and Harmonic Analysis,* North-Holland Math. Library, vol. 44, North-Holland Publ. Co., Amsterdam; Kodansha, Ltd., Tokyo 1990.

[17] G.N. WATSON, *A Treatise on the Theory of Bessel Functions,* second edition, Cambridge Univ. Press, Cambridge 1944.

[18] H. WEYL, *The Theory of Groups and Quantum Mechanics,* Dover Publ., Inc., New York 1950.

[19] M.W. WONG, *Weyl Transforms,* Universitext, Springer-Verlag, New York 1998.

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Received January 21, 2022; revised August 9, 2022.