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EXTENSIONS OF QUASIDIAGONAL C^*-ALGEBRAS AND CONTROLLING THE K_0-MAP OF EMBEDDINGS

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ABSTRACT. We study the validity of the Blackadar–Kirchberg conjecture for extensions of separable, nuclear, quasidiagonal C^*-algebras that satisfy the UCT. More specifically, we show that the conjecture for the extension has an affirmative answer if the ideal lies in a class of C^*-algebras that is closed under local approximations and contains all separable ASH-algebras, as well as certain classes of simple, unital C^*-algebras and crossed products of unital C^*-algebras with $\mathbb{Z}$.

KEYWORDS: *C^*-algebras, extensions, quasidiagonality, Blackadar–Kirchberg Conjecture*, ASH-*algebras.*

MSC (2020): 46L05.

1. INTRODUCTION

We know that every quasidiagonal C^*-algebra is stably finite [6, Propo sition 7.1.15]. The converse is not true, even for exact C^*-algebras. A counterexample is $C^*_r(\mathbb{F}_2)$. Indeed, it is stably finite because it has a faithful trace, but it is not quasidiagonal because $\mathbb{F}_2$ is not amenable [6, Corollary 7.1.17]. However, it is a very interesting question to ask under which extra conditions the converse holds. Because $C^*_r(\mathbb{F}_2)$ is not nuclear [6, Theorem 2.6.8], we should ask for a nuclearity assumption. In [2], Blackadar and Kirchberg conjectured that it is enough to assume nuclearity and separability.

CONJECTURE 1.1 ([2], Question 7.3.1). *If A is separable, stably finite and nuclear, then it is quasidiagonal.*

Although the conjecture is still open, there are some partial results confirming it. For instance, the conjecture holds when A (in addition to the conjecture assumption) has either one of the following properties:

(i) A is simple and satisfies the UCT [29, Corollary B];
(ii) A is traceless [13, Corollary C]; or

(iii) $A = B \rtimes_\sigma \mathbb{Z}$, where B is an AH-algebra of real rank zero [23, Theorem 1.1].

Because nuclearity and separability are closed under taking extensions, a positive answer on the Blackadar–Kirchberg conjecture would automatically guarantee a positive answer to the following conjecture.

CONJECTURE 1.2. *If*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

is a short exact sequence, with I, B separable, nuclear quasidiagonal, then A is quasidiagonal if and only if A is stably finite.

It has to be mentioned that under the aforementioned assumptions, A is not automatically quasidiagonal. One easy counterexample is

$$0 \longrightarrow \mathcal{K} \xrightarrow{\iota} \mathcal{T} \xrightarrow{\pi} C(\mathbb{T}) \longrightarrow 0$$

where $\mathcal{T}$ denotes the Toeplitz algebra and $\mathcal{K}$ the compact operators. Notice that $\mathcal{K}$ and $C(\mathbb{T})$ are quasidiagonal, but $\mathcal{T}$ is not. However, it is not stably finite either, as it contains a non-unitary isometry (the unilateral shift). Trying to verify Conjecture 1.2 is the starting point for this paper.

In [5], Brown and Dădârlat, connected Conjecture 1.2 with the presence of a property related to K-theory which they named *K_0-embedding property* (see Definition 2.2). More specifically, if the ideal has the K_0-embedding property and the quotient is separable, nuclear, quasidiagonal and satisfies the UCT, then the Blackadar–Kirchberg Conjecture holds for the C^*-algebra in the middle (see also Remark 2.5). If A is a separable and quasidiagonal C^*-algebra, the presence of K_0-embedding property means that for every $G \leqslant K_0(A)$ with $G \cap K_0(A)^+ = \{0\}$, there exists an embedding $\rho : A \hookrightarrow B$, where B is quasidiagonal and $\rho_*(G) = 0$. Note that $G \cap K_0(A)^+ = \{0\}$ is easily seen to be a necessary condition for the existence of such an embedding. Apart from specific easy cases (see the end of Section 2), not much has been known regarding which C^*-algebras have the K_0-embedding property.

Let $\mathcal{Y}$ be the class of C^*-algebras that can be written as a finite direct sum of algebras belonging to either one of the following classes:

(i) $D \rtimes_\sigma \mathbb{Z}$, where D is separable, nuclear, unital, quasidiagonal and satisfies the UCT, $\sigma : \mathbb{Z} \to \mathrm{Aut}(D)$ is a minimal action and D has a σ-invariant trace;

(ii) separable ASH-algebras.

In our paper, we will show that all algebras in $\mathcal{Y}$ have the K_0-embedding property. Combining this with other results from our paper (mainly Proposition 2.7 and Proposition 3.5) and the aforementioned comments, we deduce the following theorem.

THEOREM 1.3. *Assume that*

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} B \longrightarrow 0$$

is a short exact sequence where A is separable, $A \otimes \mathcal{Q}$ is locally approximated by algebras in $\mathcal{Y}$ and B is separable, nuclear, quasidiagonal, satisfying the UCT.

Then E is quasidiagonal if and only if it is stably finite.

REMARK 1.4. Note that if A is separable, simple, nuclear, unital, quasidiagonal and satisfies the UCT, then $A \otimes \mathcal{Q}$ has finite nuclear dimension by the main result of [7], so by [29, Theorem 6.2(iii)], it is an ASH-algebra. Thus A satisfies the hypothesis of Theorem 1.3.

REMARK 1.5. Note that $\mathcal{Y}$ is closed under taking matrix algebras, direct sums, as well as tensoring with $\mathcal{Q}$. So, if a C^*-algebra A satisfies the hypothesis of Theorem 1.3, then $A \otimes \mathcal{K}$ (and more generally $A \otimes D$ for every AF-algebra D) also satisfies the hypothesis of Theorem 1.3.

Let E be as in Theorem 1.3. Then it is not simple and usually it does not admit any faithful trace. For example, the latter is guaranteed if the ideal A is stable. So, Theorem 1.3 verifies the Blackadar and Kirchberg conjecture for a large class of C^*-algebras that have no faithful trace. Actually, even the case where the C^*-algebra arises as an extension of a separable, nuclear, quasidiagonal C^*-algebra with the UCT by $C(X) \otimes \mathcal{K}$, cannot be deduced in a straightforward way from any of the previous results that we could find in the literature.

For the proof of Theorem 1.3, we need results regarding ordered K-theory from [14], as well as techniques from [28], in order to construct many of the K_0 maps. We then rely on a classification theorem of Schafhauser [27, Corollary 5.4] to "lift" these maps to the C^*-algebra level, and thus achieve the existence of the embeddings needed to show the presence of K_0-embedding property. For the case when the ideal is a separable ASH-algebra, we use results and techniques from [10], [11] and [20].

The paper is structured as follows: in Section 2, we give the definitions, some needed preliminaries and mention already known examples of C^*-algebras with the K_0-embedding property. In Section 3, we show that K_0-embedding property is a local property. In Section 4, we set the stage for the last two sections, in which we produce new examples of C^*-algebras with the K_0-embedding property. More specifically, in Section 5 we show the K_0-embedding property for direct sums of certain C^*-algebras, including simple ones and crossed products with $\mathbb{Z}$ via minimal actions (Proposition 5.6). It has to be noted that the results we show in Section 4 make us realize for which direct sums we can show the K_0-embedding property when applying our strategy. In Section 6, we establish K_0-embedding property for separable ASH-algebras. There is also an appendix, on which, for the sake of completion, we present a proof of a result regarding crossed products, that is essential for the proof of Proposition 5.4.

Throughout the paper $\mathbb{N}^* = \{1, 2, 3, \dots\}$, $\mathcal{Q}$ will be the universal UHF-algebra, by $F \subset\subset A$ we will denote a finite subset of A, while $\otimes$ will be the minimal tensor product. Moreover, for an element a in a C^*-algebra, $\overline{a}$ will denote an image under some quotient map. For a C^*-algebra A, $\mathrm{sr}(A)$ and $\mathrm{RR}(A)$ will be the stable and real rank of a C^*-algebra, respectively. If F is a set and C is a C^*-algebra, with $F \subset_\varepsilon C$ we mean that for every element in $a \in F$ there is an element $b \in C$ that is ε-close to a in norm. We will use the abbreviation ccp for completely positive and contractive maps. And finally, if $\tau \in T(A)$ is a trace and $a \in M_n(A)$, we will sometimes abuse the notation and write $\tau(a) = \mathrm{Tr} \otimes \tau(a) = \sum\limits_{i=1}^{n} \tau(a_{ii})$.

2. PRELIMINARIES AND BASIC EXAMPLES

We start by giving a few definitions.

DEFINITION 2.1 ([6], Definition 7.1.1). A separable C^*-algebra is *quasidiagonal* if there exists a sequence of asymptotically multiplicative and asymptotically isometric ccp maps $\phi_n : A \to M_{k(n)}$.

DEFINITION 2.2 ([5], [5]Definition 4.4). We say that a separable and quasidiagonal C^*-algebra A has the *K_0-embedding property* if for every $G \leqslant K_0(A)$ such that $G \cap K_0^+(A) = \{0\}$, there is a faithful $*$-homomorphism $\rho : A \to C$, where C is quasidiagonal and $\rho_*(G) = 0$.

DEFINITION 2.3 ([5], Definition 4.3). We say that a separable and quasidiagonal C^* algebra A has the *QD-extension property* if for every separable, nuclear, quasidiagonal C^*-algebra B which satisfies the UCT, and for every short exact sequence

$$0 \longrightarrow A \otimes \mathcal{K} \xrightarrow{\iota} E \xrightarrow{\pi} B \longrightarrow 0$$

E is quasidiagonal if and only if E is stably finite.

We need to add more notation.

DEFINITION 2.4. Let (G, G^+) be an ordered abelian group. A subgroup $H \leqslant G$ is called *singular* if $H \cap G^+ = \{0\}$. If $x \in G$, we will say that x is *singular* if $\mathbb{Z}x \cap G^+ = \{0\}$.

REMARK 2.5. If A is separable and quasidiagonal, by [5, Proposition 4.6] it has the K_0-embedding property if and only if it has the QD-extension property. Assume that A is separable, quasidiagonal, has the K_0-embedding property and

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} B \longrightarrow 0$$

is a short exact sequence with B separable, nuclear, quasidiagonal, satisfying the UCT and E is stably finite. Then, if we tensor everything with $\mathcal{K}$, the sequence remains exact. Also $E \otimes \mathcal{K}$ is stably finite and the properties of B pass to $B \otimes \mathcal{K}$. Because A has the K_0-embedding property and hence the QD-extension property, $E \otimes \mathcal{K}$ is quasidiagonal hence E is quasidiagonal. Thus, by the aforementioned and [5, Theorem 4.11], in order to prove Conjecture 1.2 (when B satisfies the UCT), it is enough to show that every nuclear, separable, quasidiagonal C^*-algebra has the K_0-embedding property.

Let A be a C^*-algebra. Then we can write $A \otimes \mathcal{Q}$ as an inductive limit, i.e.

$$A \otimes \mathcal{Q} = \varinjlim A \otimes M_{n!}(\mathbb{C}),$$

where the connecting maps are

$$\mathrm{id} \otimes \phi_i, \text{ where } \phi_i : M_{i!}(\mathbb{C}) \to M_{(i+1)!}(\mathbb{C}) \text{ is defined by } \phi_i(a) = \mathrm{diag}(a, a, \ldots, a).$$

and $\mathrm{id} \otimes \mu_i$, where

$$\mu_i : M_{i!}(\mathbb{C}) \to \mathcal{Q}$$

is the inclusion from the definition of $(\mathcal{Q}, \mu_i) = \varinjlim M_{n!}(\mathbb{C})$ as an inductive limit.

By the stability of K_0 [25, Proposition 4.3.8] $K_0(A) \cong K_0(M_n(A))$.

By the Künneth Theorem [1, Theorem 23.1.2] (or the continuity of K_0 [25, Theorem 6.3.2]), we have

$$K_0(A \otimes \mathcal{Q}) = K_0(A) \otimes \mathbb{Q}. \tag{2.1}$$

Let $x, y \in K_0(A)$ and $a, b, c, d \in \mathbb{Z}$. Then

$$x \otimes \frac{a}{b} + y \otimes \frac{c}{d} = x \otimes \frac{ad}{bd} + y \otimes \frac{bc}{bd} = adx \otimes \frac{1}{bd} + bcy \otimes \frac{1}{bd} = (adx + bcy) \otimes \frac{1}{bd}.$$

So, every (finite) sum of elementary tensors in $K_0(A) \otimes \mathbb{Q}$ is still an elementary tensor. Thus

$$K_0(A) \otimes \mathbb{Q} = \{x \otimes y : x \in K_0(A), y \in \mathbb{Q}\}.$$

Recall that for a C^*-algebra A, the *positive cone* of $K_0(A)$ is

$$K_0(A)^+ := \{[p]_0 : p \in P_\infty(A)\}.$$

We can put an order on $K_0(A)$ as follows:

$$x \leqslant y \text{ if and only if } y - x \in K_0(A)^+.$$

The following proposition is well-known but we present a proof for the sake of completion.

PROPOSITION 2.6. *The positive cone of $K_0(A \otimes \mathcal{Q})$ is*

$$K_0(A \otimes \mathcal{Q})^+ = \{x \otimes y : x \in K_0(A)^+, y \in \mathbb{Q}_{\geqslant 0}\}.$$

Proof. First of all, note that by the continuity of K_0 we have

$$K_0(A \otimes \mathcal{Q})^+ = \bigcup_{i=1}^{\infty} (\mathrm{id} \otimes \mu_i)_*(K_0(A \otimes M_{n!}(\mathbb{C}))^+).$$

Note that $(\mathrm{id} \otimes \mu_i)_* : K_0(A) \to K_0(A) \otimes \mathbb{Q}$ is the division with i. To see this, observe that $(\mathrm{id} \otimes \phi_i)_* : K_0(A) \to K_0(A)$ is the multiplication with $i+1$ and then see that the properties of the inductive limit are satisfied. Result follows. ∎

The aforementioned will help us show our first permanence property.

PROPOSITION 2.7. *If A is separable and quasidiagonal then $A \otimes \mathcal{Q}$ has the K_0-embedding property, if and only if A has it.*

Proof. Let $\iota : A \hookrightarrow A \otimes \mathcal{Q}$ be the natural embedding. Assume first that $A \otimes \mathcal{Q}$ has the K_0-embedding property. Let $G \leqslant K_0(A)$ be singular. By Proposition 2.6, $\iota_*(G)$ is singular. Hence, we have an embedding $\rho : A \otimes \mathcal{Q} \hookrightarrow D$, where D is quasidiagonal and $\rho_*(\iota_*(G)) = 0$. After considering $\rho \circ \iota : A \hookrightarrow D$, we deduce that A has the desired property. Conversely, assume that A has the K_0-embedding property and let $G \leqslant K_0(A \otimes \mathcal{Q})$ be singular. Set $H := \{a \in K_0(A) : \exists b \in \mathbb{Q} \text{ such that } a \otimes b \in G\}$. Note that H is a singular subgroup of $K_0(A)$. By assumption there exists a quasidiagonal C^*-algebra B and an embedding $h : A \hookrightarrow B$, such that $h_*(H) = 0$. Consider the map $h \otimes \mathrm{id} : A \otimes \mathcal{Q} \hookrightarrow B \otimes \mathcal{Q}$. Note that $(h \otimes \mathrm{id})_*(G) = 0$, so $A \otimes \mathcal{Q}$ has the K_0-embedding property. ∎

If we want to show that the K_0-embedding property is satisfied for a class of C^*-algebras, the previous proposition allows us (at least in the most cases) to assume in addition that the C^*-algebras are $\mathcal{Q}$-stable. This helps a lot because for every separable, stably finite, unital, $\mathcal{Q}$-stable C^*-algebra, the ordered K_0 group is well behaved.

DEFINITION 2.8 ([25], Definition 7.2.5). An ordered abelian group (G, G^+) is called *unperforated* if every $x \in G$ for which $nx \geqslant 0$ for some $n \in \mathbb{N}$ satisfies $x \geqslant 0$.

Notice that an unperforated ordered group must be torsion free. If A is $\mathcal{Q}$-stable, then $K_0(A) \cong K_0(A) \otimes \mathbb{Q}$ by (2.1) and $K_0(A)$ is unperforated. Indeed $A \cong A \otimes \mathcal{Q}$, so if $nx \geqslant 0$, then $nx = a \otimes b$, where $a \in K_0(A)^+$ and $b \in \mathbb{R}_{\geqslant 0}$ by Proposition 2.6. Thus $x = a \otimes \frac{b}{n} \in K_0(A) \otimes \mathbb{Q}$. Hence $a \geqslant 0$ in $K_0(A)$.

For the rest of the section, all groups will be abelian, unless clearly stated otherwise. Let (G, G^+, u) be a scaled, ordered, countable group that satisfies $G \cong G \otimes \mathbb{Q}$. In this case the isomorphism is on the category of ordered groups and $(G \otimes \mathbb{Q})^+ = \{a \otimes b : a \in G^+, b \in \mathbb{Q}_{\geqslant 0}\}$. Note that $(G \otimes \mathbb{Q}, (G \otimes \mathbb{Q})^+)$ is indeed an ordered group by [15, Proposition 2.3]. It follows that it is unperforated and hence torsion free. Let $H \leqslant G$ be a singular subgroup. Set

$$\mathcal{F} = \{L \leqslant G : L \cap G^+ = 0, H \subset L\}.$$

Then $\mathcal{F} \neq \emptyset$ $(H \in \mathcal{F})$ and for every $(L_i)_{i \in I}$ increasing chain in $\mathcal{F}$, $\bigcup\limits_{i \in I} L_i \in \mathcal{F}$. Hence, Zorn's Lemma applies and $\mathcal{F}$ has a maximal element. Such subgroup will be called *maximally singular*. Observe that every singular subgroup is contained inside a maximally singular subgroup.

Let $H_0 \leqslant G$ be a singular subgroup. Consider the following property for H_0:

(2.2) If there exists $k \in \mathbb{N}^*$ such that $kx \in H_0$ then it follows that $x \in H_0$.

LEMMA 2.9. *If H_0 is maximally singular or $H_0 \cong H_0 \otimes \mathbb{Q}$, then* (2.2) *holds.*

Proof. Assume first that H_0 is maximally singular. For the sake of contradiction, assume that there exist $x \in G$ and $k \in \mathbb{N}^*$, such that $kx \in H_0$, but $x \notin H_0$. Consider $H_0' := \operatorname{span}_{\mathbb{Z}}\{H_0, x\} \leqslant G$. Then $H_0' \supsetneq H_0$ and also H_0' is singular because $kH_0' \subset H_0$, contradicting the maximality of H_0. Assume now that $H_0 \cong H_0 \otimes \mathbb{Q}$ and let $x \in G$ such that $kx \in H_0$. By assumption, $kx = a \otimes b$ for some $a \in H_0$ and $b \in \mathbb{Q}$. It follows that $x = a \otimes \frac{b}{k} \in H_0 \otimes \mathbb{Q} \cong H_0$. ∎

Let $G_0 = G/H_0$. We can put an order on G_0 as follows:

$$G_0^+ = \{\overline{x} \in G_0 : \exists y \in H_0 \text{ such that } x + y \geqslant 0 \text{ in } G\}. \tag{2.3}$$

LEMMA 2.10. *$(G_0, G_0^+, \overline{u})$, as defined in* (2.3) *is a scaled ordered group and the quotient map $\pi : G \to G_0$ is positive. If, in addition* (2.2) *holds, then it is unperforated. Moreover, if H_0 is maximally singular, then $(G_0, G_0^+, \overline{u})$ is also totally ordered, hence dimension group.*

Proof. This is essentially contained in the proof of [28, Lemma 1.14]. However, for the sake of completion, we will repeat the arguments here.

First of all, it is straightforward to check that $\pi(x) \in G_0^+$ for every $x \in G^+$. Moreover, we have:

Claim 1. $G_0^+ + G_0^+ \subset G_0^+$.

Proof. Let $\overline{x}, \overline{y} \in G_0^+$. By definition there exist $x_1, y_1 \in H_0$ such that $x + x_1 \geqslant 0$ and $y + y_1 \geqslant 0$ in G. Adding the two inequalities together, we get $(x + y) + (x_1 + y_1) \geqslant 0$. Notice that $x_1 + y_1 \in H_0$, so $x + y \in G_0^+$ by definition. ∎

Claim 2. $G_0^+ - G_0^+ = G_0$.

Proof. This follows from the facts that $G^+ - G^+ = G$ and $\pi(G^+) \subset G_0^+$. ∎

Claim 3. $G_0^+ \cap -G_0^+ = \{0\}$.

Proof. Let $x \in G$ such that $\overline{x}$ and $-\overline{x} \in G_0^+$. Then there exists $e, f \in H_0$ such that $x + e \geqslant 0$ and $-x + f \geqslant 0$. If $x + e > 0$, then adding the two relations together yields $e + f > 0$. But $e + f \in H_0$, so we have a contradiction. Thus $x + e = 0$, which implies $x \in H_0$, so $\overline{x} = 0$. ∎

Combining Claims 1, 2 and 3, we deduce that (G_0, G_0^+) is an ordered group. By the first sentence of the proof, π is positive. Let $\overline{x} \in G_0$. Because u is an order unit for (G, G^+), there exist $n \in \mathbb{N}^*$ such that $-nu \leqslant x \leqslant nu$. Because π is positive, $-n\overline{u} \leqslant \overline{x} \leqslant n\overline{u}$. Because $\overline{x}$ is arbitrary, $\overline{u}$ is an order unit for (G_0, G_0^+).

Assume now that (2.2) holds. We will show that the ordered group is unperforated. More specifically, let $n\overline{x} \in G_0^+$ for some positive integer n and $\overline{x} \in G_0$. Then there is $y \in H_0$ such that $nx + y \geqslant 0$ in G. Observe that because $G \cong G \otimes \mathbb{Q}$, $\frac{y}{n}$ is a well-defined element of G. By (2.2) $\frac{y}{n} \in H_0$ and because G is unperforated, $x + \frac{y}{n} \in G^+$. So, $\overline{x} \in G_0^+$, thus (G_0, G_0^+) is unperforated.

Assume in addition that H_0 is maximally singular. We will show that (G_0, G_0^+) is totally ordered. Let $\overline{x} \in G_0 \backslash \{0\}$. Notice that $x \notin H_0$ so the fact that H_0 is maximally singular yields that there exist $n \in \mathbb{Z}$ and $e \in H_0$ such that $nx + e > 0$. If $n > 0$, then $n\overline{x} > 0$. Because (G_0, G_0^+) is unperforated, $\overline{x} > 0$. Similarly, if $n < 0$, then $\overline{x} < 0$. Hence (G_0, G_0^+) is totally ordered. ∎

Notice that if (2.2) does not hold, then G_0 will have torsion and thus perforation.

Recall that a *state* on an ordered group (G, G^+, u) is a group homomorphism $\phi : G \to \mathbb{R}$ such that $\phi(u) = 1$ and $\phi(G^+) \subset \mathbb{R}^+$. The set of states is defined to be $S(G)$. We call a state ϕ *faithful* if $\phi(x) > 0 \;\; \forall x \in G^+ \backslash \{0\}$. Every (non-zero) scaled ordered group has a state by [14, Corollary 4.4].

Following the previous notation, we can see that if $\phi \in S(G)$ with $\phi(H_0) = 0$, then $\widetilde{\phi} : G_0 \to \mathbb{R}$ with $\widetilde{\phi}(\overline{x}) = \phi(x)$ is a state on $(G_0, G_0^+, \overline{u})$. And moreover, the map

$$\Phi : \{\phi \in S(G) : \phi(H_0) = 0\} \to S(G_0) \tag{2.4}$$

with $\Phi(\phi) = \widetilde{\phi}$ is a bijection.

If $\tau \in T(A)$, then we have an induced state $\widehat{\tau} \in S(K_0(A))$ such that $\forall p \in P_n(A)$, $\widehat{\tau}[p]_0 = \mathrm{Tr} \otimes \tau(p) = \sum_{i=1}^{n} \tau(p_{ii})$.

Thus, we have an induced map

$$\Psi : T(A) \to S(K_0(A)) \tag{2.5}$$

with $\Psi(\tau) = \widehat{\tau}$.

This map is onto if A is exact by [16, Theorem 5.11] and [3, Theorem 3.3]. If we also have $\mathrm{RR}(A) = 0$, [26, Proposition 1.1.12] and the comments below it, yield that the map is a bijection. Note that if A is an AF-algebra, $\mathrm{RR}(A) = 0$.

Moreover, for a scaled ordered group (G, G^+, u), the group of infinitesimals is

$$\mathrm{Inf}(G) = \{x \in G : \rho(x) = 0 \;\; \forall \rho \in S(G)\}. \tag{2.6}$$

If, moreover, the ordered group is unperforated, it is known that

$$\mathrm{Inf}(G) = \{x \in G : u + nx \geqslant 0 \;\; \forall n \in \mathbb{Z}\} \tag{2.7}$$

(see for instance [9, Lemma 2.5] if the ordered group is simple but not necessarily unperforated; in our case the proof is identical).

We will need the following simple lemma.

LEMMA 2.11. *Let A, B be unital and stably finite C^*-algebras and $\phi : A \to B$ a unital $*$-homomorphism. Then $\phi_*(\mathrm{Inf}(K_0(A))) \subset \mathrm{Inf}(K_0(B))$.*

Proof. Let $x \in \mathrm{Inf}(K_0(A))$ and $\rho_B \in S(K_0(B), K_0(B)^+, [1_B]_0)$. Then $\rho_A := \rho_B \circ \phi_* : K_0(A) \to \mathbb{R}$ is a state in $(K_0(A), K_0(A)^+, [1_A]_0)$. Note that we use the fact that $\phi(1_A) = 1_B$. By (2.6), $\rho_A(x) = 0$. Hence, $\rho_B(\phi_*(x)) = 0$. Because ρ_B is arbitrary, $\phi_*(x) \in \mathrm{Inf}(K_0(B))$ by (2.6). ∎

The next lemma will be very useful in Section 5.

LEMMA 2.12. *Let* (G, G^+, u) *be a scaled ordered, countable abelian group with* $G \cong G \otimes \mathbb{Q}$ *via an order isomorphism. Let also* $H_1 \subset H_2 \subset G$ *such that* $H_1 \leqslant G$ *is a subgroup with* $H_1 \cap G^+ = \{0\}$ *and* $H_2 \subset G$ *is a subsemigroup with* $H_2 \cap -G^+ = \{0\}$ *and* $H_2 \cap -H_2 = H_1$. *Then there exists a state* $\rho \in S(G)$ *such that* $\rho(H_1) = 0$ *and* $\rho(x) \geqslant 0$ *for every* $x \in H_2$.

Proof. Set $G_0 = G/H_1$ and let $\pi : G \to G_0$ be the quotient map. By Lemma 2.10, $(G_0, G_0^+, \overline{u})$ endowed with the order as defined in (2.3), is a scaled ordered group. Set $P = G_0^+ + \overline{H}_2$. We will show that $(G_0, P, \overline{u})$ is a scaled ordered group. Indeed, $P - P = G_0$, because $P \supset G_0^+$. Because H_2, and hence $\overline{H}_2$, is a semigroup, we have that $P + P \subset P$. Let $x \in P \cap -P$. Then $x = a_1 + b_1 = -a_2 - b_2$, where $a_i \in G_0^+$ and $b_i \in \overline{H}_2$. Hence

$$(a_1 + a_2) + (b_1 + b_2) = 0.$$

But $a_1 + a_2 \in G_0^+$, while $b_1 + b_2 \in \overline{H}_2$. By assumption, we have that $H_2 \cap -G^+ = \{0\}$. Thus $-G_0^+ \cap \overline{H}_2 = \{0\}$, which yields

$$a_1 + a_2 = 0 \quad \text{and} \quad b_1 + b_2 = 0. \tag{2.8}$$

Moreover, by assumption, we have that $H_2 \cap -H_2 = H_1$, which implies that $\overline{H}_2 \cap -\overline{H}_2 = \{0\}$. Also, $G_0^+ \cap -G_0^+ = \{0\}$. So, (2.8) yields that $a_1 = a_2 = b_1 = b_2 = 0$, thus $x = 0$. This means that $P \cap -P = \{0\}$. It follows that (G_0, P) is an ordered group. Because $\overline{u}$ is an order unit on (G_0, G_0^+) and $P \supset G_0^+$, $\overline{u}$ must be an order unit on (G_0, P). By [14, Corollary 4.4], $(G_0, P, \overline{u})$, has a state, call it τ. Because $P \supset G_0^+$, $\tau \in S(G_0, G_0^+)$ and $\tau(z) \geqslant 0$, for every $z \in \overline{H}_2$. Finally, because the map in (2.4) is onto, $\tau = \Phi(\rho)$, for some $\rho \in S(G)$. It is not difficult to show that ρ satisfies the desired properties. ∎

REMARK 2.13. Let A be a separable, exact, stably finite, unital and $\mathcal{Q}$-stable C^*-algebra and $G \leqslant K_0(A)$ a singular subgroup. Note that the comments after Definition 2.8 guarantee that $K_0(A)$ satisfies the hypothesis of Lemma 2.12. By Lemma 2.12 for $H_1 = H_2 = G$ and the fact that the map in (2.5) is onto, we deduce that there exists $\tau \in T(A)$ such that $\widehat{\tau}(G) = 0$. If, moreover, every state in $K_0(A)$ is induced by a faithful trace (this happens for instance if A is simple), then we can choose τ to be faithful.

The next proposition allows us to pass to unitizations.

PROPOSITION 2.14. *If* $\widetilde{A}$ *has the* K_0*-embedding property, then* A *has it, too.*

Proof. Observe that $\iota : A \hookrightarrow \widetilde{A}$ (ι is the natural inclusion) has the property that $\iota_*(x) \geqslant 0$ if and only if $x \geqslant 0$ (see [25, Chapter 4] for instance), hence sends singular elements to singular elements. Then our assumption yields the result immediately. ∎

We close the section with some basic examples of C^*-algebras with the K_0-embedding property:

(i) *AF-algebras* [28, Lemma 1.14].

(ii) $A \rtimes \mathbb{Z}$, where A is an AF-algebra, provided that $A \rtimes \mathbb{Z}$ is quasidiagonal [4, Theorem 5.5].

(iii) Every C^*-algebra A for which there exists D quasidiagonal with $K_0(D)$ being a torsion group and $\rho : A \hookrightarrow D$ faithful $*$-homomorphism. This is a direct corollary of Proposition 2.7. Note that $K_0(D \otimes \mathcal{Q}) = 0$.

This yields more examples like *suspensions*, *cones* and more generally *exact and connective C^*-algebras*. This happens because if A is exact and connective then by [13, Theorem A], A embeds to the Rørdam algebra $\mathcal{A}_{[0,1]}$ (see [24] for the construction and properties of $\mathcal{A}_{[0,1]}$). Note that $\mathcal{A}_{[0,1]}$ is quasidiagonal and has trivial K-theory.

(iv) *Separable, quasidiagonal, C^*-algebras with totally ordered K_0 group* (this is obvious from the definition).

3. K_0- EMBEDDING PROPERTY IS A LOCAL PROPERTY

Our main goal of the section is to show that K_0-embedding property is a local property. This is a very crucial result, because it will help us show that the property is satisfied by a wide variety of C^*-algebras, including the approximate subhomogeneous ones. Our first step is to show that in order to "kill" singular subgroups via an embedding, it is enough to do it via a sequence of asymptotically isometric and asymptotically multiplicative ccp maps. This is important, because, in order to pass from locally to globally, we need to extend maps. Although we cannot in general extend $*$-homomorphisms, we can use Arveson's extension Theorem [6, Theorem 1.6.1] to extend ccp maps defined on some subalgebra, to the whole algebra. But first we need the following proposition, which is almost identical to [5, Proposition 2.5].

Recall that a C^*-algebra is called *MF* if there exists a faithful $*$-homomorphism $\phi : A \hookrightarrow \prod_{i=1}^{\infty} M_{k(n)} / \bigoplus_{i=1}^{\infty} M_{k(n)}$. Notice that a quasidiagonal C^*-algebra is MF. The converse is not true in general (for instance $C^*_r(\mathbb{F}_2)$ is MF but not quasidiagonal). However the Choi–Effros Lifting Theorem [8, Theorem 3.10] implies that a nuclear, MF-algebra is quasidiagonal.

For an extension η, $E(\eta)$ will be the C^*-algebra in the middle. Moreover, with $\mathcal{M}(A)$ we will denote the multiplier algebra of A and we will use the notation $Q(A) = \mathcal{M}(\mathcal{A})/A$.

PROPOSITION 3.1. *Let*

$$(\eta) : \quad 0 \longrightarrow A \otimes \mathcal{K} \longrightarrow E \longrightarrow B \longrightarrow 0$$

be an extension, where B is separable, nuclear and quasidiagonal, A is MF *and σ-unital (i.e. it has a countable approximate unit) and $[\eta] = 0 \in \mathrm{Ext}(B, A \otimes \mathcal{K})$. Then E is* MF.

Proof. Let $\rho : B \to B(H)$ be a faithful representation such that H is separable, $\rho(B) \cap \mathcal{K}(H) = \{0\}$ and the orthogonal complement of the non-degeneracy subspace of $\rho(B)$ is infinite dimensional. Let τ be the extension from [5, Theorem 2.3]. Then τ is both trivial and absorbing. By [5, Lemma 2.2],

$$E \hookrightarrow E(\eta \oplus \tau).$$

Because $[\eta] = 0 \in \mathrm{Ext}(B, A \otimes \mathcal{K})$ and τ is absorbing,

$$E(\eta \oplus \tau) \cong E(\tau).$$

Moreover, by looking at the statement of [5, Theorem 2.3], we can see that there exists an embedding

$$E(\tau) \hookrightarrow (\rho(B) + \mathcal{K}(H)) \otimes \widetilde{A}.$$

Thus $E \hookrightarrow (\rho(B) + \mathcal{K}(H)) \otimes \widetilde{A}$. But $\rho(B) + \mathcal{K}(H)$ is quasidiagonal from [6, Theorem 7.2.5] and also nuclear, hence exact. So $(\rho(B) + \mathcal{K}(H)) \otimes \widetilde{A}$ is MF by [23, Proposition 3.6]. Thus E is MF. ∎

Now we are ready to show the result we promised.

PROPOSITION 3.2. *Assume that A is separable, nuclear, quasidiagonal, and for some $G \leqslant K_0(A)$ singular, there exists a faithful $*$-homomorphism $\rho : A \hookrightarrow \prod M_{k(n)} / \bigoplus M_{k(n)}$ such that $\rho_*(G) = 0$. Then there exists a quasidiagonal C^*-algebra B and a faithful $*$-homomorphism $\phi : A \to B$, such that $\phi_*(G) = 0$.*

Proof. The proof of this proposition is essentially contained in the proof of [5, Theorem 4.11]. However, for the sake of completion we will repeat the arguments here. Let G be a singular subgroup of $K_0(A)$. Because $\bigoplus_{\mathbb{N}} C(\mathbb{T})$ satisfies the UCT, there exists a short exact sequence

$$0 \longrightarrow A \otimes \mathcal{K} \longrightarrow E \longrightarrow \bigoplus_{\mathbb{N}} C(\mathbb{T}) \longrightarrow 0$$

satisfying $\delta_1(K_1(\bigoplus_{\mathbb{N}} C(\mathbb{T}))) = G$. Recall that δ_1 is the boundary map in the K-theory six term exact sequence. By assumption there exists an embedding $\rho_0 : A \hookrightarrow \prod M_{k(n)} / \bigoplus M_{k(n)}$ such that $(\rho_0)_*(G) = 0$. Let $D \subset \prod M_{k(n)} / \bigoplus M_{k(n)}$ be the hereditary C^*-subalgebra generated by $\rho_0(A)$. Then the $*$-homomorphism $\rho : A \otimes \mathcal{K} \to D \otimes \mathcal{K}$ satisfying $\rho(a \otimes b) = \rho_0(a) \otimes b$, is approximately unital (see [5, Definition 3.1] for the definition of this property and the paragraphs below for an explanation of why this is true). Thus $\mathcal{M}(A \otimes \mathcal{K}) \subset \mathcal{M}(D \otimes \mathcal{K})$ by [18, 3.12.12]. Hence there exists a Busby invariant $\eta_2 : \bigoplus_{\mathbb{N}} C(\mathbb{T}) \to Q(D \otimes \mathcal{K})$ and an embedding $E \hookrightarrow E(\eta_2)$ such that the following diagram

$$\begin{tikzcd}
(\eta_1): & 0 \arrow[r] & A \otimes \mathcal{K} \arrow[r, "\iota"] \arrow[d, hook] & E \arrow[r] \arrow[d, hook] & \bigoplus_{\mathbb{N}} C(\mathbb{T}) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
(\eta_2): & 0 \arrow[r] & D \otimes \mathcal{K} \arrow[r] & E(\eta_2) \arrow[r] & \bigoplus_{\mathbb{N}} C(\mathbb{T}) \arrow[r] & 0
\end{tikzcd}$$

is commutative. Moreover, $K_1(D) = 0$ by the proof of [5, Lemma 3.2] and $\rho_*(G) = 0$ by [5, Lemma 4.5] and the fact that $\prod M_{k(n)} / \bigoplus M_{k(n)}$ has real rank zero and cancellation of projections. Hence both boundary maps on the bottom short exact sequence are zero. Indeed $K_1(D) = 0$ implies $\delta_0 = 0$, while $\delta_1 = 0$ because of $\rho_*(G) = 0$ and the naturality of the six term exact sequence. From the UCT and the fact that $K_i(\bigoplus_{\mathbb{N}} C(\mathbb{T}))$ is a free $\mathbb{Z}$-module for $i = 0, 1$, we get that $[\eta_2] = 0 \in \mathrm{Ext}(\bigoplus_{\mathbb{N}} C(\mathbb{T}), D \otimes \mathcal{K})$. Moreover, the fact that A is separable, implies that $D \otimes \mathcal{K}$ is σ-unital. So, by Proposition 3.1 $E(\eta_2)$ is an MF-algebra. Because MF-algebras are closed under taking subalgebras and every nuclear MF-algebra is quasidiagonal, E is quasidiagonal. Finally by the 6-term exact sequence, $\iota_*(G) = 0$, so $A \hookrightarrow A \otimes \mathcal{K} \hookrightarrow E$ is the desired embedding. ∎

The next step is to reduce to "killing" finitely generated singular subgroups.

PROPOSITION 3.3. *Let A be separable, unital, nuclear, quasidiagonal and assume that for every finitely generated singular subgroup G of $K_0(A)$, there exists a faithful $*$-homomorphism $\rho : A \to B$, where B is quasidiagonal and $\rho_*(G) = 0$. Then A has the K_0-embedding property.*

Proof. Let $G \leqslant K_0(A)$ be any singular subgroup. Because G is countable, there is an increasing sequence of singular and finitely generated subgroups $G_n \leqslant K_0(A)$, such that $\bigcup_{n=1}^{\infty} G_n = G$. By assumption, there are faithful $*$-homomorphisms $\rho_n : A \to B_n$, where B_n is quasidiagonal and $(\rho_n)_*(G_n) = 0$. Because B_n is quasidiagonal, for each n there exists a sequence of ccp, asymptotically isometric and asymptotically multiplicative maps $\phi_{mn} : B_n \to M_{k_{mn}}$. Set $\psi_n = \phi_{d(n)n}$. Observe that we can take the $d(n)$'s to be large enough, so that $(\psi_n \circ \rho_n)_{n\in\mathbb{N}}$ is asymptotically multiplicative and asymptotically isometric. Thus if

$$\rho : A \to \prod M_{l(n)} / \bigoplus M_{l(n)} \text{ with } \rho(x) = (\psi_n \circ \rho_n(x))_{n\in\mathbb{N}} (\text{where } l(n) = k_{d(n)n})$$

then ρ is a faithful $*$-homomorphism. Let now $g \in G$. Then there is $n_0 \in \mathbb{N}$ such that $g \in G_n \ \forall n \geqslant n_0$. Hence $(\rho_n)_*(g) = 0$, which implies $(\psi_n \circ \rho_n)_*(g) = 0 \ \forall n \geqslant n_0$, so $\rho_*(g) = 0$ in $K_0(\prod M_{l(n)} / \bigoplus M_{l(n)}) = l^{\infty}(\mathbb{Z})/c_{00}(\mathbb{Z})$. Because g is arbitrary, it follows that $\rho_*(G) = 0$. Finally, Proposition 3.2 applies to yield that A has the K_0-embedding property. ∎

Now we have the tools to show that K_0-embedding property is a local property. But first we need to say explicitly what we mean by this.

DEFINITION 3.4 ([11], Definition 1.5). Let $\mathcal{C}$ be a class of C^*-algebras and A be a C^*-algebra. We say that A is *locally approximated by algebras in $\mathcal{C}$* if for every finite subset $F \subset\subset A$ and for every $\varepsilon > 0$, there exists a C^*-subalgebra $C \subset A$ such that $C \in \mathcal{C}$ and $F \subset_\varepsilon C$.

Assume that A is locally approximated by algebras in $\mathcal{C}$. Notice that by shrinking the class $\mathcal{C}$, so that it contains only the C^*-algebras needed to guarantee local approximation, we may assume that $\mathcal{C}$ consists only of C^*-subalgebras of A. Observe that $\widetilde{A}$ is locally approximated by algebras in $\widetilde{\mathcal{C}}$, where $\widetilde{\mathcal{C}} = \{B + \mathbb{C}1_{\widetilde{A}} : B \in \mathcal{C}\}$. This observation is important for managing the non-unital technicalities on the following proposition. Moreover, $A \otimes \mathcal{Q}$ can be locally approximated by $\{\mathcal{B} \otimes \mathcal{Q} : B \in \mathcal{C}\}$. Note also that by the standard picture for K_0, every element of $K_0(A)$ is of the form $[p]_0 - [s(p)]_0$, where $p \in P_m(\widetilde{A})$ and $s : \widetilde{A} \to \widetilde{A}$ satisfies $s(a + b \cdot 1) = b \cdot 1$ [25, Proposition 4.2.2].

Let $h : A \to B$ be a ccp map that is (F, ε) multiplicative, for some subset $F \subset P_\infty(A)$ and $0 < \varepsilon < \frac{1}{4}$. Let $p \in F$ be a projection. Then, $\|h(p) - h(p)^2\| < \varepsilon$, hence [25, Proposition 6.3.1] implies that there exist $q \in P_\infty(B)$ such that $\|h(p) - q\| < 2\varepsilon < \frac{1}{2}$. So, we can define $h_*([p]_0) = [q]_0$. Because every two projections that have distance < 1 give rise to the same element in K_0, $h_*([p]_0)$ is well-defined. Moreover, observe that if $p_1, p_2, p_1 \oplus p_2 \in F$, then $h_*([p_1]_0) + h_*([p_2]_0) = h_*([p_1 \oplus p_2]_0)$. In this way, we can extend the notion of the induced K_0 map to sequences of asymptotically multiplicative ccp maps (for more details see, for instance, [9, Definition 2.4 and below]).

PROPOSITION 3.5. *Let A be a separable C^*-algebra that is locally approximated by algebras in $\mathcal{C}$, where $\mathcal{C}$ is a class that contains only separable, nuclear, quasidiagonal C^*-algebras with the K_0-embedding property. Then A has the K_0-embedding property.*

Proof. Notice that because nuclearity and quasidiagonality are both local properties, A satisfies them. (For nuclearity, it is [6, Example 2.3.7]. For quasidiagonality, although we could not find an explicit reference, the result is well-known. Also the reader can deduce that the proof of this proposition implies that quasidiagonality is a local property.)

Because of Proposition 2.7 and the fact that separability, nuclearity and quasidiagonality are preserved under tensoring with $\mathcal{Q}$, we may assume that A is $\mathcal{Q}$-stable. Let $G \leqslant K_0(A)$ be finitely generated and singular. Let

$$G = \operatorname{span}_{\mathbb{Z}}\{g_1, g_2, \dots, g_m\}, \text{ where } g_i = [p_i^{(0)}]_0 - [s(p_i^{(0)})]_0, \text{ with } p_i^{(0)} \in P_w(\widetilde{A}).$$

Note that because $K_0(A)$ is torsion free and the structure theorem of finitely generated abelian groups, G is a free $\mathbb{Z}$-module, hence the g_i's are linearly independent. Because, nuclearity, quasidiagonality, separability and the ordered K_0 group are not affected when passing to matrix algebras, we may assume that $p_i^{(0)} \in \widetilde{A}$. Set

$$P = \{p_i^{(0)}, i = 1, 2, \dots, m\}$$

and fix an increasing sequence $P \subset F_n \subset\subset \widetilde{A}$ with $\overline{\bigcup F_n} = \widetilde{A}$, $\varepsilon_n \to 0$, $\varepsilon_n < \frac{1}{80}$ for every n.

By assumption (and the comments before the statement of this proposition) there exist $C_n \in \mathcal{C}, C_n \subset A$ such that $F_n \subset_{\varepsilon_n} \widetilde{C}_n$, where $\widetilde{C}_n = C_n + \mathbb{C}1_{\widetilde{A}}$. By [25, Lemma 6.3.1] we can find $p_{in} \in P(\widetilde{C}_n)$ such that $\|p_{in} - p_i^{(0)}\| < 1$. Moreover, we can find a sequence $(\overline{F}_n)_n$ of finite subsets of $\widetilde{C}_n$ such that $p_{in} \in \overline{F}_n$ and for every $a \in F_n$ there exists $b \in \overline{F}_n : \|b - a\| < \varepsilon_n < \frac{1}{80}$.

Observe that $g_i = [p_{in}]_0 - [s(p_{in})]_0$ in $K_0(A)$. Let $\iota : C_n \hookrightarrow A$ be the natural embedding. Consider $L := \operatorname{span}_{\mathbb{Z}}\{l_1, l_2, \ldots, l_m\}$, for some l_i that satisfy $\iota_*(l_i) = g_i$. Such l_i indeed exist because $p_{in} \in P(\widetilde{C}_n)$. Note that these need not be unique, because ι_* is not in general injective. Assume that there exists $v \in L$ such that $v > 0$. Then $\iota_*(v) \geqslant 0$. But we have a contradiction, because $\iota_*(v) \in G$, which is singular, and $\iota_*(v) \neq 0$ by construction of L and the fact that the g_i's are linearly independent. Thus, we can view G as a singular subgroup of $K_0(C_n)$, or even as a singular subgroup of $K_0(\widetilde{C}_n)$ (see Proposition 2.14).

By assumption, C_n has the K_0-embedding property, so there exists $\phi_n : \widetilde{C}_n \to M_{k(n)}$ ucp such that:

$$(\phi_n) \text{ is } (\overline{F}_n, \varepsilon_n) \text{ multiplicative, } \|\phi_n(a)\| \geqslant \|a\| - \frac{10\varepsilon_n}{12}$$
$$\text{for every } a \in \overline{F}_n \text{ and } (\phi_n)_*(G) = 0. \tag{3.1}$$

By Arveson's Extension Theorem, we can extend ϕ_n to $\overline{\phi}_n : \widetilde{A} \to M_{k(n)}$ ucp. Of course, $(\overline{\phi}_n)_*(G) = 0$.

Let $\varepsilon > 0, x, y \in A$. Then there exists $n_0 = n_0(\varepsilon)$ with the following two properties:

$$\forall N \geqslant n_0, \varepsilon_N < \frac{\varepsilon}{24} \quad \text{and} \tag{3.2}$$

$$\exists x_n^{(0)}, y_n^{(0)} \in F_{n_0} \subset F_N \text{ such that } \|x_n^{(0)} - x\| < \frac{\varepsilon}{24}, \|y_n^{(0)} - y\| < \frac{\varepsilon}{24}. \tag{3.3}$$

Moreover, there exist $x_N, y_N \in \overline{F}_N$ such that

$$\|x_N - x_n^{(0)}\| < \varepsilon_N \quad \text{and} \quad \|y_N - y_n^{(0)}\| < \varepsilon_N.$$

Thus

$$\|x_N - x\| < \frac{\varepsilon}{12}, \quad \|y_N - y\| < \frac{\varepsilon}{12}. \tag{3.4}$$

Now if $N \geqslant n_0$, it follows that

$$\begin{aligned}
&\|\overline{\phi}_N(xy) - \overline{\phi}_N(x)\overline{\phi}_N(y)\| \\
&\qquad \leqslant \|\overline{\phi}_N(x_N y_N) - \overline{\phi}_N(x_N)\overline{\phi}_N(y_N)\| + \|\overline{\phi}_N(xy) - \overline{\phi}_N(x_N y_N)\| \\
&\qquad\qquad + \|\overline{\phi}_N(x)\overline{\phi}_N(y) - \overline{\phi}_N(x_N)\overline{\phi}_N(y_N)\| \\
&\qquad < \varepsilon_N + \|xy - x_N y_N\| + \|\overline{\phi}_N(x)\overline{\phi}_N(y) - \overline{\phi}_N(x_N)\overline{\phi}_N(y_N)\|
\end{aligned}$$

$$< \varepsilon_N + 4\frac{\varepsilon}{12} + 4\frac{\varepsilon}{12} < \varepsilon \tag{3.5}$$

where on the second inequality we use (3.1), on the third we use (3.4) and on the fourth we use (3.2). The fact that $\overline{\phi}$ is contractive is used on the second and third inequalities. Moreover, we have

$$\begin{aligned}\|\overline{\phi}_N(x)\| &\geqslant \|\overline{\phi}_N(x_N)\| - \|\overline{\phi}_N(x - x_N)\| \\ &\geqslant \|x_N\| - \frac{10\varepsilon}{12} - \|x - x_N\|\|x\| - \frac{\varepsilon}{12} - \frac{10\varepsilon}{12} - \frac{\varepsilon}{12} = \|x\| - \varepsilon.\end{aligned}$$

Hence $(\overline{\phi}_n)_{n\in\mathbb{N}}$ is asymptotically multiplicative and isometrically isometric. The result yields from Proposition 3.2 and Proposition 3.3. ∎

REMARK 3.6. Let $\mathcal{C}$ be a class that contains only nuclear, separable, quasidiagonal C^*-algebras that have the K_0-embedding property. Let also $A = \varinjlim A_n$, where $A_n \in \mathcal{C}$ for every n. Assume that the connecting maps are injective. Then A is locally approximated by algebras in $\mathcal{C}$. Thus, by Proposition 3.5, A has the K_0-embedding property.

4. CONSTRUCTING AF-EMBEDDINGS WITH CONTROL ON K-THEORY

Let A be a unital, separable, nuclear, C^*-algebra with a faithful trace that satisfies the UCT and $G \leqslant K_0(A)$ a singular subgroup. By [27, Theorem A] A is AF-embeddable. In order to show the K_0-embedding property for A, it is enough to find (for every such G) an AF-algebra B and an embedding $\rho : A \hookrightarrow B$ such that $\rho_*(G)$ is singular. Indeed, because AF-algebras have the K_0-embedding property, there exists an embedding $\phi : B \hookrightarrow D$, where D is a quasidiagonal C^*-algebra and $\phi_*(\rho_*(G)) = 0$. Our desired embedding is $\phi \circ \rho$. However, constructing B and ρ can be very difficult. On the other hand, it is easier to construct the K_0 map. Then, if A is "nice enough" and the K_0 map is chosen suitably, we can "lift" it to the C^*-algebra level. Our main tool is the following theorem, which is a direct corollary of [27, Corollary 5.4]. We would like to thank Jose Carrion and Chris Schafhauser for pointing us out how can it be deduced.

THEOREM 4.1 (cf. Corollary 5.4, [27]). *Assume that A, B are unital, separable C^*-algebras such that: A is nuclear and satisfies the* UCT, *B is a $\mathcal{Q}$-stable* AF-*algebra that has a unique trace τ_B. Assume also that there exist $\sigma : K_0(A) \to K_0(B)$ positive group homomorphism and $\tau_A \in T(A)$ faithful, such that $\sigma([1_A]_0) = [1_B]_0$ and $\widehat{\tau}_A = \widehat{\tau}_B \circ \sigma$. Then there exists a unital, faithful $*$-homomorphism $\phi : A \to B$ such that $\phi_* = \sigma$.*

Proof. Let A, B, τ_A, τ_B and σ as in the hypothesis. Set

$$P := \{x \in K_0(B) : \widehat{\tau}_B(x) > 0\} \cup \{0\}.$$

Then $(K_0(B), P, [1_B]_0)$ is a simple ordered dimension group with unique state, so by the Effros–Handelman–Shen Theorem [25, Theorem 7.2.6] as well as [26, Corollary 1.5.4, Proposition 1.5.5], there exists a unital, separable, simple AF-algebra C with unique trace τ_C such that $(K_0(C), K_0(C)^+, [1_C]_0) \cong (K_0(B), P, [1_B]_0)$. via an order isomorphism $\gamma : K_0(C) \to K_0(B)$. Denote with $\beta : (K_0(C), K_0(C)^+) \to (K_0(B), K_0(B)^+)$ the map that satisfies $\beta(x) = \gamma(x)$ for every $x \in K_0(C)$. Denote also with $\alpha : K_0(A) \to K_0(C)$ the (unique) group homomorphism such that $\beta \circ \alpha = \sigma$. Let $y \in K_0(A)^+ \backslash \{0\}$. Because τ_A is faithful, $\widehat{\tau}_A(y) > 0$, which implies $\widehat{\tau}_B(\sigma(y)) > 0$, so $\sigma(y) \in P \backslash \{0\}$, which implies that $\gamma^{-1} \circ \sigma(y) \in K_0(C)^+ \backslash \{0\}$. Because γ^{-1} and β are set theoretic inverses, we get $a(y) = \gamma^{-1} \circ \sigma(y)$, so α is a positive group homomorphism. Moreover $\alpha[1_A]_0 = [1_C]_0$. By [27, Corollary 5.4], there is a unital, faithful $*$-homomorphism $\rho : A \to C$ such that $\rho_* = \alpha$. Let $x \in K_0(C) \backslash \{0\}$. Then $\widehat{\tau}_C(x) > 0$ which implies $\widehat{\tau}_B(\beta(x)) > 0$. Because $\widehat{\tau}_B$ is the unique state in $(K_0(B), K_0(B)^+)$, it follows from [14, Corollary 4.13] that $\beta(x) > 0$. So β is a positive group homomorphism. Moreover $\beta[1_C]_0 = [1_B]_0$. By Elliott's classification of AF-algebras [25, Theorem 7.3.4], there exists a unital $*$-homomorphism $\psi : C \to B$ such that $\psi_* = \beta$. Because is injective and C simple, ψ is automatically faithful. Thus $\phi := \psi \circ \rho : A \to B$ is a unital, faithful $*$-homomorphism such that $\phi_* = \sigma$. ∎

A natural question to ask is whether K_0-embedding property is preserved under direct sums.

Let A, B separable, unital, nuclear and quasidiagonal with the K_0-embedding property and $G \leqslant K_0(A \oplus B) \simeq K_0(A) \oplus K_0(B)$ be a singular subgroup. If $G = G_1 \oplus G_2$, where $G_1 \leqslant K_0(A), G_2 \leqslant K_0(B)$ singular subgroups, then everything works fine. The problem, however, is that G can be way more complicated. Let $x \in K_0(A)$ singular with the property that $mx \neq 0$ for every $m \in \mathbb{N}^*$. Then $(x, -[1]_0), (x, [1]_0), (x, 0)$ are all singular. Thus, if $A \oplus B$ has the K_0-embedding property, then there have to be $\phi_i : A \hookrightarrow B_i, i = 1, 2, 3$, where B_i is quasidiagonal, such that $(\phi_1)_*(x) > 0$, $(\phi_2)_*(x) < 0$ and $(\phi_3)_*(x) = 0$. We will show that this indeed happens if there exists $\tau \in T(A)$ faithful such that $\widehat{\tau}(x) = 0$ (Proposition 4.3). But first we need a simple lemma.

LEMMA 4.2. *Let G be a countable abelian group that is also a $\mathbb{Q}$-vector space. Then there exists a total order on G.*

Proof. Because G is countable and a $\mathbb{Q}$-vector space, it has a countable (Hamel) basis, call it $\mathcal{B}$. If the basis is finite, then $G \cong \mathbb{Q}^n$ for some $n \in \mathbb{N}$, while if the basis is countably infinite, then $G \cong c_{00}(\mathbb{Q})$. In any case, we may see the elements of G as sequences. We will put an order as follows.

If $x = (x_n)_n$ and $y = (y_n)_n$, $x, y \in G$, then we initially look at the first coordinate. If $x_1 < y_1$, set $x \preceq y$. If $x_1 > y_1$, set $y \preceq x$. If $x_1 = y_1$, we look at the second coordinate. If $x_2 < y_2$, set $x \preceq y$. If $x_2 > y_2$, set $y \preceq x$. If $x_2 = y_2$, we look at the third coordinate, etc. In this way we have defined a total order $\preceq$ on G. Note that this order is heavily used and is called the lexicographic order. ∎

Now we can prove the result we promised.

PROPOSITION 4.3. *Let A be a separable, unital, nuclear C^*-algebra that satisfies the* UCT. *Let also $\tau \in T(A)$ be a faithful trace and $x \in K_0(A)$ such that x is non-torsion and $\widehat{\tau}(x) = 0$. Then, there exist faithful, unit preserving $*$-homomorphisms $\phi_i : A \to B_i$, where $i = 1, 2, 3$, and B_i are unital* AF*-algebras, such that*

$$(\phi_1)_*(x) > 0, \quad (\phi_2)_*(x) < 0 \quad \textit{and} \quad (\phi_3)_*(x) = 0.$$

Proof. After tensoring with the universal UHF-algebra, Proposition 2.7 allows us to assume that A is $\mathcal{Q}$-stable. Then $K_0(A)$ is countable and also a $\mathbb{Q}$-vector space. Hence, by Lemma 4.2, there exists a total order on $K_0(A)$, call it $\succeq$. Fix a faithful trace $\tau \in T(A)$ and set

$$P = \{a \in K_0(A) : \widehat{\tau}(a) > 0\} \cup \{a \in K_0(A) : \widehat{\tau}(a) = 0 \text{ and } a \succeq 0\} \subset K_0(A).$$

Observe that because $\succeq$ is a total order, $(K_0(A), P, [1_A]_0)$ is a scaled, totally ordered (hence dimension) group. Indeed, let $a, b \in P$. Then $\widehat{\tau}(a), \widehat{\tau}(b) \geqslant 0$. Thus $\widehat{\tau}(a+b) \geqslant 0$. If $\widehat{\tau}(a+b) > 0$, then by construction $a + b \in P$. If $\widehat{\tau}(a+b) = 0$, we must have $\widehat{\tau}(a) = \widehat{\tau}(b) = 0$ which implies $a \succeq 0$ and $b \succeq 0$. Thus $a + b \succeq 0$, which, along with $\widehat{\tau}(a+b) = 0$, implies that $a + b \in P$. Moreover, if $a \in P \cap -P$, we deduce that $\widehat{\tau}(a) \geqslant 0$ and $\widehat{\tau}(a) \leqslant 0$. Hence $\widehat{\tau}(a) = 0$. So $a \in P$ implies $a \succeq 0$ while $a \in -P$ implies $0 \succeq a$. Thus $a = 0$. This shows that $P \cap -P = \{0\}$. If $a \in K_0(A)$, then we have three cases. We either have $\widehat{\tau}(a) > 0$, which yields $a \in P$, or $\widehat{\tau}(a) < 0$, which yields $a \in -P$, or $\widehat{\tau}(a) = 0$. In this case, if $a \succeq 0$, then $a \in P$, while if $0 \succeq a$, it follows that $a \in -P$. So, the order is total. Moreover, if $x \in K_0(A)^+ \backslash \{0\}$, then, because τ is faithful, we have that $\widehat{\tau}(x) > 0$ which implies $x \in P$. So $K_0(A)^+ \subset P$. Hence, $[1_A]_0$ is still an order unit and

$$\beta : (K_0(A), K_0(A)^+, [1_A]_0) \to (K_0(A), P, [1_A]_0)$$

is a positive group homomorphism sending the order unit to the order unit.

Claim. $S(K_0(A), P, [1_A]_0) = \{\widehat{\tau}\}$.

Proof. Let $\rho \in S(K_0(A), P, [1_A]_0)$ and $y \in K_0(A)$. For better notation set also $u = [1_A]_0$. Assume that $\rho(y) \neq \widehat{\tau}(y)$. Assume first that $\rho(y) < \widehat{\tau}(y)$. Then $\exists q \in \mathbb{Q} : \rho(y) < q < \widehat{\tau}(y)$. Thus

$$\widehat{\tau}(y - qu) > 0 \Rightarrow y - qu \geqslant 0 \Rightarrow y \geqslant qu \xRightarrow{\rho \in S(K_0(A), P^+)} \rho(y) \geqslant \rho(qu) \Rightarrow \rho(y) \geqslant q,$$

contradiction. The case $\rho(y) > \widehat{\tau}(y)$ is contradicted in an identical way. Because y is arbitrary, $\rho = \widehat{\tau}$. ∎

By the Effros–Handelman–Shen Theorem [25, Theorem 7.2.6], there is a unital, separable, AF-algebra B_1 such that $(K_0(B_1), K_0(B_1)^+, [1_{B_1}]_0) \cong (K_0(A), P, [1_A]_0)$ via an order isomorphism $\gamma : K_0(A) \to K_0(B_1)$. Because of the Claim and the fact that the map Ψ in (2.5) is a bijection, B_1 has a unique trace, call it τ_{B_1} that satisfies $\widehat{\tau}_{B_1} \circ \beta = \widehat{\tau}$. Also B_1 is $\mathcal{Q}$-stable because $K_0(A) \cong K_0(A) \otimes \mathbb{Q}$. Indeed,

$$(K_0(B_1), K_0(B_1)^+, [1]_0) \cong (K_0(B_1 \otimes \mathcal{Q}), K_0(B_1 \otimes \mathcal{Q})^+, [1]_0)$$

which implies

$$B_1 \cong B_1 \otimes \mathcal{Q}$$

by Elliott's Classification Theorem for AF-algebras [25, Theorem 7.3.4]. Furthermore, we have the following commutative diagram:

$$\begin{tikzcd} K_0(A) \arrow[r, "\gamma\circ\beta"] \arrow[dr, "\widehat{\tau}"'] & K_0(B_1) \arrow[d, "\widehat{\tau}_{B_1}"] \\ & \mathbb{R} \end{tikzcd}$$

and we also have $\beta[1_A]_0 = [1_{B_1}]_0$.

So by Theorem 4.1, there is a unital, faithful $*$-homomorphism $\phi_1 : A \hookrightarrow B_1$ such that $(\phi_1)_* = \beta$.

For $x, y \in K_0(A)$ we define a new order $\preceq$ such that $x \preceq y$ if and only if $y \succeq x$. Notice that $\preceq$ is also a total order on $K_0(A)$. If we use $\preceq$, instead of $\succeq$ as our total order and do the same work as before, we can find B_2 unital, AF and $\phi_2 : A \to B_2$ faithful $*$-homomorphism. Fix any $x \neq 0$ such that $\widehat{\tau}(x) = 0$. Then, out of $(\phi_1)_*(x)$ and $(\phi_2)_*(x)$, one is positive and the other negative. Without loss of generality $x \succeq 0$. Then $(\phi_1)_*(x) > 0$ and $(\phi_2)_*(x) < 0$. Finally, the existence of ϕ_3 is already known from the proof of [27, Theorem A]. ∎

A natural question to ask is how this total order on $K_0(A)$ looks like. To get some idea, we will exhibit the following (basic) example.

EXAMPLE 4.4. Let $A = \mathcal{Q} \oplus \mathcal{Q}$ and consider the faithful trace τ such that $\tau(a, b) = \frac{\sigma(a)+\sigma(b)}{2}$, where σ is the unique trace on $\mathcal{Q}$. Let us also use the following total order on $K_0(A) = \mathbb{Q} \oplus \mathbb{Q}$ (which will help us order the elements in $\ker(\widehat{\tau})$): define $(x, y) \succeq (a, b)$ if and only if either $x > a$ or $x = a$ and $y \geqslant b$. Then $P = \{(a, b) : a + b > 0\} \cup \{(a, -a) : a \geqslant 0\}$. So if we see $\mathbb{Q} \oplus \mathbb{Q}$ as the points on the Euclidean plane with rational coefficients, then P contains everything to the right of the line with equation $x + y = 0$ plus the bottom part of the line.

The following corollary is immediate from the proof of Proposition 4.3, but we write it down, as it has its own independent interest, and we will also need it in Section 6.

COROLLARY 4.5. *Let A be a separable, unital, nuclear C^*-algebra that satisfies the* UCT *and $\tau \in T(A)$ is faithful trace. Then there exists a unital, faithful $*$-homomorphism $\phi : A \to B$, where B is unital* AF*-algebra that is $\mathcal{Q}$-stable, and $\phi_*(x)$ is non-zero singular for every $x \in K_0(A)$ non-torsion with $\widehat{\tau}(x) = 0$.*

Proof. Take $\phi = \phi_1 \oplus \phi_2$, where ϕ_1, ϕ_2 are as in the proof of Proposition 4.3. ∎

From the result of Corollary 4.5, the following question arises naturally: if A is a separable, quasidiagonal, nuclear, unital C^*-algebra and $x \in K_0(A)$ singular, when is it true that x can be "killed" by a state induced by a faithful trace? If A is also simple, this always happens because of Remark 2.13. Unfortunately, in the non-simple case this fails even for a commutative C^*-algebra.

EXAMPLE 4.6. Consider $A = C(S^2) \oplus \mathbb{C}$.

In [1, Example 6.3.4, (d)], it is shown that

$$K_0(C(S^2)) = \mathbb{Z}^2, K_0(C(S^2))^+ = \{(x, y), x > 0\} \cup \{(0, 0)\} \quad \text{and} \quad [1_{C(S^2)}] = (1, 0).$$

Set $x = (0, 1, -1) \in K_0(A)$. Obviously $\mathbb{Z}x \cap K_0(A)^+ = \{0\}$.

Moreover, $u = (1, 0, 1)$ is an order unit of $K_0(A)$ and if $y = (0, 1, 0)$, it follows that $u + ny > 0 \ \ \forall n \in \mathbb{Z}$. By (2.6) and (2.7), we have that

$$\rho(y) = 0 \quad \forall \rho \in S(K_0(A)). \tag{4.1}$$

Assume that $\rho_0(x) = 0$ for some $\rho_0 \in S(K_0(A))$. Then (4.1) implies

$$0 = \rho_0(x) = \rho_0(x - y) = \rho_0(0, 0, -1).$$

But $(0, 0, -1) < 0$, so ρ_0 cannot be induced by a faithful trace.

5. NEW EXAMPLES OF C^*-ALGEBRAS WITH THE K_0-EMBEDDING PROPERTY

Let $\mathcal{G}$ be the class of all separable, unital, nuclear, quasidiagonal C^*-algebras that satisfy the UCT, and have the property that every state in $K_0(A)$ is induced by a faithful trace.

Because

$$T(A \otimes \mathcal{Q}) = \{\tau \otimes \sigma : \tau \in T(A)\},$$

where σ is the unique trace on $\mathcal{Q}$, it follows that $A \in \mathcal{G}$ if and only if $A \otimes \mathcal{Q} \in \mathcal{G}$.

A natural question to ask is how big this class is.

First of all, it has to be noted that by [27, Theorem A] all the algebras in the class, are embeddable (in a unit preserving way) into simple, AF-algebras.

PROPOSITION 5.1. *Every $A \in \mathcal{G}$ has the K_0-embedding property.*

Proof. Let $A \in \mathcal{G}$. After tensoring with the Universal UHF-algebra, we may assume that A is $\mathcal{Q}$-stable because of Proposition 2.7. Recall that the aforementioned comments imply $A \otimes \mathcal{Q} \in \mathcal{G}$. Fix a singular subgroup $H \leqslant K_0(A)$. By Lemma 2.12 for $G = K_0(A)$, $H_1 = H_2 = H$ and the assumption that every state in $K_0(A)$ can be induced by a faithful trace, we deduce that there exists $\tau \in T(A)$ faithful, such that $\widehat{\tau}(H) = 0$ (see Remark 2.13). Result follows from Corollary 4.5 and the fact that AF-algebras have the K_0-embedding property. ∎

We will now exhibit some examples of C^*-algebras in $\mathcal{G}$.

PROPOSITION 5.2. *If X is a separable, compact, Hausdorff and connected topological space, then $C(X) \in \mathcal{G}$.*

Proof. By [1, Example 6.10.3 and Corollary 6.3.6], $K_0(C(X))$ is a simple ordered group with unique state, call it ρ. Because the map in (2.5) is onto, ρ is induced by (any) faithful trace in $C(X)$. Finally it is well-known that $C(X)$ is nuclear, quasidiagonal and satisfies the UCT. Hence $C(X) \in \mathcal{G}$. ∎

PROPOSITION 5.3. *If A is separable, nuclear, unital, simple, satisfies the* UCT *and has a trace, then $A \in \mathcal{G}$.*

Proof. First, notice that by [29, Corollary B], A is automatically quasidiagonal. Because the map in (2.5) is onto, every state in $K_0(A)$ can be induced by a trace. But A is simple, so all traces are automatically faithful. Result follows. ∎

PROPOSITION 5.4. *Let A be unital, separable, nuclear, satisfies the* UCT *and $\sigma : \mathbb{Z} \to \mathrm{Aut}(A)$ be a minimal action. Assume that A has a σ-invariant trace. Then $A \rtimes_\sigma \mathbb{Z} \in \mathcal{G}$.*

Proof. $A \rtimes_\sigma \mathbb{Z}$ is unital and separable. By [6, Theorem 4.2.4] $A \rtimes_\sigma \mathbb{Z} \simeq A \rtimes_{\sigma,\mathrm{r}} \mathbb{Z}$ and $A \rtimes_\sigma \mathbb{Z}$ is nuclear. Because A has a σ-invariant trace, $A \rtimes_\sigma \mathbb{Z}$ is quasidiagonal by [27, Corollary 6.5]. Because the bootstrap class in closed under taking crossed products with $\mathbb{Z}$ [1, 22.3.5], $A \rtimes_\sigma \mathbb{Z}$ satisfies the UCT. Let now $\rho \in S(K_0(A \rtimes_\sigma \mathbb{Z}))$. Because $A \rtimes_\sigma \mathbb{Z}$ is nuclear hence exact, the map in (2.5) is onto, so there is $\tau \in T(A \rtimes_\sigma \mathbb{Z})$ such that $\widehat{\tau} = \rho$. The restriction $\tau|_A$ is an invariant trace. Notice that $N_{\tau|_A} = \{x \in A : \tau|_A(x^*x) = 0\}$ is a σ-invariant ideal. Because the action is minimal, $N_{\tau|_A} = 0$, so $\tau|_A$ is faithful. Hence $\tau|_A \circ E \in T(A \rtimes_\sigma \mathbb{Z})$, where $E : A \rtimes_\sigma \mathbb{Z} \to A$ is the conditional expectation that sends $\sum\limits_{g \in \mathbb{Z}} a_g g$ to a_0, is also faithful. From the comments in [1, p. 84], we have that $\widehat{\tau|_A \circ E} = \rho$ (see appendix for a detailed proof of this). Thus $A \rtimes_\sigma \mathbb{Z} \in \mathcal{G}$. ∎

If the action σ is trivial, then it is minimal if and only if A is simple. In this case, $A \rtimes \mathbb{Z} = A \otimes C(\mathbb{T})$. Thus, if A is simple, $A \otimes C(\mathbb{T}) \in \mathcal{G}$. Actually, in order to deduce that the tensor product is in $\mathcal{G}$, we can weaken the simplicity condition for A, and assume $A \in \mathcal{G}$ instead.

PROPOSITION 5.5. *Let $A \in \mathcal{G}$. Then $A \otimes C(\mathbb{T}) \in \mathcal{G}$.*

Proof. Assume that $A \in \mathcal{G}$. There is a split exact sequence

$$0 \longrightarrow SA \xrightarrow{\iota} A \otimes C(\mathbb{T}) \xrightarrow{\pi} A \longrightarrow 0 ,$$

where we identify $A \otimes C(\mathbb{T})$ with $C(\mathbb{T}, A)$, ι is the inclusion map and $\pi(f) = f(1)$. So, $K_0(A \otimes C(\mathbb{T})) \cong K_0(A) \oplus K_1(A)$. By [23, Proposition 5.7], we have that $([1_A]_0, b) \in K_0(A \otimes C(\mathbb{T}))^+$ for every $b \in K_1(A)$. Hence $(0, b) \in \mathrm{Inf}\, K_0(A \otimes C(\mathbb{T}))$ for every $b \in K_1(A)$. Let $\rho \in S(K_0(C(\mathbb{T}) \otimes A))$. Then $\rho(a, b) = \rho_0(a)$, where $\rho_0 \in S(K_0(A))$. By assumption, $\rho_0 = \widehat{\tau}_0$, where τ_0 is a faithful trace on A. Observe that $\rho = \widehat{\tau}$, where $\tau = \tau_0 \otimes \sigma$ and σ is (any) faithful trace on $C(\mathbb{T})$. So τ is faithful and hence $A \otimes C(\mathbb{T}) \in \mathcal{G}$. ∎

On the other hand, notice that non-trivial direct sums of unital, quasidiagonal C^*-algebras cannot be on $\mathcal{G}$.

So, it is natural to set

$$\mathcal{O} = \Big\{ \bigoplus_{i=1}^{n} A_i : n \in \mathbb{N}^* \text{ and for every } i, \text{ either } A_i \in \mathcal{G} \text{ or } A_i \text{ is an AF-algebra} \Big\}.$$

Corollary 4.5 gives us an indication that the elements of this class should have the K_0-embedding property. We will show that this indeed happens.

PROPOSITION 5.6. *If $A \in \mathcal{O}$, then A has the K_0-embedding property.*

Proof. Let $A \in \mathcal{O}$. Then $A = \bigoplus\limits_{i=1}^{n} A_i$, for some $A_i \in \mathcal{G}$ or AF-algebra. After tensoring with the Universal UHF-algebra, Proposition 2.7, allows us to assume that A_i is $\mathcal{Q}$-stable for every A_i. Recall that $K_0(A) = \bigoplus\limits_{i=1}^{n} K_0(A_i)$. Let $G \leqslant K_0(A)$ singular. Because A_i is separable, $K_0(A_i)$ is countable. Because of Proposition 2.14 and its proof we may assume that if A_i is an AF-algebra, then it is unital. Because every singular subgroup is contained in a maximally singular subgroup, we may assume that G is maximally singular. Because of Lemma 2.9, we have that G satisfies (2.2).

We want to find an embedding ϕ of A into an AF-algebra B such that $\phi_*(G)$ is singular. The easiest way to do this, is to find embeddings $\phi_i : A_i \hookrightarrow B_i$ and take ϕ to be their direct sum. Because it is not easy to construct the AF-algebras B_i explicitly, we will first construct the maps on the K_0-level and then use Theorem 4.1 to "lift" to the C^*-algebra level. Note that if some A_i is an AF-algebra, we might not be able to use Theorem 4.1, as it is even possible that A_i does not have any faithful trace. However, in this case every positive group homomorphism $\pi_i : K_0(A_i) \to K_0(B_i)$ with $\pi_i[1] = [1]$, lifts to the C^*-algebra level (recall that A_i is an AF-algebra in this case). So, we need to find positive group homomorphisms $\pi_i : K_0(A_i, K_0(A_i)^+, [1]_0) \to (H_i, P_i, u_i)$ with $\pi_i([1]) = u_i$ and (H_i, P_i) dimension groups. One way to construct a dimension group, is to construct a totally ordered group. We will also want to secure that (H_i, P_i, u_i) has a unique state. Finally, we want to define traces $\tau_i \in T(A_i)$ such that the unique state on (H_i, P_i, u_i) is the composition of $\widehat{\tau}_i$ with π_i.

For every $i = 1, 2, \ldots, n$ set

$$G_i^{\text{zero}} = \{x_i \in K_0(A_i) \text{ such that } (0, 0, \ldots, x_i, 0, \ldots, 0) \in G\}.$$

We have that $G_i^{\text{zero}} \cap K_0(A_i)^+ = \{0\}$ and also G_i^{zero} satisfies (2.2) because G satisfies (2.2). Thus, if $H_i = K_0(A_i)/G_i^{\text{zero}}$ is endowed with the order as defined in (2.3), $(H_i, H_i^+, \overline{[1_{A_i}]_0})$ is a scaled ordered, unperforated group by Lemma 2.10

and $H_i \cong H_i \otimes \mathbb{Q}$. The latter holds because $K_0(A_i) \cong K_0(A_i) \otimes \mathbb{Q}$ and $G_i^{\text{zero}} \cong G_i^{\text{zero}} \otimes \mathbb{Q}$.

If we want to achieve our goal, we must kill the elements of G_i^{zero}.

Let $\pi_i : K_0(A_i) \to H_i$ be the quotient map and

$$H = \bigoplus_{i=1}^{n} H_i, \quad H^+ = \bigoplus_{i=1}^{n} H_i^+, \quad \pi = \bigoplus_{i=1}^{n} \pi_i : K_0(A) \to H.$$

Our first claim allows us to make sure that after moving to the quotient our group is still maximally singular and there are no more nonzero elements of the form $(0,0,\dots,a,0,\dots,0)$.

Claim 1. $\pi(G)$ is maximally singular and if $y_i \in H_i$ with $(0,\dots,0,y_i,0,\dots,0) \in \pi(G)$, then $y_i = 0$.

Proof of Claim 1. Assume that there exists $y \in H^+ \cap \pi(G)$ with $y \neq 0$. Then

$$y = \pi(x) \text{ for some } x = (x_1,\dots,x_n) \in G. \tag{5.1}$$

Because $y \in H^+$, for each $i = 1,2,\dots,n$ there exists

$$r_i \in G_i^{\text{zero}} : x_i + r_i \geqslant 0. \tag{5.2}$$

Because $y \neq 0$, there exists i_0 such that

$$x_{i_0} + r_{i_0} > 0. \tag{5.3}$$

Because $r_i \in G_i^{\text{zero}}$,

$$(0,0,\dots,0,r_i,0,\dots,0) \in G. \tag{5.4}$$

(5.1), (5.2), (5.3), (5.4) yield that

$$0 < (x_1 + r_1, \dots, x_n + r_n) \in G$$

contradicting the fact that G is singular. Hence, $\pi(G)$ is singular.

To show maximality assume for the sake of contradiction that

$$L \supsetneq \pi(G), \quad L \cap H^+ = \{0\}. \tag{5.5}$$

This yields that

$$\pi^{-1}(L) \cap K_0(A)^+ = \{0\}. \tag{5.6}$$

Indeed, if there exists $z \in \pi^{-1}(L), z > 0$, then we have that $\pi(z) \in L$ and $\pi(z) \geqslant 0$. However, if $\pi(z) = 0$, then $z = (z_1,\dots,z_n)$ for some $z_i \in G_i^{\text{zero}}$, which contradicts the fact that $z > 0$. So, $\pi(z) > 0$ which contradicts $L \cap H^+ = \{0\}$.

(5.5), (5.6), along with the maximality of G give that $\pi^{-1}(L) = G$ so $\pi(\pi^{-1}(L)) = \pi(G)$ which implies $L = \pi(G)$, contradiction. So, $\pi(G)$ is maximally singular.

To prove the last statement, assume for the sake of contradiction that

$$(0,\dots,0,y_i,0,\dots,0) \in \pi(G) \tag{5.7}$$

where $y_i \in H_i$ for some i. Then there exist

$$x = (x_1,\dots,x_n) \in G \tag{5.8}$$

such that

$$\pi(x) = (0, \dots, 0, y_i, 0, \dots, 0).$$

If $j \neq i$, then $\pi_j(x_j) = 0$. Thus $x_j \in G_j^{\text{zero}}$. Moreover, $\pi_i(x_i) = y_i$.

Because $x_j \in G_j^{\text{zero}}$, it follows that

$$(0, \dots, 0, x_j, 0, \dots, 0) \in G \tag{5.9}$$

for every $j \neq i$. Finally, (5.8), (5.9) $\Rightarrow (0, \dots, 0, x_i, 0, \dots, 0) \in G \Rightarrow x_i \in G_i^{\text{zero}} \Rightarrow y_i = 0$ as desired. ∎

Set

$H_i^{\text{neg}} = \{x_i \in H_i,$ such that $\exists (a_i, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n) \in \pi(G),$ and $a_j \geqslant 0\ \forall j \neq i\}$ and

$H_i^{\text{pos}} = \{x_i \in H_i,$ such that $\exists (a_i, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n) \in \pi(G),$ and $a_j \leqslant 0\ \forall j \neq i\}$.

By construction of H_i,

$$H_i^{\text{pos}} \cap H_i^{\text{neg}} = \{0\}, \quad H_i^{\text{pos}} = -H_i^{\text{neg}}, \quad H_i^{\text{pos}} \text{ and } H_i^{\text{neg}} \text{ are semigroups and } H_i^{\text{neg}} \cap H_i^{+} = \{0\}. \tag{5.10}$$

In order to achieve our goal, we must make (on the totally ordered group) the elements of H_i^{pos} positive and the elements of H_i^{neg} negative.

On the next claim, we will define our traces.

Claim 2. For every $i = 1, 2, \dots, n$, there exists $\tau_i \in T(A_i)$, which can be taken faithful if $A_i \in \mathcal{G}$, such that:

(i) $\widehat{\tau}_i(G_i^{\text{zero}}) = 0$;

(ii) if $\overline{\tau}_i$ is the induced state on H_i, then $\overline{\tau}(z) \geqslant 0$ for every $z \in H_i^{\text{pos}}$ and thus $\overline{\tau}(z) \leqslant 0$ for every $z \in H_i^{\text{neg}}$.

Proof of Claim 2. The fact that G_i^{zero} is singular, (5.10) and $K_0(A_i) \cong K_0(A_i) \otimes \mathbb{Q}$ guarantee that the assumptions of Lemma 2.12 are satisfied for $G_i^{\text{zero}} = H_1$ and $\pi_i^{-1}(H_i^{\text{pos}}) = H_2$. Result follows from the fact that the map in (2.5) is onto and Lemma 2.12 itself. Note that if $A_i \in \mathcal{G}$ for some i, then every state can be induced by a faithful trace, which allows us to choose τ_i to be faithful (see Remark 2.13). ∎

Now, we are ready to define our total orders.

Let $x_i \neq 0, x_i \in H_i$. Then by maximality of $\pi(G)$ and the fact that $x = (0, 0, \dots, x_i, 0, \dots, 0) \notin \pi(G)$, we deduce that

$$\exists k \in \mathbb{Z} \text{ and } a = (a_1, a_2, \dots, a_n) \in \pi(G), \text{ such that } a + kx > 0 \text{ . Thus}$$

$$a_i + kx_i \geqslant 0. \tag{5.11}$$

Observe that $\forall l \neq i$, we have $a_l \geqslant 0$, so

$$a_i \in H_i^{\mathrm{neg}}. \tag{5.12}$$

Define $\Phi_i : H_i \to \{0, -1, 1\}$ such that

$$\Phi_i(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \neq 0 \text{ and } \exists k \in \mathbb{N}^* \text{ and } a \in H_i^{\mathrm{neg}} : a + kx \geqslant 0, \\ -1 & \text{if } x \neq 0 \text{ and } \exists k \in \mathbb{Z}_{<0} \text{ and } a \in H_i^{\mathrm{neg}} : a + kx \geqslant 0. \end{cases} \tag{5.13}$$

As expected, the elements that take value 1, will be our positive elements (on the totally ordered groups), while the elements that take value -1 will be our negative elements. In Claims 3–5 we will show the properties of Φ_i needed to make sure that we will end up with total orders.

Claim 3. Φ_i is well-defined.

Proof of Claim 3. First of all, (5.11) and (5.12) yield that $\forall x \in H_i, \Phi_i(x)$ can take at least one value, according to the definition. Assume that $\exists x \in H_i \backslash \{0\}, n_1 > 0, n_2 < 0 \, (n_i \in \mathbb{Z}), a, b \in H_i^{\mathrm{neg}}$, such that $a + n_1 x \geqslant 0$ and $b + n_2 x \geqslant 0$. Notice that if $a = 0$ and $b = 0$, we get that $x = 0$, contradiction. So at least one of a, b is nonzero. In addition, we have

$$\begin{cases} a + n_1 x \geqslant 0, \\ b + n_2 x \geqslant 0, \end{cases} \Rightarrow \begin{cases} |n_2| a + n_1 |n_2| x \geqslant 0, \\ n_1 b + n_1 n_2 x \geqslant 0, \end{cases} \xRightarrow[\text{add together}]{n_2 < 0} |n_2| a + n_1 b \geqslant 0.$$

Because $a, b \in H_i^{\mathrm{neg}}, n_1, n_2 \neq 0$ and by (5.10), we deduce that $a = b = 0$, contradiction. So Φ_i is well-defined. ∎

Claim 4. (i) If $\Phi_i(x) = \Phi_i(y) = 1$, then $\Phi_i(x + y) = 1$.
(ii) If $\Phi_i(x) = \Phi_i(y) = -1$, then $\Phi_i(x + y) = -1$.

Proof of Claim 4. We will only prove the first statement. The second one can be proved in an identical way.

Assume that $\Phi_i(x) = \Phi_i(y) = 1$. Then $x, y \neq 0$ and $\exists a, b \in H_i^{\mathrm{neg}}, k, m \in \mathbb{N}^*$, such that

$$\begin{cases} a + kx \geqslant 0, \\ b + my \geqslant 0, \end{cases} \Rightarrow \begin{cases} mkx + ma \geqslant 0, \\ mky + kb \geqslant 0, \end{cases} \Rightarrow ma + kb + mk(x + y) \geqslant 0. \tag{5.14}$$

If

$$x + y = 0, \tag{5.15}$$

then it has to be $ma + kb \geqslant 0$. Because $a, b \in H_i^{\mathrm{neg}}, m, k \in \mathbb{N}^*$ and by (5.10) we deduce that $a = b = 0$. But this yields that $x, y \geqslant 0$. By (5.15) it follows that $x = y = 0$, contradiction. So $x + y \neq 0$ and (5.14) gives $\Phi_i(x + y) = 1$. ∎

Claim 5. $\Phi_i(-x) = -\Phi_i(x)$.

Proof. This is obvious. ∎

Claim 6 will allow us to guarantee that the image on G under the K_0 map of the AF-embedding will be singular.

Claim 6. There is no $(x_1, x_2, \dots, x_n) \in \pi(G)\backslash\{0\}$ such that for every $i = 1, 2, \dots, n$, we have $\Phi_i(x_i) \geqslant 0$ (hence the same statement with negatives is true).

Proof of Claim 6. Assume the contrary. Then there exists

$$(x_1, x_2, \dots, x_n) \in \pi(G)\backslash\{0\} \text{ such that } \forall i, \Phi_i(x_i) \geqslant 0. \tag{5.16}$$

Let

$$\mathcal{J} := \{i : x_i \neq 0\}.$$

By assumption $\mathcal{J} \neq \varnothing$. By (5.16), for each $i \in \mathcal{J}$,

$$(a_1^{(i)}, a_2^{(i)}, \dots, a_{i-1}^{(i)}, m_i x_i + a_i^{(i)}, a_{i+1}^{(i)}, \dots, a_n^{(i)}) > 0 \tag{5.17}$$

for some $m_i \in \mathbb{N}^*, a_i^{(i)} \in H_i^{\text{neg}}, a_j^{(i)} \geqslant 0 \ \forall j \neq i$ and $(a_1^{(i)}, \dots, a_n^{(i)}) \in \pi(G)$. It follows that

$$l_i := \Big(\frac{N}{m_i} a_1^{(i)}, \dots, \frac{N}{m_i} a_{i-1}^{(i)}, \frac{N}{m_i} a_i^{(i)} + N x_i, \dots, \frac{N}{m_i} a_n^{(i)}\Big) > 0, \quad \text{where } N = \prod_{i \in \mathcal{J}} m_i. \tag{5.18}$$

But $\frac{N}{m_i}(a_1^{(i)}, \dots, a_n^{(i)}) \in \pi(G)$ and $N(x_1, \dots, x_n) \in \pi(G)$, so after taking the sum, we have that

$$l := \Big(N x_1 + \sum_{i \in \mathcal{J}} \frac{N}{m_i} a_1^{(i)}, \dots, N x_n + \sum_{i \in \mathcal{J}} \frac{N}{m_i} a_n^{(i)}\Big) \in \pi(G). \tag{5.19}$$

But $l = \sum_{i \in \mathcal{J}} l_i > 0$, contradicting the fact that $\pi(G)$ is singular. ∎

Our next claim, allows us to make sure that the group homomorphism (on the K_0-level) will be positive.

Claim 7. For $x \in H_i$, the following hold:
(i) if $\Phi_i(x) \geqslant 0$, then $\overline{\tau}_i(x) \geqslant 0$;
(i) if $\Phi_i(x) \leqslant 0$, then $\overline{\tau}_i(x) \leqslant 0$.

Proof of Claim 7. It is enough to prove the first statement for $x \neq 0$. Assume that $\Phi_i(x) = 1$. Then there is $k \in \mathbb{N}^*$ and $a \in H_i^{\text{neg}}$ such that $kx + a > 0$. Thus $\overline{\tau}_i(kx) + \overline{\tau}_i(a) \geqslant 0$. But, because $a \in H_i^{\text{neg}}$, it follows that $\overline{\tau}_i(a) \leqslant 0$ by Claim 2, hence $\overline{\tau}_i(kx) \geqslant 0$ which implies $\overline{\tau}_i(x) \geqslant 0$. ∎

Define $P_i = \{x \in H_i, \ \ \Phi_i(x) \geqslant 0\}$. Notice that (H_i, P_i) is a countable, totally ordered group (by Claims 4 and 5), hence it is a dimension group. Claim 7 verifies that

$$\{x \in H_i, \ \overline{\tau}_i(x) > 0\} \subset P_i \quad \text{and} \quad \{x \in H_i, \ \overline{\tau}_i(x) < 0\} \subset -P_i. \tag{5.20}$$

Because $\overline{\tau}_i(\overline{[1_{A_i}]_0}) = \widehat{\tau}_i([1_{A_i}]_0) = 1$, $\overline{[1_{A_i}]_0}$ is an order unit on this new ordered group.

Claim 8 guarantees that each of the totally ordered groups has a unique state, and actually the one needed so that the assumptions of Theorem 4.1 are satisfied.

Claim 8. $S(H_i, P_i, \overline{[1_{A_i}]_0}) = \{\overline{\tau}_i\}$.

Proof of Claim 8. Let $\rho \in S(H_i, P_i, \overline{[1_{A_i}]_0})$ and $x \in H_i$. For better notation set also $u = \overline{[1_{A_i}]_0}$. Assume that $\rho(x) \neq \overline{\tau}_i(x)$. Assume first that $\rho(x) < \overline{\tau}_i(x)$. Then $\exists q \in \mathbb{Q} : \rho(x) < q < \overline{\tau}_i(x)$. Hence we have

$$\overline{\tau}_i(x - qu) > 0 \xRightarrow{(5.20)} x - qu \geqslant 0 \Rightarrow x \geqslant qu \xRightarrow{\rho \in S(H_i)} \rho(x) \geqslant \rho(qu) \Rightarrow \rho(x) \geqslant q,$$

contradiction. Similarly, we can contradict the case $\rho(x) > \overline{\tau}_i(x)$. So $\rho(x) = \overline{\tau}_i(x)$. Because x is arbitrary, $\rho = \overline{\tau}_i$. ∎

By the Effros–Handelman–Shen Theorem [25, Theorem 7.2.6], there are B_i unital, AF-algebras, such that

$$(K_0(B_i), K_0(B_i)^+, [1_{B_i}]_0) \cong (H_i, P_i, \overline{[1_{A_i}]_0}). \tag{5.21}$$

Because of Claim 8, B_i has a unique trace, τ_{B_i} and $\widehat{\tau}_{B_i} = \overline{\tau}_i$. Also B_i is $\mathcal{Q}$-stable because $H_i \cong H_i \otimes \mathbb{Q}$. Furthermore, we have the following commutative diagram:

$$\begin{tikzcd} K_0(A_i) \arrow[r, "\overline{\pi}_i"] \arrow[dr, "\widehat{\tau}_i"'] & K_0(B_i) \arrow[d, "\widehat{\tau}_{B_i}"] \\ & \mathbb{R} \end{tikzcd}$$

where $\overline{\pi}_i$ is the group homomorphism arising from π_i after composing with the order isomorphism implementing (5.21). By (5.20), it is positive. Moreover, we have $\overline{\pi}_i[1_{A_i}]_0 = [1_{B_i}]_0$.

Pick $i \in \{1, 2, 3, \ldots, n\}$. If $A_i \in \mathcal{G}$, τ_i is faithful, so Theorem 4.1 applies and hence there is a faithful $*$-homomorphism $\phi_i : A_i \hookrightarrow B_i$ such that $(\phi_i)_* = \pi_i$.

If A_i is an AF-algebra, then by Elliott's Classification Theorem for AF-algebras [25, Theorem 7.3.4], π_i lifts to a $*$-homomorphism $\phi_i : A_i \to B_i$. Note that if $\ker \phi_i$ is non-zero, then it should contain a non-zero projection p (recall that because A_i is an AF-algebra, it has property (SP) which means that every non-zero hereditary C^*-subalgebra has a non-zero projection). So $\phi_i(p) = 0$, which implies $\pi_i[p]_0 = 0$. But this contradicts the fact that $\ker \phi_i \cap K_0(A_i)^+ = \{0\}$, so ϕ_i is injective for every i. If we set $B = \bigoplus_{i=1}^{n} B_i$, $\phi = \bigoplus_{i=1}^{n} \phi_i$, then $\phi : A \hookrightarrow B$ is faithful. Also B is an AF-algebra and $\phi_* = \pi$. Finally, by Claim 6, $\phi_*(G)$ is singular so there exists D quasidiagonal and $\psi : B \hookrightarrow D$ such that $\psi_*(\phi_*(G)) = 0$. By composing the two maps, we have that $\psi \circ \phi : A \hookrightarrow D$ satisfies $(\psi \circ \phi)_*(G) = 0$. So A has the K_0-embedding property. ∎

Actually, Proposition 5.6 still holds if, when defining $\mathcal{G}$, we replace the condition that all states should be induced by faithful traces, with the following weaker K-theoretic condition:

For every $G_1 \subset G_2 \subset K_0(A \otimes \mathcal{Q})$ with $G_1 \leqslant K_0(A \otimes \mathcal{Q})$ singular subgroup and $G_2 \subset K_0(A \otimes \mathcal{Q})$ subsemigroup with $G_2 \cap -K_0(A \otimes \mathcal{Q})^+ = \{0\}$ and $G_2 \cap -G_2 = G_1$, there exists $\tau \in T(A \otimes \mathcal{Q})$ faithful such that $\widehat{\tau}(G_1) = 0$ and $\widehat{\tau}(z) \geqslant 0$ for every $z \in G_2$.

However, this condition is very technical and we have not managed to find any interesting C^*-algebras that have states that are not induced by any faithful trace, but still satisfy the condition.

Moreover, notice that Proposition 5.3 and Proposition 5.6 yield another proof of the following part of Theorem 1.3: if $A = \bigoplus_{i=1}^{n} A_i$, where each A_i is unital, separable, simple, nuclear, quasidiagonal and satisfies the UCT, B is separable, nuclear, quasidiagonal and satisfies the UCT and

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} B \longrightarrow 0$$

is a short exact sequence, then E is quasidiagonal if and only if is stably finite. This proof does not use either the results of Section 6, or any of the classification results in [29] or [7].

Now we will use the aforementioned results to get new examples of C^*-algebras with the K_0-embedding property.

COROLLARY 5.7. *If A is a separable and commutative C^*-algebra, then A has the K_0-embedding property.*

Proof. First of all, by Proposition 2.14 we may assume that A is unital. Observe that A can be written as an inductive limit $A = \varinjlim C(X_n)$, where the connecting maps are injective and all X_n are separable, compact, Hausdorff with $\dim(X_n) < \infty$. Indeed, let $\{x_n,\ n \in \mathbb{N}\}$ be a dense subset of A, with $x_0 = 1$ and consider $A_n = C^*(1, x_1, \dots, x_n)$. For every $n \geqslant 1$, A_n is unital and commutative so $A_n = C(X_n)$, where X_n is separable, compact and Hausdorff. Moreover, $A = \varinjlim A_n$ where the connecting maps are the inclusions. Finally, by [17, Proposition 1.4], $\dim(X_n) < \infty$. So Remark 3.6 allows us to assume that $A = C(X)$, where $n = \dim(X) < \infty$. But by [12, Theorem 1.10.16], A is locally approximated by algebras $C(X_i)$, where X_i are finite CW-complexes with finitely many connected components. By Proposition 5.6, $C(X_i)$ has the K_0-embedding property for every i. So, A has the K_0-embedding property by Proposition 3.5. ∎

COROLLARY 5.8. *Let $A = \bigoplus_{i=1}^{n} A_i$ be a direct sum of C^*-algebras and $\sigma : \mathbb{Z} \to \mathrm{Aut}(A)$ be an action such that $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_n$ for some $n \in \mathbb{N}$ and minimal actions $\sigma_i : \mathbb{Z} \to \mathrm{Aut}(A_i)$. Assume that each A_i is separable, nuclear, unital, satisfies the UCT and has a σ_i-invariant trace. Then $A \rtimes_\sigma \mathbb{Z}$ has the K_0-embedding property.*

Proof. By [19, Lemma 2.8.2], $A \rtimes_\sigma \mathbb{Z} = \bigoplus_{i=1}^{n} A_i \rtimes_{\sigma_i} \mathbb{Z}$. The result follows from Proposition 5.4 and Proposition 5.6. ∎

6. ASH-ALGEBRAS

Recall that a C^*-algebra is called *subhomogeneous* if there exists a positive integer n, such that every irreducible representation of A is on a Hilbert space of dimension less or equal to n. A C^*-algebra is called *approximately subhomogeneous (ASH)* if it is an inductive limit of subhomogeneous C^*-algebras. We know that ASH-algebras are nuclear, quasidiagonal [26, Chapter 3.4] and satisfy the UCT (every ASH-algebra is an inductive limit of type I C^*-algebras, but these algebras satisfy the UCT by [1, 22.3.5]). In [11] Elliott, Niu, Santiago and Tikuisis defined an important subclass of ASH-algebras.

DEFINITION 6.1 (Definition 2.1, [11]). A *non-commutative cell complex (NCCC)* is a C^*-algebra given by the following recursive definition:

(i) a finite dimensional C^*-algebra is a NCCC;

(ii) if A is a NCCC, $n, k \in \mathbb{N}$, $\phi : A \to C(S^{n-1}, M_k)$ is any unital $*$-homomorphism, and $\psi : C(D^n, M_k) \to C(S^{n-1}, M_k)$ is the restriction homomorphism, then the pullback $A \bigoplus_{C(S^{n-1},M_k)} C(D^n, M_k) = \{(a, f) \in A \oplus C(D^n, M_k) : \phi(a) = \psi(f)\}$ is also a NCCC.

We will allow $n = 0$. In this case we will use the conventions $D^0 = \{pt\}$ and $S^{-1} = \emptyset$. So ϕ will be the zero map in this case.

Note that every NCCC is unital. NCCC are a subclass of recursive subhomogeneous (RSH) -algebras, which were introduced by C. Phillips in [20]. The name "non-commutative cell complex" comes from the fact that in the commutative case, this is equivalent to A being isomorphic to $C(X)$, where X is a (finite) cell complex.

The following definition is on the style of [20, Definition 1.2], but for NCCC.

DEFINITION 6.2. From the previous definition it is clear that every NCCC is of the form

$$A = \Big[\cdots\Big[F_0 \bigoplus_{C(S^{n_1-1},M_{r_1})} C(D^{n_1}, M_{r_1})\Big] \bigoplus_{C(S^{n_2-1},M_{r_2})} C(D^{n_2}, M_{r_2}) \cdots\Big] \bigoplus_{C(S^{n_l-1},M_{r_l})} C(D^{n_l}, M_{r_l}) \tag{6.1}$$

with F_0 finite dimensional. Note that by definition it can be $F_0 = 0$. Set $A_0 = F_0$ and A_i to be the i-th pullback, for every $i > 0$. We will denote with $\phi_i = \phi_i^{(A)} : A_i \to C(S^{n_{i+1}-1}, M_{r_i}), \psi_i = \psi_i^{(A)} : C(D^{n_i}, M_{r_i}) \to C(S^{n_i-1}, M_{r_i})$ the $*$-homomorphisms defining the pullback. Recall that $\phi_i(a) = \psi_{i+1}(f)$ for every $(a, f) \in A_{i+1}$.

An expression of this type will be called a *decomposition* of A. Note that such a decomposition is not unique.

Associated with this decomposition are:

(i) its *length* l;

(ii) its *base spaces* $X_0, X_1, \ldots, X_l$ (where X_0 is a disjoint union of singletons and $X_i = D^{n_i}$, for $i \geqslant 1$) and *total space* $X := \coprod X_k$;

(iii) its *i-th stage algebra* A_i $(i = 0, 1, 2, \ldots, l)$;

(iv) its *(topological) dimension* $\dim(A) = \max\limits_k \dim(X_k)$;

(v) its *standard representation* $\sigma = \sigma^{(A)} : A \to F_0 \oplus (\bigoplus_{i=1}^{l} C(D^{n_i}, M_{r_i}))$ defined by forgetting the restriction to a subalgebra in each of the fibered products in the decomposition;

(vi) the associated *evaluation maps* $\mathrm{ev}_x : A \to M_{r_i}$ for $x \in X$;

(vii) the *rank function* $\mathrm{rank} : P_\infty(A) \to C(X, \mathbb{N})$ via the natural definition (recall that X is the total space).

Before going forward, let us clarify some notation. For the rest of the section, unless clearly stated otherwise, A will be a NCCC of length l with decomposition as in (6.1). Let $p \in P_\infty(A)$. Because of the standard representation, we can view p inside $P_\infty(F_0 \oplus (\bigoplus_{i=1}^{l} C(D^{n_i}, M_{r_i})))$, so we may write $p = (p_0, \ldots, p_l)$, where $p_0 \in P_\infty(F_0)$ and $p_i \in P_\infty(C(D^{n_i}))$ for every $i > 0$. Moreover, for every $i > 0$ we will write $\mathrm{rank}\, p_i$ instead of $\mathrm{rank}\, p_i(x)$, because D^{n_i} is connected, so rank is constant on each D^{n_i}.

When denoting $y = [p]_0 - [q]_0$ we will mean that $p = (p_0, \ldots, p_l)$, $q = (q_0, \ldots, q_l) \in P_\infty(A)$, and we will also write $\overline{p}_s = (p_0, \ldots, p_s), \overline{q}_s = (q_0, \ldots, q_s), y_s = [\overline{p}_s]_0 - [\overline{q}_s]_0$, for $s = 0, 1, 2, \ldots, l$.

The reason why we chose to work with NCCC is the fact that every unital separable, subhomogeneous algebra is locally approximated by NCCC.

PROPOSITION 6.3 (Theorem C, [11]). *Let A be a unital separable subhomogeneous algebra. Then A is locally approximated by* NCCC.

Let A be a NCCC. Our goal is to show that A has the K_0-embedding property (see Proposition 6.12). We first present a sketch of the proof. Fix $y = [p]_0 - [q]_0$, where $p = (p_0, \ldots, p_l), q = (q_0, \ldots, q_l) \in P_\infty(A)$. If there exist $x_1, x_2 \in X$ with $\mathrm{rank}\, p(x_1) > \mathrm{rank}\, q(x_1)$ and $\mathrm{rank}\, p(x_2) < \mathrm{rank}\, q(x_2)$, then $\sigma_*(y)$ is non-zero singular.

If $\mathrm{rank}\, p(x) > \mathrm{rank}\, q(x)$ for every x, then by [20, Proposition 4.3] we have that there exists $M > 0$ such that $My > 0$ in $K_0(A)$.

Assume that $\mathrm{rank}\, p(x) \geqslant \mathrm{rank}\, q(x)$ for every x, but there exists x_0 such that $\mathrm{rank}\, p(x_0) = \mathrm{rank}\, q(x_0)$. Then it follows that $\sigma_*(x) \geqslant 0$, but the previous conclusion is not true anymore, as we cannot use [20, Proposition 4.3]. For example, consider $A = A_1 \oplus A_2$, where A_i is a NCCC for $i = 1, 2$ (it is not difficult to see that A is a NCCC), $p = (p_1, p_2)$ and $q = (q_1, q_2) \in P_\infty(A)$, where rank $p_1(x) >$ rank $q_1(x)$ for every x, $[p_2]_0 - [q_2]_0 \in K_0(A_2)$ is not a torsion element and $[p_2]_0 - [q_2]_0 \in \ker \sigma_*^{(A_2)}$. Then $y = [p]_0 - [q]_0 \in K_0(A)$ is singular but $\sigma_*^{(A)}(y) > 0$. In order to bypass this problem, for every $y = [p]_0 - [q]_0 \in K_0(A)$, with $\sigma_*^{(A)}(y) \geqslant 0$, we will construct a $*$-homomorphism $h_y : A \to D_y(A)$, where $D_y(A)$ is an AF-algebra with the property that if $(h_y)_*(y) \geqslant 0$, then there exists $M > 0$ such that $My \geqslant 0$. This map will be constructed in two steps. First, we will show that we can construct a NCCC $R_y(A)$, which can be obtained from A after deleting all the coordinates $j > 0$ such that rank $p_j >$ rank q_j. To show this, our main tool is Lemma 6.7 while the construction is made right after it. Then we will construct (in Proposition 6.11) a $*$-homomorphism $\Psi_y : A \to R_y(A)$, with the property that if $(\Psi_y)_*(y) \geqslant 0$, then there exists $M > 0$ with $My \geqslant 0$. The key why the latter property will hold is Lemma 6.8. The second step is to construct the $*$-homomorphism to the AF-algebra. This will be achieved by showing the existence of a decomposition $R_y(A) = L_1 \oplus L_2$, such that the first coordinate of $(\Psi_y)_*(y) \in K_0(R_y(A)) \cong K_0(L_1) \oplus K_0(L_2)$ is positive while the second one is an infinitesimal (Lemma 6.10 and Proposition 6.11). Then the desired $*$-homomorphism will be the composition of the projection to the second coordinate with the embedding from Corollary 4.5. Moreover, we will make all such Ψ_y (recall that y runs through the elements of $K_0(A)$ such that rank $p(x) \geqslant$ rank $q(x)$ for every x) to belong to a finite set of $*$-homomorphisms. So, if we take their (finite) direct sum and then add the standard representation, we get a faithful $*$-homomorphism $h : A \to E(A)$. It would not be difficult to show that h sends singular elements to singular elements. Notice that $E(A) \in \mathcal{O}$. So Proposition 5.6 will give us that A has the K_0-embedding property.

Before starting the detailed proof, we need to introduce some more notation.

DEFINITION 6.4. Let A be a NCCC of length l and $y = [p]_0 - [q]_0 \in K_0(A)$, where $p = (p_0, \dots, p_l), q = (q_0, \dots, q_l) \in P_\infty(A)$. We will say that y is *almost positive* if rank $p(x) \geqslant$ rank $q(x)$ for every x on the total space of A.

It is known what the trace simplex $T(A)$ looks like.

LEMMA 6.5 (Corollary 2.5, [10]). *Any trace $\tau \in T(A)$ is of the form*

$$\tau(f_0, \dots, f_l) = a_0 \tau_0(f_0) + \sum_{i=1}^{l} a_i \int\limits_{D^{n_i} \setminus S^{n_i - 1}} \operatorname{tr}(f_i(x)) \mathrm{d}\mu_i(x),$$

where τ_0 is a trace in F_0, μ_i is a probability measure in $D^{n_i} \setminus S^{n_i-1}$, $a_i \in [0,1]$ and $a_0 + a_1 + \cdots + a_l = 1$.

Recall that if X is a contractible topological space, then $(K_0(C(X)), K_0(C(X))^+) \cong (\mathbb{Z}, \mathbb{Z}^+)$. More specifically the order isomorphism is $f \mapsto$ rank f (see for instance [25, Example 3.3.6]). But D^{n_i} is contractible for every $i > 0$. So, $K_0(F_0 \oplus (\bigoplus_{i=1}^{l} C(D^{n_i}, M_{r_i})))$ has no non-zero infinitesimals and equality on the K_0-group coincides with equality of the ranks. This observation allows us to deduce the following lemma.

LEMMA 6.6. *The group of infinitesimals of $K_0(A)$ is*

$$\begin{aligned}\operatorname{Inf}(K_0(A)) &= \ker(\sigma_*)\\ &= \{[f]_0 - [g]_0 : \operatorname{rank} f(x) = \operatorname{rank} g(x) \textit{ for every } x \textit{ in the total space}.\end{aligned}$$

Proof. Let $y = [p]_0 - [q]_0 \in \operatorname{Inf}(K_0(A))$. By Lemma 2.11, $\sigma_*(y) \in \operatorname{Inf} K_0(F_0 \oplus (\bigoplus_{i=1}^{l} C(D^{n_i}, M_{r_i})))$. Hence $\sigma_*(y) = 0$, by the comments above. If $y \in \ker(\sigma_*)$, then, again by the comments above, we have $\operatorname{rank} p(x) = \operatorname{rank} q(x)$ for every x in the total space. Finally, assume that $\operatorname{rank} p(x) = \operatorname{rank} q(x)$ for every x. Then by Lemma 6.5, $\widehat{\tau}(y) = 0$ for every $\tau \in T(A)$. Hence $y \in \operatorname{Inf}(K_0(A))$. ∎

Let $y = [p]_0 - [q]_0 \in K_0(A)$. Define

$$\mathcal{W} = \mathcal{W}_A(y) := \{i \geqslant 1 : \operatorname{rank} p_i > \operatorname{rank} q_i\}.$$

LEMMA 6.7. *Let $y = [p]_0 - [q]_0$ be almost positive with* $\operatorname{rank} p_l = \operatorname{rank} q_l$. *Assume that $\mathcal{W}_A(y) \neq \emptyset$ and set $j := \max \mathcal{W}_A(y)$. Then $\phi_i(0,\ldots,0,a,0,\ldots,0) = 0$ for every $i \geqslant j$ and $a \in \ker \psi_j$.*

Proof. Let $y = [p]_0 - [q]_0$ be almost positive with $\operatorname{rank} p_l = \operatorname{rank} q_l$. Assume that $\mathcal{W}_A(y) \neq \emptyset$ and set $j := \max \mathcal{W}_A(y)$. By assumption, we have that $0 < j < l$. We have $A_{j+1} = \Big[A_{j-1} \bigoplus_{C(S^{n_j-1}, M_{r_j})} C(D^{n_j}, M_{r_j})\Big] \bigoplus_{C(S^{n_{j+1}-1}, M_{r_{j+1}})} C(D^{n_{j+1}}, M_{r_{j+1}})$.

Pick $s \in S^{n_{j+1}-1}$ and denote $\widetilde{\phi}_j := \mathrm{ev}_s \circ \phi_j : A_j \to M_{r_{j+1}}$. Then by [10, Remark 4.7], we get that $\widetilde{\phi}_j$ factors (up to unitary equivalence which we may ignore because it does not affect the K_0 map) through the direct sum via the following commutative diagram:

$$\begin{tikzcd} A_j \arrow[r, "\widetilde{\phi}_j"] \arrow[d, "\sigma"] & M_{r_{j+1}} \\ A_{j-1} \oplus C(D^{n_j}, M_{r_j}) \arrow[r, "\phi_B \oplus \phi_D"] & M_q \oplus M_w \arrow[u, "\iota"'] \end{tikzcd}$$

where σ is the standard representation, ϕ_B, ϕ_D are $*$-homomorphisms (not necessarily unital; they could be even zero), $q + w = r_{j+1}$ and ι is the diagonal embedding. By maximality of j, we have that $\operatorname{rank} p_{j+1} = \operatorname{rank} q_{j+1}$. But $D^{n_{j+1}}$ is contractible, so $[p_{j+1}]_0 = [q_{j+1}]_0$ in $K_0(C(D^{n_{j+1}}))$. Notice that $(\phi_j)_*(y_j) = (\psi_{j+1})_*([p_{j+1}]_0 - [q_{j+1}]_0) = 0$. So,

$$(\widetilde{\phi}_j)_*(y_j) = 0. \tag{6.2}$$

Let $w = \operatorname{rank} p_j - \operatorname{rank} q_j > 0$ and consider $(\phi_D)_* : \mathbb{Z} \to \mathbb{Z}$. By the commutativity of the diagram, as well as (6.2), we can see that $(\phi_D)_*(w) = 0$. Hence $(\phi_D)_* = 0$.

Thus it has to be zero. This means that $\phi_D = 0$. If now $a \in \ker \psi_j$, then from all the aforementioned we deduce that

$$\widetilde{\phi}_j(0,a) = 0 \Rightarrow \phi_j(0,a)(s) = 0.$$

But s is arbitrary, so $\phi(0,a) = 0$.

Let now $l > i > j$ and assume that the result holds for up to $i-1 \geqslant j$. We will show it for i. By repeating the arguments for the base case we get the following commutative diagram (after fixing $s \in S^{n_{i+1}-1}$)

$$\begin{tikzcd} A_i \arrow[rr, "\widetilde{\phi}_i"] \arrow[d, "\sigma"] & & M_{r_{i+1}} \\ A_{j-1} \oplus C(D^{n_j}, M_{r_j}) \oplus \cdots \oplus C(D^{n_i}, M_{r_i}) \arrow[rr, "h_{j-1}\oplus\cdots\oplus h_i"] & & M_{q(j-1)} \oplus \cdots \oplus M_{q(i)} \arrow[u, "\iota"'] \end{tikzcd}$$

where $\widetilde{\phi}_i = \mathrm{ev}_y \circ \phi_i$. Once again σ is the standard representation, ι the diagonal embedding and $h_{j-1}, \ldots, h_i$ are $*$-homomorphisms.

By the same arguments we get $\widetilde{\phi}_i(y_i) = 0$ and $h_j = 0$. By the commutativity of the diagram and the fact that y is arbitrary, $\phi_i(0, \ldots, 0, a, 0, \ldots, 0) = 0$, for every $a \in \ker(\psi_j)$ as desired. Note that the fact that $(0, 0, \ldots, 0, a, 0, \ldots, 0)$ is a well-defined element in A_i is guaranteed by the inductive hypothesis. ∎

Let $y = [p]_0 - [q]_0 \in K_0(A)$ be almost positive and $l > j = \max \mathcal{W}_A(y)$. Note that we still assume $\mathcal{W}_A(y) \neq \emptyset$. Assume that $j < l$. Define

$$\overline{\phi}_j : A_{j-1} \to C(S^{n_{j+1}-1}, M_{r_{j+1}})$$

via $(\overline{\phi}_j)(a) = \phi_j(a,b)$ for every $a \in A_{j-1}$ and some (all) $b \in C(D^{n_j}, M_{r_j})$ such that $(a,b) \in A_j$.

Notice that $\overline{\phi}_j$ is a well-defined, unital $*$-homomorphism (or zero if $\phi_j = 0$). Indeed, if $(a, b_1), (a, b_2) \in A_j$, then

$$\phi_j(a, b_1) - \phi_j(a, b_2) = \phi_j(0, b_1 - b_2) = 0$$

by Lemma 6.7. Hence, we can define the pullback

$$\overline{A}_{j+1} := A_{j-1} \bigoplus_{C(S^{n_{j+1}-1}, M_{r_{j+1}})} C(D^{n_{j+1}}, M_{r_{j+1}}).$$

Note that the maps defining the pullback are $\overline{\phi}_j$ and ψ_{j+1}. Set

$$\Phi_{j+1} : A_{j+1} \to \overline{A}_{j+1}$$

via $\Phi_{j+1}(f, g, h) = (f, h)$, where $f \in A_{j-1}, g \in C(D^{n_j}, M_{r_j}), h \in C(D^{n_{j+1}}, M_{r_{j+1}})$.

Note that $\overline{\phi}_j(f) = \phi_j(f, g) = \psi_{j+1}(h)$. Thus $(f, h) \in \overline{A}_{j+1}$. So, Φ_{j+1} is a well-defined unital $*$-homomorphism.

Actually, we can generalize this construction.

If $i > j$, then we can inductively define the pullback

$$\overline{A}_{ij} := \left[A_{j-1} \bigoplus_{C(S^{n_{j+1}-1}, M_{r_{j+1}})} C(D^{n_{j+1}}, M_{r_{j+1}})\right] \oplus \cdots \oplus \bigoplus_{C(S^{n_i-1}, M_{r_i})} C(D^{n_i}, M_{r_i}) \tag{6.3}$$

by using the map

$$\begin{aligned}\overline{\phi}_{ij} : &\left[A_{j-1} \bigoplus_{C(S^{n_{j+1}-1}, M_{r_{j+1}})} C(D^{n_{j+1}}, M_{r_{j+1}})\right] \\ &\oplus \cdots \oplus \bigoplus_{C(S^{n_{i-1}-1}, M_{r_{i-1}})} C(D^{n_{i-1}}, M_{r_{i-1}}) \to C(S^{n_i-1}, M_{r_i})\end{aligned}$$

with

$$\overline{\phi}_{ij}(\overline{f}_{j-1}, f_{j+1}, \ldots, f_{i-1}) = \phi_i(\overline{f}_{j-1}, f_j, f_{j+1}, \ldots, f_{i-1})$$

for some (any) $f_j \in C(D^{n_j, M_{r_j}})$ such that the right hand side is well-defined. Notice that Lemma 6.7 guarantees that the map is well defined. Furthermore set

$$\Phi_{ij} : A_i \to \overline{A}_{ij} \tag{6.4}$$

where the map takes an element of A_i and "removes" it is j-th component.

LEMMA 6.8. *If $(\Phi_{ij})_*(y_i) \geqslant 0$, then there is $M \in \mathbb{N}^*$ such that $My_i > 0$ in $K_0(A_i)$.*

Proof. Suppose that $(\Phi_{ij})_*(y_i) \geqslant 0$. Also, (after replacing p, q with $p \oplus 1_s$ and $q \oplus 1_s$ for large enough s; note that we can do this because $\operatorname{rank}((p \oplus 1_s)(x)) - \operatorname{rank}((q \oplus 1_s)(x)) = \operatorname{rank} p(x) - \operatorname{rank} q(x)$ for every x), we may assume that there exists a partial isometry $v = (\overline{v}_{j-1}, v_{j+1}, \ldots, v_i) \in M_\infty(\overline{A}_{ij})$ such that $v^*v = (\overline{q}_{j-1}, q_{j+1}, \ldots, q_i)$ and $vv^* \leqslant (\overline{p}_{j-1}, p_{j+1}, \ldots, p_i)$. Recall that $\overline{v}_{j-1} \in M_\infty(A_{j-1})$ and $v_k \in M_\infty(C(D^{n_k}))$ for every $k = j+1, \ldots, i$. Because $j = \max \mathcal{W}_A(y)$, we have $\operatorname{rank} p_j > \operatorname{rank} q_j$. Again, we may assume that $\operatorname{rank} p_j - \operatorname{rank} q_j > \frac{\dim(A)-1}{2}$ (if this is not true, then we can take direct sums $p_j \oplus p_j \oplus \cdots \oplus p_j$ and $q_j \oplus q_j \oplus \cdots \oplus q_j$ as large needed to achieve this). By using [20, Proposition 4.2] for $p_j, q_j, S^{n_j-1} \subset D^{n_j}$ and the partial isometry $\phi_{j-1}(\overline{v}_{j-1}) \in M_\infty(C(S^{n_j-1}))$, we get that there exists a partial isometry $v_j \in M_\infty(C(D^{n_j}))$, such that $\widetilde{v} := (\overline{v}_{j-1}, v_j) \in M_\infty(A_j)$ and also satisfies

$$\widetilde{v}\widetilde{v}^* < \overline{p}_j \quad \text{and} \quad \widetilde{v}^*\widetilde{v} = \overline{q}_j.$$

Let $w := (\overline{v}_{j-1}, v_j, v_{j+1}, \ldots, v_i) \in M_\infty(A_i)$. Then

$$ww^* < \overline{p}_i \quad \text{and} \quad w^*w = \overline{q}_i$$

so $y_i > 0$. ∎

We are left to deal with the case $\mathcal{W}_A(y) = \emptyset$. On our next lemma, we will show that this case (under mild extra assumptions) leads to a direct sum decomposition with nice properties. But first we need to recall some notation from [26, Chapter 1.5].

Let E be a unital and stably finite C^*-algebra. Let $I \leqslant K_0(E)$ and $I^+ = K_0(E)^+ \cap I$. We say that (I, I^+) is an *ideal* in $(K_0(E), K_0(E)^+)$ if:

(i) $I = I^+ - I^+$ and

(ii) for all $x, y \in K_0(E)$, if $0 \leqslant x \leqslant y$ and $y \in I^+$, then $x \in I^+$.

If $S \subset K_0(E)^+$ is a subsemigroup and (I, I^+) is an ideal, we say that (I, I^+) is *generated by* S if $I^+ = \{a \in K_0(E)^+ : 0 \leqslant a \leqslant b \text{ for some } b \in S\}$.

LEMMA 6.9. *Let F be a finite dimensional C^*-algebra and $\Gamma \subset K_0(F)^+$ be a subsemigroup. Assume that (I, I^+) is the ideal of $(K_0(F), K_0(F)^+)$ generated by Γ. Then there exists $a \in \Gamma$ such that (I, I^+) is generated by a.*

Proof. First of all, F is a direct sum of matrix algebras, so $(K_0(F), K_0(F)^+) \cong (\mathbb{Z}^n, \mathbb{Z}^n_+)$ for some n. Note that I must be of the form

$$I = \{(b_1, \dots, b_n) : b_j = 0 \text{ for every } j \in J\}$$

with J being some subset of $\{1, 2, \dots, n\}$. Moreover, for every $j \notin J$, there is $a^{(j)} = (a_1^{(j)}, \dots, a_n^{(j)}) \in \Gamma$ with the property that $a_j^{(j)} > 0$. Take $a = \sum\limits_{j \notin J} a^{(j)}$ and observe that this is our generator. ∎

LEMMA 6.10. *Let A be a* NCCC *with length $l > 0$ and decomposition as in* (6.1). *Let also $y = [p]_0 - [q]_0$ almost positive, such that $\mathcal{W}_A(y) = \emptyset$ and assume that there exists $x_0 \in X_0$ such that* $\operatorname{rank} p_0(x) > \operatorname{rank} q_0(x)$. *Then there exist F_1, F_2 finite dimensional C^*-algebras with $F_0 = F_1 \oplus F_2$ that satisfy the following properties:*

(i) *$A = F_1 \oplus B$, where*

$$B = \Big[\cdots \Big[F_2 \bigoplus_{C(S^{n_1-1}, M_{r_1})} C(D^{n_1}, M_{r_1})\Big] \bigoplus_{C(S^{n_2-1}, M_{r_2})} C(D^{n_2}, M_{r_1}) \cdots\Big] \bigoplus_{C(S^{n_l-1}, M_{r_l})} C(D^{n_l}, M_{r_l})$$

is a NCCC.

(ii) $\Gamma_2 := \{a \in K_0(F_2)^+ : (a, 0, \dots, 0) \in \sigma_*(K_0(B))\} = \{0\}$.

Proof. Define

$$\Gamma := \{a \in K_0(F_0)^+ : (a, 0, \dots, 0) \in \sigma_*(K_0(A))\}. \tag{6.5}$$

Note that Γ is a semigroup and $\Gamma \neq \{0\}$ by assumption. Let (I, I^+) be the ideal generated by Γ. By Lemma 6.9, there exists $a \in \Gamma$ such that (I, I^+) is generated (as an ideal) by a.

It is known (see [26, Proposition 1.5.3]) that there exists F_1 ideal of F_0 such that $(K_0(F_1), K_0(F_1)^+) \cong (I, I^+)$. But in finite dimensional C^*-algebras, ideals are always summands, so there exists F_2 such that

$$F_0 = F_1 \oplus F_2. \tag{6.6}$$

Let $z = [f]_0 - [g]_0$, where $f = (f_0, \dots, f_l)$ and $g = (g_0, \dots, g_l) \in P_\infty(A)$ such that $\sigma_*(z) = (a, 0, \dots, 0)$ (such z exists because $a \in \Gamma$). Then

$$\operatorname{rank}(f_i) = \operatorname{rank}(g_i) \quad \text{for every } i \geqslant 1. \tag{6.7}$$

We have that $[f_0]_0 - [g_0]_0 = a = [h]_0$ for some $h \in P_\infty(F_0)$. It follows that

$$\begin{aligned}
[f_0]_0 = [h \oplus g_0]_0 &\Rightarrow [\phi_0(f_0)]_0 = [\phi_0(h \oplus g_0)]_0 \Rightarrow \operatorname{rank} \phi_0(f_0) = \operatorname{rank} \phi_0(h \oplus g_0) \\
&\Rightarrow \operatorname{rank}(f_1) = \operatorname{rank}(\phi_0(h)) + \operatorname{rank}(g_1) \overset{(6.7)}{\Longrightarrow} \phi_0(h) = 0 \Rightarrow \phi_0(1_{F_1}) = 0 \\
&\Rightarrow (1_{F_1}, 0) \in A_1
\end{aligned}$$

where we used the facts that $[h] = a$ and (I, I^+) is generated by a to get the second to last equation.

We will use induction to show that $\phi_i(1_{F_1}, 0, \dots, 0) = 0$ for every i. We have already shown it for $i = 0$. Assume that it holds for $i - 1$. We will show it for i.

We have:

$$\begin{aligned}
&[(f_0, \dots, f_i)]_0 - [(h \oplus g_0, g_1, \dots, g_i)]_0 \in \operatorname{Inf} K_0(A) \\
&\Rightarrow [\phi_i(f_0, \dots, f_i)]_0 - [\phi_i(h \oplus g_0, g_1, \dots, g_i)]_0 \in \operatorname{Inf} K_0(C(S^{n_{i+1}-1}, M_{r_{i+1}})) \\
&\Rightarrow \operatorname{rank} \phi_i(f_0, f_1, \dots, f_i) = \operatorname{rank} \phi_i(h \oplus g_0, g_1, \dots, g_i) \\
&\Rightarrow \operatorname{rank}(f_{i+1}) = \operatorname{rank}(\phi_0(h, 0, \dots, 0)) + \operatorname{rank}(g_{i+1}) \Rightarrow \phi_i(h, 0, \dots, 0) = 0 \\
&\Rightarrow \phi_i(1_{F_1}, 0, \dots, 0) = 0.
\end{aligned}$$

We used Lemma 2.11 and the fact that if $x = [p] - [q] \in \operatorname{Inf} K_0(C(X))$, then $\operatorname{rank}(p) = \operatorname{rank}(q)$. This follows immediately from (2.6). Finally, note that because of the induction hypothesis everything is well-defined.

So, we have

$$A \cong F_1 \oplus B \tag{6.8}$$

where

$$B = \Big[\cdots \Big[F_2 \bigoplus_{C(S^{n_1-1}, M_{r_1})} C(D^{n_1}, M_{r_1}) \Big] \bigoplus_{C(S^{n_2-1}, M_{r_2})} C(D^{n_2}, M_{r_1}) \cdots \Big] \bigoplus_{C(S^{n_l-1}, M_{r_l})} C(D^{n_l}, M_{r_l})$$

is a NCCC. The isomorphism is via the natural map and the relation $\phi_i(1_{F_1}, 0, \dots, 0) = 0$ for every i guarantees that everything is well-defined. Moreover, let

$$\Gamma_2 = \{a \in K_0(F_2)^+ : (a, 0, \dots, 0) \in \sigma_*(K_0(B))\}.$$

We will show that $\Gamma_2 = \{0\}$. Indeed if $a \in K_0(F_2)^+ \setminus \{0\}$ such that $\sigma_*^{(B)}(x) = (a, 0, \dots, 0)$ for some $x \in K_0(B)$, then

$$\sigma_*^{(A)}(0, x) = ((0, a), 0, \dots, 0) \Rightarrow (0, a) \in \Gamma \subset I^+ \subset K_0(F_1)$$

which clearly cannot happen. ∎

PROPOSITION 6.11. *Let A be a* NCCC *and $y = [p]_0 - [q]_0 \in K_0(A)$ almost positive. There exists $R = R_y(A)$ a* NCCC *and $\Psi = \Psi_y^{(A)} : A \to R_y(A)$ unital $*$-homomorphism that has the following properties:*

(i) *If $(\Psi_y^{(A)})_*(y) \geqslant 0$, then there exists $M > 0$ such that $My \geqslant 0$.*

(ii) *$R_y(A) = F_1^{y,A} \oplus B_y(A)$, where $F_1^{y,A}$ is a finite dimensional C^*-algebra and $B_y(A)$ is a* NCCC. *Moreover, if $(\Psi_y^{(A)})_*(y) = (y^{(1)}, y^{(2)})$, then $y^{(1)} \geqslant 0$ and $y^{(2)} \in$* Inf $K_0(B_y(A))$. *(It has to be noted that we allow any of the summands to be* 0.*)*

(iii) *For given A,*

$$\begin{aligned}|\{B_y(A) : y \text{ is almost positive }\}| &\leqslant 2^n |\{R_y(A) : y \text{ is almost positive }\}| \\ &= 2^n |\{\Psi_y^{(A)} : y \text{ is almost positive }\}| < \infty,\end{aligned}$$

where n is the number of disjoint singletons in X_0.

Proof. We will define $R_y(A), \Psi_y^{(A)}$ with induction on the length of A.

If the length of A is zero, then define $R_y(A) = A$ and $\Psi_y^{(A)} = \mathrm{id}$ for every y. Properties (i)–(iii) hold trivially.

Suppose that we have defined $R_y(L), \Psi_y^{(L)}$ for every L NCCC with length less or equal than $l-1$ and for every $y \in K_0(L)$ almost positive. Let A be a NCCC with length l and $y = [p]_0 - [q]_0 \in K_0(A)$ almost positive. We have four cases.

Case 1. $\operatorname{rank} p_l > \operatorname{rank} q_l$.

Denote $\pi : A \to A_{l-1}$ to be the projection to the $l-1$-th stage algebra. Set

$$R_y(A) = R_{\pi_*(y)}(A_{l-1}) \quad \text{and} \quad \Psi_y^{(A)} = \Psi_{\pi_*(y)}^{A_{l-1}} \circ \pi.$$

Note that the length of A_{l-1} is $l-1$, so everything is well-defined by induction hypothesis.

Assume that $(\Psi_y^{(A)})_*(y) \geqslant 0$. This means that

$$(\Psi_{\pi_*(y)}^{A_{l-1}} \circ \pi)_*(y) \geqslant 0.$$

By induction hypothesis there exists $N \in \mathbb{N}$ such that

$$N\pi_*(y) \geqslant 0.$$

Let now $M \in \mathbb{N} : N|M$ and $\dim(A) < M$. After replacing p, q with $p \oplus 1_s$ and $q \oplus 1_s$ for large enough s, we may assume that

$$\overline{p}_{l-1}^M := \overline{p}_{l-1} \oplus \cdots \oplus \overline{p}_{l-1} \succeq \overline{q}_{l-1} \oplus \cdots \oplus \overline{q}_{l-1} := \overline{q}_{l-1}^M$$
(each summand is taken M times) .

So there is a partial isometry $v \in M_\infty(A_{l-1})$ such that

$$vv^* = \overline{q}_{l-1}^M \quad \text{and} \quad v^*v \leqslant \overline{p}_{l-1}^M.$$

Similarly define

$$p_l^M := p_l \oplus \cdots \oplus p_l \quad \text{and} \quad q_l \oplus \cdots \oplus q_l := q_l^M.$$

Because $\operatorname{rank} p_l > \operatorname{rank} q_l$, $\operatorname{rank} p_l^M - \operatorname{rank} q_l^M \geqslant M > \frac{\dim(A)-1}{2}$. So the hypothesis of [20, Proposition 4.2] is satisfied for $p_l^M, q_l^M, S^{n_l-1} \subset D^{n_l}$ and the partial isometry $\phi_{l-1}(v)$. Thus, $\phi_{l-1}(v)$ can be extended to a partial isometry w on $M_\infty(C(D^{n_l}))$ such that

$$ww^* = q_l \oplus \cdots \oplus q_l \quad \text{and} \quad w^*w < p_l \oplus \cdots \oplus p_l.$$

Hence, by considering the partial isometry $(v, w) \in M_\infty(A)$, we get that $My > 0$, so (i) is satisfied.

Case 2. $\operatorname{rank} p(x) = \operatorname{rank} q(x)$ for every x in the total space.

Set

$$R_y(A) = A \quad \text{and} \quad \Psi_y^{(A)} = \mathrm{id}.$$

Property (i) holds trivially and property (ii) holds for $R_y(A) = 0 \oplus A$ because $y \in \operatorname{Inf}(K_0(A))$ by Lemma 6.6.

Case 3. $\mathcal{W}_A(y) = \varnothing$ and there is $x_0 \in X_0$ such that $\operatorname{rank} p_0(x_0) > \operatorname{rank} q_0(x_0)$.

Set

$$R_y(A) = A \quad \text{and} \quad \Psi_y^{(A)} = \mathrm{id}.$$

By Lemma 6.10, $A = F_1 \oplus B$, where F_1 is a finite dimensional C^*-algebra and B is a NCCC. Let $y = (y^{(1)}, y^{(2)})$. Because y is almost positive, $y^{(1)} \geqslant 0$. Because $\Gamma_2 = \{0\}$ (Γ_2 is as defined in Lemma 6.10), it follows that $\sigma_*^{(B)}(y^{(2)}) = 0$. Thus, Lemma 6.6 yields that $y^{(2)} \in \operatorname{Inf}(K_0(B))$. So, (ii) holds. Moreover, (i) holds trivially.

Case 4. $\mathcal{W}_A(y) \neq \varnothing$ and $\operatorname{rank} p_l = \operatorname{rank} q_l$.

Let $j = \max \mathcal{W}_A(y)$. By assumption, $j < l$. Recall that in (6.4) we defined a map

$$\Phi_{lj} : A \to \overline{A}_{lj}$$

where $\overline{A}_{lj}$, is as defined in (6.3) and has length $l-1$. Set

$$R_y(A) = \overline{A}_{lj} \quad \text{and} \quad \Psi_y^{(A)} = \Psi^{(\overline{A}_{lj})}_{(\Phi_{lj})_*(y)} \circ \Phi_{lj}.$$

By induction hypothesis, everything is well-defined. Assume that $(\Psi_y^{(A)})_*(y) \geqslant 0$. This means that

$$(\Psi^{(\overline{A}_{lj})}_{(\Phi_{lj})_*(y)} \circ \Phi_{lj})_*(y) \geqslant 0.$$

By induction hypothesis there is $N \in \mathbb{N}^*$ such that

$$M(\Phi_{lj})_*(y) \geqslant 0.$$

Thus by Lemma 6.8, there exists $M' > 0$ such that $M'y > 0$, as desired. So (i) holds.

We will now show that (ii) holds for Cases 1 and 4.

In both cases, notice that on the inductive step the cardinality of $\mathcal{W}_A(y)$ decreases by 1. Moreover, the inductive step preserves almost positivity, and as long as the cardinality remains non-zero, the presence in one of these two cases. So, when we reach $R_y(A)$, the cardinality should become zero, which means that we now lie on one of the other two cases. But property (ii) is about $R_y(A)$ and we have already showed it for Cases 2 and 3. So, it holds for Cases 1 and 4 as well.

We are left to show property (iii).

Note that for every $y \in K_0(A)$ almost positive, $R_y(A)$ is formed from A after "deleting" the coordinates of the set $\mathcal{W}_A(y)$. But this is done uniquely, so $R_y(A)$ depends only on the elements of $\mathcal{W}_A(y)$. Observe that $\Psi_y^{(A)}$ is completely determined by $R_y(A)$. Note that $R_y(A) = F_1^{y,A} \oplus B_y(A)$ as in Lemma 6.10, if $y_0 > 0$, while $R_y(A) = 0 \oplus R_y(A)$ if $y_0 = 0$. By looking at the proof of Lemma 6.10, F_1 depends only on the ideal (I, I^+). So, $B_y(A)$ is completely determined by the elements of $\mathcal{W}_A(y)$ plus what the ideal (I, I^+) is. But $(K_0(F_0), K_0(F_0)^+) \cong (\mathbb{Z}^n, \mathbb{Z}^n_+)$, where n is the number of disjoint singletons in X_0. So, we have 2^n choices for (I, I^+) (see the proof of Lemma 6.9). Hence

$$
\begin{aligned}
|\{B_y(A) : y \text{ is almost positive}\}| &\leqslant 2^n |\{R_y(A) : y \text{ is almost positive }\}| \\
&= 2^n |\{\Psi_y^{(A)} : y \text{ is almost positive}\}| \leqslant 2^{l+n} < \infty. \quad \blacksquare
\end{aligned}
$$

Now we are ready to show that NCCC have the K_0-embedding property.

PROPOSITION 6.12. *Let A be a* NCCC. *Then A has the K_0-embedding property.*

Proof. Let A be a NCCC. For given $y = [p]_0 - [q]_0 \in K_0(A)$ almost positive, consider the following sequence of maps:

$$
A \xrightarrow{\Psi_y^{(A)}} R_y(A) \xrightarrow{\pi} B_y(A) \xrightarrow{\rho} D_y(A)
$$

where $\pi : R_y(A) = F_1^{y,A} \oplus B_y(A) \to B_y(A)$ is the projection to the second coordinate, ρ is the embedding from Corollary 4.5 with respect to (any) $\tau \in T(B_y(A))$ faithful and $D_y(A)$ is an AF-algebra. Set

$$
h_y^{(A)} := \rho \circ \pi \circ \Psi_y^{(A)} : A \to D_y(A).
$$

By Proposition 6.11(iii),

$$
|\{D_y(A) : y \text{ is almost positive }\}| = |\{h_y^{(A)} : y \text{ is almost positive }\}| < \infty. \tag{6.9}
$$

Denote

$$
h_0^{(A)} := \bigoplus_y h_y^{(A)} \quad \text{and} \quad D(A) = \bigoplus_y D_y(A)
$$

where the sum is over all y that are almost positive. By (6.9) the sums can be taken to be finite if we never count the same summand twice, so we may assume that $D(A)$ is AF. Moreover, set

$$h^{(A)} = \sigma^{(A)} \oplus h_0^{(A)} : A \to F_0 \oplus \Big(\bigoplus_{i=1}^{l} C(D^{n_i}, M_{r_i})\Big) \oplus D(A) := E(A).$$

We will show that $h^{(A)}$ sends singular elements to singular elements. Indeed, suppose that

$$(h^{(A)})_*(y) > 0 \tag{6.10}$$

for some $y \in K_0(A)$. Obviously, y has to be almost positive. Moreover,

$$(h_y^{(A)})_*(y) \geqslant 0. \tag{6.11}$$

Because of the construction of the embedding ρ and (6.11), it follows that

$$y^{(2)} \notin \operatorname{Inf} K_0(B_y(A)) \backslash \{\text{torsion elements}\},$$

where $y^{(2)}$ is as in statement of Proposition 6.11. But by Proposition 6.11(ii),

$$y^{(2)} \in \operatorname{Inf} K_0(B_y(A)).$$

Hence, there is $N \in \mathbb{N}^*$ such that $Ny^{(2)} = 0$. Thus

$$N(\Psi_y^{(A)})_*(y) \geqslant 0.$$

Finally, by Proposition 6.11(i), there is $M > 0$ such that $My \geqslant 0$. But, because of (6.10), it cannot be $My = 0$. Thus, it follows that $My > 0$ as desired.

The result follows from the fact that $E(A) \in \mathcal{O}$ so it has the K_0-embedding property by Proposition 5.6. ∎

Assume that there exists $y \in \ker((h^{(A)})_*)$ that is non-torsion and singular. Then $(\sigma_y^{(A)})_*(y) = 0$, so y is almost positive. Notice that we are in Case 2 of the proof of Proposition 6.11 so $\Psi_y^{(A)} = \mathrm{id}$ and y is a non-torsion infinitesimal. Thus $(h_y^{(A)})_*(y) = \rho_*(y) \neq 0$ because ρ is the map from Corollary 4.5. This contradicts $y \in \ker((h^{(A)})_*)$. It follows that h sends non-torsion singular elements to non-torsion singular elements.

Note that because the class of subhomogeneous C^*-algebras is closed under taking quotients, every ASH-algebra can be written as an inductive limit of subhomogeneous C^*-algebras with injective connecting maps. So, if we combine Proposition 2.14, Proposition 6.12, Proposition 6.3, Proposition 3.5 and Remark 3.6, we get the result we are aiming for the next.

PROPOSITION 6.13. *Let A be a separable* ASH-*algebra. Then A has the K_0-embedding property.*

Now Theorem 1.3 can be deduced from all the aforementioned.

Proof of Theorem 1.3. Let A and $\mathcal{Y}$ as in the hypothesis. Due to Remark 2.5, it is enough to show that A has the K_0-embedding property. Because of Proposition 2.7 it is enough to show that every A that can be locally approximated by algebras in $\mathcal{Y}$, has the K_0-embedding property. Note that $\mathcal{Y}$ contains only separable, nuclear and quasidiagonal C^*-algebras, so by Proposition 3.5 it is enough to show that all C^*-algebras in $\mathcal{Y}$ have the K_0-embedding property. But if R is a NCCC, then by Proposition 6.12 and the observation right after its proof, there exists a faithful $*$-homomorphism $h : R \to E(R)$ such that $E(R) \in \mathcal{O}$ and h_* sends non-torsion singular elements to non-torsion singular elements. So, Proposition 3.5, Proposition 5.4 and Proposition 5.6 yield that every C^*-algebra in $\mathcal{Y}$ has the K_0-embedding property. ∎

APPENDIX A.

For the sake of completion, we will prove the following proposition, which is mentioned (without proof) on [1, p. 84]. This proposition is essential for the proof of Proposition 5.4.

PROPOSITION A.1. *Let A be a unital and separable C^*-algebra, $\sigma : \mathbb{Z} \to \mathrm{Aut}(A)$ be an action and $\tau \in T(A)$ a σ-invariant trace. Then every trace extending τ induces the same state in $K_0(A \rtimes_\sigma \mathbb{Z})$.*

Let $E : A \rtimes_\sigma \mathbb{Z} \to A$ be the conditional expectation that sends $\sum\limits_{g\in\mathbb{Z}} a_g g$ to a_0. For every σ-invariant trace $\tau \in T(A)$, $\tau \circ E \in T(A \rtimes_\sigma \mathbb{Z})$. Thus invariant traces can always be extended to traces in the crossed product, so the statement of the aforementioned proposition makes sense.

For the definition of the functions Δ_τ and $\underline{\Delta}_\tau$ that we will use throughout this appendix, we refer the reader to [21, p. 378] and [21, p. 379], respectively. Our starting point is the following known proposition.

PROPOSITION A.2 (cf. Proposition 2, [21]). *Let*

$$0 \longrightarrow J \xrightarrow{\ i\ } B \xrightarrow{\ \pi\ } A \longrightarrow 0$$

be a short exact sequence of C^*-algebras and $\tau \in T(A)$. Then

$$0 \longrightarrow \widehat{\tau} \circ \pi_*(K_0(B)) \longrightarrow \widehat{\tau}(K_0(A)) \xrightarrow{\ q\ } \underline{\Delta}_\tau(\ker(i_*)) \longrightarrow 0$$

where the first map is the inclusion of the two subgroups of $\mathbb{R}$ and q is the restriction of the quotient map

$$q : \mathbb{R} \to \mathbb{R}/\widehat{\tau} \circ \pi_*(K_0(B))$$

to $\widehat{\tau}(K_0(A))$, is a short exact sequence. Moreover, for every $p \in P_\infty(A)$, $q(\tau(p)) = \underline{\Delta}_\tau(-\delta_0[p])$, where δ_0 is the boundary map in K-theory.

The first part of the statement is contained on the statement of [21, Proposition 2], while the second one is (explicitly) shown in its proof.

Because the result we want to show concerns K-theory of crossed products with the integers, we need to recall the Pimsner–Voiculescu 6-term exact sequence [22].

THEOREM A.3 (Pimsner–Voiculescu 6-term exact sequence). *Let A be a unital C^*-algebra and $\sigma : \mathbb{Z} \to \mathrm{Aut}(A)$ be an action. Then there exists a short exact sequence*

$$0 \longrightarrow A \otimes \mathcal{K} \xrightarrow{\phi} T_\sigma \xrightarrow{\psi} A \rtimes \mathbb{Z} \longrightarrow 0 \tag{A.1}$$

for a C^*-algebra T_σ (see [30, 2.1] *for its definition). Moreover, if $\iota : A \hookrightarrow A \rtimes_\sigma \mathbb{Z}$ and $j : A \hookrightarrow T_\sigma$ are the natural inclusions, they satisfy $\iota = \psi \circ j$. In addition j induces isomorphisms on both K_0 and K_1. Furthermore, the following diagram*

$$\begin{tikzcd} K_i(A) \arrow[r, "1-\sigma_*"] \arrow[d, "\beta_*"] & K_i(A) \arrow[r, "\iota_*"] \arrow[d, "j_*"] & K_i(A \rtimes_\sigma \mathbb{Z}) \arrow[d, "\mathrm{id}"] \\ K_i(A \otimes \mathcal{K}) \arrow[r, "\phi_*"] & K_i(T_\sigma) \arrow[r, "\psi_*"] & K_i(A \rtimes_\sigma \mathbb{Z}) \end{tikzcd} \tag{A.2}$$

where $i = 0, 1$ and $\beta : A \hookrightarrow A \otimes \mathcal{K}$ is the natural embedding which yields isomorphisms in K-theory by stability, is commutative.

Finally the short exact sequence (A.1) *induces the following* 6*-term exact sequence in K-theory*

$$\begin{tikzcd} K_0(A) \arrow[r, "1-\sigma_*"] & K_0(A) \arrow[r, "\iota_*"] & K_0(A \rtimes_\sigma \mathbb{Z}) \arrow[d, "\delta_0"] \\ K_1(A \rtimes_\sigma \mathbb{Z}) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "\iota_*"] & K_1(A) \arrow[l, "1-\sigma_*"] \end{tikzcd}$$

In order to be precise, we need to mention that ϕ and ψ are defined in [30, Lemma 2.3] and [30, Lemma 2.4] respectively, (A.1) is [30, Proposition 2.7] and (A.2) is deduced after combining [30, Proposition 2.14] with $\iota = \psi \circ j$.

Let now $\tau \in T(A)$ be a σ-invariant trace and $\tau_1 \in T(A \rtimes_\sigma \mathbb{Z})$ a trace extending τ. Consider the map

$$\underline{\Delta}^\sigma_\tau : \ker(1 - \sigma_*) \leqslant K_1(A) \to \mathbb{R}/\widehat{\tau}(K_0(A))$$

via

$$\underline{\Delta}^\sigma_\tau([u]_1) = \Delta_\tau(u\sigma(u^{-1})).$$

Moreover, by commutativity of the diagram (A.2), $\widehat{\tau}_1 \circ \psi_*(K_0(T_\sigma)) = \widehat{\tau} \circ j_*^{-1}(K_0(T_\sigma)) = \widehat{\tau}(K_0(A))$. The latter holds because j_* is an isomorphism. Furthermore, again by commutativity, $\beta_*(\ker(1 - \sigma_*)) = \ker(\phi_*)$.

By the proof of [21, Theorem 3], $\underline{\Delta}^\sigma_\tau = \underline{\Delta}_{\tau_1} \circ \beta_*$ (this is essentially what Pimsner shows on the proof; note slightly different notation).

So, by applying Proposition A.2 to the exact sequence (A.1), we get, for every $p \in P_\infty(A \rtimes_\sigma \mathbb{Z})$,

$$q(\tau_1(p)) = \underline{\Delta}_{\tau_1}(\beta_*([u]_1)) = \underline{\Delta}^\sigma_\tau([u]_1) \tag{A.3}$$

for some $u \in U_\infty(K_1(A))$. Notice that $q(\tau_1(p))$ is independent of the trace extension τ_1. Moreover, we have the following exact sequence

$$0 \longrightarrow \widehat{\tau}(K_0(A)) \longrightarrow \widehat{\tau}_1(K_0(A \rtimes_\sigma \mathbb{Z})) \xrightarrow{q} \underline{\Delta}_\tau^\sigma(\ker(1-\sigma_*)) \longrightarrow 0. \tag{A.4}$$

Now we can prove the result we are aiming for the next.

Proof of Proposition A.1. Let A be a separable and unital C^*-algebra, $\sigma : \mathbb{Z} \to \mathrm{Aut}(A)$ an action, $\tau \in T(A)$ a σ-invariant trace. For fixed $x \in K_0(A \rtimes_\sigma \mathbb{Z})$, consider the set

$$L_x := \{\widehat{\tau}_1(x) - \widehat{\tau}_2(x) : \tau_1, \tau_2 \text{ extend } \tau\}.$$

Of course $0 \in L_x$. Assume that for some x, $L_x \neq \{0\}$. Then, there exist τ_1, τ_2 extending τ such that $\widehat{\tau}_1(x) - \widehat{\tau}_2(x) \neq 0$. By considering convex combinations $w\tau_1 + (1-w)\tau_2$, we can see that L_x has to contain an interval around zero, so it has to be uncountable. On the other hand, by (A.3), $q(\widehat{\tau}_3(x)) = q(\widehat{\tau}_4(x))$ for every $\tau_3, \tau_4 \in T(A \rtimes_\sigma \mathbb{Z})$ that extend τ. By the exactness of (A.4), $\widehat{\tau}_3(x) - \widehat{\tau}_4(x) \in \widehat{\tau}(K_0(A))$. Thus $L_x \subset \widehat{\tau}(K_0(A))$. But A is separable, hence $\widehat{\tau}(K_0(A))$ is countable, contradiction. Hence $L_x = \{0\}$ for every x. Proof is complete. ∎

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