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GEOMETRY OF HOLOMORPHIC VECTOR BUNDLES AND SIMILARITY OF COMMUTING TUPLES OF OPERATORS

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In memory of R.G. Douglas

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ABSTRACT. In this paper, a new criterion for the similarity of commuting tuples of operators on Hilbert spaces is introduced. As an application, we obtain a geometric similarity invariant of tuples in the Cowen–Douglas class which gives a partial answer to a question raised by R.G. Douglas in Complex geometry and operator theory, *Acta Math.* **141**(1978), 187–261 and Operator theory and complex geometry, *Extracta Math.* **24**(2009), 135–165 about the similarity of quasi-free Hilbert modules. Moreover, a new subclass of commuting tuples of Cowen–Douglas class is obtained.

KEYWORDS: *Commuting tuple, Cowen–Douglas operator, curvature, holomorphic bundle, similarity.*

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1. INTRODUCTION

Let $\mathcal{H}$ be a complex separable Hilbert space, and $\mathcal{L}(\mathcal{H})$ the collection of bounded linear operators on $\mathcal{H}$. Problems of operator theory often involve unitary and similarity equivalences of operators (operator tuples). For a positive integer m, let $\mathbb{T} = (T_1, \dots, T_m)$ be an m-tuple of bounded operators acting on $\mathcal{H}$. If $\mathbb{T}$ satisfies $T_iT_j = T_jT_i$ for all $1 \leqslant i, j \leqslant m$, $\mathbb{T}$ will be referred to as a commuting tuple. Let $\mathbb{S} = (S_1, \dots, S_m)$ be a commuting tuple and $S_i \in \mathcal{L}(\widetilde{\mathcal{H}}), 1 \leqslant i \leqslant m$. The equation $X\mathbb{T} = \mathbb{S}X$ for some $X \in \mathcal{L}(\mathcal{H}, \widetilde{\mathcal{H}})$ means $XT_i = S_iX$ for all $1 \leqslant i \leqslant m$. If X is a unitary operator, $\mathbb{T}$ is unitarily equivalent to $\mathbb{S}$ (denoted by $\mathbb{T} \sim_{\mathrm{u}} \mathbb{S}$). If X is invertible, $\mathbb{T}$ is similar to $\mathbb{S}$ ($\mathbb{T} \sim_{\mathrm{s}} \mathbb{S}$).

Let Ω be a connected open subset of $\mathbb{C}$ and n be a positive integer. In [15], M.J. Cowen and R.G. Douglas introduced a class of bounded linear operators, denoted by $\mathcal{B}_n^1(\Omega)$, which contains Ω as eigenvalues of constant multiplicity n. In [16], they also pointed out that some results of [15] can be directly extended to $\Omega \subset \mathbb{C}^m, m > 1$, i.e. the class of operators $\mathcal{B}_n^1(\Omega)$ may be generalized to an operator tuple class $\mathcal{B}_n^m(\Omega)$. Each Cowen–Douglas tuple naturally determines a Hermitian holomorphic vector bundle, and such two tuples are unitarily equivalent if and only if there is an isometric and connection-preserving bundle map between the bundles [15, 16]. In particular, the unitary classification of tuples in $\mathcal{B}_1^m(\Omega)$ involves only the curvature of Hermitian holomorphic bundles.

In [48], G. Misra has introduced and discussed homogeneous operators in $\mathcal{B}_1^1(\mathbb{D})$. By using the curvature as the invariant, these homogeneous operators have been completely characterized. For homogeneous operators in $\mathcal{B}_n^1(\mathbb{D}), n > 1$, A. Koranyi and G. Misra analyzed their structure and proved a classification theorem (see [45]). In [49], G. Misra estimated the curvatures of operators in $\mathcal{B}_1^1(\Omega)$, and further obtained a widely used curvature inequality, stating that the curvature $\mathcal{K}_{S^*}$ of the backward shift operator dominates the curvature $\mathcal{K}_T$ if T is contractive. Subsequently, G. Misra and N.S.N. Sastry [51, 52] proved that the inequality holds also for curvatures of tuples in $\mathcal{B}_1^m(\Omega)$. Conversely, the fact that the curvature inequality implies that the operator has a stronger contraction than usual case has been proved by S. Biswas, D.K. Keshari and G. Misra in [5]. Other properties of curvature inequality have been discussed in [6, 21, 22, 24, 50, 64].

In [19], R.E. Curto and N. Salinas linked the above Cowen–Douglas operator theory to the generalized reproducing kernel theory. They also discussed the correspondence between the analytic functional Hilbert space with coordinate multiplication $\mathbb{M}_z = (M_{z_1}, \ldots, M_{z_m})$ and the canonical module of Cowen–Douglas tuples, proving the following result.

THEOREM 1.1 ([19]). *Under mild conditions, the tuples* $\mathbb{M}_z = (M_{z_1}, \ldots, M_{z_m})$ *acting on two analytic functional Hilbert spaces are unitarily equivalent if and only if their normalized reproducing kernel functions are intertwined by a constant unitary matrix.*

In [12], intertwining operators of the multiplication operator on Hilbert spaces were characterized by using matrix-valued reproducing kernels.

It is well known that unitary operators maintain rigidity, while general invertible operators destroy rigidity. Taking this into account, we expect that the study of operator similarity is challenging, even in one variable. The model theorem is given in Chapter 0.2 of [55], in the view of complex geometry, and shows that the eigenvector bundle induced by contraction in $\mathcal{B}_n^1(\Omega)$ has a kind of tensor structure. By using the main result of [62] and the model theorem for contractions, H. Kwon and S. Treil proved a theorem which allows one to decide whether a contractive operator T is similar to the n times copies of M_z^* on Hardy space or not, which is

$$\left\| \frac{\partial P(w)}{\partial w} \right\|_{HS}^2 - \frac{n}{(1-|w|^2)^2} \leqslant \frac{\partial^2}{\partial w \partial \overline{w}} \psi(w), \quad w \in \mathbb{D}$$

for projection-valued function P with $\operatorname{ran} P(w) = \ker(T-w)$ and a bounded subharmonic function ψ. Then, the result was generalized to the case of weighted

Bergman shift by R.G. Douglas, H. Kwon and S. Treil [46]. Subsequently, the quantity $-\|\frac{\partial P(w)}{\partial w}\|^2_{HS}$ has been proved to be the trace of the curvature of T (cf. [29]). Currently, this result does not have a version for commuting m-tuples. Although there exist plenty of model theorems about the commuting operator tuples [1, 3, 4, 54], the techniques cannot be easily generalized for the lack of proper conditions for the Corona theorem in several variables.

In infinite-dimensional separable Hilbert spaces, strongly irreducible operators can be regarded as a natural generalization of Jordan block matrix. Strong irreducibility is a similarity invariant of operators. In [11], Y. Cao, J.S. Fang and C.L. Jiang introduced the K_0-group into the similarity classification of operators and characterized when operators have a unique strongly irreducible decomposition up to similarity. Consequently, C.L. Jiang, X.Z. Guo and the second author gave a similarity theorem of Cowen–Douglas operators by using the ordered K-group of the commutant algebra as an invariant [38]. From the perspective of complex geometry, the similarity of Cowen–Douglas operators is described through the equivalence of two families of eigenvectors in [37]. Using the eigenvector bundle associated to $T \in \mathcal{B}^1_n(\Omega)$, M. Uchiyama discussed when T is similar or quasi-similar to the unilateral shift [63].

In [31], W.W. Hastings provided a function-theoretic characterization of subnormal tuples quasi-similar to the Cauchy tuple. Concerning absolute equivalence, virtual unitary equivalence, and almost unitarily equivalence of tuples, readers are referred to [18, 41, 61].

In 2009, R.G. Douglas raised an open question [20, Question 4] which has not been completely solved so far. The open question is the following.

Question. Can one give conditions involving the curvatures which imply that two quasi-free Hilbert modules of multiplicity one are similar?

In this note, the main result is above the geometric similarity invariant of arbitrary Cowen–Douglas tuples without the assumptions of n-hypercontraction and the help of the Corona theorem. To some extent, it gives a partial answer to the question above.

The paper is organized as follows. In Section 1, we recall some notions and basic results about tuples in the Cowen–Douglas class. In Section 2, we obtain an equivalence condition for the similarity of commuting operator tuples. Furthermore, a similarity classification theorem for tuples in $\mathcal{B}^m_1(\Omega)$ is given by using the local equivalence of the holomorphic bundles associated with some Cowen–Douglas tuples of index two. In Section 3, we introduce a new class of commuting tuples in the Cowen–Douglas class (notice that the unitary intertwining operator is not diagonal in this case). In Section 4, some weakly homogeneous operators are investigated.

2. PRELIMINARIES

In this section we will recall some notations and basic results of tuples in the Cowen–Douglas class. Let $\mathcal{L}(\mathcal{H})^m$ be the collection of all commuting m-tuples of bounded operators on $\mathcal{H}$. For $\mathbb{T} = (T_1, \dots, T_m) \in \mathcal{L}(\mathcal{H})^m$, we define $\mathbb{T}x = (T_1x, \dots, T_mx)$, $x \in \mathcal{H}$ and $\mathbb{T} - w = (T_1 - w_1, \dots, T_m - w_m)$, then $\ker(\mathbb{T} - w) = \bigcap\limits_{i=1}^{m} \ker(T_i - w_i)$ with $w = (w_1, \dots, w_m)$ in Ω. The class of Cowen–Douglas tuple of operators with rank n over Ω: $\mathcal{B}_n^m(\Omega)$ is defined as follows [15, 16]:

$$\mathcal{B}_n^m(\Omega) := \{\mathbb{T} \in \mathcal{L}(\mathcal{H})^m : \quad \begin{array}{l} \text{(i) } \bigvee\limits_{w\in\Omega} \ker(\mathbb{T} - w) = \mathcal{H}, \\ \text{(ii) } \operatorname{ran}(\mathbb{T} - w) \text{ is closed for all } w \in \Omega, \\ \text{(iii) } \dim \ker(\mathbb{T} - w) = n \text{ for all } w \in \Omega\}. \end{array}$$

It follows that for each $w \in \Omega$, $\ker(\mathbb{T} - w)$ is an n-dimensional vector subspace of $\mathcal{H}$. Define $E_{\mathbb{T}} := \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker(\mathbb{T} - w)\}$ with a projection map $\pi : E_{\mathbb{T}} \to \Omega$ such that $\pi^{-1}(w) = \ker(\mathbb{T} - w)$. It is a sub-bundle of $\Omega \times \mathcal{H}$ and its Hermitian structure comes from $\mathcal{H}$. Thus, $E_{\mathbb{T}}$ associated with $\mathbb{T}$ is an n-dimensional Hermitian holomorphic vector bundle.

THEOREM 2.1 ([15, 16]). *Let $\mathbb{T}, \mathbb{S} \in \mathcal{B}_n^m(\Omega)$. Then $\mathbb{T} \sim_{\mathrm{u}} \mathbb{S}$ if and only if the Hermitian holomorphic vector bundles $E_{\mathbb{T}}$ and $E_{\mathbb{S}}$ are congruent* (*denoted by $E_{\mathbb{T}} \sim_{\mathrm{u}} E_{\mathbb{S}}$*) *over some open subset Ω_0 of $\Omega \subset \mathbb{C}^m$.*

When $m = 1$, the above theorem is proved in Theorem 2.6 of [15], in the case of $m > 1$, Theorem 2.1 is also valid ([16], pp. 16) due to M.J. Cowen and R.G. Douglas. They make a rather detailed study of certain aspects of complex geometry and introduce the following concepts.

Let E be a C^∞-vector bundle over Ω. A connection D is a differential operator, which takes sections of E to sections with 1-form coefficients and satisfies the Leibnitz rule $D(fs) = (\mathrm{d}f)s + fDs$ for section s and function f. Similarly, D^2 can be defined, $D^2 s = \mathcal{K}s\mathrm{d}z\mathrm{d}\bar{z}$ for section s, bundle map $\mathcal{K}$ determined by D^2 is called the curvature of the bundle E.

For every Hermitian holomorphic vector bundle E over Ω, there is a unique canonical connection Θ, which is a Chern connection metric-preserving and compatible with the holomorphic structure. Given a holomorphic frame $\gamma = \{\gamma_i\}_{i=1}^n$ of E, we have the metric $h(w) = ((\langle \gamma_j(w), \gamma_i(w) \rangle))_{n\times n}$ and $D\gamma = \gamma\Theta$, $\Theta = (\Theta_{ij})_{i,j=1}^n$ is the matrix of connection 1-form. The curvature of E can be defined as

$$\mathcal{K}(w) = -\sum_{i,j=1}^{m} \frac{\partial}{\partial \overline{w}_j}\Big(h^{-1}(w)\frac{\partial h(w)}{\partial w_i}\Big)\mathrm{d}w_i \wedge \mathrm{d}\overline{w}_j \tag{2.1}$$

for $w = (w_1, \dots, w_m) \in \Omega$. When E is a line bundle, equation (2.1) is equivalent to $\mathcal{K}(w) = -\sum\limits_{i,j=1}^{m} \frac{\partial^2 \log \|\gamma(w)\|^2}{\partial \overline{w}_j \partial w_i} \mathrm{d}w_i \wedge \mathrm{d}\overline{w}_j$, where γ is a non-zero section of E.

For any C^∞-bundle map ϕ on E and given frame σ of E, we have that:

(i) $\phi_{\overline{w}}(\sigma) = \frac{\partial}{\partial \overline{w}}(\phi(\sigma))$;

(ii) $\phi_w(\sigma) = \frac{\partial}{\partial w}(\phi(\sigma)) + [h^{-1}\frac{\partial}{\partial w}h, \phi(\sigma)]$.

Since the curvature can also be regarded as a bundle map, we obtain covariant derivatives $\mathcal{K}_{w^I\overline{w}^J}$, $I, J \in \mathbb{Z}_+^m$ of the curvature by using the inductive formulaes above, where $\mathbb{Z}_+^m$ is the collection of m-tuples of nonnegative integers. The curvature $\mathcal{K}$ and its covariant derivatives $\mathcal{K}_{w^I\overline{w}^J}$ are the unitarily invariants of Hermitian holomorphic vector bundle E (see [15, 16]).

THEOREM 2.2 ([15, 16]). *Let $\mathbb{T}, \mathbb{S} \in \mathcal{B}_n^m(\Omega)$. Then $E_{\mathbb{T}} \sim_{\mathrm{u}} E_{\mathbb{S}}$ if and only if there exists an isometry $V : E_{\mathbb{T}} \to E_{\mathbb{S}}$ and a number k depending on $E_{\mathbb{T}}, E_{\mathbb{S}}$ such that*

$$V\mathcal{K}_{\mathbb{T},w^I\overline{w}^J} = \mathcal{K}_{\mathbb{S},w^I\overline{w}^J}V, \quad I, J \in \mathbb{Z}_+^m, |I|, |J| < k.$$

3. ON THE SIMILARITY OF COMMUTING OPERATOR TUPLES

The classification of similarities of commuting operator tuples has always been a challenging problem. Even in the operator case, it is not yet clear how to describe the similarity of Cowen–Douglas operators in $\mathcal{B}_1^1(\Omega)$ using only geometric quantities, such as the curvature. M.J. Cowen and R.G. Douglas put forward the following conjecture in 4.35 of [15]: if $\overline{\mathbb{D}}$ (the closure of unit disc $\mathbb{D}$) is a k-spectral set for $T, S \in \mathcal{B}_1^1(\mathbb{D})$, then $T \sim_{\mathrm{s}} S$ if and only if their curvatures $\mathcal{K}_T$ and $\mathcal{K}_S$ satisfy $\lim\limits_{w \to \partial\mathbb{D}} \frac{\mathcal{K}_T(w)}{\mathcal{K}_S(w)} = 1$. In [13, 14], two counterexamples were constructed by D.N. Clark and G. Misra. Instead of the quotient of the curvatures, they considered the quotient of metrics h_T and h_S of E_T and E_S denoted by a_w. It was then proved in [14] that contraction T is similar to S_α (with weight sequence $\{(\frac{n+1}{n+2})^{\alpha/2}\}_{n=0}^\infty$) if and only if a_w is bounded and bounded below by 0. This result can be regarded as a geometric version of the classical result for the weighted shifts given by A.L. Shields (see [60]). For recent developments concerning the similarity of Cowen–Douglas operators, the reader is referred to [21, 22, 23, 34, 46].

Although there are many studies on the similarity classification of Cowen–Douglas operators, the similarity classification of commuting tuples is not yet fully solved. In this chapter, we provide a different necessary and sufficient condition for the similarity of commuting operator tuples. We introduce the following definition of $\sigma_{\mathbb{T}_0,\mathbb{T}_1}$, and the notation is adopted from the next definition.

DEFINITION 3.1. Let $\mathbb{T}_i \in \mathcal{L}(\mathcal{H}_i)^m, i = 0, 1$. Define $\sigma_{\mathbb{T}_0,\mathbb{T}_1} : \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0) \to \mathcal{L}(\mathcal{H}_0)^m$ to be the tuple

$$\sigma_{\mathbb{T}_0,\mathbb{T}_1}(X) = \mathbb{T}_0X - X\mathbb{T}_1, \quad X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0).$$

Let $\sigma_{\mathbb{T}_0} : \mathcal{L}(\mathcal{H}_0) \to \mathcal{L}(\mathcal{H}_0)^m$ be the tuple $\sigma_{\mathbb{T}_0,\mathbb{T}_0}$.

3.1. ON THE SIMILARITY OF COMMUTING OPERATOR TUPLES. In order to describe clearly Theorem 3.2, Lemma 3.3, Corollary 3.4 and Theorem 3.6, we need to introduce the following notations. Unless otherwise specified, we always assume that

$$\begin{aligned}&\mathbb{T}_{ij}=(T_{ij}^1,\dots,T_{ij}^m),\quad \mathbb{S}_{ij}=(S_{ij}^1,\dots,S_{ij}^m),\quad 0\leqslant i\leqslant j\leqslant 1\quad\text{and}\\ &\mathbb{T}=(T_{11}^1,\dots,T_{11}^m),\quad \mathbb{S}=(S_{00}^1,\dots,S_{00}^m)\end{aligned}$$

for some positive integer m. The main theorem of this paper is the following one.

THEOREM 3.2. *Let* $\mathbb{T}$, $\mathbb{S}\in\mathcal{L}(\mathcal{H})^m$. *Suppose that* $\{\mathbb{S}_{11}\in\mathcal{L}(\mathcal{H})^m:\ker\sigma_{\mathbb{S}_{11},\mathbb{T}}=\{0\}\}\neq\emptyset$. *Then* $\mathbb{T}\sim_{\mathrm{s}}\mathbb{S}$ *if and only if there exist two operator tuples* $\widetilde{\mathbb{T}}=(T_1,\dots,T_m)$, $\widetilde{\mathbb{S}}=(S_1,\dots,S_m)\in\mathcal{L}(\mathcal{H}\oplus\mathcal{H})^m$ *such that:*

(i) $T_i=\begin{pmatrix}T_{00}^i & T_{01}^i\\ 0 & T_{11}^i\end{pmatrix}, S_i=\begin{pmatrix}S_{00}^i & S_{01}^i\\ 0 & S_{11}^i\end{pmatrix}, 1\leqslant i\leqslant m$, *where* $\mathbb{T}_{01}\in\operatorname{ran}\sigma_{\mathbb{T}_{00},\mathbb{T}},\mathbb{S}_{01}\in\operatorname{ran}\sigma_{\mathbb{S},\mathbb{S}_{11}}$ *and* $\ker\sigma_{\mathbb{T}_{00},\mathbb{S}}=\{0\}$;

(ii) $\widetilde{\mathbb{T}}\sim_{\mathrm{u}}\widetilde{\mathbb{S}}$.

In order to prove our main theorem, we first need a lemma which characterizes the unitary operator which intertwines two special commuting operator tuples.

LEMMA 3.3. *Let* $\widetilde{\mathbb{T}}=(T_1,\dots,T_m),\widetilde{\mathbb{S}}=(S_1,\dots,S_m)\in\mathcal{L}(\mathcal{H}\oplus\mathcal{H})^m$, *where* $T_i=\begin{pmatrix}T_{00}^i & T_{01}^i\\ 0 & T_{11}^i\end{pmatrix}, S_i=\begin{pmatrix}S_{00}^i & S_{01}^i\\ 0 & S_{11}^i\end{pmatrix}, 1\leqslant i\leqslant m$, *and* $\mathbb{T}_{01}=-\sigma_{\mathbb{T}_{00},\mathbb{T}}(X),\mathbb{S}_{01}=-\sigma_{\mathbb{S},\mathbb{S}_{11}}(Y)$ *for some* $X,Y\in\mathcal{L}(\mathcal{H})$. *Suppose that* $\ker\sigma_{\mathbb{T}_{00},\mathbb{S}}=\ker\sigma_{\mathbb{S}_{11},\mathbb{T}}=\{0\}$, *then there exists a unitary operator* $U=((U_{i,j}))_{2\times 2}$ *such that* $U\widetilde{\mathbb{T}}=\widetilde{\mathbb{S}}U$ *if and only if the following statements hold:*

(i) $U_{10}T_{00}^iU_{10}^{-1}=S_{11}^i$, $U_{01}^{*-1}T_{11}^iU_{01}^*=S_{00}^i$, $1\leqslant i\leqslant m$;

(ii) $(I+XX^*)^{-1}=U_{10}^*U_{10},(I+X^*X)^{-1}=U_{01}^*U_{01}$;

(iii) $Y-U_{01}X^*U_{10}^{-1}\in\ker\sigma_{\mathbb{S},\mathbb{S}_{11}}$.

Proof. Let $U=\begin{pmatrix}U_{00} & U_{01}\\ U_{10} & U_{11}\end{pmatrix}$. From $U\widetilde{\mathbb{T}}=\widetilde{\mathbb{S}}U$, we have:

$$U_{10}XT_{11}^i-U_{10}T_{00}^iX=S_{11}^iU_{11}-U_{11}T_{11}^i,\tag{3.1}$$

$$U_{00}T_{00}^i-YS_{11}^iU_{10}=S_{00}^iU_{00}-S_{00}^iYU_{10},\quad T_{00}^iU_{00}^*+(XT_{11}^i-T_{00}^iX)U_{01}^*=U_{00}^*S_{00}^i,\tag{3.2}$$

$$U_{10}T_{00}^i=S_{11}^iU_{10},\quad T_{11}^iU_{01}^*=U_{01}^*S_{00}^i,\quad 1\leqslant i\leqslant m.\tag{3.3}$$

First of all, we will prove that U_{01} and U_{10} are invertible. By (3.1) and (3.3), we have $U_{10}XT_{11}^i-S_{11}^iU_{10}X=S_{11}^iU_{11}-U_{11}T_{11}^i$ and $(U_{10}X+U_{11})T_{11}^i=S_{11}^i(U_{10}X+U_{11})$, $1\leqslant i\leqslant m$. From (3.2) and (3.3), we also have

$$T_{00}^i(U_{00}^*-XU_{01}^*)=(U_{00}^*-XU_{01}^*)S_{00}^i,\quad 1\leqslant i\leqslant m.$$

It follows that

$$U_{10}X + U_{11} \in \bigcap_{i=1}^{m} \ker \sigma_{S_{11}^i, T_{11}^i} = \ker \sigma_{\mathbb{S}_{11}, \mathbb{T}} \quad \text{and}$$
$$U_{00}^* - XU_{01}^* \in \bigcap_{i=1}^{m} \ker \sigma_{T_{00}^i, S_{00}^i} = \ker \sigma_{\mathbb{T}_{00}, \mathbb{S}}.$$

Note that $\ker \sigma_{\mathbb{S}_{11}, \mathbb{T}} = \ker \sigma_{\mathbb{T}_{00}, \mathbb{S}} = \{0\}$. We see

$$U_{00} = U_{01}X^*, \quad U_{11} = -U_{10}X. \tag{3.4}$$

So the form of the unitary operator U is $\begin{pmatrix} U_{01}X^* & U_{01} \\ U_{10} & -U_{10}X \end{pmatrix}$. By using the fact $UU^* = U^*U = I \oplus I$, we have the following equations:

$$U_{01}(I + X^*X)U_{01}^* = I, \quad U_{10}(I + XX^*)U_{10}^* = I, \tag{3.5}$$
$$XU_{01}^*U_{01} = U_{10}^*U_{10}X, \tag{3.6}$$
$$XU_{01}^*U_{01}X^* + U_{10}^*U_{10} = I, \quad X^*U_{10}^*U_{10}X + U_{01}^*U_{01} = I. \tag{3.7}$$

By equations (3.5)–(3.7), we have $U_{10}^*U_{10}(I + XX^*) = I$ and $(I + X^*X)U_{01}^*U_{01} = I$. Moreover, combining equation (3.5), we obtain U_{01} and U_{10}^* are invertible and also $(I + X^*X)U_{01}^*$ and $U_{10}(I + XX^*)$. Since $I + XX^*$ and $I + X^*X$ are invertible, it is easy to see that $U_{10}^*U_{10} = (I + XX^*)^{-1}$ and $U_{01}^*U_{01} = (I + X^*X)^{-1}$. From equation (3.3) and the invertibility of U_{01} and U_{10}, we imply $U_{10}T_{00}^iU_{10}^{-1} = S_{11}^i$ and $U_{01}^{*-1}T_{11}^iU_{01}^* = S_{00}^i$ for all $1 \leqslant i \leqslant m$.

By equation (3.2), for any $1 \leqslant i \leqslant m$, we have

$$S_{00}^iYU_{10} - S_{00}^iU_{00} = YS_{11}^iU_{10} - U_{00}T_{00}^i = YS_{11}^iU_{10} - U_{00}U_{10}^{-1}S_{11}^iU_{10}. \tag{3.8}$$

Multiplying U_{10}^{-1} on the right side of equation (3.8), we obtain

$$S_{00}^iY - S_{00}^iU_{00}U_{10}^{-1} = YS_{11}^i - U_{00}U_{10}^{-1}S_{11}^i, \quad 1 \leqslant i \leqslant m.$$

From equation (3.4), it follows that $S_{00}^i(Y - U_{01}X^*U_{10}^{-1}) = (Y - U_{01}X^*U_{10}^{-1})S_{11}^i$ for all $1 \leqslant i \leqslant m$. That is, $Y - U_{01}X^*U_{10}^{-1} \in \bigcap_{i=1}^{m} \ker \sigma_{S_{00}^i, S_{11}^i} = \ker \sigma_{\mathbb{S}, \mathbb{S}_{11}}$.

For the sufficient part, let $U = \begin{pmatrix} U_{01}X^* & U_{01} \\ U_{10} & -U_{10}X \end{pmatrix}$ which satisfies the conditions (i)–(iii). It implies U is a unitary operator. In the following, we will check $U\widetilde{\mathbb{T}}U^* = \widetilde{\mathbb{S}}$, that is, $UT_iU^* = S_i$, $1 \leqslant i \leqslant m$. Note that UT_iU^* has the following form:

$$\begin{pmatrix} U_{01}(X^*X+I)T_{11}^iU_{01}^* & U_{01}X^*T_{00}^iU_{10}^* - (U_{01}X^*(XT_{11}^i - T_{00}^iX) + U_{01}T_{11}^i)X^*U_{10}^* \\ 0 & U_{10}T_{00}^i(XX^*+I)U_{10}^* \end{pmatrix}.$$

Since $Y - U_{01}X^*U_{10}^{-1} \in \ker \sigma_{\mathbb{S}, \mathbb{S}_{11}}$ and

$$YS_{11}^i - S_{00}^iY = (U_{01}X^*U_{10}^{-1})S_{11}^i - S_{00}^i(U_{01}X^*U_{10}^{-1})$$

for all $1 \leqslant i \leqslant m$. By statements (i) and (ii), we have:

$$\begin{aligned} & U_{01}X^*T^i_{00}U^*_{10} - (U_{01}X^*(XT^i_{11} - T^i_{00}X) + U_{01}T^i_{11})X^*U^*_{10} \\ &\qquad = U_{01}X^*T^i_{00}(I + XX^*)U^*_{10} - U_{01}(I + X^*X)T^i_{11}X^*U^*_{10} \\ &\qquad = U_{01}X^*U^{-1}_{10}S^i_{11} - S^i_{00}U^{*-1}_{01}X^*U^*_{10} \\ &\qquad = YS^i_{11} - S^i_{00}Y, \quad 1 \leqslant i \leqslant m. \end{aligned}$$

Based on a routine computation, we obtain $U\widetilde{\mathbb{T}}U^* = \widetilde{\mathbb{S}}$. The proof of these equations also use that fact $X^*U^*_{10}U_{10} = X^*(I + XX^*)^{-1} = (I + X^*X)^{-1}X^* = U^*_{01}U_{01}X^*$. These equalities finish the proof of sufficient part. ∎

Proof of Theorem 3.2. Sufficiency. Firstly, there is $\mathbb{S}_{11}$ such that $\ker \sigma_{\mathbb{S}_{11},\mathbb{T}} = \{0\}$. Suppose that two commuting tuples of operators $\widetilde{\mathbb{T}} = (T_1, \dots, T_m)$ and $\widetilde{\mathbb{S}} = (S_1, \dots, S_m) \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})^m$ are unitarily equivalent, that is, there exists a unitary operator $U = \left(\begin{smallmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{smallmatrix}\right)$ such that $U\widetilde{\mathbb{T}} = \widetilde{\mathbb{S}}U$. If condition (i) in the theorem is satisfied by $\widetilde{\mathbb{T}}, \widetilde{\mathbb{S}}$, by Lemma 3.3, we know that U^*_{01} is invertible and $T^i_{11}U^*_{01} = U^*_{01}S^i_{00}$ for all $1 \leqslant i \leqslant m$. It follows that $\mathbb{T}U^*_{01} = U^*_{01}\mathbb{S}$. Thus, $\mathbb{T}$ is similar to $\mathbb{S}$.

Necessity. Since $\mathbb{T}$ is similar to $\mathbb{S}$, there exists an invertible operator X_1 such that

$$\mathbb{T} = X_1\mathbb{S}X_1^{-1}, \tag{3.9}$$

that is, $T^i_{11} = X_1S^i_{00}X_1^{-1}$ for all $1 \leqslant i \leqslant m$. Without loss of generality, we assume that $(X_1^{-1})^*X_1^{-1} - I \geqslant 0$. Otherwise, let $\alpha = \inf\{x : x \in \sigma((X_1^{-1})^*X_1^{-1})\}$, we have $\frac{(X_1^{-1})^*X_1^{-1}}{\alpha} - I \geqslant 0$, since X_1 is invertible. Then notice that $\alpha > 0$, upon replacing X_1^{-1} by $\frac{X_1^{-1}}{\sqrt{\alpha}}$, we obtain that $(X_1^{-1})^*X_1^{-1} - I \geqslant 0$. Therefore, we find a bounded linear operator X, such that

$$I + X^*X = (X_1^{-1})^*X_1^{-1}. \tag{3.10}$$

Obviously, $I + XX^*$ is also invertible and positive. In the same way as constructing X, we know that there exists X_2 satisfies

$$(I + XX^*)^{-1} = X_2^*X_2, \tag{3.11}$$

and X_2 is an invertible operator.

Choose a non-zero commuting tuple of operators $\mathbb{S}_{11} \in \mathcal{L}(\mathcal{H})^m$ such that $\ker \sigma_{\mathbb{S}_{11},\mathbb{T}} = \{0\}$, that is, $\bigcap\limits_{i=1}^{m} \ker \sigma_{S^i_{11},T^i_{11}} = \{0\}$. Next, we will construct another tuple $\mathbb{T}_{00} \in \mathcal{L}(\mathcal{H})^m$. Let

$$T^i_{00} := X_2^{-1}S^i_{11}X_2, \tag{3.12}$$

for $1 \leqslant i \leqslant m$. Then $\mathbb{T}_{00} = X_2^{-1}\mathbb{S}_{11}X_2$.

We claim that $\ker \sigma_{\mathbb{T}_{00},\mathbb{S}} = \{0\}$. If $Z \in \ker \sigma_{\mathbb{T}_{00},\mathbb{S}}$, then for all $1 \leqslant i \leqslant m$, we have $T^i_{00}Z = ZS^i_{00}$, equivalently, $S^i_{11}X_2ZX_1^{-1} = X_2ZX_1^{-1}T^i_{11}$, since equations (3.9) and (3.12) hold. By $\bigcap\limits_{i=1}^{m} \ker \sigma_{S^i_{11}, T^i_{11}} = \{0\}$, we obtain $X_2 Z X_1^{-1} = 0$. Note that X_1, X_2 are both invertible. It follows that $Z = 0$ and $\bigcap\limits_{i=1}^{m} \ker \sigma_{T^i_{00}, S^i_{00}} = \ker \sigma_{\mathbb{T}_{00}, \mathbb{S}} = \{0\}$.

For any bounded operator $W \in \bigcap\limits_{i=1}^{m} \ker \sigma_{S^i_{00}, S^i_{11}}$, let $Y := W + X_1^* X^* X_2^{-1}$. This implies that

$$Y - X_1^* X^* X_2^{-1} \in \bigcap_{i=1}^{m} \ker \sigma_{S^i_{00}, S^i_{11}} = \ker \sigma_{\mathbb{S}, \mathbb{S}_{11}}. \tag{3.13}$$

Based on the above discussion, we assume that $\widetilde{\mathbb{T}} = (T_1, \dots, T_m), \widetilde{\mathbb{S}} = (S_1, \dots, S_m)$ with $T_i = \begin{pmatrix} T^i_{00} & T^i_{01} \\ 0 & T^i_{11} \end{pmatrix}, S_i = \begin{pmatrix} S^i_{00} & S^i_{01} \\ 0 & S^i_{11} \end{pmatrix}, 1 \leqslant i \leqslant m$ and $\mathbb{T}_{01} = -\sigma_{\mathbb{T}_{00}, \mathbb{T}}(X), \mathbb{S}_{01} = -\sigma_{\mathbb{S}, \mathbb{S}_{11}}(Y)$. Then a simple calculation shows that $\widetilde{\mathbb{T}}$ and $\widetilde{\mathbb{S}}$ are commuting tuples and satisfy the condition (i).

Set $U := \begin{pmatrix} X_1^* X^* & X_1^* \\ X_2 & -X_2 X \end{pmatrix}$. From Lemma 3.3 and equations (3.9)–(3.13), we obtain U is unitary and $U\widetilde{\mathbb{T}} = \widetilde{\mathbb{S}}U$. Hence, $\widetilde{\mathbb{T}} \sim_{\mathrm{u}} \widetilde{\mathbb{S}}$. ∎

Given an m-tuple of operators $\mathbb{T} = (T_1, \dots, T_m) \in \mathcal{B}^m_n(\Omega)$, by Subsection 2.2 in [42], we know that $\mathbb{T}$ is unitarily equivalent to the adjoint of an m-tuple of multiplication operators $\mathbb{M}_z = (M_{z_1}, \dots, M_{z_m})$ by coordinate functions on some Hilbert space $\mathcal{H}_K$ of holomorphic functions on $\Omega^* = \{\overline{w} : w \in \Omega\}$ possessing a reproducing kernel K. It is expressed equivalently as $\mathbb{T} \sim_{\mathrm{u}} (\mathbb{M}_z^*, \mathcal{H}_K)$. Define e_w to be the evaluation function of $\mathcal{H}_K$ at w. Given a vector $\xi \in \mathbb{C}^n$, the function $e^*_{\overline{w}}\xi \in \mathcal{H}_K$ and is denoted by $K(\cdot, \overline{w})\xi$, which has the reproducing property $\langle f, K(\cdot, \overline{w})\xi \rangle_{\mathcal{H}_K} = \langle f(\overline{w}), \xi \rangle_{\mathbb{C}^n}$. In addition, we have $\ker(\mathbb{M}_z^* - w) = \{K(\cdot, \overline{w})\xi, \xi \in \mathbb{C}^n\}$.

In order to find the minimal order m of covariant partial derivatives in Theorem 2.2, M.J. Cowen and R.G. Douglas introduced the concept of coalescing set [17]. The algebra $\mathscr{A}(w)$ is generated by the curvatures and their covariant derivatives at w. The coalescing set of $\mathscr{A}(x)$ is the set where the dimension of $\mathscr{A}(x)$ (as a function of x) is not locally constant. It is trivially closed and nowhere dense. Furthermore, they proved that two bundles are locally equivalent on at least one dense open set, i.e. the complement of the coalescing set for the curvature corresponding to one of bundles, which means that the two bundles are equivalent. In Corollary 3.4 and Theorem 3.6, using this geometric quantity, we characterize the similarity classification of tuples in the Cowen–Douglas class. In other words, we give a partial answer to R.G. Douglas's question about the geometric similarity of Cowen–Douglas class for multivariable case. Our results allow one to use geometric quantities of Cowen–Douglas tuples with index one to assess whether they are similar or not.

COROLLARY 3.4. *Let* $\mathbb{T},\mathbb{S} \in \mathcal{B}_n^m(\Omega)$ *and* $\mathbb{T} \sim_{\mathrm{u}} (\mathbb{M}_z^*, \mathcal{H}_{K_{\mathbb{T}}}), \mathbb{S} \sim_{\mathrm{u}} (\mathbb{M}_z^*, \mathcal{H}_{K_{\mathbb{S}}})$. *Suppose that* $\{\mathbb{S}_{11} \in \mathcal{L}(\mathcal{H})^m : \ker \sigma_{\mathbb{S}_{11},\mathbb{T}} = \{0\}\} \neq \emptyset$. *Then* $\mathbb{T} \sim_{\mathrm{s}} \mathbb{S}$ *if and only if there exist two tuples* $\widetilde{\mathbb{T}} = (T_1, \ldots, T_m), \widetilde{\mathbb{S}} = (S_1, \ldots, S_m) \in \mathcal{B}_{2n}^m(\Omega)$ *such that:*

(i) $T_i = \begin{pmatrix} T_{00}^i & T_{01}^i \\ 0 & T_{11}^i \end{pmatrix}, S_i = \begin{pmatrix} S_{00}^i & S_{01}^i \\ 0 & S_{11}^i \end{pmatrix}, 1 \leqslant i \leqslant m$, *where* $\mathbb{T}_{01} \in \operatorname{ran} \sigma_{\mathbb{T}_{00},\mathbb{T}}, \mathbb{S}_{01} \in \operatorname{ran} \sigma_{\mathbb{S},\mathbb{S}_{11}}$ *and* $\ker \sigma_{\mathbb{T}_{00},\mathbb{S}} = \{0\}$;

(ii) *the bundles* $E_{\widetilde{\mathbb{T}}}$ *and* $E_{\widetilde{\mathbb{S}}}$ *of* $\widetilde{\mathbb{T}}$ *and* $\widetilde{\mathbb{S}}$ *are locally equivalent on an open dense subset of* Ω, *the complement of the coalescing set for the curvature of* $E_{\widetilde{\mathbb{T}}}$.

Proof. Let Ω_0 be the complement of the coalescing set for the curvature of $E_{\widetilde{\mathbb{T}}}$. Clearly, $\Omega_0 \subset \Omega$. If the bundles $E_{\widetilde{\mathbb{T}}}$ and $E_{\widetilde{\mathbb{S}}}$ are locally equivalent on Ω_0, by using the main theorem of [17] due to M.J. Cowen and R.G. Douglas, we obtain that metric-preserving connections $D_{\widetilde{\mathbb{T}}}$ and $D_{\widetilde{\mathbb{S}}}$ of $E_{\widetilde{\mathbb{T}}}$ and $E_{\widetilde{\mathbb{S}}}$ are equivalent to order $2n$ on Ω. From the proof of the sufficiency in Theorem 3.2, we see that $\mathbb{T}$ is similar to $\mathbb{S}$.

From the proof of Theorem 3.2 and the main theorem of [17], we only need to prove $\widetilde{\mathbb{T}}, \widetilde{\mathbb{S}} \in \mathcal{B}_{2n}^m(\Omega)$.

Suppose that there exist X, Y such that $\mathbb{T}_{01} = \sigma_{\mathbb{T}_{00},\mathbb{T}}(-X), \mathbb{S}_{01} = \sigma_{\mathbb{S},\mathbb{S}_{11}}(-Y)$, that is, $T_{01}^j = XT_{11}^j - T_{00}^j X, S_{01}^j = YS_{11}^j - S_{00}^j Y$ for all $1 \leqslant j \leqslant m$. Without losing generality, we assume that $\mathbb{T} = (\mathbb{M}_z^*, \mathcal{H}_{K_{\mathbb{T}}})$, and then

$$\ker(\mathbb{T} - w) = \{K_{\mathbb{T}}(\cdot, \overline{w})\xi, \xi \in \mathbb{C}^n\}, \quad w \in \Omega.$$

For fixed but arbitrary $w \in \Omega$ and $\xi \in \mathbb{C}^n$, we have

$$\begin{aligned} T_{01}^j K_{\mathbb{T}}(\cdot, \overline{w})\xi &= (XT_{11}^j - T_{00}^j X)K_{\mathbb{T}}(\cdot, \overline{w})\xi = w_j X K_{\mathbb{T}}(\cdot, \overline{w})\xi - T_{00}^j X K_{\mathbb{T}}(\cdot, \overline{w})\xi \\ &= (T_{00}^j - w_j)(-XK_{\mathbb{T}}(\cdot, \overline{w})\xi), \quad 1 \leqslant j \leqslant m. \end{aligned}$$

It follows that $T_{01}^j(\ker(\mathbb{T} - w)) \subset \operatorname{ran}(\mathbb{T}_{00} - w), 1 \leqslant j \leqslant m$ and $\mathbb{T}_{01}(\ker(\mathbb{T} - w)) \subset \operatorname{ran}(\mathbb{T}_{00} - w)$. Thus, $\widetilde{\mathbb{T}} \in \mathcal{B}_{2n}^m(\Omega)$. Similarly, we also have $\widetilde{\mathbb{S}} \in \mathcal{B}_{2n}^m(\Omega)$. This completes the proof. ∎

C.L. Jiang, D.K. Keshari, G. Misra and the second author in [35] showed that for $T, \widetilde{T} \in \mathcal{B}_1^1(\Omega)$, if $XT = \widetilde{T}X$, then either $X = 0$ or X has a dense range. In fact, we see that this result is also true when $\mathbb{T}, \widetilde{\mathbb{T}}$ are tuples in the Cowen–Douglas class with index one. The next lemma shows that the conditions in Lemma 3.3 can be satisfied in many cases.

LEMMA 3.5. *Let* $\mathbb{T}, \mathbb{S} \in \mathcal{B}_1^m(\Omega)$ *and* $\mathbb{T} \sim_{\mathrm{u}} (\mathbb{M}_z^*, \mathcal{H}_{K_0}), \mathbb{S} \sim_{\mathrm{u}} (\mathbb{M}_z^*, \mathcal{H}_{K_1})$. *If*

$$\lim_{\operatorname{dist}(w,\partial\Omega)\to 0} \frac{K_0(w,w)}{K_1(w,w)} = 0,$$

then there exists no non-zero bounded intertwining operator X *such that* $X\mathbb{T} = \mathbb{S}X$, *i.e.* $\ker \sigma_{\mathbb{S},\mathbb{T}} = \{0\}$.

Proof. Without loss of generality, we set $\mathbb{T} = (\mathbb{M}_z^*, \mathcal{H}_{K_0}), \mathbb{S} = (\mathbb{M}_z^*, \mathcal{H}_{K_1})$, where $\mathcal{H}_{K_i}$ are vector-valued analytic functional Hilbert spaces with reproducing kernels $K_i, i = 0, 1$, respectively. Suppose that $X\mathbb{T} = \mathbb{S}X$ for a bounded operator X. This means that $X(\ker(\mathbb{T} - w)) \subset \ker(\mathbb{S} - w)$, $w \in \Omega$. Since $K_0(\cdot, \overline{w}) \in \ker(\mathbb{T} - w)$, $K_1(\cdot, \overline{w}) \in \ker(\mathbb{S} - w)$, there exists a holomorphic function ϕ on Ω such that $X(K_0(\cdot, \overline{w})) = \phi(w) K_1(\cdot, \overline{w}), w \in \Omega$ (see details in Proposition 2.4 from [56]). Note that $\left\| X\left(\frac{K_0(\cdot,\overline{w})}{\|K_0(\cdot,\overline{w})\|}\right) \right\| = |\phi(w)| \frac{\|K_1(\cdot,\overline{w})\|}{\|K_0(\cdot,\overline{w})\|} \leqslant \|X\|$, we have $|\phi(w)| \leqslant \|X\| \frac{\|K_0(\cdot,\overline{w})\|}{\|K_1(\cdot,\overline{w})\|}$. By using $\lim\limits_{\text{dist}(w,\partial\Omega)\to 0} \frac{K_0(w,w)}{K_1(w,w)} = 0$, we obtain that $|\phi|$ will go to zero when $\text{dist}(w, \partial\Omega)$ goes to zero. By the maximum modulus principle for holomorphic function, we have $\phi(w)$ is equal to zero for all $w \in \Omega$, so does $X(K_0(\cdot, \overline{w}))$. According to the spanning property $\ker(\mathbb{T} - w) = \bigvee\{K_0(\cdot, \overline{w})\}$, we infer $X = 0$. That means $\ker \sigma_{\mathbb{S},\mathbb{T}} = \{0\}$. ∎

THEOREM 3.6. *Let $\mathbb{T}, \mathbb{S} \in \mathcal{B}_1^m(\Omega)$, and $\mathbb{T} \sim_{\mathrm{u}} (\mathbb{M}_z^*, \mathcal{H}_{K_\mathbb{T}}), \mathbb{S} \sim_{\mathrm{u}} (\mathbb{M}_z^*, \mathcal{H}_{K_\mathbb{S}})$. Then $\mathbb{T} \sim_{\mathrm{s}} \mathbb{S}$ if and only if there exist two operator tuples $\widetilde{\mathbb{T}} = (T_1, \ldots, T_m), \widetilde{\mathbb{S}} = (S_1, \ldots, S_m) \in \mathcal{B}_2^m(\Omega)$ such that:*

(i) $T_i = \begin{pmatrix} T_{00}^i & T_{01}^i \\ 0 & T_{11}^i \end{pmatrix}, S_i = \begin{pmatrix} S_{00}^i & S_{01}^i \\ 0 & S_{11}^i \end{pmatrix}, 1 \leqslant i \leqslant m$, *where* $\mathbb{T}_{01} \in \operatorname{ran} \sigma_{\mathbb{T}_{00},\mathbb{T}}, \mathbb{S}_{01} \in \operatorname{ran} \sigma_{\mathbb{S},\mathbb{S}_{11}}$ *and* $\ker \sigma_{\mathbb{T}_{00},\mathbb{S}} = \ker \sigma_{\mathbb{S}_{11},\mathbb{T}} = \{0\}$;

(ii) *the bundles $E_{\widetilde{\mathbb{T}}}$ and $E_{\widetilde{\mathbb{S}}}$ of $\widetilde{\mathbb{T}}$ and $\widetilde{\mathbb{S}}$ are locally equivalent on an open dense subset of Ω, the complement of the coalescing set for the curvature of $E_{\widetilde{\mathbb{T}}}$.*

Proof. From the proof of Theorem 3.2 and Corollary 3.4, we only need to prove that there exists an m-tuple $\mathbb{S}_{11} \in \mathcal{B}_1^m(\Omega)$ satisfying $\ker \sigma_{\mathbb{S}_{11},\mathbb{T}} = \{0\}$.

We first choose a generalized Bergman kernel $K_{\widehat{\mathbb{S}}}$ on $\Omega \times \Omega$ (this concept was introduced by R.E. Curto and N. Salinas in [19]), which satisfies

$$\lim_{\text{dist}(w,\partial\Omega)\to 0} K_{\widehat{\mathbb{S}}}(w, w) = \infty.$$

Set $K_{\mathbb{S}_{11}} := K_{\widehat{\mathbb{S}}} \cdot K_{\mathbb{T}}$. By [19] and Theorem 2.6 in [58], we know that $K_{\mathbb{S}_{11}}(w, w)$ is also a generalized Bergman kernel and there exists $\mathbb{S}_{11} \in \mathcal{B}_1^m(\Omega)$ such that $\mathbb{S}_{11} \sim_{\mathrm{u}} (\mathbb{M}_z^*, \mathcal{H}_{K_{\mathbb{S}_{11}}})$. Furthermore, we have that

$$\lim_{\text{dist}(w,\partial\Omega)\to 0} \frac{K_{\mathbb{T}}(w, w)}{K_{\mathbb{S}_{11}}(w, w)} = \lim_{\text{dist}(w,\partial\Omega)\to 0} \frac{1}{K_{\widehat{\mathbb{S}}}(w, w)} = 0.$$

By Lemma 3.5, we know $\ker \sigma_{\mathbb{S}_{11},\mathbb{T}} = \bigcap\limits_{i=1}^{m} \ker \sigma_{S_{11}^i, T_{11}^i} = \{0\}$. This completes the proof. ∎

3.2. APPLICATION. Let T be a bounded operator on some Hilbert $\mathcal{M}$, and $\mathcal{M}$ be a subspace of Hilbert space $\mathcal{N}$. A bounded operator S on $\mathcal{N}$ is a dilation of T if $P_{\mathcal{M}} S|_{\mathcal{M}} = T$. We know that the adjoint of multiplication operator M_z on Hardy space $\mathcal{H}$ is the Cowen–Douglas operator with index one over $\mathbb{D}$. Due to the fact

$M_z \sim_{\mathrm{s}} M_z|_{\mathcal{K}}$ for any invariant subspace $\mathcal{K}$ of M_z, we have that there exist plenty of operators such that their dilation is M_z^* and they are all similar to M_z^*. Thus the following question is natural:

Question. For any Cowen–Douglas operator S, is there a Cowen–Douglas operator T such that S is a dilation of T but not similar to T?

By using the main theorem of this paper, we give lots of positive examples for this question.

In [25], J.S. Fang, C.L. Jiang and the second author introduced an operator in the form of $T_x = \begin{pmatrix} T & x\otimes e_0 \\ 0 & M_z^* \end{pmatrix}$, where M_z^* is the adjoint of multiplication operator on Hardy space $\mathcal{H}$, $M_z^* e_0 = 0$ and $T \in \mathcal{L}(\mathcal{H})$ with spectral radius $r(T) < 1$, $x \in \mathcal{H}$. They proved that T_x is a Cowen–Douglas operator with index one over Σ, a connected component of $\mathbb{D} \setminus \sigma(T)$ which contains $\{w \in \mathbb{D} : r(T) < |w| < 1\}$. Here $T_x \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ is a dilation of M_z^*, since $P_{0\oplus\mathcal{H}} T_x|_{0\oplus\mathcal{H}} = M_z^*$. In the following lemma, we replace the adjoint of the multiplication operator on Hardy space with a general Cowen–Douglas operator, and find the result is still valid.

LEMMA 3.7 ([25]). *Let $S \in \mathcal{B}_n^1(\mathbb{D}) \cap \mathcal{L}(\mathcal{H})$ and $S \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_K)$. Suppose that $T \in \mathcal{L}(\mathcal{H})$ with spectral radius $r(T) < 1$ and $T_{S,x} = \begin{pmatrix} T & x\otimes e_0 \\ 0 & S \end{pmatrix}$, where $e_0 = K(\cdot, 0)\xi_0$ for some $\xi_0 \in \mathbb{C}^n$, $x \in \mathcal{H}$. Let Σ be the connected component of $\mathbb{D} \setminus \sigma(T)$ which contains $\{w \in \mathbb{D} : r(T) < |w| < 1\}$. Then we have the following:*

(i) *for $w \in \mathbb{D} \setminus \sigma(T)$,* $\dim \ker(T_{S,x} - w) = n$ *and* $\operatorname{ran}(T_{S,x} - w) = \mathcal{H} \oplus \mathcal{H}$;

(ii) $T_{S,x} \in \mathcal{B}_n^1(\Sigma)$ *if and only if x is a cyclic vector of T, i.e., $\bigvee\limits_{n\geqslant 0} \{T^n x\} = \mathcal{H}$.*

Proof. In the following, we will describe $\ker(T_{S,x} - w)$ for $w \in \mathbb{D} \setminus \sigma(T)$. Suppose that $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \ker(T_{S,x} - w), w \in \mathbb{D} \setminus \sigma(T)$. This is equivalent to $(T - w)y_1 + (x \otimes e_0)y_2 = 0$ and $(S - w)y_2 = 0, w \in \mathbb{D} \setminus \sigma(T)$. Without loss of generality, we assume that $S = (M_z^*, \mathcal{H}_K)$. Then $\ker(S - w) = \{K(\cdot, \overline{w})\xi, \xi \in \mathbb{C}^n\}$. That means $y_2 = K(\cdot, \overline{w})\xi$ for some $\xi \in \mathbb{C}^n$. Note that $T - w$ is invertible when $w \in \mathbb{D} \setminus \sigma(T)$. We have

$$y_1 = -(T - w)^{-1}(x \otimes e_0)y_2 = -(T - w)^{-1}\langle K(0, \overline{w})\xi, \xi_0\rangle x.$$

Thus, $-(T - w)^{-1}\langle K(0, \overline{w})\xi, \xi_0\rangle x \oplus K(\cdot, \overline{w})\xi$ is an eigenvector of $T_{S,x}$ with eigenvalue $w \in \mathbb{D} \setminus \sigma(T)$ for $\xi \in \mathbb{C}^n$. Since $\dim\ker(S - w) = n$, we infer $\dim\ker(T_{S,x} - w) = n$.

For any $w \in \mathbb{D} \setminus \sigma(T)$, $T_{S,x} - w$ is surjective if for every $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H}$, there exists $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H}$ such that $\begin{pmatrix} T-w & x\otimes e_0 \\ 0 & S-w \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} (T-w)y_1 + (x\otimes e_0)y_2 \\ (S-w)y_2 \end{pmatrix} = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$. The existence of y_2 is clear, since S is a Cowen–Douglas operator. If we take $y_1 = (T - w)^{-1}(y_1' - (x \otimes e_0)y_2)$, then the last equation holds. This proves the statement (i).

In order to get statement (ii), by (i), we only need to prove that

$$\bigvee_{w\in\Sigma}\{-(T-w)^{-1}\langle K(0,\overline{w})\xi,\xi_0\rangle x\oplus K(\cdot,\overline{w})\xi,\xi\in\mathbb{C}^n\}=\mathcal{H}\oplus\mathcal{H}$$

and $\bigvee_{n\geqslant 0}\{T^n x\}=\mathcal{H}$ are equivalent. Suppose that there exists an $x_1\oplus x_2\in\mathcal{H}\oplus\mathcal{H}$ such that $\langle -(T-w)^{-1}\langle K(0,\overline{w})\xi,\xi_0\rangle x\oplus K(\cdot,\overline{w})\xi,x_1\oplus x_2\rangle=0$. Then

$$\langle (w-T)^{-1}\langle K(0,\overline{w})\xi,\xi_0\rangle x,x_1\rangle=-\langle K(\cdot,\overline{w})\xi,x_2\rangle.$$

Note that $K(\cdot,\overline{w})\xi$ is analytic on $\mathbb{D}$, $(w-T)^{-1}=\frac{1}{w}\sum\limits_{n=0}^{\infty}(\frac{T}{w})^n$ for $|w|>r(T)$ and $\langle (w-T)^{-1}\langle K(0,\overline{w})\xi,\xi_0\rangle x,x_1\rangle$ is analytic when $|w|>r(T)$. Then $\langle (w-T)^{-1}\langle K(0,\overline{w})\xi,\xi_0\rangle x,x_1\rangle=-\langle K(\cdot,\overline{w})\xi,x_2\rangle, r(T)<|w|<1$. Thus, by the analytic continuation theorem, we know that $\langle (w-T)^{-1}\langle K(0,\overline{w})\xi,\xi_0\rangle x,x_1\rangle$ is analytic on $\mathbb{C}$. Since $\lim\limits_{|w|\to\infty}\left\langle \sum\limits_{n=0}^{\infty}\left(\frac{T^n x}{w^{n+1}}\right),x_1\right\rangle=0$, $\langle (w-T)^{-1}\langle K(0,\overline{w})\xi,\xi_0\rangle x,x_1\rangle$ is a bounded entire function on $\mathbb{C}$. Then $\sum\limits_{n=0}^{\infty}\langle T^n x,x_1\rangle\frac{1}{w^{n+1}}=0$. Therefore, $\langle T^n x,x_1\rangle=0, n\geqslant 0$. Suppose x is a cyclic vector of T. This implies $x_1=0$. From $\bigvee_{w\in\Sigma}\{K(\cdot,\overline{w})\xi,\xi\in\mathbb{C}^n\}=\mathcal{H}$, we see that $x_2=0$. That means $\bigvee_{w\in\Sigma}\ker(T_{S,x}-w)=\mathcal{H}\oplus\mathcal{H}$. Suppose x is not a cyclic vector of T. Let $0\neq x_1\perp\{T^n x : n\geqslant 0\}$. Then $(x_1\oplus 0)\perp\bigvee_{w\in\Sigma}\ker(T_{S,x}-w)$ and therefore $\bigvee_{w\in\Sigma}\ker(T_{S,x}-w)\neq\mathcal{H}\oplus\mathcal{H}$. This completes the proof. ∎

PROPOSITION 3.8. *Let $T_{S,x}$ be the operator in Lemma 3.7 and $T\in\mathcal{L}(\mathcal{H})$ with spectral radius $r(T)<1$. Suppose that $\lim\limits_{|w|\to r(T)}\|(T-w)^{-1}x\|=\infty$ and x is a cyclic vector of T, then $T_{S,x}$ is not similar to S.*

Proof. Suppose that $T_{S,x}$ is similar to S. By Theorem 3.2, without losing generality, there exists a bounded linear operator X such that $Y=(I+X^*X)^{1/2}$ and $T_{S,x}Y=YS$. By Lemma 3.7 and x is a cyclic vector of T, we have that $T_{S,x}\in\mathcal{B}_n^1(\Sigma)$, where Σ is the connected component of $\mathbb{D}\setminus\sigma(T)$ which contains $\{w\in\mathbb{D}: r(T)<|w|<1\}$. Note that $-(T-w)^{-1}\langle K(\cdot,\overline{w})\xi,e_0\rangle x\oplus K(\cdot,\overline{w})\xi\in\ker(T_{S,x}-w)$ for $\xi\in\mathbb{C}^n$ and $w\in\Sigma$. By Proposition 2.4 in [56] and $T_{S,x}Y=YS$, we find $\xi_w\in\mathbb{C}^n$ such that

$$(I+X^*X)^{1/2}K(\cdot,\overline{w})\xi_w=-(T-w)^{-1}\langle K(0,\overline{w})\xi_w,\xi_0\rangle x\oplus K(\cdot,\overline{w})\xi_w.$$

This implies that $\|(I+X^*X)^{1/2}K(\cdot,\overline{w})\xi_w\|^2\leqslant(1+\|X\|^2)\langle K(\overline{w},\overline{w})\xi_w,\xi_w\rangle$ and

$$\begin{aligned}\|(I+X^*X)^{1/2}K(\cdot,\overline{w})\xi_w\|^2&=\|(T-w)^{-1}\langle K(0,\overline{w})\xi_w,\xi_0\rangle x\|^2+\langle K(\overline{w},\overline{w})\xi_w,\xi_w\rangle\\&=|\langle K(0,\overline{w})\xi_w,\xi_0\rangle|^2\|(T-w)^{-1}x\|^2+\langle K(\overline{w},\overline{w})\xi_w,\xi_w\rangle.\end{aligned}$$

Then we have $0 \leqslant |\langle K(0,\overline{w})\xi_w, \xi_0\rangle|^2 \|(T-w)^{-1}x\|^2 \leqslant \|X\|^2\langle K(\overline{w},\overline{w})\xi_w,\xi_w\rangle$. Note that $\langle K(\overline{w},\overline{w})\xi_w,\xi_w\rangle, w \in \mathbb{D}$ and X are bounded, then

$$|\langle K(0,\overline{w})\xi_w, \xi_0\rangle|^2 \|(T-w)^{-1}x\|^2$$

is bounded. Since $\lim\limits_{|w|\to r(T)} \|(T-w)^{-1}x\| = \infty$, $|\langle K(0,\overline{w})\xi_w,\xi_0\rangle| \to 0$ when $|w| \to r(T)$. We know that $\langle K(0,\overline{w})\xi_w,\xi_0\rangle$ is holomorphic, by the maximum modulus principle for holomorphic function, then $\langle K(0,\overline{w})\xi_w,\xi_0\rangle = 0$ for all $w \in \Sigma$. This is a contradiction. So $T_{S,x}$ is not similar to S. ▮

Upon using our main theorem (Theorem 3.2), a new proof of the sufficiency of A.L. Shields' similarity theorem in [60] is given.

EXAMPLE 3.9. Let $T, S \in \mathcal{B}_1^1(\mathbb{D})$ and $T \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_{K_0}), S \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_{K_1})$, where $K_i(z,w) = \sum\limits_{j=0}^{\infty} a_j^i z^j \overline{w}^j$ and $a_j^i > 0$ for $i = 0,1, j \geqslant 0$. If $\frac{a_j^1}{a_j^0}$ is bounded and bounded from 0 for all $j \geqslant 0$, then $T_0 \sim_{\mathrm{s}} T_1$.

Proof. By Subsection 2.1 of [47] due to Q. Lin and Theorem 3.6 in this paper, we find that there exist T_0, S_1 such that $\ker \sigma_{T_0,S} = \ker \sigma_{S_1,T} = \{0\}$. If $\frac{a_j^1}{a_j^0}$ is bounded and bounded from 0 and $a_j^i > 0$, then there exists $l > 0$ such that $l^2 a_j^1 - a_j^0 > 0$ and $b_j = \sqrt{l^2 \frac{a_j^1}{a_j^0} - 1}$ for all $j \geqslant 0$. Let $X = \operatorname{diag}\{b_0, b_1, b_2, \ldots\}$. Then X is a bounded operator. Without losing generality, we assume that $T = (M_z^*, \mathcal{H}_{K_0}), S = (M_z^*, \mathcal{H}_{K_1})$. Selecting the non-zero holomorphic sections of T and S as $K_0(\cdot,\overline{w})$ and $lK_1(\cdot,\overline{w})$, respectively. Thus, $l^2K_1(\overline{w},\overline{w}) = K_0(\overline{w},\overline{w}) + \|XK_0(\cdot,\overline{w})\|^2 = \|(1+X^*X)^{1/2}K_0(\cdot,\overline{w})\|^2$. Note that $I + X^*X$ and $I + XX^*$ are positive and invertible. There exist invertible operators U_{01}, U_{10} such that $(1+XX^*)^{-1} = U_{10}^*U_{10}, (1+X^*X)^{-1} = U_{01}^*U_{01}$. Choosing $\widetilde{Y} \in \ker \sigma_{S,S_1}$. Set $Y = \widetilde{Y} + U_{01}X^*U_{10}^{-1}$, $\widetilde{T} = \begin{pmatrix} T_0 & \sigma_{T_0,T}(-X) \\ 0 & T \end{pmatrix}$, $\widetilde{S} = \begin{pmatrix} S & \sigma_{S,S_1}(-Y) \\ 0 & S_1 \end{pmatrix}$ and $U = \begin{pmatrix} U_{01}X^* & U_{01} \\ U_{10} & -U_{10}X \end{pmatrix}$. By Theorem 3.2, we know that U is a unitary operator and $U\widetilde{T} = \widetilde{S}U$. Thus we have $T \sim_{\mathrm{s}} S$. ▮

EXAMPLE 3.10. Let $T_i \in \mathcal{B}_1^1(\mathbb{D})$, $T_i \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_{K_i}), i = 0,1$. If $K_1(z,w) - K_0(z,w) = P(z,\overline{w})$ is a polynomial and positive over $\mathbb{D} \times \mathbb{D}$, then $T_0 \sim_{\mathrm{s}} T_1$.

Proof. From the symmetry of $K_i, i = 0,1$, P is also symmetric. Without losing generality, we assume that $P(z,\overline{w}) = \sum\limits_{p,q=0}^{m} a_{pq} z^p \overline{w}^q$, $z, \overline{w} \in \mathbb{D}$ for some positive m. This implies that matrix $A := (a_{pq})_{p,q=0}^m$ is positive. By diagonalization of A, there exist $\{\phi_i\}_{i=0}^m \subset H^\infty(\mathbb{D})$, a set of bounded holomorphic function on $\mathbb{D}$, such that $P(z,\overline{w}) = \sum\limits_{j=0}^{m} \phi_j(z)\overline{\phi_j(w)}, z, \overline{w} \in \mathbb{D}$. Let $K_0(z,w) = \sum\limits_{n=0}^{\infty} e_n(z)e_n^*(w)$ for some orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of $\mathcal{H}_{K_0}$. Set $\phi_j(z) = \sum\limits_{i=0}^{m} b_{ji}e_i(z)$, $b_{ji} \in \mathbb{C}, 0 \leqslant j \leqslant m$. A linear operator X is defined as the following:

$$X(e_i) := \begin{cases} \sum\limits_{j=0}^{m} b_{ji}e_j, & 0 \leqslant i \leqslant m, \\ 0, & i > m. \end{cases}$$

Then

$$\begin{aligned} \|X\| &= \sup_{\|y\|=1} \|Xy\| = \sup_{\|y\|=1} \Big\| X \sum_{n=0}^{\infty} b_n e_n \Big\| = \sup_{\|y\|=1} \Big\| \sum_{n=0}^{m} b_n \Big(\sum_{j=0}^{m} b_{jn} e_j \Big) \Big\| \\ &= \sup_{\|y\|=1} \Big\| \sum_{j=0}^{m} \sum_{n=0}^{m} b_n b_{jn} e_j \Big\| \leqslant \Big(\sum_{n=0}^{m} b_n^2 \Big)^{1/2} \sum_{j=0}^{m} \Big(\Big\| \sum_{n=0}^{m} b_{jn}^2 \Big\|^{1/2} \Big) \leqslant (m+1)M, \end{aligned}$$

where $y = \sum\limits_{n=0}^{\infty} b_n e_n \in \mathcal{H}$, $M = \max\limits_{0 \leqslant i \leqslant m} \{\|\phi_i\|\}$. Thus, X is bounded. Note that $XK_0(z,\overline{w}) = \sum\limits_{i=0}^{\infty} Xe_i(w)e_i(z) = \sum\limits_{i=0}^{m} \sum\limits_{j=0}^{m} b_{ji}e_j(w)e_i(z) = \sum\limits_{j=0}^{m} \sum\limits_{i=0}^{m} b_{ji}e_i(z)e_j(w) = \sum\limits_{j=0}^{m} \phi_j(z)e_j(w)$, we then have $\|XK_0(z,\overline{w})\|^2 = \sum\limits_{i=0}^{m} |\phi_i(z)|^2$ and $\|XK_0(\cdot,\overline{w})\|^2 = P(w,\overline{w})$. Similarly to the proof of Example 3.9, we deduce that $T_0 \sim_{\mathrm{s}} T_1$. ∎

EXAMPLE 3.11. Let $\mathbb{T}_i = (T_1^i, \ldots, T_m^i) \in \mathcal{B}_1^m(\Omega) \cap \mathcal{L}(\mathcal{H}_i)$, $\mathbb{T}_i \sim_{\mathrm{u}} (\mathbb{M}_z^*, \mathcal{H}_{K_i})$ for $i = 0, 1$. If there exists a uniformly bounded positive sequence $\{\lambda_\alpha\}_{\alpha \in \mathbb{Z}_+^m}$ such that $K_1(z,w) - K_0(z,w) = \sum\limits_{\alpha \in \mathbb{Z}_+^m} \lambda_\alpha e_\alpha(z) e_\alpha^*(w)$ for some orthonormal basis $\{e_\alpha\}_{\alpha \in \mathbb{Z}_+^m}$ of $\mathcal{H}_{K_0}$, then $\mathbb{T}_0 \sim_{\mathrm{s}} \mathbb{T}_1$.

Proof. Without loss of generality, we assume that $\mathbb{T}_i = (\mathbb{M}_z^*, \mathcal{H}_{K_i}), i = 0, 1$. Since $\mathbb{T}_i \in \mathcal{B}_1^m(\Omega)$, we know that $K_i(\cdot, \overline{w}) \in \ker(\mathbb{T}_i - w)$ for $w \in \Omega$ and $i = 0, 1$. Then

$$\begin{aligned} K_1(w,w) - K_0(w,w) &= \sum_{\alpha \in \mathbb{Z}_+^m} \lambda_\alpha |e_\alpha(w)|^2 = \sum_{\alpha \in \mathbb{Z}_+^m} \lambda_\alpha |\langle K_0(\cdot,\overline{w}), e_\alpha \rangle|^2 \\ &= \sum_{\alpha \in \mathbb{Z}_+^m} \lambda_\alpha \langle K_0(\cdot,\overline{w}), e_\alpha \rangle \langle e_\alpha, K_0(\cdot,\overline{w}) \rangle \\ &= \Big\langle \sum_{\alpha \in \mathbb{Z}_+^m} \lambda_\alpha \langle K_0(\cdot,\overline{w}), e_\alpha \rangle e_\alpha, K_0(\cdot,\overline{w}) \Big\rangle. \end{aligned} \tag{3.14}$$

Let $X_1 := \sum\limits_{\alpha \in \mathbb{Z}_+^m} \lambda_\alpha e_\alpha \otimes e_\alpha$. Further, equation (3.14) can be written as $K_1(w,w) - K_0(w,w) = \langle X_1 K_0(\cdot,\overline{w}), K_0(\cdot,\overline{w}) \rangle$. It is easy to see that X_1 is positive, since $\lambda_\alpha > 0$ for all $\alpha \in \mathbb{Z}_+^m$. Thus, there exists an operator X_2 such that $X_1 = X_2^* X_2$ and $K_1(w,w) - K_0(w,w) = \|X_2 K_0(\cdot,\overline{w})\|^2$. Similarly to the proof of Example 3.9, we deduce that $\mathbb{T}_0 \sim_{\mathrm{s}} \mathbb{T}_1$. ∎

EXAMPLE 3.12. Let $\mathbb{T}_i = (T_1^i, \dots, T_m^i) \in \mathcal{B}_1^m(\Omega) \cap \mathcal{L}(\mathcal{H}_i)$, $\mathbb{T}_i \sim_{\mathrm{u}} (\mathbb{M}_z^*, \mathcal{H}_{K_i})$ for $i = 0, 1$. Suppose that $L^2(X, \mu)$ is separable for some σ-finite measure space (X, μ). If there exists $\phi \in L^\infty(X, \mu)$ (need not to be holomorphic) such that $K_1(w, w) = (1 + |\phi(w)|^2)K_0(w, w)$, then $\mathbb{T}_0 \sim_{\mathrm{s}} \mathbb{T}_1$.

Proof. From $\phi \in L^\infty(X, \mu)$ for some σ-finite measure space (X, μ), then there is a multiplication operator $M_\phi : L^2(X, \mu) \to L^2(X, \mu)$ defined by $M_\phi f(x) = \phi(x) f(x)$, and it satisfies $\|M_\phi f\|_2 = \Big(\int\limits_X |\phi f|^2 \mathrm{d}\mu\Big)^{1/2} \leqslant \Big(\int\limits_X (\|\phi\|_\infty |f|)^2 \mathrm{d}\mu\Big)^{1/2} \leqslant \|\phi\|_\infty \|f\|, f \in L^2(X, \mu)$, thus M_ϕ is bounded. For $f, g \in L^2(X, \mu)$, we have $\langle M_\phi f, g\rangle = \int\limits_X (\phi f)\overline{g} \mathrm{d}\mu = \int\limits_X f\overline{(\overline{\phi} g)} \mathrm{d}\mu = \langle f, M_{\overline{\phi}} g\rangle$, which implies $M_\phi^* = M_{\overline{\phi}}$.

From $K_1(w, w) = (1 + |\phi(w)|^2)K_0(w, w)$, we have

$$\|K_1(\cdot, \overline{w})\|^2 = (1 + |\phi(w)|^2)\|K_0(\cdot, \overline{w})\|^2.$$

Since $L^2(X, \mu)$ is separable, there is a unitary operator $U : \mathcal{H}_{K_0} \to L^2(X, \mu)$ such that

$$\begin{aligned}\|K_1(\cdot, \overline{w})\|^2 &= (1 + |\phi(w)|^2)\|UK_0(\cdot, \overline{w})\|^2 = \langle (1 + |\phi(w)|^2)UK_0(\cdot, \overline{w}), UK_0(\cdot, \overline{w})\rangle \\ &= \langle (I + M_\phi^* M_\phi)UK_0(\cdot, \overline{w}), UK_0(\cdot, \overline{w})\rangle = \|(I + U^* M_\phi^* M_\phi U)^{1/2} K_0(\cdot, \overline{w})\|^2.\end{aligned}$$

Without loss of generality, we assume that $\mathbb{T}_i = (\mathbb{M}_z^*, \mathcal{H}_{K_i})$, then

$$K_i(\cdot, \overline{w}) \in \ker(\mathbb{T}_i - w)$$

for $w \in \Omega$ and $i = 0, 1$. Similarly to the proof of Example 3.9, we deduce that $\mathbb{T}_0 \sim_{\mathrm{s}} \mathbb{T}_1$. ∎

4. A SUBCLASS $N\mathcal{FB}_{n_0,n_1}^m(\Omega)$ OF COWEN–DOUGLAS TUPLES

Let *Möb* denote the group of all biholomorphic automorphisms of $\mathbb{D}$. Recall that a bounded operator T is said to be homogeneous if the spectrum $\sigma(T)$ of T is contained in $\overline{\mathbb{D}}$ and for every $\phi \in$ *Möb*, $\phi(T)$ is unitarily equivalent to T. The concept of homogeneous operator can be extended to the commuting operator tuple. When $\mathcal{D}$ is a bounded symmetric domain, an m-tuple $\mathbb{T} = (T_1, \dots, T_m)$ of commuting bounded operators is said to be homogeneous with respect to G if their joint Taylor spectrum is contained in $\overline{\mathcal{D}}$ and for every holomorphic automorphism $\phi \in G$, $\phi(\mathbb{T})$ is unitarily equivalent to $\mathbb{T}$ (see [7, 53]). The topic of homogeneous operators and tuples received much attention [7, 8, 9, 32, 44, 45, 48, 57], mostly by using representation theory of Lie groups and complex geometry. G. Misra in [48] has fully characterized the homogeneous operators in $\mathcal{B}_1^1(\mathbb{D})$, e.g. proving that for any $j > 0$, the unilateral shift operator with weight sequence $\left\{\sqrt{\frac{i+1}{i+j}}\right\}_{i=0}^\infty$ is homogeneous. The homogeneous operator in $\mathcal{B}_1^1(\mathbb{D})$ not only provides us with a model, but also helps us to study the properties and similarity of other operators. In [45], A. Koranyi and G. Misra completed the classification of irreducible homogeneous operators in $\mathcal{B}_n^1(\mathbb{D})$.

For $\alpha \in \mathbb{C}$ and homogeneous operators T_0, T_1 acting on $\mathcal{H}$, $\begin{pmatrix} T_0 & \alpha(T_0 - T_1) \\ 0 & T_1 \end{pmatrix}$ is homogeneous if T_0 and T_1 have the same associated unitary representation given by A. Koranyi in Lemma 2.1 from [43]. Let $t_n = t_n(a,b) = \sqrt{\frac{n+a}{n+b}}, n \in \mathbb{Z}$ for $a, b \in (0,1), a \neq b$. Define the operator $T = T(a,b)$ as $Te_n = t_n e_{n+1}$ for the natural basis $\{e_n\}_{n\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$. It is shown to be homogeneous. Let $\alpha > 0$. Defining $\widetilde{T} = \widetilde{T}(a,b,\alpha) = \begin{pmatrix} T(a,b) & \alpha(T(a,b) - T(b,a)) \\ 0 & T(b,a) \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$ and rearranging the bases, we obtain a block matrix so that $\widetilde{T}_n = \widetilde{T}_n(a,b,\alpha)$ at $(n+1,n)$-position and the rest are 0. It is shown that $\widetilde{T}$ is homogeneous and irreducible in [43]. This is the first example of irreducible bi-lateral homogeneous 2-shifts with three parameters due to A. Koranyi. Next, another bi-lateral homogeneous 2-shift introduced by S. Hazra in [32]. Let $B(s)$ and B be bi-lateral shifts, the weight sequence of $B(s)$ be $w_n = \frac{n + \frac{1+\lambda}{2} + s}{n + \frac{1+\lambda}{2} - s}$ $(s \neq 0)$, and B be unweighted. Then operators $B(s)$ and B are homogeneous in Theorem 5.2 of [9]. For $\alpha > 0$, define $B(\lambda, s, \alpha) = \begin{pmatrix} B(s) & \alpha(B(s) - B) \\ 0 & B \end{pmatrix}$. It is also homogeneous by Lemma 2.1 of [43]. It is proved in [32] that $B(\lambda, s, \alpha)$ is irreducible and the homogeneous operators defined by A. Koranyi and S. Hazra, respectively, are mutually unitarily inequivalent.

Inspired by the above results, we here define a new class of tuples of commuting bounded operators. With the help of this class, we discuss the similarity of commuting tuples. In addition, we know that the Cowen–Douglas class is a very rich operator and tuple class, including many homogeneous operators, normal operators and so on. The structure of the elements in $\mathcal{B}_n^m(\Omega)$ is very complicated, so that we still cannot clearly describe their similarity. Therefore, it is necessary to investigate a subclass of $\mathcal{B}_n^m(\Omega)$.

4.1. DEFINITIONS.

In what follows, we assume that n_0, n_1 are positive integers.

DEFINITION 4.1. Let $\mathbb{T}_{ii} = (T_{ii}^1, \ldots, T_{ii}^m) \in \mathcal{B}_{n_i}^m(\Omega), i = 0, 1$ and $\mathbb{T}_{01} = (T_{01}^1, \ldots, T_{01}^m)$ be a commuting m-tuple of bounded operators. Suppose that the m-tuple $\mathbb{T} = (T_1, \ldots, T_m)$ satisfies $T_j = \begin{pmatrix} T_{00}^j & T_{01}^j \\ 0 & T_{11}^j \end{pmatrix}$ for $1 \leqslant j \leqslant m$. We call $\mathbb{T} \in N\mathcal{FB}_{n_0,n_1}^m(\Omega)$, if $\mathbb{T}_{01} \in \operatorname{ran} \sigma_{\mathbb{T}_{00},\mathbb{T}_{11}}$.

By Corollary 3.4, we see that tuples in $N\mathcal{FB}_{n_0,n_1}^m(\Omega)$ are Cowen–Douglas tuples with index $n_0 + n_1$ over Ω. If $n_0 = n_1 = n$, the class $N\mathcal{FB}_{n_0,n_1}^m(\Omega)$ can be expressed as $N\mathcal{FB}_{2n}^m(\Omega)$.

REMARK 4.2. Suppose that tuple $\mathbb{T}$ satisfies the conditions of Definition 4.1 and there exists an operator X such that $\mathbb{T}_{01} = \sigma_{\mathbb{T}_{00},\mathbb{T}_{11}}(-X)$. Then $\mathbb{T}_{ii}, i = 0, 1$ are commuting m-tuples, which means $\mathbb{T}$ is a commuting m-tuple, since

$$\begin{aligned}T_pT_q &= \begin{pmatrix} T^p_{00}T^q_{00} & -T^p_{00}T^q_{00}X + XT^p_{11}T^q_{11} \\ 0 & T^p_{11}T^q_{11} \end{pmatrix} = \begin{pmatrix} T^q_{00}T^p_{00} & -T^q_{00}T^p_{00}X + XT^q_{11}T^p_{11} \\ 0 & T^q_{11}T^p_{11} \end{pmatrix} \\ &= T_qT_p \quad \text{for all } 1 \leqslant p,q \leqslant m.\end{aligned}$$

Based on Theorem 1.49 in [40], C. Jiang, D.K. Keshari, G. Misra and the second author introduced an operator class, denoted by $\mathcal{FB}^1_n(\Omega)$ in [34, 35], which is norm dense in $\mathcal{B}^1_n(\Omega)$. They also showed that the complete unitary invariants of operators in $\mathcal{FB}^1_n(\Omega)$ include the curvatures and the second fundamental forms of the diagonal operators. We will give the commuting tuple version of this kind of operator.

DEFINITION 4.3. Let $\mathbb{T}_{ii} = (T^1_{ii}, \ldots, T^m_{ii}) \in \mathcal{B}^m_1(\Omega) \cap \mathcal{L}(\mathcal{H}_i)^m, i = 0, 1$. Suppose that there exists $T_{01} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$ such that $\mathbb{T} = (T_1, \ldots, T_m)$ is a commuting m-tuple with $T_j = \begin{pmatrix} T^i_{00} & T_{01} \\ 0 & T^i_{11} \end{pmatrix}$, $1 \leqslant j \leqslant m$. We call $\mathbb{T} \in \mathcal{FB}^m_2(\Omega)$, if $T_{01} \in \ker \sigma_{\mathbb{T}_{00}, \mathbb{T}_{11}}$.

In order to show that the tuples in $N\mathcal{FB}^m_2(\Omega)$ may not belong to $\mathcal{FB}^m_2(\Omega)$, we need to introduce the following concept which is first defined in [39].

DEFINITION 4.4 ([39], Property (H)). Let $T_{ii} \in \mathcal{L}(\mathcal{H}_i), i = 0, 1$ and

$$T = \begin{pmatrix} T_{00} & XT_{11} - T_{00}X \\ 0 & T_{11} \end{pmatrix} \in \mathcal{B}^1_2(\Omega).$$

We say that T satisfies the Property (H) if and only if the following statements hold: If $Y \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_0)$ satisfies:

(i) $T_{00}Y = YT_{11}$;

(ii) $Y = T_{00}Z - ZT_{11}$ for some Z.

Then $Y = 0$. That is equivalent to $\ker \sigma_{T_{00}, T_{11}} \cap \operatorname{ran} \sigma_{T_{00}, T_{11}} = \{0\}$.

By Definition 4.4, we see if $T = \begin{pmatrix} T_{00} & XT_{11} - T_{00}X \\ 0 & T_{11} \end{pmatrix}$ satisfies the Property (H), and $XT_{11} \neq T_{00}X$, then T does not belong to $\mathcal{FB}^1_2(\Omega)$. Otherwise, $XT_{11} - T_{00}X \in \ker \sigma_{T_{00}, T_{11}} \cap \operatorname{ran} \sigma_{T_{00}, T_{11}}$. That means $XT_{11} = T_{00}X$. It is a contradiction. In the following, we will give two results to show when T would satisfy the Property (H).

PROPOSITION 4.5 ([39]). *Let $T_0, T_1 \in \mathcal{L}(\mathcal{H})$ and S_1 be the right inverse of T_1. If $\lim\limits_{n \to \infty} \frac{\|T^n_0\| \cdot \|S^n_1\|}{n} = 0$, then the Property* (H) *holds, i.e. if there exists $X \in \mathcal{L}(\mathcal{H})$ such that $T_0X = XT_1$ and $X = T_0Y - YT_1$, then $X = 0$ (i.e.* $\ker \sigma_{T_0, T_1} \cap \operatorname{ran} \sigma_{T_0, T_1} = \{0\}$*).*

EXAMPLE 4.6 ([39]). Let $A, B \in \mathcal{B}^1_1(\mathbb{D})$ be backward shift operators with weighted sequences $\{a_i\}^{\infty}_{i=1}$ and $\{b_i\}^{\infty}_{i=1}$. If $\lim\limits_{n \to \infty} n \frac{\prod^n_{k=1} b_k}{\prod^n_{k=1} a_k} = \infty$, then $\ker \sigma_{A,B} \cap \operatorname{ran} \sigma_{A,B} = \{0\}$.

In [35], it is proved that the unitary operator intertwining two operators T and $\widetilde{T}$ in $\mathcal{FB}^1_n(\Omega)$ should be a diagonal matrix. From the proof of Lemma 3.3, it can be seen that a unitary operator intertwining the two tuples in the class $N\mathcal{F}\mathcal{B}_n^m(\Omega)$ could be non-diagonal. This is another reason why we study this new class. Although the structures of tuples in the classes $N\mathcal{F}\mathcal{B}_2^m(\Omega)$ and $\mathcal{F}\mathcal{B}_2^m(\Omega)$ are quite different, the following proposition shows that they are also closely related. The unitary equivalence of the tuples in $N\mathcal{F}\mathcal{B}_2^m(\Omega)$ can always be related to the similarity of the tuples in $\mathcal{F}\mathcal{B}_2^m(\Omega)$.

PROPOSITION 4.7. *For $i = 0, 1$, let $\mathbb{T}_{ii}, \mathbb{S}_{ii} \in \mathcal{B}_1^m(\Omega), \mathbb{T}_{01}, \mathbb{S}_{01} \in \mathcal{L}(\mathcal{H})^m$. Let $\widetilde{\mathbb{T}} = (T_1, \ldots, T_m)$, $\widetilde{\mathbb{S}} = (S_1, \ldots, S_m) \in N\mathcal{F}\mathcal{B}_2^m(\Omega)$ with $T_j = \begin{pmatrix} T^j_{00} & T^j_{01} \\ 0 & T^j_{11} \end{pmatrix}$, $S_j = \begin{pmatrix} S^j_{00} & S^j_{01} \\ 0 & S^j_{11} \end{pmatrix}$, $1 \leqslant j \leqslant m$. Suppose that $\ker \sigma_{\mathbb{T}_{00}, \mathbb{S}_{00}} = \ker \sigma_{\mathbb{S}_{11}, \mathbb{T}_{11}} = \{0\}$. If $\widetilde{\mathbb{T}} \sim_{\mathrm{u}} \widetilde{\mathbb{S}}$, then there exist operators S_0, S_1 and tuples $\widehat{\mathbb{T}} = (\widehat{T}_1, \ldots, \widehat{T}_m), \widehat{\mathbb{S}} = (\widehat{S}_1, \ldots, \widehat{S}_m)$ with $\widehat{T}_i = \begin{pmatrix} S^i_{00} & S_0 \\ 0 & T^i_{00} \end{pmatrix}$, $\widehat{S}_i = \begin{pmatrix} T^i_{11} & S_1 \\ 0 & S^i_{11} \end{pmatrix}$, $1 \leqslant i \leqslant m$, such that $\widehat{\mathbb{T}}, \widehat{\mathbb{S}} \in \mathcal{F}\mathcal{B}_2^m(\Omega)$ and $\widehat{\mathbb{T}} \sim_{\mathrm{s}} \widehat{\mathbb{S}}$.*

Proof. Suppose that there exist X, Y such that $\mathbb{T}_{01} = \sigma_{\mathbb{T}_{00}, \mathbb{T}_{11}}(-X), \mathbb{S}_{01} = \sigma_{\mathbb{S}_{00}, \mathbb{S}_{11}}(-Y)$. Since $\ker \sigma_{\mathbb{T}_{00}, \mathbb{S}_{00}} = \ker \sigma_{\mathbb{S}_{11}, \mathbb{T}_{11}} = \{0\}$ and $\widetilde{\mathbb{T}} \sim_{\mathrm{u}} \widetilde{\mathbb{S}}$, by Lemma 3.3, we will find a unitary operator $U = ((U_{i,j}))_{2\times 2}$ such that

$$T^i_{00} = U_{10}^{-1} S^i_{11} U_{10}, \quad T^i_{11} = U_{01}^* S^i_{00} U_{01}^{*-1}, \tag{4.1}$$

and $S^i_{00}(Y - U_{01}X^*U_{10}^{-1}) = (Y - U_{01}X^*U_{10}^{-1})S^i_{11}$, $1 \leqslant i \leqslant m$. Multiplying U_{10} on the right side of the equation above, by equation (4.1), we have

$$S^i_{00}(YU_{10} - U_{01}X^*) = (YU_{10} - U_{01}X^*)T^i_{00}, \quad 1 \leqslant i \leqslant m. \tag{4.2}$$

Then multiplying U_{01}^* on the left side of the last equation above, by equation (4.1) again and $XU_{01}^*U_{01} = U_{10}^*U_{10}X$ due to Lemma 3.3, we obtain

$$T^i_{11}(U_{01}^*Y - X^*U_{10}^*) = (U_{01}^*Y - X^*U_{10}^*)S^i_{11}, \quad 1 \leqslant i \leqslant m. \tag{4.3}$$

Set $S_0 = YU_{10} - U_{01}X^*$ and $S_1 = U_{01}^*Y - X^*U_{10}^*$. By equations (4.2) and (4.3), we see that $S_0 \in \ker\sigma_{\mathbb{S}_{00}, \mathbb{T}_{00}}$ and $S_1 \in \ker\sigma_{\mathbb{T}_{11}, \mathbb{S}_{11}}$. Let $\widehat{T}_i = \begin{pmatrix} S^i_{00} & S_0 \\ 0 & T^i_{00} \end{pmatrix}$, $\widehat{S}_i = \begin{pmatrix} T^i_{11} & S_1 \\ 0 & S^i_{11} \end{pmatrix}$, $1 \leqslant i \leqslant m$ and $\widehat{\mathbb{T}} = (\widehat{T}_1, \ldots, \widehat{T}_m), \widehat{\mathbb{S}} = (\widehat{S}_1, \ldots, \widehat{S}_m)$. That means $\widehat{\mathbb{T}}, \widehat{\mathbb{S}} \in \mathcal{F}\mathcal{B}_2^m(\Omega)$ from Definition 4.3. Set $Z := U_{01}^* \oplus U_{10}$. Then Z is invertible. Using the equations $XU_{01}^*U_{01} = U_{10}^*U_{10}X$ and (4.1) again, we imply that $Z\widehat{T}_iZ^{-1} = \widehat{S}_i$ for all $1 \leqslant i \leqslant m$. Hence, $Z\widehat{\mathbb{T}} = \widehat{\mathbb{S}}Z$ and $\widehat{\mathbb{T}} \sim_{\mathrm{s}} \widehat{\mathbb{S}}$. ∎

Let $\{\mathbb{T}\}' = \{X : X\mathbb{T} = \mathbb{T}X\}$, $\{\mathbb{T}, \mathbb{T}^*\}' = \{X : X\mathbb{T} = \mathbb{T}X,\ X\mathbb{T}^* = \mathbb{T}^*X\}$. The commuting tuple $\mathbb{T}$ is said to be irreducible, if there are no nontrivial orthogonal idempotents in $\{\mathbb{T}\}'$. The following lemma is given by J. Fang, C. Jiang and P. Wu in Lemma 3.3 of [26], which shows that the double commutant $\{T, T^*\}'$ of irreducible operator T contains only scalar operators. We will prove that this result also holds for irreducible operator tuples.

LEMMA 4.8 ([26]). *If $\mathbb{T} \in \mathcal{L}(\mathcal{H})^m$ is irreducible and there is $X \in \mathcal{L}(\mathcal{H})$ such that $X \in \{\mathbb{T}, \mathbb{T}^*\}'$, then X is a scalar multiple of identity.*

Proof. Since $X\mathbb{T} = \mathbb{T}X$, $X\mathbb{T}^* = \mathbb{T}^*X$, we have $X^*X\mathbb{T} = \mathbb{T}X^*X$. Then, for any spectral projection P of X^*X, this implies $P\mathbb{T} = \mathbb{T}P$. From the irreducibility of $\mathbb{T}$, it follows that $P = 0$ or I. Furthermore, $\sigma(X^*X) = \{\alpha\}$ and $X^*X = \alpha I$. Note that $X\mathbb{T}(\ker X) = \mathbb{T}X(\ker X) = 0$ and $X\mathbb{T}^*(\ker X) = \mathbb{T}^*X(\ker X) = 0$. We know that $\ker X$ is a reducing subspace for $\mathbb{T}$, then $\ker X = \{0\}$ or $\mathcal{H}$, since $\mathbb{T}$ is irreducible. So, either X is injective or X is 0. Suppose that X is injective with dense range. By the polar decomposition of X, we have $X = U(X^*X)^{1/2}$, U is a unitary operator. We assume that $\alpha \neq 0$, then $U\mathbb{T} = \mathbb{T}U, U\mathbb{T}^* = \mathbb{T}^*U$. Repeating the above assumption, we have $U = \beta I$. Thus $X = \sqrt{\alpha}\beta I$ is a scalar multiple of identity. ∎

Let T_1, T_2 be two bounded operators acting on $\mathcal{H}$ and $\alpha \in \mathbb{C}$. For $\widetilde{T} = \begin{pmatrix} T_1 & \alpha(T_1-T_2) \\ 0 & T_2 \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$, in Lemma 2.1 of [43], A. Koranyi proved that the operator $\widetilde{T}$ is unitarily equivalent to $\begin{pmatrix} T_2 & \alpha(T_2-T_1) \\ 0 & T_1 \end{pmatrix}$ through intertwining unitary operator $\frac{1}{\sqrt{1+\alpha^2}} \begin{pmatrix} -\alpha I & I \\ I & \alpha I \end{pmatrix}$. In the following two propositions, we discuss the conditions that the tuples in $\mathcal{NFB}_{n_0,n_1}$ make this conclusion hold, which are similar to the above.

PROPOSITION 4.9. *Let $\mathbb{T}_{ii} \in \mathcal{B}_n^m(\Omega) \cap \mathcal{L}(\mathcal{H})^m$, $i = 0, 1$, $\mathbb{T}_{01} = \sigma_{\mathbb{T}_{00},\mathbb{T}_{11}}(-X)$ and $\mathbb{S}_{01} = \sigma_{\mathbb{T}_{11},\mathbb{T}_{00}}(-Y)$ for $X, Y \in \mathcal{L}(\mathcal{H})$. Let $\widetilde{\mathbb{T}} = (T_1, \dots, T_m), \widetilde{\mathbb{S}} = (S_1, \dots, S_m) \in \mathcal{NFB}_{2n}^m(\Omega)$ with $T_i = \begin{pmatrix} T_{00}^i & T_{01}^i \\ 0 & T_{11}^i \end{pmatrix}$, $S_i = \begin{pmatrix} T_{11}^i & S_{01}^i \\ 0 & T_{00}^i \end{pmatrix}$, $1 \leqslant i \leqslant m$. Suppose that $\ker \sigma_{\mathbb{T}_{00},\mathbb{T}_{11}} = \{0\}$, $\mathbb{T}_{00}, \mathbb{T}_{11}$ are irreducible and $XX^* \in \{\mathbb{T}_{00}\}'$, $X^*X \in \{\mathbb{T}_{11}\}'$. Then $\widetilde{\mathbb{T}} \sim_{\mathrm{u}} \widetilde{\mathbb{S}}$ if and only if there exists $\theta \in \mathbb{R}$, such that $\mathbb{S}_{01} = \mathrm{e}^{\mathrm{i}\theta}\, \sigma_{\mathbb{T}_{11},\mathbb{T}_{00}}(-X^*)$.*

Proof. Let $U = ((U_{i,j}))_{2\times 2}$ be a unitary operator which satisfies that $U\widetilde{\mathbb{T}} = \widetilde{\mathbb{S}}U$. By Lemma 3.3, we have U_{01}, U_{10} are invertible and $(I + X^*X)^{-1} = U_{01}^*U_{01}$, $(I + XX^*)^{-1} = U_{10}^*U_{10}$. Since X is a bounded linear operator, then $I + X^*X$ and $I + XX^*$ are positive and invertible. Furthermore, we have that $U_1 := (I + X^*X)^{1/2}U_{01}^*$, $U_2 := U_{10}(I + XX^*)^{1/2}$ are unitary. By using the statement (i) of Lemma 3.3, we also have $U_{10}T_{00}^i = T_{00}^iU_{10}$, $T_{11}^iU_{01}^* = U_{01}^*T_{11}^i$, $1 \leqslant i \leqslant m$. It follows that

$$U_2(I + XX^*)^{-1/2}T_{00}^i = T_{00}^iU_2(I + XX^*)^{-1/2} \quad \text{and} \tag{4.4}$$

$$T_{11}^i(I + X^*X)^{-1/2}U_1 = (I + X^*X)^{-1/2}U_1T_{11}^i, \quad 1 \leqslant i \leqslant m. \tag{4.5}$$

From the conditions $XX^* \in \{T_{00}^i\}'$, $X^*X \in \{T_{11}^i\}'$, we obtain that $(I + XX^*)T_{00}^i = T_{00}^i(I + XX^*)$ and $(I + X^*X)T_{11}^i = T_{11}^i(I + X^*X)$, $1 \leqslant i \leqslant m$. By functional calculus of positive operators, we have

$$(I + XX^*)^{-1/2}T_{00}^i = T_{00}^i(I + XX^*)^{-1/2}, \quad (I + X^*X)^{-1/2}T_{11}^i = T_{11}^i(I + X^*X)^{-1/2}$$

for $1 \leqslant i \leqslant m$. Combining with equations (4.4) and (4.5), we imply that $U_2 T^i_{00} = T^i_{00} U_2$, $U_1 T^i_{11} = T^i_{11} U_1$, $1 \leqslant i \leqslant m$. Thus, $U_2 \in \{\mathbb{T}_{00}, \mathbb{T}^*_{00}\}'$, $U_1 \in \{\mathbb{T}_{11}, \mathbb{T}^*_{11}\}'$, since U_1, U_2 are unitary. From $\mathbb{T}_{00}, \mathbb{T}_{11}$ are irreducible and Lemma 4.8, we obtain $U_1 = \mathrm{e}^{\mathrm{i}\theta_1} I, U_2 = \mathrm{e}^{\mathrm{i}\theta_2} I$ for some $\theta_1, \theta_2 \in \mathbb{R}$. By the statement (iii) of Lemma 3.3, we have $Y - U_{01} X^* U^{-1}_{10} \in \ker \sigma_{\mathbb{T}_{11}, \mathbb{T}_{00}}$. It follows that $T^i_{11}(Y - U_{01} X^* U^{-1}_{10}) = T^i_{11}(Y - \mathrm{e}^{-\mathrm{i}(\theta_1+\theta_2)} X^*) = (Y - \mathrm{e}^{-\mathrm{i}(\theta_1+\theta_2)} X^*) T^i_{00}$ and $Y T^i_{00} - T^i_{11} Y = \mathrm{e}^{-\mathrm{i}(\theta_1+\theta_2)} (X^* T^i_{00} - T^i_{11} X^*)$, $1 \leqslant i \leqslant m$. This finishes the proof of necessary part.

For the proof of the sufficient part, choose any $\theta_1, \theta_2 \in \mathbb{R}$ such that $\theta_1 + \theta_2 = -\theta$. Define the operator U as follows

$$U = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta_1}(I + X^*X)^{-1/2} X^* & \mathrm{e}^{-\mathrm{i}\theta_1}(I + X^*X)^{-1/2} \\ \mathrm{e}^{\mathrm{i}\theta_2}(I + XX^*)^{-1/2} & -\mathrm{e}^{\mathrm{i}\theta_2}(I + XX^*)^{-1/2} X \end{pmatrix}.$$

Using the fact of $X(I + X^*X)^{-1} = (I + XX^*)^{-1} X$, we obtain U is a unitary operator. By a simple calculation, we imply $UT_i = S_i U$ for $1 \leqslant i \leqslant m$, then $U\widetilde{\mathbb{T}} = \widetilde{\mathbb{S}} U$. ∎

PROPOSITION 4.10. *For $i = 0, 1$, let $\mathbb{T}_{ii} \in \mathcal{B}^m_n(\Omega)$ and $\mathbb{T}_{01} = \sigma_{\mathbb{T}_{00}, \mathbb{T}_{11}}(-X)$ for some self-adjoint operator X. Let $\widetilde{\mathbb{T}} = (T_1, \dots, T_m) \in N\mathcal{F}\mathcal{B}^m_{2n}(\Omega)$ with $T_i = \begin{pmatrix} T^i_{00} & T^i_{01} \\ 0 & T^i_{11} \end{pmatrix}$, $1 \leqslant i \leqslant m$. Suppose that $X \in \{\mathbb{T}_{00}\}' \cap \{\mathbb{T}_{11}\}'$. Then the operator $\widehat{\mathbb{T}}$ obtained by interchanging the roles of $\mathbb{T}_{00}$ and $\mathbb{T}_{11}$ is unitarily equivalent to $\widetilde{\mathbb{T}}$.*

Proof. Let $\widetilde{X} = \begin{pmatrix} X(I+X^2)^{-1/2} & (I+X^2)^{-1/2} \\ (I+X^2)^{-1/2} & -X(I+X^2)^{-1/2} \end{pmatrix}$. By functional calculus of positive operators, we have $X(I + X^2)^{-1/2} = (I + X^2)^{-1/2} X$, thus $\widetilde{X}$ is self-adjoint. Note that

$$\widetilde{X}\widetilde{X}^* = \widetilde{X}^*\widetilde{X} = \begin{pmatrix} X(I+X^2)^{-1}X + (I+X^2)^{-1} & X(I+X^2)^{-1} - (I+X^2)^{-1}X \\ (I+X^2)^{-1}X - X(I+X^2)^{-1} & (I+X^2)^{-1} + X(I+X^2)^{-1}X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

we see that $\widetilde{X}$ is unitary. From $X \in \{\mathbb{T}_{00}\}' \cap \{\mathbb{T}_{11}\}'$, then for any $1 \leqslant i \leqslant m$, we have $XT^i_{jj} = T^i_{jj} X, j = 0, 1$. By functional calculus of $I + X^2$, we also have

$$(I + X^2)^{-1/2} T^i_{jj} = T^i_{jj} (I + X^2)^{-1/2} \quad \text{and} \quad (I + X^2)^{1/2} T^i_{jj} = T^i_{jj} (I + X^2)^{1/2}$$

for $1 \leqslant i \leqslant m$ and $j = 0, 1$. Based on a simple calculation, $\widetilde{X}\widetilde{\mathbb{T}} = \widehat{\mathbb{T}}\widetilde{X}$ can be obtained. Hence, $\widetilde{\mathbb{T}}$ is unitarily equivalent to $\widehat{\mathbb{T}}$. ∎

4.2. SOME PROPERTIES OF TUPLES IN $N\mathcal{F}\mathcal{B}^m_{n_0,n_1}(\Omega)$. The commuting operator tuple $\mathbb{T}$ is said to be strongly irreducible if there are no nontrivial idempotents in $\{\mathbb{T}\}'$. Otherwise, it is strongly reducible. A strongly irreducible operator can be regarded as a natural generalization of a Jordan block matrix on the infinite dimensional case. In [36], C. Jiang proved that for any strongly irreducible Cowen–Douglas operator T, $\{T\}'/\mathrm{rad}(\{T\}')$ is commutative, where $\mathrm{rad}(\{T\}')$ denotes the Jacobson radical of $\{T\}'$. Based on this, C. Jiang gave a similarity classification of strongly irreducible Cowen–Douglas operators by using the K_0-group of their commutant algebra as an invariant (see more details in [36]). These results are also generalized to the case of direct integrals of strongly irreducible operators by R. Shi (cf. [59]). The following proposition shows the strong reducibility of tuples in $\mathcal{NFB}^m_{n_0,n_1}(\Omega)$, that is, every tuple in $\mathcal{NFB}^m_{n_0,n_1}(\Omega)$ can be written as the direct sum of two tuples in $\mathcal{B}^m_{n_i}(\Omega), i=0,1$ up to similarity. For m-tuples $\mathbb{T}_{00}$ and $\mathbb{T}_{11}$, $\mathbb{T}_{00}\oplus\mathbb{T}_{11}=(T^1_{00}\oplus T^1_{11},\dots,T^m_{00}\oplus T^m_{11})$.

PROPOSITION 4.11. *For* $i=0,1$, *let* $\mathbb{T}_{ii}\in\mathcal{B}^m_{n_i}(\Omega)\cap\mathcal{L}(\mathcal{H}_i)^m$ *and* $\mathbb{T}_{01}=\sigma_{\mathbb{T}_{00},\mathbb{T}_{01}}(-X)$ *for* $X\in\mathcal{L}(\mathcal{H}_1,\mathcal{H}_0)$. *Let* $\widetilde{\mathbb{T}}=(T_1,\dots,T_m)\in\mathcal{NFB}^m_{n_0,n_1}(\Omega)$ *with* $T_i=\begin{pmatrix} T^i_{00} & T^i_{01}\\ 0 & T^i_{11}\end{pmatrix}$, $1\leqslant i\leqslant m$. *Then* $\widetilde{\mathbb{T}}$ *is strongly reducible. What is more,* $\widetilde{\mathbb{T}}$ *is similar to* $\mathbb{T}_{00}\oplus\mathbb{T}_{11}$.

Proof. Let $W=\begin{pmatrix} I & -X\\ 0 & I\end{pmatrix}$. We have that

$$WT_j=\begin{pmatrix} T^j_{00} & -T^j_{00}X\\ 0 & T^j_{11}\end{pmatrix}=\begin{pmatrix} T^j_{00} & 0\\ 0 & T^j_{11}\end{pmatrix}\begin{pmatrix} I & -X\\ 0 & I\end{pmatrix}=(T^j_{00}\oplus T^j_{11})W,\quad 1\leqslant j\leqslant m$$

and $W\widetilde{\mathbb{T}}=(\mathbb{T}_0\oplus\mathbb{T}_1)W$. Note that W is invertible and $W^{-1}=\begin{pmatrix} I & X\\ 0 & I\end{pmatrix}$. Then we finish the proof. ∎

The characterization of irreducibility of tuples in $\mathcal{NFB}^m_2(\Omega)$ is as follows.

PROPOSITION 4.12. *Let* $\mathbb{T}_{ii}\in\mathcal{B}^m_1(\Omega)$, $\mathbb{T}_{ii}\sim_{\mathrm{u}}(\mathbb{M}^*_z,\mathcal{H}_{K_i}), i=0,1$ *and* $\mathbb{T}_{01}=\sigma_{\mathbb{T}_{00},\mathbb{T}_{11}}(-X)$ *for some* X. *Suppose that* $\widetilde{\mathbb{T}}=(T_1,\dots,T_m)\in\mathcal{NFB}^m_2(\Omega)$ *with* $T_i=\begin{pmatrix} T^i_{00} & T^i_{01}\\ 0 & T^i_{11}\end{pmatrix}$, $1\leqslant i\leqslant m$. *If* $\lim\limits_{\operatorname{dist}(w,\partial\Omega)\to 0}\frac{K_0(w,w)}{K_1(w,w)}=0$, *then* $\widetilde{\mathbb{T}}$ *is irreducible.*

Proof. Suppose that $\widetilde{\mathbb{T}}$ is reducible, then there exists a nontrivial orthogonal projection $P=\begin{pmatrix} P_{00} & P_{01}\\ P_{10} & P_{11}\end{pmatrix}\in\{\widetilde{\mathbb{T}}\}'$, such that

$$\begin{pmatrix} P_{00}T^i_{00} & P_{00}(XT^i_{11}-T^i_{00}X)+P_{01}T^i_{11}\\ P_{10}T^i_{00} & P_{10}(XT^i_{11}-T^i_{00}X)+P_{11}T^i_{11}\end{pmatrix}=\begin{pmatrix} T^i_{00}P_{00}+(XT^i_{11}-T^i_{00}X)P_{10} & T^i_{00}P_{01}+(XT^i_{11}-T^i_{00}X)P_{11}\\ T^i_{11}P_{10} & T^i_{11}P_{11}\end{pmatrix}\tag{4.6}$$

for all $1\leqslant i\leqslant m$. It follows that $P_{10}\in\bigcap\limits_{i=1}^{m}\ker\sigma_{T^i_{11},T^i_{00}}=\ker\sigma_{\mathbb{T}_{11},\mathbb{T}_{00}}$.

By Lemma 3.5, if we have $\lim\limits_{\operatorname{dist}(w,\partial\Omega)\to 0}\frac{K_0(w,w)}{K_1(w,w)}=0$, then $\ker\sigma_{\mathbb{T}_{11},\mathbb{T}_{00}}=\{0\}$ and $P_{10}=0$. Note that P is a self-adjoint idempotent, we obtain $P_{01}=0$ and $P_{ii}=P^*_{ii}=P^2_{ii}$, $i=0,1$. From equation (4.6), we infer $P_{00}T^i_{00}=T^i_{00}P_{00}$, $P_{11}T^i_{11}=T^i_{11}P_{11}$, $1\leqslant i\leqslant m$. Then $P_{00}\mathbb{T}_{00}=\mathbb{T}_{00}P_{00}$, $P_{11}\mathbb{T}_{11}=\mathbb{T}_{11}P_{11}$. Since tuples in $\mathcal{B}^m_1(\Omega)$ are irreducible, we have $P_{ii}=0$ or I. According to $P_{00}(XT^i_{11}-T^i_{00}X)=(XT^i_{11}-T^i_{00}X)P_{11}$, we have $P_{00}=P_{11}=0$ or I, that is, P is trivial. This is a contradiction. Hence, $\widetilde{\mathbb{T}}$ is irreducible. ∎

By the following proposition, the Hermitian holomorphic vector bundles corresponding to the tuples in $\mathcal{NFB}^m_{n_0,n_1}(\Omega)$ are given.

PROPOSITION 4.13. *Let* $\mathbb{T}_{ii} \in \mathcal{B}^m_{n_i}(\Omega)$, $\mathbb{T}_{ii} \sim_{\mathrm{u}} (\mathbb{M}^*_z, \mathcal{H}_{K_i}), i = 0, 1$ *and* $\mathbb{T}_{01} = \sigma_{\mathbb{T}_{00},\mathbb{T}_{11}}(-X)$ *for some* X. *Suppose that* $\widetilde{\mathbb{T}} = (T_1, \dots, T_m) \in \mathcal{NFB}^m_{n_0+n_1}(\Omega)$ *with* $T_i = \begin{pmatrix} T^i_{00} & T^i_{01} \\ 0 & T^i_{11} \end{pmatrix}, 1 \leqslant i \leqslant m$. *Then for all* $w \in \Omega$,

$$E_{\mathbb{T}}(w) = \overline{\operatorname{span}}\{K_0(\cdot, \overline{w})\xi_0, X(K_1(\cdot, \overline{w})\xi_1) + K_1(\cdot, \overline{w})\xi_1, \xi_0 \in \mathbb{C}^{n_0}, \xi_1 \in \mathbb{C}^{n_1}\}.$$

Proof. Since $E_{\mathbb{T}_{ii}}(w) = \overline{\operatorname{span}}\{K_i(\cdot, \overline{w})\xi_i, \xi_i \in \mathbb{C}^{n_i}\}$ and the dimension of $E_{\mathbb{T}_{ii}}(w)$ is n_i, $i = 0, 1$, it is easy to see that

$$K_0(\cdot, \overline{w})\xi_0, \quad X(K_1(\cdot, \overline{w})\xi_1) + K_1(\cdot, \overline{w})\xi_1 \in \ker(\widetilde{\mathbb{T}} - w), \quad w \in \Omega, \xi_0 \in \mathbb{C}^{n_0}, \xi_1 \in \mathbb{C}^{n_1}.$$

Note that $\dim \ker(\widetilde{\mathbb{T}} - w) = n_0 + n_1, w \in \Omega$, then we only need to prove that for each $\xi_0 \in \mathbb{C}^{n_0}, \xi_1 \in \mathbb{C}^{n_1}$, $K_0(\cdot, \overline{w})\xi_0$ and $X(K_1(\cdot, \overline{w})\xi_1) + K_1(\cdot, \overline{w})\xi_1$ are linearly independent. For fixed but arbitrary $\xi_0 \in \mathbb{C}^{n_0}, \xi_1 \in \mathbb{C}^{n_1}$, suppose that there exist $x_0, x_1 \in \mathbb{C}$ such that

$$x_0 K_0(\cdot, \overline{w})\xi_0 + x_1(X(K_1(\cdot, \overline{w})\xi_1) + K_1(\cdot, \overline{w})\xi_1) = 0.$$

By taking the inner product with $K_1(\cdot, \overline{w})\xi', \xi' \in \mathbb{C}^{n_1}$ on both sides, we have that $\langle x_1 K_1(\cdot, \overline{w})\xi_1, K_1(\cdot, \overline{w})\xi'\rangle = 0$. With the spanning property of $\{K_1(\cdot, \overline{w})\xi', \xi' \in \mathbb{C}^{n_1}\}$, we infer $x_1 K_1(\cdot, \overline{w})\xi_1 = 0$, then $x_1 = 0$, since $K_1(\cdot, \overline{w})\xi_1$ is non-zero. Thus, we obtain $x_0 K_0(\cdot, \overline{w})\xi_0 = 0$. Similarly, we have $x_0 = 0$. This completes the proof. ∎

EXAMPLE 4.14. For $i = 0, 1$, let $\mathbb{T}_{ii}, \mathbb{S}_{ii} \in \mathcal{B}^m_1(\Omega)$, $\mathbb{T}_{ii} = (\mathbb{M}^*_z, \mathcal{H}_{K_i})$, $\mathbb{S}_{ii} = (\mathbb{M}^*_z, \mathcal{H}_{\widetilde{K}_i})$ and $\mathbb{T}_{01} = \sigma_{\mathbb{T}_{00},\mathbb{T}_{11}}(-X), \mathbb{S}_{01} = \sigma_{\mathbb{S}_{00},\mathbb{S}_{11}}(-Y)$ for some X, Y. Let $\widetilde{\mathbb{T}} = (T_1, \dots, T_m), \widetilde{\mathbb{S}} = (S_1, \dots, S_m) \in \mathcal{NFB}^m_2(\Omega)$ with $T_i = \begin{pmatrix} T^i_{00} & T^i_{01} \\ 0 & T^i_{11} \end{pmatrix}$, $S_i = \begin{pmatrix} S^i_{00} & S^i_{01} \\ 0 & S^i_{11} \end{pmatrix}$ for $1 \leqslant i \leqslant m$. By Lemma 4.13, we have $\{K_0(\cdot, \overline{w}), X(K_1(\cdot, \overline{w})) + K_1(\cdot, \overline{w})\}$ is a frame of $E_{\widetilde{\mathbb{T}}}(w)$. Similarly, a frame of $E_{\widetilde{\mathbb{S}}}(w)$ is obtained.

Define $K_\gamma, K_{\widetilde{\gamma}}$ to be the function on $\Omega^* \times \Omega^*$ taking values in the 2×2 matrices $\mathcal{M}_2(\mathbb{C})$:

$$K_\gamma(z, w) = \begin{pmatrix} K_0(z, w) & \langle X(K_1(\cdot, w)), K_0(\cdot, z)\rangle \\ \langle K_0(\cdot, w), X(K_1(\cdot, z))\rangle & \langle X(K_1(\cdot, w)), X(K_1(\cdot, z))\rangle + K_1(z, w) \end{pmatrix},$$

$$K_{\widetilde{\gamma}}(z, w) = \begin{pmatrix} \widetilde{K}_0(z, w) & \langle Y(\widetilde{K}_1(\cdot, w)), \widetilde{K}_0(\cdot, z)\rangle \\ \langle \widetilde{K}_0(\cdot, w), Y(\widetilde{K}_1(\cdot, z))\rangle & \langle Y(\widetilde{K}_1(\cdot, w)), Y(\widetilde{K}_1(\cdot, z))\rangle + \widetilde{K}_1(z, w) \end{pmatrix}.$$

By Subsection 2.2 in [42], we know that $\widetilde{\mathbb{T}}$ and $\widetilde{\mathbb{S}}$ are unitarily equivalent to the adjoint of multiplication operator tuple $\mathbb{M}_z$ on some analytic functional spaces $\mathcal{H}_{K_\gamma}$ and $\mathcal{H}_{K_{\widetilde{\gamma}}}$ with reproducing kernel $K_\gamma(z, w)$ and $K_{\widetilde{\gamma}}(z, w)$, respectively. That means $\widetilde{\mathbb{T}} \sim_{\mathrm{u}} (\mathbb{M}^*_z, \mathcal{H}_{K_\gamma}), \widetilde{\mathbb{S}} \sim_{\mathrm{u}} (\mathbb{M}^*_z, \mathcal{H}_{K_{\widetilde{\gamma}}})$. R.E. Curto and N. Salinas gave a necessary and sufficient condition for the unitary equivalence of commuting operator tuples acting on reproducing kernel Hilbert spaces (see Remark 3.8,

[19]), that is, $\mathbb{M}_z$ acting on $\mathcal{H}_{K_\gamma}$ and $\mathcal{H}_{K_{\widetilde{\gamma}}}$ are unitarily equivalent if and only if $\Phi(z)K_\gamma(z,w)\overline{\Phi^{\mathrm{T}}(w)} = K_{\widetilde{\gamma}}(z,w)$ for some holomorphic and invertible function Φ.

Now if there exist holomorphic functions ϕ and ψ such that

$$\Phi(w) := \begin{pmatrix} 0 & \phi(w) \\ \psi(w) & 0 \end{pmatrix}$$

which satisfies $\Phi(z)K_\gamma(z,w)\overline{\Phi^{\mathrm{T}}(w)} = K_{\widetilde{\gamma}}(z,w)$, then $\mathbb{T}$ is unitarily equivalent to $\widetilde{\mathbb{T}}$, that is,

$$\begin{aligned}
&\begin{pmatrix} 0 & \phi(z) \\ \psi(z) & 0 \end{pmatrix} \begin{pmatrix} K_0(z,w) & \langle X(K_1(\cdot,w)),K_0(\cdot,z)\rangle \\ \langle K_0(\cdot,w),X(K_1(\cdot,z))\rangle & \langle X(K_1(\cdot,w)),X(K_1(\cdot,z))\rangle+K_1(z,w) \end{pmatrix} \begin{pmatrix} 0 & \overline{\psi(w)} \\ \overline{\phi(w)} & 0 \end{pmatrix} \\
&\quad = \begin{pmatrix} \phi(z)(\langle X(K_1(\cdot,w)),X(K_1(\cdot,z))\rangle+K_1(z,w))\overline{\phi(w)} & \phi(z)(\langle K_0(\cdot,w),X(K_1(\cdot,z))\rangle)\overline{\psi(w)} \\ \psi(z)(\langle X(K_1(\cdot,w)),K_0(\cdot,z)\rangle)\overline{\phi(w)} & \psi(z)K_0(z,w)\overline{\psi(w)} \end{pmatrix} \\
&\quad = \begin{pmatrix} \widetilde{K}_0(z,w) & \langle Y(\widetilde{K}_1(\cdot,w)),\widetilde{K}_0(\cdot,z)\rangle \\ \langle \widetilde{K}_0(\cdot,w),Y(\widetilde{K}_1(\cdot,z))\rangle & \langle Y(\widetilde{K}_1(\cdot,w)),Y(\widetilde{K}_1(\cdot,z))\rangle+\widetilde{K}_1(z,w) \end{pmatrix}.
\end{aligned}$$

Choosing $z = w$, we have that

$$\begin{aligned}
\widetilde{K}_0(w,w) &= |\phi(w)|^2(\|XK_1(\cdot,w)\|^2 + K_1(w,w)) = \|(I + X^*X)^{1/2}(\phi(w)K_1(\cdot,w))\|^2, \\
|\psi(w)|^2K_0(w,w) &= \|Y(\widetilde{K}_1(\cdot,w))\|^2+\widetilde{K}_1(w,w) = \|(I+Y^*Y)^{1/2}\widetilde{K}_1(\cdot,w)\|^2, \quad w\in\Omega.
\end{aligned}$$

By Theorem 3.2 and the proof of Example 3.9, we have that $\mathbb{T}_{00} \sim_{\mathrm{s}} \mathbb{S}_{11}, \mathbb{S}_{00} \sim_{\mathrm{s}} \mathbb{T}_{11}$. Thus, $\mathbb{T}_{00} \oplus \mathbb{S}_{00} \sim_{\mathrm{s}} \mathbb{T}_{11} \oplus \mathbb{S}_{11}$.

Let $T, S \in \mathcal{B}_1^1(\Omega)$ and $T \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_{K_0}), S \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_{K_1})$. By Lemma 3.5, we know that if $T_0 \sim_{\mathrm{s}} T_1$, then $\frac{K_0(w,w)}{K_1(w,w)}$ is bounded and bounded below from zero. In the following proposition, we will prove that there is a similar result in the operator class $N\mathcal{F}\mathcal{B}_2^1(\Omega)$. For the case of index two, $\frac{K_0(w,w)}{K_1(w,w)}$ is replaced by the ratio of the determinants of the metrics corresponding to the two bundles.

PROPOSITION 4.15. *For $i = 0,1$, let $T_{ii}, S_{ii} \in \mathcal{B}_1^1(\Omega)$ and $T_{ii} \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_{K_i})$, $S_{ii} \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_{\widetilde{K}_i})$. Suppose that $T = \begin{pmatrix} T_{00} & T_{01} \\ 0 & T_{11} \end{pmatrix}$, $S = \begin{pmatrix} S_{00} & S_{01} \\ 0 & S_{11} \end{pmatrix} \in N\mathcal{F}\mathcal{B}_2^1(\Omega)$ and there exist X, Y such that $T_{01} = \sigma_{T_{00},T_{11}}(-X), S_{01} = \sigma_{S_{00},S_{11}}(-Y)$. If $T \sim_{\mathrm{s}} S$, then there exist metrics h_T, h_S corresponding to E_T, E_S such that $m \leqslant \frac{\det h_T(\omega)}{\det h_S(\omega)} \leqslant M, \omega \in \Omega$, for positive numbers m and M.*

Proof. Without loss of generality, we assume that $T_{ii} = (M_z^*, \mathcal{H}_{K_i}), S_{ii} = (M_z^*, \mathcal{H}_{\widetilde{K}_i}), i = 0,1$. Then $K_i(\cdot,\overline{w}), \widetilde{K}_i(\cdot,\overline{w})$ are the sections of $E_{T_{ii}}$ and $E_{S_{ii}}, i = 0,1$, respectively. By Lemma 4.13, we know that

$$\{K_0(\cdot,\overline{w}), XK_1(\cdot,\overline{w}) + K_1(\cdot,\overline{w})\}, \quad \{\widetilde{K}_0(\cdot,\overline{w}), Y\widetilde{K}_1(\cdot,\overline{w}) + \widetilde{K}_1(\cdot,\overline{w})\}$$

are frames of $E_T(w), E_S(w)$, respectively. It follows that

$$h_T(\omega) = \begin{pmatrix} K_0(\overline{w},\overline{w}) & \langle X(K_1(\cdot,\overline{w})), K_0(\cdot,\overline{w})\rangle \\ \langle K_0(\cdot,\overline{w}), X(K_1(\cdot,\overline{w}))\rangle & \|X(K_1(\cdot,\overline{w}))\|^2 + K_1(\overline{w},\overline{w}) \end{pmatrix},$$

$$\det h_T(\omega) = K_0(\overline{w},\overline{w})(K_1(\overline{w},\overline{w}) + \|X(K_1(\cdot,\overline{w}))\|^2) - |\langle K_0(\cdot,\overline{w}), X(K_1(\cdot,\overline{w}))\rangle|^2.$$

Similarly, we have

$$\det h_S(\omega) = \widetilde{K}_0(\overline{w},\overline{w})(\widetilde{K}_1(\overline{w},\overline{w}) + \|Y(\widetilde{K}_1(\cdot,\overline{w}))\|^2) - |\langle \widetilde{K}_0(\cdot,\overline{w}), Y(\widetilde{K}_1(\cdot,\overline{w}))\rangle|^2.$$

By Proposition 4.11, we know that operators in $N\mathcal{FB}_2^1(\Omega)$ are strongly reducible and $T \sim_s T_{00} \oplus T_{11}$, $S \sim_s S_{00} \oplus S_{11}$. If $T \sim_s S$, then $T_{00} \oplus T_{11} \sim_s S_{00} \oplus S_{11}$. By the main theorem of [38], we know that every Cowen–Douglas operator has a unique strongly irreducible decomposition up to similarity. Thus, the equivalence relation is either $T_{00} \sim_s S_{00}, T_{11} \sim_s S_{11}$ or $T_{00} \sim_s S_{11}, T_{11} \sim_s S_{00}$. In either case, according to Lemma 3.5, there exist positive numbers m_1 and M_1 such that $m_1 \leqslant \frac{\det h_{T_{00}\oplus T_{11}}}{\det h_{S_{00}\oplus S_{11}}} = \frac{K_0(\overline{w},\overline{w})K_1(\overline{w},\overline{w})}{\widetilde{K}_0(\overline{w},\overline{w})\widetilde{K}_1(\overline{w},\overline{w})} \leqslant M_1$. By using Cauchy–Schwarz inequality, we have $K_0(\overline{w},\overline{w})\|X(K_1(\cdot,\overline{w}))\|^2 - |\langle K_0(\cdot,\overline{w}), X(K_1(\cdot,\overline{w}))\rangle|^2 \geqslant 0$. Thus,

$$\frac{\det h_T(\omega)}{\det h_{T_{00}\oplus T_{11}}(\omega)} \geqslant 1.$$

On the other hand, since X is a bounded linear operator, we have

$$\frac{\det h_T(\omega)}{\det h_{T_{00}\oplus T_{11}}(\omega)} \leqslant \frac{K_0(\overline{w},\overline{w})(K_1(\overline{w},\overline{w}) + \|X(K_1(\cdot,\overline{w}))\|^2)}{K_0(\overline{w},\overline{w})K_1(\overline{w},\overline{w})} \leqslant 1 + \|X\|^2.$$

Similarly, we obtain $1 \leqslant \frac{\det h_S}{\det h_{S_{00}\oplus S_{11}}} \leqslant 1 + \|Y\|^2$. Note that

$$\frac{\det h_T}{\det h_S} = \frac{\det h_T}{\det h_{T_{00}\oplus T_{11}}} \cdot \frac{\det h_{T_{00}\oplus T_{11}}}{\det h_{S_{00}\oplus S_{11}}} \cdot \frac{\det h_{S_{00}\oplus S_{11}}}{\det h_S}.$$

Let $m := \frac{m_1}{1+\|Y\|^2}, M := M_1(1+\|X\|^2)$. We have $m \leqslant \frac{\det h_T(\omega)}{\det h_S(\omega)} \leqslant M$. This completes the proof. ∎

5. CONCLUDING REMARKS

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be weakly homogeneous if $\sigma(T) \subset \overline{\mathbb{D}}$ and $\phi(T)$ is similar to T for each ϕ in *Möb*. When $\mathcal{D}$ is a bounded symmetric domain, a commuting m-tuple $\mathbb{T} = (T_1, \ldots, T_m)$ of bounded operators is said to be weakly homogeneous with respect to G if their joint Taylor spectrum is contained in $\overline{\mathcal{D}}$ and $\phi(\mathbb{T})$ is similar to $\mathbb{T}$ for every holomorphic automorphism $\phi \in G$. Given a Hilbert space $\mathcal{H}$ with sharp reproducing kernel K on $\mathbb{D} \times \mathbb{D}$, S. Ghara in [27] obtains an equivalent condition that the multiplication operator M_z on $(\mathcal{H}, K)$ is weakly homogeneous.

PROPOSITION 5.1. *Let* $\mathbb{T}_{ii} \in \mathcal{B}^m_{n_i}(\mathbb{D}^m), i = 0, 1$ *and* $\mathbb{T}_{01} \in \operatorname{ran} \sigma_{\mathbb{T}_{00},\mathbb{T}_{11}}$. *Suppose that* $\widetilde{\mathbb{T}} = (T_1, \dots, T_m) \in \mathcal{NFB}^m_{n_0+n_1}(\mathbb{D}^m)$ *with* $T_i = \begin{pmatrix} T^j_{00} & T^j_{01} \\ 0 & T^j_{11} \end{pmatrix}, 1 \leqslant j \leqslant m$. *If* $\mathbb{T}_{00}, \mathbb{T}_{11}$ *are both weakly homogeneous with respect to Möb*^{m}*, then* $\widetilde{\mathbb{T}}$ *is also Möb*^{m}*-weakly homogeneous.*

Proof. Suppose that there exists X such that $\mathbb{T}_{01} = \sigma_{\mathbb{T}_{00},\mathbb{T}_{11}}(-X)$. By Proposition 4.11, we know that $\widetilde{\mathbb{T}}$ is similar to $\mathbb{T}_{00} \oplus \mathbb{T}_{11}$ and $\left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right) \begin{pmatrix} T^j_{00} & T^j_{01} \\ 0 & T^j_{11} \end{pmatrix} \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) = \begin{pmatrix} T^j_{00} & 0 \\ 0 & T^j_{11} \end{pmatrix}$ for $1 \leqslant j \leqslant m$. Further, we have $\left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right) \begin{pmatrix} T^j_{00} & T^j_{01} \\ 0 & T^j_{11} \end{pmatrix}^n \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) = \begin{pmatrix} T^j_{00} & 0 \\ 0 & T^j_{11} \end{pmatrix}^n$ for any positive integer n and

$$\left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right) \phi_{\alpha_j}\left(\begin{pmatrix} T^j_{00} & T^j_{01} \\ 0 & T^j_{11} \end{pmatrix}\right) \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) = \begin{pmatrix} \phi_{\alpha_j}(T^j_{00}) & 0 \\ 0 & \phi_{\alpha_j}(T^j_{11}) \end{pmatrix}, \quad \phi_{\alpha_j} \in \text{Möb}, 1 \leqslant j \leqslant m.$$

Let $\phi_\alpha = (\phi_{\alpha_1}, \phi_{\alpha_2}, \dots, \phi_{\alpha_m})$. Then $\phi_\alpha \in \text{Möb}^m$ and

$$\left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right) \phi_\alpha(\widetilde{\mathbb{T}}) \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) = \phi_\alpha(\mathbb{T}_{00} \oplus \mathbb{T}_{11}).$$

Since $\mathbb{T}_{00}, \mathbb{T}_{11}$ are both weakly homogeneous, it follows that there exists an invertible operator Y_α depending on α, such that $Y_\alpha^{-1} \left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right) \phi_\alpha(\widetilde{\mathbb{T}}) \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) Y_\alpha = \mathbb{T}_{00} \oplus \mathbb{T}_{11}$. By using Proposition 4.11 again, we obtain that

$$\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) Y_\alpha^{-1} \left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right) \phi_\alpha(\widetilde{\mathbb{T}}) \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) Y_\alpha \left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right) = \widetilde{\mathbb{T}}.$$

Hence, $\widetilde{\mathbb{T}}$ is weakly homogeneous. ∎

PROPOSITION 5.2. *Let* $\mathbb{T}_{ii} \in \mathcal{B}^m_{n_i}(\mathbb{D}^m), i = 0, 1$ *and* $\mathbb{T}_{01} \in \operatorname{ran} \sigma_{\mathbb{T}_{00},\mathbb{T}_{11}}$. *Suppose that* $\widetilde{\mathbb{T}} = (T_1, \dots, T_m) \in \mathcal{NFB}^m_{n_0+n_1}(\mathbb{D}^m)$ *with* $T_i = \begin{pmatrix} T^j_{00} & T^j_{01} \\ 0 & T^j_{11} \end{pmatrix}, 1 \leqslant j \leqslant m$. *If* $\widetilde{\mathbb{T}}$ *is weakly homogeneous with respect to Möb*^{m}*, then* $\mathbb{T}_{00} \oplus \mathbb{T}_{11}$ *is also Möb*^{m}*-weakly homogeneous.*

Proof. Suppose that there exists X such that $\mathbb{T}_{01} = \sigma_{\mathbb{T}_{00},\mathbb{T}_{11}}(-X)$. If $\widetilde{\mathbb{T}}$ is weakly homogeneous with respect to *Möb*^{m}, then there exists an invertible operator Y_α depending on $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{D}^m$ and $\phi_\alpha = (\phi_{\alpha_1}, \phi_{\alpha_2}, \dots, \phi_{\alpha_m}) \in$ *Möb*^{m}, such that $Y_\alpha^{-1}\phi_\alpha(\widetilde{\mathbb{T}})Y_\alpha = \widetilde{\mathbb{T}}$. By using the strong reducibility of $\widetilde{\mathbb{T}}$ in Proposition 4.11, we have

$$\left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right) Y_\alpha^{-1} \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) \phi_\alpha(\mathbb{T}_{00} \oplus \mathbb{T}_{11}) \left(\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right) Y_\alpha \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right) = \mathbb{T}_{00} \oplus \mathbb{T}_{11}.$$

Thus, $\mathbb{T}_{00} \oplus \mathbb{T}_{11}$ is weakly homogeneous with respect to *Möb*^{m}. ∎

PROPOSITION 5.3. *If* $\left(\begin{smallmatrix} T_{00} & T_{01} \\ 0 & T_{11} \end{smallmatrix}\right) \in \mathcal{NFB}^1_2(\mathbb{D})$ *is weakly homogeneous and for any* $\phi_\alpha \in$ *Möb,* $\ker \sigma_{\phi_\alpha(T_{00}),T_{11}} = \{0\}$*. Then* T_{00}, T_{11} *are both weakly homogeneous.*

Proof. Suppose that there exists X such that $T_{01} = \sigma_{T_{00},T_{11}}(-X)$. If $\left(\begin{smallmatrix} T_{00} & T_{01} \\ 0 & T_{11} \end{smallmatrix}\right)$ is weakly homogeneous, by Proposition 5.2, we see that $T_{00} \oplus T_{11}$ is weakly homogeneous, that is, $\phi_\alpha(T_{00}) \oplus \phi_\alpha(T_{11})$ is similar to $T_{00} \oplus T_{11}$ for any $\phi_\alpha \in$ *Möb*. Note that $\phi_\alpha(T_{ii}) \in \mathcal{B}_1^1(\mathbb{D}), i = 0, 1$, by the main theorem of [38], we know that every Cowen–Douglas operator has a unique strongly irreducible decomposition up to similarity, then either $\phi_\alpha(T_{ii}) \sim_{\rm s} T_{ii}, i = 0, 1$ or $\phi_\alpha(T_{00}) \sim_{\rm s} T_{11}, \phi_\alpha(T_{11}) \sim_{\rm s} T_{00}$ for $\phi_\alpha \in$ *Möb*. Since $\ker \sigma_{\phi_\alpha(T_{00}),T_{11}} = \{0\}$, we have $\phi_\alpha(T_{ii}) \sim_{\rm s} T_{ii}, i = 0, 1$ for any $\phi_\alpha \in$ *Möb*. Hence, T_{00}, T_{11} are both weakly homogeneous. ∎

PROPOSITION 5.4. *Let $T \in \mathcal{B}_1^1(\mathbb{D})$. If T is weakly homogeneous, then there exists $\Psi(\cdot,\cdot) : \mathbb{D} \times \mathbb{D} \to \mathbb{R}^+$ such that $\mathcal{K}_\Psi(\alpha, \phi_\alpha(w))|\phi'_\alpha(w)|^2 + \mathcal{K}_\Psi(\alpha, w) = 0$, where $\mathcal{K}_\Psi(\alpha, w) = -\frac{\partial^2}{\partial w \partial \overline{w}} \log \Psi(\alpha, w)$ and $\phi_\alpha \in$ Möb. In particular, $\Psi(\alpha, w)$ also satisfies $\mathcal{K}_\Psi(w, w) = -\frac{\mathcal{K}_\Psi(w,0)}{(1-|w|^2)^2}$ and $\mathcal{K}_\Psi(0, -w) = -\mathcal{K}_\Psi(0, w)$.*

Proof. Without losing generality, we assume that $\phi_\alpha(w) = \frac{\alpha - w}{1 - \overline{\alpha} w}, \alpha, w \in \mathbb{D}$. If T is weakly homogeneous, then for any $\alpha \in \mathbb{D}$, T is similar to $\phi_\alpha(T)$. Let e be a non-zero section of E_T associated with T. Note that $\phi_\alpha(T) \in \mathcal{B}_1^1(\mathbb{D})$ and $e(\phi_\alpha(w)) \in \ker(\phi_\alpha(T) - w)$. By Theorem 3.2, we find that a bounded operator X_α and $\psi_\alpha \in H^\infty(\mathbb{D})$ depending on α, such that $\|e(\phi_\alpha(w))\|^2 = |\psi_\alpha(w)|^2(\|e(w)\|^2 + \|X_\alpha(e(w))\|^2), w \in \mathbb{D}$. Further, we have

$$\frac{\partial^2}{\partial w \partial \overline{w}} \log \|e(\phi_\alpha(w))\|^2 = \frac{\partial^2}{\partial w \partial \overline{w}} \log \|e(w)\|^2 + \frac{\partial^2}{\partial w \partial \overline{w}} \log \Big(1 + \frac{\|X_\alpha(e(w))\|^2}{\|e(w)\|^2}\Big)$$

for $w \in \mathbb{D}$. Define $\Psi(\cdot,\cdot) : \mathbb{D} \times \mathbb{D} \to \mathbb{R}^+$ as $\Psi(\alpha, w) := 1 + \frac{\|X_\alpha(e(w))\|^2}{\|e(w)\|^2}$ and $\mathcal{K}_\Psi(\alpha, w) = -\frac{\partial^2}{\partial w \partial \overline{w}} \log \Psi(\alpha, w)$. Thus we infer

$$\mathcal{K}_{\phi_\alpha(T)}(w) = \mathcal{K}_T(w) + \mathcal{K}_\Psi(\alpha, w). \tag{5.1}$$

From the arbitrariness of $w \in \mathbb{D}$ in equation (5.1), we replace w with $\phi_\alpha(w)$. It follows that

$$\mathcal{K}_{\phi_\alpha(T)}(\phi_\alpha(w)) = \mathcal{K}_T(\phi_\alpha(w)) + \mathcal{K}_\Psi(\alpha, \phi_\alpha(w)). \tag{5.2}$$

It can be obtained by a simple application of the chain rule that

$$\mathcal{K}_T(w) = \mathcal{K}_{\phi_\alpha(T)}(\phi_\alpha(w))|\phi'_\alpha(w)|^2.$$

Then equation (5.2) can be transformed into

$$\mathcal{K}_T(w)|\phi'_\alpha(w)|^{-2} = \mathcal{K}_T(\phi_\alpha(w)) + \mathcal{K}_\Psi(\alpha, \phi_\alpha(w)). \tag{5.3}$$

Using the chain rule again, we have $\mathcal{K}_{\phi_\alpha(T)}(w) = -\frac{\partial^2}{\partial w \partial \overline{w}} \log \|e(\phi_\alpha(w))\|^2 = \mathcal{K}_T(\phi_\alpha(w))|\phi'_\alpha(w)|^2$. Then equation (5.1) is equivalent to

$$\mathcal{K}_T(\phi_\alpha(w)) = \mathcal{K}_T(w)|\phi'_\alpha(w)|^{-2} + \mathcal{K}_\Psi(\alpha, w)|\phi'_\alpha(w)|^{-2}. \tag{5.4}$$

Combining equations (5.3) and (5.4), we obtain that

$$\mathcal{K}_\Psi(\alpha, \phi_\alpha(w))|\phi'_\alpha(w)|^2 + \mathcal{K}_\Psi(\alpha, w) = 0. \tag{5.5}$$

Note that $\phi'_\alpha(w) = \frac{|\alpha|^2-1}{(1-\bar{\alpha}w)^2}$ and $\phi_\alpha(0) = \alpha$. By (5.5), we have $\mathcal{K}_\Psi(\alpha,\alpha)(1-|\alpha|^2)^2 + \mathcal{K}_\Psi(\alpha,0) = 0$ for all $\alpha \in \mathbb{D}$, that is, $\mathcal{K}_\Psi(w,w) = -\frac{\mathcal{K}_\Psi(w,0)}{(1-|w|^2)^2}$ for all $w \in \mathbb{D}$. Since $\phi_0(w) = -w$, we imply $\mathcal{K}_\Psi(0,-w) + \mathcal{K}_\Psi(0,w) = 0$. ∎

PROPOSITION 5.5. *Let $T \in \mathcal{B}_1^1(\mathbb{D})$ and e be a non-zero section of E determined by T. For any $\phi_\alpha \in$ Möb, E_α is the Hermitian holomorphic vector bundle associated with $\phi_\alpha(T)$. If T is weakly homogeneous, then there exists vector bundle F^α with $F^\alpha(w) = \bigvee\left\{ \begin{pmatrix} I \\ X_\alpha \end{pmatrix} e(w)\right\}$, such that $E \otimes E_\alpha \sim_{\mathrm{u}} F^\alpha \otimes F^\alpha_\alpha$, where X_α is a bounded operator depending on α and $F^\alpha_\alpha(w) = \bigvee\left\{ \begin{pmatrix} I \\ X_\alpha \end{pmatrix} e(\phi_\alpha(w))\right\}, w \in \mathbb{D}$.*

Proof. Without losing generality, we assume that $\phi_\alpha(w) = \frac{\alpha-w}{1-\bar{\alpha}w}$. If T is weakly homogeneous, then $T \sim_{\mathrm{s}} \phi_\alpha(T), \alpha \in \mathbb{D}$. It is easy to see that $\phi_\alpha(T) \in \mathcal{B}_1^1(\mathbb{D})$ and $e(\phi_\alpha(w)) \in \ker(\phi_\alpha(T) - w)$. By Theorem 3.2, we know that there exists a bounded operator X_α and $\psi_\alpha \in H^\infty(\mathbb{D})$ depending on α, such that

$$\|e(\phi_\alpha(w))\|^2 = |\psi_\alpha(w)|^2(\|e(w)\|^2 + \|X_\alpha(e(w))\|^2), \quad w \in \mathbb{D}.$$

This is equivalent to $\frac{\|e(\phi_\alpha(w))\|^2}{\|e(w)\|^2} = |\psi_\alpha(w)|^2\Big(1 + \frac{\|X_\alpha(e(w))\|^2}{\|e(w)\|^2}\Big)$ and $\frac{\|e(w)\|^2}{\|e(\phi_\alpha(w))\|^2} = |\psi_\alpha(\phi_\alpha(w))|^2\Big(1 + \frac{\|X_\alpha(e(\phi_\alpha(w)))\|^2}{\|e(\phi_\alpha(w))\|^2}\Big)$, since $\phi_\alpha(\phi_\alpha(w)) = 1, \alpha, w \in \mathbb{D}$. Then

$$\begin{aligned} |\psi_\alpha(w)\psi_\alpha(\phi_\alpha(w))|^2 &\cdot \Big(1 + \frac{\|X_\alpha(e(w))\|^2}{\|e(w)\|^2}\Big)\Big(1 + \frac{\|X_\alpha(e(\phi_\alpha(w)))\|^2}{\|e(\phi_\alpha(w))\|^2}\Big) \\ &= |\psi_\alpha(w)\psi_\alpha(\phi_\alpha(w))|^2 \cdot \frac{\left\| \begin{pmatrix} e(w) \\ X_\alpha(e(w)) \end{pmatrix} \otimes \begin{pmatrix} e(\phi_\alpha(w)) \\ X_\alpha(e(\phi_\alpha(w))) \end{pmatrix} \right\|^2}{\|e(w) \otimes e(\phi_\alpha(w))\|^2} \\ &= 1, \quad \alpha, w \in \mathbb{D}. \end{aligned}$$

Further, we have

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log \frac{\left\| \begin{pmatrix} e(w) \\ X_\alpha(e(w)) \end{pmatrix} \otimes \begin{pmatrix} e(\phi_\alpha(w)) \\ X_\alpha(e(\phi_\alpha(w))) \end{pmatrix} \right\|^2}{\|e(w) \otimes e(\phi_\alpha(w))\|^2} = 0, \quad \alpha, w \in \mathbb{D}. \tag{5.6}$$

Let $F^\alpha(w) = \bigvee\{\begin{pmatrix} I \\ X_\alpha \end{pmatrix} e(w)\}$ and $F^\alpha_\alpha(w) = \bigvee\{\begin{pmatrix} I \\ X_\alpha \end{pmatrix} e(\phi_\alpha(w))\}$ for $\alpha, w \in \mathbb{D}$. By Theorem 2.2 and equation (5.6), we infer $E \otimes E_\alpha \sim_{\mathrm{u}} F^\alpha \otimes F^\alpha_\alpha$. ∎

REMARK 5.6. Let T be a Cowen–Douglas operator with index one. By Proposition 3.2 in [30], the first and second authors joint with L. Zhao proved that for any $\phi_\alpha \in$ *Möb*, if the holomorphic Hermitian vector bundle E_α associated with $\phi_\alpha(T)$ is congruent to $E_T \otimes \mathcal{L}_\alpha$ for some line bundle $\mathcal{L}_\alpha$, then T is homogeneous. In this case, when the index of T is one, the $\mathcal{K}_\Psi$ in Proposition 5.4 is zero, since $\|e(\phi_\alpha(w))\|^2 = |\varphi_\alpha(w)|^2\|e(w)\|^2$ for some holomorphic function φ and $\alpha, w \in \mathbb{D}$. But if $\|e(\phi_\alpha(w))\|^2 = (1 + |\varphi_\alpha(w)|^2)\|e(w)\|^2$, then T is not a homogeneous operator, since $1 + |\varphi_\alpha(w)|^2$ is not the square of the Modulus of some holomorphic function. Although it can be regarded as $X_\alpha e(w) = \varphi_\alpha(w)e(w)$, the equation

$\mathcal{K}_\Psi(\alpha,\phi_\alpha(w))|\phi_\alpha'(w)|^2+\mathcal{K}_\Psi(\alpha,w)=0$, a necessary condition of T to be a weakly homogeneous operator, does not hold. We have $\mathcal{K}_\Psi(\alpha,w)=-\frac{|\varphi_\alpha'(w)|^2}{(1+|\varphi_\alpha(w)|^2)^2}$ and $\mathcal{K}_\Psi(\alpha,\phi_\alpha(w))=-\frac{|\varphi_\alpha'(\phi_\alpha(w))|^2}{(1+|\varphi_\alpha(\phi_\alpha(w))|^2)^2}$. The reason for this phenomenon may be $(I+X_\alpha^*X_\alpha)^{1/2}$ is not intertwining T and $\phi_\alpha(T)$.

In what follows, we assume that the Hilbert space $\mathcal{H}_i, i\geqslant 0$ is an analytical function space with reproducing kernel $K_i(z,w)$, where

$$K_0(z,w)=-\frac{\ln(1-z\overline{w})}{z\overline{w}},\quad K_n(z,w)=\frac{1}{(1-z\overline{w})^n},\quad n\geqslant 1, z,w\in\mathbb{D}.$$

Let T be a Cowen–Douglas operator and $T\sim_{\mathrm{u}}(M_z^*,\mathcal{H}_K)$. For $T\in\mathcal{B}_n^1(\mathbb{D})$ and contractive, M. Uchiyama [63] has provided a necessary and sufficient condition for T to be similar to the n times copies of M_z^* on Hardy space, which is equivalent to say that there exist positive constants m,M such that $m\sum\limits_{i=1}^n|x_i|^2\leqslant(1-|w|^2)\langle K(\overline{w},\overline{w})\xi,\xi\rangle\leqslant M\sum\limits_{i=1}^n|x_i|^2$ for any $w\in\mathbb{D}$ and $\xi=\sum\limits_{i=1}^n x_i\xi_i, x_i\in\mathbb{C},\xi_i=(0,\dots,0,1,0,\dots,0)^{\mathrm{T}}$ with 1 on the ith position. When $T\in\mathcal{B}_1^1(\mathbb{D})$ and is n-hypercontractive, the second and third authors have shown [33] that T is similar to M_z^* on $(\mathcal{H}_n,K_n)$ if and only if $\frac{K(w,w)}{K_n(w,w)}$ is bounded and bounded below from zero. For each $n\geqslant 1$, we know that the multiplication operator on $(\mathcal{H}_n,K_n)$ is homogeneous [48]. It is easy to see that an operator similar to a homogeneous operator is weakly homogeneous. The n-hypercontraction T, $n\geqslant 1$ determined by the similarity mentioned above is thus weakly homogeneous. For some positive definite kernels K, see Theorem 5.3 of [27], it is shown that the multiplication operator M_z on $(\mathcal{H},KK_n), n>0$, is a weakly homogeneous operator. In the following, we give a few weakly homogeneous operators according to Example 3.11, some of which are non-contractive.

EXAMPLE 5.7. Let $T\in\mathcal{B}_1^1(\mathbb{D})$ and $T\sim_{\mathrm{u}}(M_z^*,\mathcal{H}_K)$. Suppose that $K(z,w)=K_1(z,w)+iK_0(z,w)$ for some positive constant i. Then T is a non-contractive weakly homogeneous operator.

Proof. Let $\lambda_k=\frac{i}{k+1}, k\geqslant 0$. Then $K(z,w)=\sum\limits_{k=0}^\infty(1+\lambda_k)z^k\overline{w}^k$. By Lemma 3.1 of [33], we know that T is unitarily equivalent to a weighted backward shift operator with weight sequence $\left\{\sqrt{\frac{1+\lambda_k}{1+\lambda_{k+1}}}\right\}_{k\geqslant 0}$. Since $\frac{1+\lambda_k}{1+\lambda_{k+1}}>1$ for all $k>0$ and unitary operators preserve contractivity, we have T is non-contractive. From $\lambda_k\geqslant\lambda_{k+1}$, $\lambda_k\to 0\,(k\to\infty)$ and Example 3.11, we infer that T is similar to the adjoint of multiplication operator M_z on $(\mathcal{H}_1,K_1)$. Thus, T is weakly homogeneous. ∎

EXAMPLE 5.8. Let $T \in \mathcal{B}_1^1(\mathbb{D})$ and $T \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_K)$. Suppose that $K(z,w) = K_i(z,w) + K_j(z,w), i > j \geqslant 0$. Then T is weakly homogeneous.

Proof. When $i = 1, j = 0$, by Example 5.7, we know that T is weakly homogeneous.

When $i > 1, j = 0$, we have

$$\begin{aligned} K(z,w) = K_i(z,w) + K_0(z,w) &= \sum_{k=0}^{\infty} \frac{(k+1)\cdots(k+i-1)}{(i-1)!} z^k \overline{w}^k + \sum_{k=0}^{\infty} \frac{1}{k+1} z^k \overline{w}^k \\ &= \sum_{k=0}^{\infty} \Big(1 + \frac{(i-1)!}{(k+1)^2(k+2)\cdots(k+i-1)}\Big) \frac{(k+1)\cdots(k+i-1)}{(i-1)!} z^k \overline{w}^k. \end{aligned}$$

Let $\lambda_k = \frac{(i-1)!}{(k+1)^2(k+2)\cdots(k+i-1)}, k \geqslant 0$. Then $\lambda_k \geqslant \lambda_{k+1}$ and $\lambda_k \to 0\,(k \to \infty)$. From Example 3.11, we infer that T is similar to the adjoint of multiplication operator M_z on $(\mathcal{H}_i, K_i)$. Thus, T is weakly homogeneous.

When $j \geqslant 1$, we have

$$\begin{aligned} K(z,w) &= K_i(z,w) + K_j(z,w) \\ &= \sum_{k=0}^{\infty} \frac{(k+1)\cdots(k+i-1)}{(i-1)!} z^k \overline{w}^k + \sum_{k=0}^{\infty} \frac{(k+1)\cdots(k+j-1)}{(j-1)!} z^k \overline{w}^k \\ &= \sum_{k=0}^{\infty} \Big(1 + \frac{(i-1)!}{(j-1)!(k+j)\cdots(k+i-1)}\Big) \frac{(k+1)\cdots(k+i-1)}{(i-1)!} z^k \overline{w}^k. \end{aligned}$$

Let $\widetilde{\lambda}_k = \frac{(i-1)!}{(j-1)!(k+j)\cdots(k+i-1)}, k \geqslant 0$. Then $\widetilde{\lambda}_k \geqslant \widetilde{\lambda}_{k+1}$ and $\widetilde{\lambda}_k \to 0\,(k \to \infty)$. From Example 3.11, we infer that T is similar to the adjoint of multiplication operator M_z on $(\mathcal{H}_i, K_i)$. Thus, T is weakly homogeneous. ∎

EXAMPLE 5.9. Let $T \in \mathcal{B}_1^1(\mathbb{D})$ and $T \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_K)$. Suppose that $K(z,w) = K_1(z,w) + \frac{1}{2} + \frac{2+z\overline{w}}{(2-z\overline{w})^2}$. Then T is a non-contractive weakly homogeneous operator.

Proof. Let $\lambda_0 = 1, \lambda_k = \frac{2k+1}{2^{k+1}}, k \geqslant 1$. Then $K(z,w) = K_1(z,w) + \sum\limits_{k=0}^{\infty} \lambda_k z^k \overline{w}^k$. When $k > 1$, we have $1 + \lambda_k > 1 + \lambda_{k+1}$. It follows that T is non-contractive. Note that $\lambda_k \geqslant \lambda_{k+1}$ and $\lambda_k \to 0\,(k \to \infty)$, from Example 3.11, we infer that T is similar to the adjoint of multiplication operator M_z on $(\mathcal{H}_1, K_1)$. Thus, T is weakly homogeneous. ∎

EXAMPLE 5.10. Let $T \in \mathcal{B}_1^1(\mathbb{D})$ and $T \sim_{\mathrm{u}} (M_z^*, \mathcal{H}_K)$. Suppose that $K(z,w) = K_1(z,w) + 2 + z\overline{w} + z\overline{w}\mathrm{e}^{z\overline{w}} - \mathrm{e}^{z\overline{w}} + \frac{1}{z\overline{w}}(\mathrm{e}^{z\overline{w}} - 1)$. Then T is a non-contractive weakly homogeneous operator.

Proof. Let $\lambda_0 = 2, \lambda_1 = \frac{3}{2}, \lambda_k = \frac{k^2}{(k+1)!}, k \geqslant 2$. Then $K(z,w) = K_1(z,w) + \sum\limits_{k=0}^{\infty} \lambda_k z^k \overline{w}^k$. When $k > 1$, we have $1 + \lambda_k > 1 + \lambda_{k+1}$. It follows that T is non-contractive. Note that $\lambda_k \geqslant \lambda_{k+1}$ and $\lambda_k \to 0\,(k \to \infty)$, from Example 3.11, we infer that T is similar to the adjoint of multiplication operator M_z on $(\mathcal{H}_1, K_1)$. Thus, T is weakly homogeneous. ∎

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